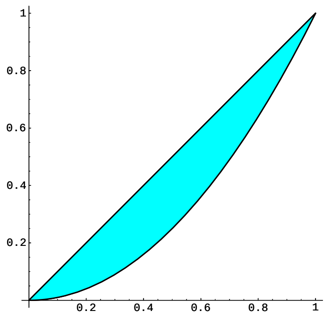
Università di Studi Roma Tre - Corso di Laurea in Matematica

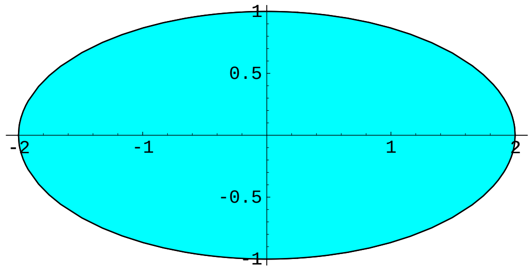
AM3 Soluzioni Tutorato 4

A.A. 2007-2008
Docente: Prof. P. Esposito
Tutori: G. Mancini, D. Piras
Tutorato 4 del 18 Marzo 2008

Esercizio 1 $\int_A y - x^2 dxdydz = \int_0^1 dx \int_{x^2}^{x} dy\, y - x^2 = \int_0^1 dx\, \frac{1}{2}y^2 - x^2 y\Big|_{x^2}^{x} =$
$\int_0^1 \frac{1}{2}x^2 - \frac{1}{2}x^4 - x^3 + x^4 dx = \int_0^1 \frac{1}{2}x^2 + \frac{1}{2}x^4 - x^3 dx = \frac{1}{6} + \frac{1}{10} - \frac{1}{4} = \frac{1}{60}$



Esercizio 2 Consideriamo una ellisse di equazione $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$. Notiamo che l'interno dell' ellisse può essere scritto come un insieme normale nelle y
$E = \{(x,y) \in \mathbb{R}^2 | -a \le x \le a,\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}\}$ dunque l'area



dell' ellisse sarà

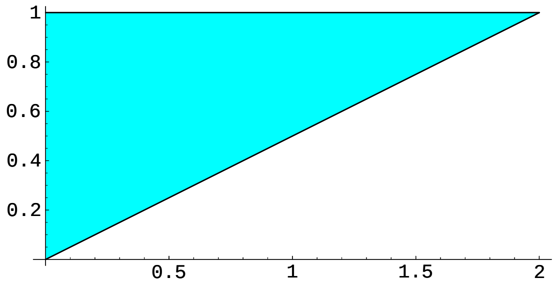
$$\int_E 1 dxdy = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} 1 = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}} dx = 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}} dx =$$

$$\text{(poniamo } x = a\sin t \quad dx = a\cos x dx) \ = 4ab\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t dt =$$

$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2 t dt = 4ab\frac{\pi}{4} = \pi ab$$

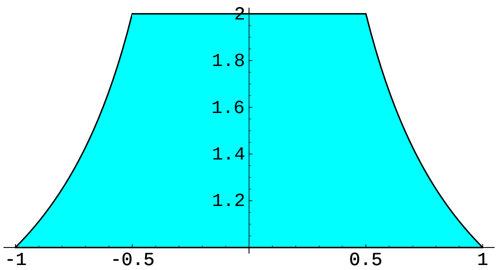
Esercizio 3 $\int_T x^2 e^{y^2} dxdy$.

T è un insieme normale in x infatti $T = \{(x,y) \in \mathbb{R}^2 | 0 \le y \le 1,\ 0 \le x \le 2y\}$



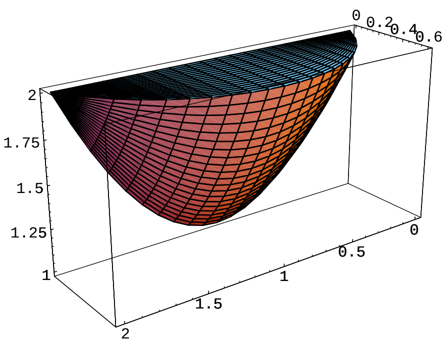
dunque $\int_T x^2 e^{y^2} dxdy = \int_0^1 dy \int_0^{2y} dx x^2 e^{y^2} = \frac{1}{3}\int_0^1 dy\, x^3 e^{y^2}\Big|_0^{2y} =$
$\frac{8}{3}\int_0^1 y^3 e^{y^2} dy = \frac{4}{3}\int_0^1 y^2 \frac{d}{dy} e^{y^2} = \frac{4}{3}\left(y^2 e^{y^2}\Big|_0^1 - \int_0^1 2y e^{y^2} dy \right) = \frac{4}{3}(e - e + 1)$
$= \frac{4}{3}$

Esercizio 4 Calcoliamo $\int_A \cos xy dxdy$ dove $A = \{(x,y) \in \mathbb{R}^2 | 1 \le y \le 2,\ -\frac{1}{y} \le x \le \frac{1}{y}\}$



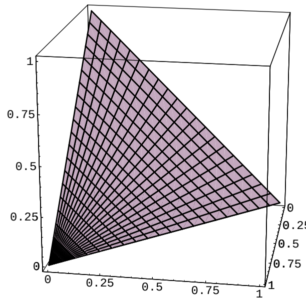
$\int_A \cos xy dxdy = \int_1^2 dy \int_{-\frac{1}{y}}^{\frac{1}{y}} dx \cos xy = \int_1^2 dy\, \frac{\sin xy}{y}\Big|_{-\frac{1}{y}}^{\frac{1}{y}} = 2\int_1^2 \frac{\sin 1}{y} dy =$
$= 2\sin 1 \log y|_1^2 = 2 \sin 1 \log 2$

Esercizio 5 Notiamo che $x^2 + y^2 - 2x + y + 2 \le 6 - 2x + y \Longrightarrow x^2 + y^2 \le 4$ dunque
$S = \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 - 2x + y + 2 \le z \le 6 - 2x + y,\ y \ge 0, x^2 + y^2 \le 4\}$
$= \{(x,y,z) \in \mathbb{R}^3 |\ |x| \le 2,\ 0 \le y \le \sqrt{4 - x^2},\ x^2 + y^2 - 2x + y + 2 \le z \le 6 - 2x + y\}$



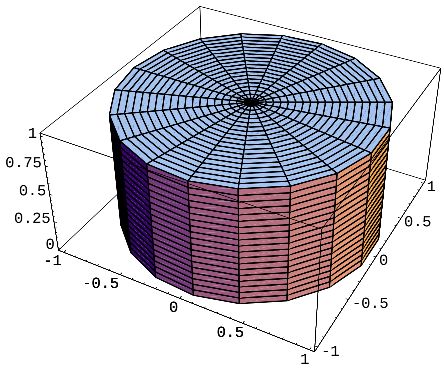
$$Vol(S) = \int_S 1 dxdydz = \int_{-2}^{2} dx \int_0^{\sqrt{4-x^2}} dy \int_{x^2+y^2-2x+y+2}^{6-2x+y} dz =$$
$$= \int_{-2}^{2} dx \int_0^{\sqrt{4-x^2}} dy 4 - x^2 - y^2 = \int_{-2}^{2} dx (4-x^2)^{\frac{3}{2}} - \frac{1}{3}(4-x^2)^{\frac{3}{2}} =$$
$$= \frac{2}{3}\int_{-2}^{2} (4-x^2)^{\frac{3}{2}} dx = \frac{4}{3}\int_0^2 (4-x^2)^{\frac{3}{2}} dx = \ (x = 2\sin t,\ dx = 2\cos t dt)$$
$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} (4 - 4\cos^2 t)^{\frac{3}{2}}\ 2\cos t dt = \frac{4\ 16}{3}\int_0^{\frac{\pi}{2}} \cos^4 t dt = 4\frac{16}{3}\frac{3}{16}\pi = 4\pi$$

Esercizio 6 $\int_D x + z\ dxdydz$
$D = \{(x, y, z) \in \mathbb{R}^3 | 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}$



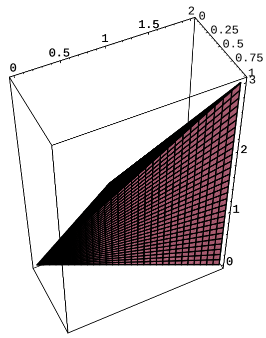
$$\int_D x + z\ dxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz\ x + z =$$
$$\int_0^1 dx \int_0^{1-x} dy\ xz + \frac{1}{2}z^2\Big|_0^{1-x-y} = \int_0^1 dx \int_0^{1-x} x(1-x-y) + \frac{1}{2}(1-x-y)^2$$
$$\int_0^1 dx -\frac{1}{2}x(1-x-y)^2 - \frac{1}{6}(1-x-y)^3\Big|_0^{1-x} = \int_0^2 \frac{1}{2}x(1-x)^2 + \frac{1}{6}(1-x)^3$$
$$= \int_0^1 \frac{1}{2}x^3 - x^2 + \frac{1}{2}x - \frac{1}{6}(1-x)^3 dx = \frac{1}{8} - \frac{1}{3} + \frac{1}{4} + \frac{1}{24} = \frac{1}{12}$$

Esercizio 7 $\int_C z\sqrt{1-y^2} dxdydz$
$C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \le 1,\ 0 \le z \le 1\}$
$C = \{(x, y, z) \in \mathbb{R}^3 || y| \le 1,\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2},\ 0 \le z \le 1\}$



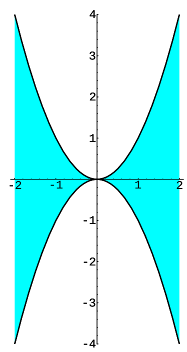
$$\int_C z\sqrt{1-y^2}dxdydz = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_0^1 dzz\sqrt{1-y^2} =$$
$$= \frac{1}{2}\int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-y^2}dx = \int_{-1}^{1} 1-y^2dy = 2\int_0^1 1-y^2dy =$$
$$= 2\left(1-\frac{1}{3}\right) = \frac{4}{3}$$

Esercizio 8 $\int_A x+y+z \; dxdydz$
$A = \{(x,y,z) \in \mathbb{R}^2 | \; 0 \le x \le 1, \; 2x \le y \le x+1, \; 0 \le z \le x+y\}$



$$\int_A x+y+z \; dxdydz = \int_0^1 dx \int_{2x}^{x+1} dy \int_0^{x+y} dz \; x+y+z =$$
$$= \int_0^1 dx \int_{2x}^{x+1} dy(x+y)^2 + \frac{1}{2}(x+y)^2 = \frac{3}{2}\int_0^1 dx \int_{2x}^{x+1} dy(x+y)^2 =$$
$$\frac{3}{2}\int_0^1 dx \; \frac{1}{3}(x+y)^3\Big|_{2x}^{x+1} = \frac{1}{2}\int_0^1 (2x+1)^3 - 27x^3dx = \frac{1}{16}(2x+1)^4 - \frac{27}{8}x^4\Big|_0^1$$
$$= \frac{81}{16} - \frac{1}{16} - \frac{27}{8} = \frac{26}{16} = \frac{13}{8}$$

Esercizio 9 $\int_D \frac{\sin x \cos x^2}{1-y^2+y^4} dxdy \;\; D = \{(x,y) \in \mathbb{R}^2 | x \in [-\pi,\pi], \; -x^2 \le y \le x^2\}$



Basta notare che D è un insieme simmetrico rispetto all'asse delle y e che $\frac{\sin x \cos x^2}{1-y^2+y^4}$ è una funzione dispari in x dunque
$$\int_D \frac{\sin x \cos x^2}{1-y^2+y^4} dxdy = 0$$

Esercizio 10 $\displaystyle\int_A e^{2x+z}\frac{y}{(1+y^2)^2}dxdydz$

$A=\{(x,y,z)\in\mathbb{R}^3|\ 0\le x,\ z\ge 0,\ 0\le y\le e^{(x+z)},\ x+z\le 1\ \}$

$A=\{(x,y,z)\in\mathbb{R}^3|\ 0\le x\le 1,\ 0\le z\le 1-x,\ 0\le y\le e^{(x+z)}\}$

$$\int_A e^{2x+z}\frac{y}{(1+y^2)^2}dxdydz=\int_0^1 dx\int_0^{1-x}dz\int_0^{e^{x+z}}e^{2x+z}\frac{y}{(1+y^2)^2}dy=$$

$$-\frac{1}{2}\int_0^1 dx\int_0^{1-x}dz\ e^{2x+z}\frac{1}{1+y^2}\Big|_0^{e^{x+z}}=\frac{1}{2}\int_0^1 dx\int_0^{1-x}dze^{2x+z}-\frac{e^{2x+z}}{1+(e^{x+z})^2}$$

$$=\frac{1}{2}\int_0^1 dx\int_0^{1-x}dze^{2x+z}-\frac{1}{2}\int_0^1 dx\int_0^{1-x}dze^x\frac{e^{x+z}}{1+(e^{x+z})^2}=\frac{1}{2}\int_0^1 dx\ e^{2x+z}\Big|_0^{1-x}$$

$$-\frac{1}{2}\int_0^1 e^x\arctan e^{x+z}\Big|_0^{1-x}=\frac{1}{2}\int_0^1(e^{x+1}-e^{2x})dx-\frac{1}{2}\int_0^1 dxe^x\arctan e+$$

$$+\frac{1}{2}\int_0^1 e^x\arctan e^x dx=\frac{1}{2}e^2-\frac{1}{2}e-\frac{1}{4}e^2+\frac{1}{4}-\frac{1}{2}e\arctan e+\frac{1}{2}\arctan e$$

$$+\frac{1}{2}\int_0^1 e^x\arctan e^x dx.$$

Ma $\displaystyle\int_0^1 e^x\arctan e^x dx=\int_1^e\arctan t dt=t\arctan t|_1^e-\int_1^e\frac{t}{1+t^2}dt=$

$$=e\arctan e-\frac{\pi}{4}-\frac{1}{2}\log(1+e^2)+\frac{1}{2}\log 2=e\arctan e-\frac{\pi}{4}+\log\sqrt{\frac{2}{1+e^2}}$$

Quindi

$$\int_A e^{2x+z}\frac{y}{(1+y^2)^2}dxdydz=\frac{1}{4}e^2-\frac{1}{2}e+\frac{1}{4}+\frac{1}{2}\arctan e-\frac{\pi}{8}+\frac{1}{2}\log\sqrt{\frac{2}{1+e^2}}$$

Esercizio 11 Sia $A=\{a_j\}_{j=1}^{\infty}$ un insieme numerabile. $\forall\varepsilon>0$ consideriamo la famiglia di cubi $\{Q_j\}_{j=1}^{\infty}$ dove Q_j è un cubo di centro a_j e volume $\frac{\varepsilon}{2^j}$ (ad esempio $Q_j=\Pi_{i=1}^N[a_{ji}-\frac{1}{2}(\frac{\varepsilon}{2^j})^{\frac{1}{n}},a_{ji}+\frac{1}{2}(\frac{\varepsilon}{2^j})^{\frac{1}{n}}])$. Per costruzione $A\subseteq\bigcup_{j=1}^{\infty}Q_j$ e $\displaystyle\sum_{j=1}^{n}|Q_j|=\sum_{j=1}^{\infty}\frac{\varepsilon}{2^j}=\varepsilon$. Dunque A ha misura nulla