Am1c – Soluzioni Tutorato IX

Integrali IV

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Esercizio 1 Tramite le sostituzioni suggerite si ottiene:

(1) $x = a\sin t \Rightarrow t = \arcsin\frac{x}{a} \Rightarrow dx = a\cos t dt$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{a^2-a^2\sin^2 t}}a\cos t dt = \int dt = t+k = \arcsin\frac{x}{a}+k$$

(2) $x = a\sinh t \Rightarrow t = \sinh^{-1}\frac{x}{a} \Rightarrow dx = a\cosh t dt$

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = \int \frac{1}{\sqrt{a^2+a^2\sinh^2 t}}a\cosh t dt = \int dt = t+k = \sinh^{-1}\frac{x}{a}+k = \ln\left(x+\sqrt{x^2+a^2}\right)+k$$

(3) $x = a\cosh t \Rightarrow t = \cosh^{-1}\frac{x}{a} \Rightarrow dx = a\sinh t dt$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{a^2\cosh^2 t-a^2}}a\sinh t dt = \int dt = t+k = \cosh^{-1}\frac{x}{a}+k = \ln\left|x+\sqrt{x^2-a^2}\right|+k$$

(4) $x = a\sin t \Rightarrow t = \arcsin\frac{x}{a} \Rightarrow dx = a\cos t dt$

$$\int \sqrt{a^2-x^2}dx = \int a\cos t\sqrt{a^2-a^2\sin^2 t}dt = a^2\int \cos^2 t dt = \frac{a^2}{2}(t+\sin t\cos t)+k =$$
$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{a^2}\right)+k$$

(5) $x = a\sinh t \Rightarrow t = \sinh^{-1}\frac{x}{a} \Rightarrow dx = a\cosh t dt$

$$\int \sqrt{a^2+x^2}dx = \int a\cosh t\sqrt{a^2+a^2\sinh^2 t}dt = a^2\int \cosh^2 t dt$$

Si ricava facilmente, integrando per parti, che $\int \cosh^2 t dt = \frac{1}{2}(\cosh t\sinh t+t)+k$ e quindi

$$\int \sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\left(\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{x^2+a^2}}{a^2}\right) + k = \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + \frac{x\sqrt{x^2+a^2}}{2} + k$$

(6) $x = a\cosh t \Rightarrow t = \cosh^{-1}\frac{x}{a} \Rightarrow dx = a\sinh t\,dt$

$$\int \sqrt{x^2-a^2}\,dx = \int a\sinh t\sqrt{a^2\cosh^2 t - a^2}\,dt = a^2\int \sinh^2 t\,dt$$

come prima, integrando per parti, si ha $\int \sinh^2 t\,dt = \frac{1}{2}(\cosh t\sinh t - t) + k$ e quindi

$$\int \sqrt{x^2-a^2}\,dx = \frac{a^2}{2}\left(\frac{x}{a^2}\sqrt{x^2-a^2} - \cosh^{-1}\frac{x}{a}\right) + k = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + k$$

Esercizio 2 Calcolare i seguenti integrali irrazionali:

(1) Posto $x = t^{12} \Rightarrow t = \sqrt[12]{x} \Rightarrow dx = 12t^{11}dt$ si ha

$$\int \frac{1-\sqrt[3]{x}}{\sqrt{x}+\sqrt[4]{x}}\,dx = \int \frac{1-t^4}{t^6+t^3}12t^{11}dt =$$
$$= -\frac{6}{5}\sqrt[6]{x^5} + \frac{12}{7}\sqrt[12]{x^7} + 2\sqrt{x} - 3\sqrt[3]{x} - 4\sqrt[4]{x} + 12\sqrt[12]{x} + 6\ln\left(\sqrt[6]{x} - \sqrt[12]{x} + 1\right) - 4\sqrt{3}\arctan\frac{2\sqrt[12]{x}-1}{\sqrt{3}} + k$$

(lo svolgimento dell'integrale di funzione razionale viene omesso)

(2) Posto $x+1 = t^6 \Rightarrow t = \sqrt[6]{x+1} \Rightarrow dx = 6t^5dt$ si ha

$$\int \frac{1-\sqrt[3]{x+1}}{\sqrt{x+1}+\sqrt[3]{x+1}}\,dx = \int \frac{1-t^2}{t^3+t^2}6t^5dt = \frac{3}{2}\sqrt[3]{(x+1)^2} - \frac{6}{5}\sqrt[6]{(x+1)^5} + k$$

(3) $$\int \frac{1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{2}{\sqrt{(2x-3)^2-1}}\,dx = \int \frac{1}{\sqrt{z^2-1}}\,dz = \ln\left|z+\sqrt{z^2-1}\right| + k =$$
$$= \ln\left|2x-3+\sqrt{(2x-3)^2-1}\right| + k$$

(4) $$\int \frac{x}{\sqrt{-x^2+x+2}}\,dx = -\frac{1}{2}\left(\int \frac{-2x+1}{\sqrt{-x^2+x+2}}\,dx - \int \frac{1}{\sqrt{-x^2+x+2}}\,dx\right) =$$
$$= -\sqrt{-x^2+x+2} + \int \frac{1}{\sqrt{9-(2x-1)^2}}\,dx = -\sqrt{-x^2+x+2} + \frac{1}{2}\arcsin\frac{2x-1}{3} + k$$

(5) $$\int \sqrt{x^2+4x+13}\,dx = \int \sqrt{(x+2)^2+9}\,dx = \frac{1}{2}\left[9\ln\left(x+2+\sqrt{x^2+4x+13}\right) + (x+2)\sqrt{x^2+4x+13}\right] + k$$

(6) $\int\sqrt{-x^2-x+1}dx=\frac{1}{2}\int\sqrt{5-(2x+1)^2}dx=\frac{1}{8}\left[5\arcsin\frac{2x+1}{\sqrt{5}}+(2x+1)\sqrt{-x^2-x+1}\right]+k$

(7) $\int\frac{x+3}{\sqrt{x^2+2x+10}}dx=\frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+10}}dx+2\int\frac{1}{\sqrt{x^2+2x+10}}dx=$

$$=\sqrt{x^2+2x+10}+2\int\frac{1}{\sqrt{(x+1)^2+9}}dx=\sqrt{x^2+2x+10}+2\ln\left(\frac{x+1}{3}+\sqrt{\left(\frac{x+1}{3}\right)^2+1}\right)+k$$

(8) $\int\frac{x^3+x}{\sqrt{-x^4+3x^2-2}}dx=\frac{1}{2}\int\frac{2x(x^2+1)}{\sqrt{-x^4+3x^2-2}}dx$

ora posto $t=x^2\Rightarrow dt=2xdx$ si ottiene

$$\frac{1}{2}\int\frac{2x(x^2+1)}{\sqrt{-x^4+3x^2-2}}dx=\frac{1}{2}\int\frac{t+1}{\sqrt{-t^2+3t-2}}dt=-\frac{1}{4}\int\frac{-2t+3}{\sqrt{-t^2+3t-2}}dt+\frac{5}{4}\int\frac{1}{\sqrt{-t^2+3t-2}}dt=$$

$$=-\frac{1}{2}\sqrt{-t^2+3t-2}+\frac{5}{4}\int\frac{2}{\sqrt{1-(2t-3)^2}}dt=$$

$$=-\frac{1}{2}\sqrt{-x^4+3x^2-2}+\frac{5}{4}\arcsin\left(2x^2-3\right)+k$$