Soluzioni II

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Esercizio 1. (a) Sia $\mathbf{y} = y_1, ...y_n$. Scriviamo la funzione di verosimiglianza

$$L(\theta|\mathbf{y}) = \prod_{i=1}^{n} P(Y_i = y_i) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!} = \frac{1}{\prod_{i=1}^{n} y_i!} e^{-n\theta}\theta^{\sum y_i}.$$

Quindi $L(\theta|\mathbf{y}) = h(\mathbf{y})g(T_1(\mathbf{y}), \theta)$ dove

$$h(\mathbf{y}) = \frac{1}{\prod_{i=1}^{n} y_i!} \qquad \text{e} \qquad g(T_1(\mathbf{y}), \theta) = e^{-n\theta}\theta^{T_1(\mathbf{y})}.$$

(b) Posto $T_2(\mathbf{y}) = (T_2^{(1)}(\mathbf{y}), T_2^{(2)}(\mathbf{y}))$ dove $T_2^{(1)}(\mathbf{y}) = \sum_{i=1}^{n} y_i$ e $T_2^{(2)}(\mathbf{y}) = \sum_{i=1}^{n} y_i^2$, si ha: $L(\theta|\mathbf{y}) = h(\mathbf{y})g(T_2(\mathbf{y}), \theta)$ con

$$h(\mathbf{y}) = \frac{1}{\prod_{i=1}^{n} y_i!} \qquad \text{e} \qquad g(T_2, \theta) = e^{-n\theta}\theta^{T_2^{(1)}(\mathbf{y})}.$$

(c) Consideriamo due campioni $\mathbf{y}$ e $\mathbf{y}'$ dove $\mathbf{y} = (y_1, \ldots, y_{n-1}, y_n)$ mentre $\mathbf{y}' = (y_1, \ldots, y_{n-1}, y'_n)$ con $y_n \neq y'_n$, ovvero i due campioni differiscono solo per l'ultimo elemento.
Si ha

$$T_3(\mathbf{y}) = T_3(\mathbf{y}')$$

Se T_3 fosse sufficiente allora $L(\theta|\mathbf{y}) = h(\mathbf{y})g(T_3(\mathbf{y}), \theta)$ e $L(\theta|\mathbf{y}') = h(\mathbf{y}')g(T_3(\mathbf{y}'), \theta)$ e quindi $L(\theta|\mathbf{y}) \propto L(\theta|\mathbf{y}')$.
Ma

$$L(\theta|\mathbf{y}) \propto e^{-n\theta}\theta^{T_3(\mathbf{y})+y_n} = e^{-n\theta}\theta^{T_3(\mathbf{y})}\theta^{y_n}$$

e

$$L(\theta|\mathbf{y}') \propto e^{-n\theta}\theta^{T_3(\mathbf{y}')}\theta^{y'_n}.$$

Quindi, poiché θ^{y_n} e $\theta^{y'_n}$ non sono tra loro proporzionali, T_3 non è sufficiente.

Esercizio 2.

$$L(\theta|\mathbf{y}) = \prod_{i=1}^{n} P(Y_i = y_i) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$$

per cui $L(\theta|\mathbf{y}) = h(\mathbf{y})g(T(\mathbf{y}),\theta)$ dove

$$h(\mathbf{y}) = 1 \qquad \text{e} \qquad g(T(\mathbf{y}),\theta) = \theta^{T(\mathbf{y})}(1-\theta)^{n-T(\mathbf{y})}.$$

Esercizio 3.

$$L(\theta|\mathbf{y}) = \prod_{i=1}^{n} \frac{1}{B(\alpha,\beta)} y_i^{\alpha-1}(1-y_i)^{\beta-1} = \left(\frac{1}{B(\alpha,\beta)}\right)^n \prod_{i=1}^{n} y_i^{\alpha-1}(1-y_i)^{\beta-1} =$$

$$= \left(\frac{1}{B(\alpha,\beta)}\right)^n \left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} \left(\prod_{i=1}^{n}(1-y_i)\right)^{\beta-1}.$$

Quindi $L(\theta|\mathbf{y}) = h(\mathbf{y})g(T(\mathbf{y}),\theta)$ dove

$$h(\mathbf{y}) = 1 \qquad \text{e} \qquad g(T(\mathbf{y}),\theta) = \left(\frac{1}{B(\alpha,\beta)}\right)^n T_1(\mathbf{y})^{\alpha-1} T_2(\mathbf{y})^{\beta-1}.$$

Esercizio 4.

$$f(\mathbf{y}) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{\lambda^5}{\Gamma(5)} e^{-\lambda y_i} y_i^{5-1} = \left(\frac{\lambda^5}{\Gamma(5)}\right)^n e^{-\lambda \sum y_i} \prod_{i=1}^{n} y_i^4.$$

Allora $L(\theta|\mathbf{y}) = h(\mathbf{y})g(T(\mathbf{y}),\theta)$ con

$$h(\mathbf{y}) = \frac{\left(\prod_{i=1}^{n} y_i\right)^4}{\Gamma(5)^n} \qquad \text{e} \qquad g(T(\mathbf{y}),\theta) = \lambda^{5n} e^{-\lambda T(\mathbf{y})}.$$

Esercizio 5. (a)

$$\int_{\mathbb{R}} f(y)dy = \int_{\lambda}^{+\infty} \frac{1}{\theta} e^{-\frac{1}{\theta}(y-\lambda)} dy$$

ponendo $t = y - \lambda$ ovvero $y = t + \lambda$ si ottiene

$$\int_{0}^{+\infty} \frac{1}{\theta} e^{-\frac{1}{\theta}t} dt = 1$$

(b) La densità congiunta è

$$f(\mathbf{y}|(\lambda,\theta)) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-(y_i-\lambda)\frac{1}{\theta}} I_{[\lambda,+\infty)}(y_i) = \frac{1}{\theta^n} \exp\left\{-(\sum_{i=1}^{n} y_i - n\lambda)\frac{1}{\theta}\right\} I_{[\lambda,+\infty)}(y_{(1)})$$

dove $y_{(1)} = \min\{y_1, ..., y_n\}$.
Quindi la funzione di verosimiglianza è

$$L((\lambda,\theta)|\mathbf{y}) = \frac{1}{\theta^n} \exp\left\{-(\sum_{i=1}^{n} y_i - n\lambda)\frac{1}{\theta}\right\} I_{(-\infty,y_{(1)}]}(\lambda)$$

Per cui la statistica sufficiente è $T(y) = (y_{(1)}, \sum_{i=1}^{n} y_i)$.

Esercizio 6. Si ha che la densità congiunta di $\mathbf{y} = y_1, \ldots, y_n$ è data da

$$f(\mathbf{y}|\theta) = \frac{1}{(e^{-\theta} - e^{-1})^n} \exp\left\{-\sum_{i=1}^{n} y_i\right\} I_{(\theta,1)}(y_{(1)})$$

Quindi

$$L(\theta|\mathbf{y}) = \frac{1}{(e^{-\theta} - e^{-1})^n} \exp\left\{-\sum_{i=1}^{n} y_i\right\} I_{(-\infty,y_{(1)})}(\theta)$$

da cui segue che $T(y) = y_{(1)}$.

Esercizio 7. La densità congiunta del campione è data da

$$f(\mathbf{y}|\theta) = \frac{1}{(2\theta)^n} \exp\left\{-\frac{\sum_{i=1}^{n} |y_i|}{\theta}\right\}$$

quindi $T(y) = \sum_{i=1}^{n} |y_i|$.