Università di Studi Roma Tre - Corso di Laurea in Matematica

AM3 Soluzioni Tutorato 6

A.A. 2007-2008
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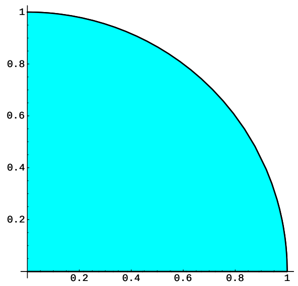
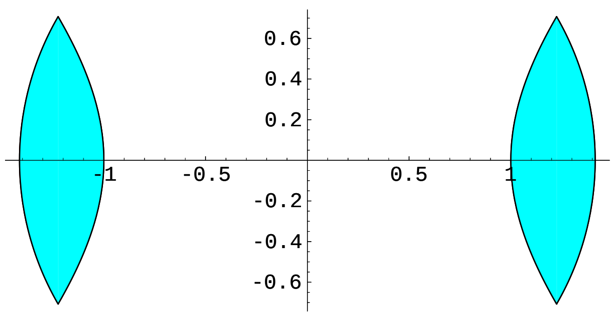
Esercizio 1 $\int_A x^2 e^{x^2+y^2} dxdy \qquad A = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0,\ x^2 + y^2 < 1\}$



Passiamo in cordinate polari $(x,y) = \Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$.
$|\det J\Phi| = \rho \quad \Phi^{-1}(A) = \{(\rho,\theta) \mid 0 < \rho < 1,\ 0 < \theta < \frac{\pi}{2}\}$

$$\int_A x^2 e^{x^2+y^2} dxdy = \int_{\Phi^{-1}(A)} \rho^2 \cos^2\theta e^{\rho^2} |\det J\Phi| d\rho d\theta = \int_0^1 d\rho \int_0^{\frac{\pi}{2}} d\theta \rho^3 \cos^2\theta e^{\rho^2}$$
$$= \frac{\pi}{4}\int_0^1 \rho^3 e^{\rho^2} d\rho = \frac{\pi}{8}\int_0^1 \rho^2 \left(2\rho e^{\rho^2}\right) d\rho = \frac{\pi}{8}\left(\rho^2 e^{\rho^2}\Big|_0^1 - 2\int_0^1 \rho e^{\rho^2} d\rho\right) =$$
$$= \frac{\pi}{8}\left(e - e^{\rho^2}\Big|_0^1\right) = \frac{\pi}{8}(e - e + 1) = \frac{\pi}{8}$$

Esercizio 2 $f(x,y) = |xy|(x^4 - y^4)\log(x^2 + y^2)$
$A = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2,\ 1 < x^2 - y^2 < 2\}$



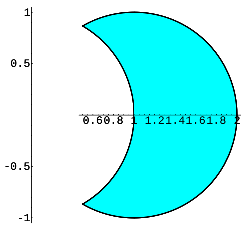
$$\int_A f(x,y)dxdy = 4\int_{A^+} f(x,y)dxdy$$
$A^+ = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2,\ 1 < x^2 - y^2 < 2,\ x > 0\ , y > 0\}$
Poniamo $(u,v) = \Phi(x,y) = (x^2 + y^2, x^2 - y^2)$

$\Phi(A) = \{(u,v) \mid 1 \le u \le 2,\ 1 \le v \le 2\}$

$J\Phi(x,y) = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix} \qquad |\det J\Phi| = 8|xy| \qquad |\det \Phi^{-1}| = \frac{1}{8|xy|}$

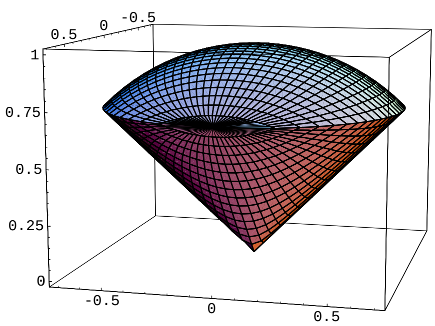
$$4\int_{A^+} f(x,y)dxdy = 4\int_{\Phi(A^+)} f(\Phi^{-1}(u,v))|\det \Phi^{-1}|dudv =$$
$$= \frac{1}{2}\int_1^2 du \int_1^2 dv\ uv\log u = \frac{3}{4}\int_1^2 du\ u\log u = \frac{3}{4}\left(\frac{1}{2}u^2\log u\Big|_1^2 - \frac{1}{2}\int_1^2 u\ du\right)$$
$$= \frac{3}{4}\left(2\log 2 - \frac{1}{4}u^2\Big|_1^2\right) = \frac{3}{4}\left(2\log 2 - \frac{3}{4}\right) = \frac{3}{2}\log 2 - \frac{9}{16}$$

Esercizio 3 $f(x,y) = \frac{|y|}{x}$
$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 1,\ (x-1)^2+y^2 < 1\}$



$$\int_A \frac{|y|}{x}\ dx\ dy = 2\int_{A^+} \frac{y}{x}\ dx\ dy \text{ dove}$$
$A^+ = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 > 1,\ (x-1)^2+y^2 < 1,\ y > 0\}$
Passiamo in cordinate polari $(x,y) = \Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$.
$|\det J\Phi| = \rho$
$\Phi^{-1}(A) = \{(\rho,\theta) \mid 0 \le \theta \le \frac{\pi}{3},\ 1 \le \rho \le 2\cos\theta\}$
$$2\int_{A^+} \frac{y}{x}\ dx\ dy = 2\int_{\Phi^{-1}(A^+)} \frac{\sin\theta}{\cos\theta}\rho\ d\rho\ d\theta = 2\int_0^{\frac{\pi}{3}} d\theta \int_1^{2\cos\theta} \rho\frac{\sin\theta}{\cos\theta}\ d\rho =$$
$$= \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta}4\cos^2\theta - \frac{\sin\theta}{\cos\theta}\ d\theta = 2\int_0^{\frac{\pi}{3}} \sin 2\theta\ d\theta + \log(\cos\theta)|_0^{\frac{\pi}{3}} =$$
$$= -\cos 2\theta|_0^{\frac{\pi}{3}} + \log\frac{1}{2} = 1 + \frac{1}{2} - \log 2 = \frac{3}{2} - \log 2$$

Esercizio 4 $\displaystyle\int_T zx^2dxdydz \qquad T = \{x^2+y^2+z^2 \le 1, 0 \le z, x^2+y^2 < z^2\}$



Passiamo in cordinate sferiche
$(x,y,z) = \Phi(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

Sappiamo che $|\det J\Phi(\rho,\theta,\phi)| = \rho^2 \sin\phi$

Inoltre $\Phi^{-1}(T) = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, \ , 0 \le \theta \le 2\pi, \ 0 \le \phi < \frac{\pi}{4}\}$

$$\int_T zx^2 dxdydz = \int_{\Phi^{-1}(T)} \rho\cos\phi\rho^2\cos^2\theta\sin^2\phi |\det J\Phi| \, d\rho \, d\theta \, d\phi =$$

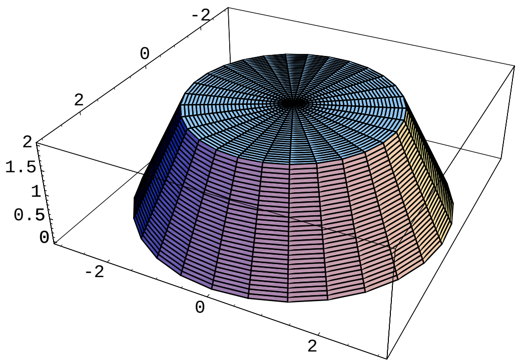
$$= \int_{\Phi^{-1}(T)} \rho^5 \cos^2\theta \sin^3\phi\cos\phi \, d\rho \, d\theta \, d\phi =$$

$$= \int_0^1 d\rho \int_0^{\frac{\pi}{4}} d\phi \int_0^{2\pi} \rho^5\cos^2\theta\sin^3\phi\cos\phi \, d\theta =$$

$$= \pi \int_0^1 d\rho \int_0^{\frac{\pi}{4}} \rho^5 \sin^3\phi\cos\phi \, d\phi = \pi \int_0^1 d\rho\rho^5 \, \frac{1}{4}\sin^4\phi \Big|_0^{\frac{\pi}{4}} =$$

$$= \frac{\pi}{16}\int_0^1 \rho^5 \, d\rho = \frac{\pi}{96}$$

Esercizio 5 .



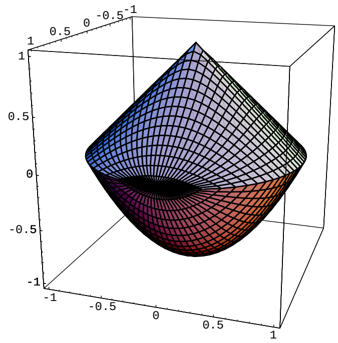
Un tronco di cono di di altezza h e raggi di base $R > r$ si può vedere come il solido di rotazione ottenuto ruotando il sottografico nel piano yz della funzione $y = \frac{r-R}{h}z + R$. Pertanto il suo volume sarà

$$Vol(T) = \pi\int_0^h (\frac{r-R}{h}z + R)^2 dz = \pi\frac{h}{r-R}\int_R^r t^2 dt = \frac{1}{3}\pi\frac{r^3 - R^3}{r-R} =$$

$$\frac{1}{3}\pi(r^2 + rR + R^2)$$

Esercizio 6 $A = \{(x,y,z) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 < z < 1 - \sqrt{x^2+y^2}\}$

A è il solido di rotazione che si ottiene ruotando il sottografico nel piano yz della funzione $f(z) = \begin{cases} 1 - z \text{ se } 0 \le z \le 1 \\ \sqrt{z+1} \text{ se } -1 \le z \le 0 \end{cases}$

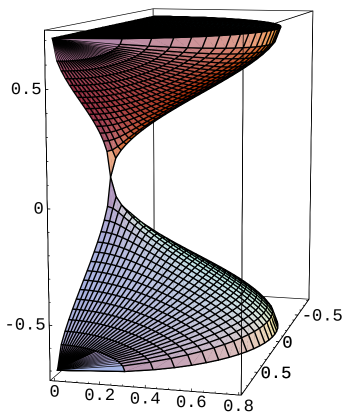


quindi

$$\mathrm{Vol}(A) = \pi \int_{-1}^{1} f(z)^2\, dz = \pi \int_{-1}^{0} (z+1)\, dz + \pi \int_{0}^{1} (1-z)^2\, dz$$
$$= \pi\, \frac{1}{2}(z+1)^2\Big|_{-1}^{0} - \pi \frac{1}{3}(1-z)^3\Big|_{0}^{1} = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5}{6}\pi$$

Esercizio 7 $\displaystyle\int_S x^2(\sin z + \cos z) dxdydz$

$S = \{(x,y,z) \mid x^2+y^2 \le \sin^4 z,\ y \ge 0,\ -\frac{\pi}{2} \le z \le \frac{\pi}{2}\}$



Passiamo in cordinate cilindriche $(x,y,z) = \Phi(\rho,\theta,z) = (\rho\cos\theta, \rho\sin\theta, z)$

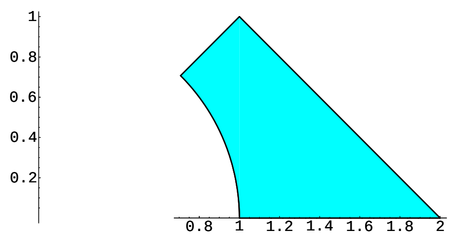
$|\det J\Phi| = \rho$

$\Phi^{-1}(S) = \{(\rho,\theta,z) \mid 0 \le \theta \le \pi,\ z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ 0 \le \rho \le \sin^2 z\}$

$$\int_S x^2(\sin z + \cos z) dxdydz = \int_{\Phi^{-1}(S)} \rho^2 \cos^2\theta(\sin z + \cos z)\rho\, d\rho d\theta dz =$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \int_0^{\pi} d\theta \int_0^{\sin^2 z} (\rho^3 \cos^2\theta(\sin z + \cos z))d\rho =$$
$$\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \int_0^{\pi} d\theta \cos^2\theta(\sin z + \cos z)\sin^8 z dz = \frac{\pi}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^9 z + \cos z \sin^8 z\, dz$$
$$= \frac{\pi}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos z \sin^8 z\, dz = \frac{\pi}{8}\frac{\sin^9 z}{9}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{36}$$

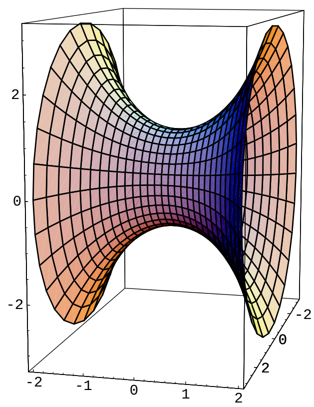
Esercizio 8 $\displaystyle\int_D \frac{y}{x^2+y^2} dxdy$

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 1,\ x+y < 2,\ y \le x,\ x > 0,\ y > 0\}$



Passiamo in cordinate polari $(x,y)=\Phi(\rho,\theta)=(\rho\cos\theta,\rho\sin\theta)$.
$|\det J\Phi|=\rho$
$\Phi^{-1}(A)=\{(\rho,\theta)\mid 0<\theta\le \frac{\pi}{4},\ 1<\rho<\frac{2}{cos\theta+\sin\theta}\}$
$\int_D \frac{y}{x^2+y^2}dxdy=\int_{\Phi^{-1}(A)}\frac{\rho\sin\theta}{\rho^2}\rho\,d\rho\,d\theta=\int_{\Phi^{-1}(A)}\sin\theta\,d\rho\,d\theta=$
$\int_0^{\frac{\pi}{4}}d\theta\int_1^{\frac{2}{cos\theta+\sin\theta}}\sin\theta\,d\rho=\int_0^{\frac{\pi}{4}}d\theta\frac{2\sin\theta}{cos\theta+\sin\theta}-\sin\theta=\int_0^{\frac{\pi}{4}}d\theta\frac{2\sin\theta}{cos\theta+\sin\theta}-$
$\int_0^{\frac{\pi}{4}}d\theta\sin\theta$
$\int_0^{\frac{\pi}{4}}d\theta\sin\theta=-\cos\theta|_0^{\frac{\pi}{4}}=1-\frac{1}{\sqrt{2}}$
$\int_0^{\frac{\pi}{4}}d\theta\frac{\sin\theta}{cos\theta+\sin\theta}=\int_0^{\frac{\pi}{4}}d\theta\frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+1}=\int_0^{\frac{\pi}{4}}\frac{\tan\theta}{\tan\theta+1}\,d\theta=\int_0^1\frac{t}{(1+t)(1+t^2)}\,dt$
$\frac{t}{(1+t)(1+t^2)}=-\frac{1}{2}\frac{1}{1+t}+\frac{1}{2}\frac{t+1}{t^2+1}$ quindi
$\int_0^1\frac{1}{(1+t)(1+t^2)}\,dt=-\frac{1}{2}\int_0^1\frac{1}{1+t}\,dt+\frac{1}{2}\int_0^1\frac{t}{t^2+1}\,dt+\frac{1}{2}\int_0^1\frac{1}{1+t^2}=$
$=-\frac{1}{2}\log(1+t)+\frac{1}{4}\log(1+t^2)+\frac{1}{2}\arctan t\Big|_0^1=-\frac{1}{4}\log 2+\frac{\pi}{8}$. Dunque
$\int_D\frac{y}{x^2+y^2}dxdy=2\int_0^{\frac{\pi}{4}}d\theta\frac{2\sin\theta}{cos\theta+\sin\theta}-\int_0^{\frac{\pi}{4}}d\theta\sin\theta=-\frac{1}{2}\log 2+\frac{\pi}{4}+\frac{1}{\sqrt{2}}-1$

Esercizio 9 $\int_D x^2+z^2\,dxdydz$
$D=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2>1,\ x^2+z^2\le\cosh^2 y, y\in[-2,2]\}$



$D=C-S$ dove $C=\{(x,y,z)\mid x^2+z^2\le\cosh^2 y,\ y\in[-2,2]\}$ e S è la sfera unitaria di centro l'origine
$\int_D x^2+z^2\,dxdydz=\int_C x^2+z^2\,dxdydz-\int_S x^2+z^2\,dxdydz$
Per calcolare l'integrale su S passiamo in cordianate sferiche
$(x,y,z)=\Phi(\rho,\theta,\phi)=(\rho\cos\theta\sin\phi,\rho\sin\theta\sin\phi,\rho\cos\phi)$
Sappiamo che $|\det J\Phi(\rho,\theta,\phi)|=\rho^2\sin\phi$
Inoltre $\Phi^{-1}(S)=\{(\rho,\theta,\phi)\mid 0\le\rho\le1,\ ,0\le\theta\le2\pi,\ 0\le\phi\le\pi\}$
$\int_S x^2+z^2\,dxdydz=\int_0^1 d\rho\int_0^{2\pi}d\theta\int_0^{\pi}(\rho^2\cos^2\theta\sin^2\phi+\rho^2\cos^2\phi)\rho^2\sin\phi\,d\phi$
$=\int_0^1 d\rho\int_0^{2\pi}d\theta\int_0^{\pi}\rho^4\cos^2\theta\sin^3\phi+\rho^4\cos^2\phi\sin\phi\,d\phi=$

$$= \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \rho^4(\cos^2\theta \sin\phi(1-\cos^2\phi) + \cos^2\phi \sin\phi)\, d\phi =$$
$$\int_0^{2\pi} d\theta \int_0^1 d\rho\rho^4(\cos^2\theta\, (\frac{1}{3}\cos^3\phi - \cos\phi)\Big|_0^{\pi} - \frac{1}{3}\cos^3\phi\Big|_0^{\pi}) =$$
$$\int_0^{2\pi} d\theta \int_0^1 d\rho\rho^4(\frac{4}{3}\cos^2\theta + \frac{2}{3}) = \frac{2}{15}\int_0^{2\pi} d\theta 2\cos^2 + 1 = \frac{8}{15}\pi$$
Calcoliamo ora $\int_C x^2 + z^2\; dxdydz$

Passiamo in cordinate cilindriche $(x,y,z) = \Phi(\rho, y, \theta) = (\rho\cos\theta, y, \rho\sin\theta)$

$|\det J\Phi| = \rho$

$\Phi^{-1}(C) = \{(\rho, y, \theta) \mid 0 \le \theta \le 2\pi,\; y \in [-2,2],\; 0 \le \rho \le \cosh y\}$

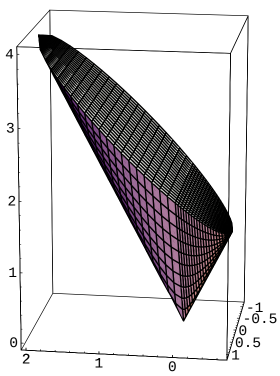
$$\int_C x^2 + z^2\; dxdydz = \int_{-2}^{2} dy \int_0^{2\pi} d\theta \int_0^{\cosh y} \rho^3 d\rho = \frac{1}{4}\int_{-2}^{2} dy \int_0^{2\pi} d\theta \cosh^4 y$$
$$= \frac{\pi}{32}\int_{-2}^{2} dy(e^{4x} + e^{-4x} + 4 + 4e^{2x} + 4e^{-2x} + 2) =$$
$$\frac{\pi}{32}\left(\frac{1}{4}e^{4x} - \frac{1}{4}e^{-4x} + 2e^{2x} - 2e^{-2x} + 6y\right)\Big|_{-2}^{2} = \frac{\pi}{16}(\frac{1}{4}(e^8 - e^{-8}) + 2(e^4 - e^{-4}) + 12)$$
$$= \frac{\pi}{64}(e^8 - e^{-8}) + \frac{\pi}{8}(e^4 - e^{-4}) + \frac{3}{4}\pi$$
Quindi $\int_D x^2 + z^2\; dxdydz = \frac{\pi}{64}(e^8 - e^{-8}) + \frac{\pi}{8}(e^4 - e^{-4}) + \frac{3}{4}\pi - \frac{8}{15}\pi$

Esercizio 10 $\int_T 2z dxdydz\; T = \{2\sqrt{x^2+y^2} \le z \le x+2\}$

Notiamo che $2\sqrt{x^2+y^2} \le x+2 \Leftrightarrow 4(x^2+y^2) \le x^2+4x+4 \Leftrightarrow 3x^2+4y^2-4x \le 4 \Leftrightarrow 3(x^2 - \frac{4}{3}x) + 4y^2 \le 4 \Leftrightarrow 3(x-\frac{2}{3})^2 + 4y^2 \le \frac{16}{3} \Leftrightarrow \frac{(x-\frac{2}{3})^2}{\frac{16}{9}} + \frac{y^2}{\frac{4}{3}} \le 1.$

Dunque T è un insieme normale della forma

$T = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in E,\; 2\sqrt{x^2+y^2} \le z \le x+2)\}$ dove E è l'ellisse di equazione $\frac{(x-\frac{2}{3})^2}{\frac{16}{9}} + \frac{y^2}{\frac{4}{3}} \le 1$



$$\int_T 2z dxdydz = \int_E dxdy \int_{2\sqrt{x^2+y^2}}^{x+2} 2z dz = \int_E (x+2)^2 - 4(x^2+y^2) dxdx$$
Passiamo in cordinate ellittiche $(x,y) = \Phi(\rho,\theta) = (\frac{2}{3} + \frac{4}{3}\rho\cos\theta, \frac{2}{\sqrt{3}}\rho\sin\theta)$

$\Phi^{-1}(E) = \{(\rho,\theta) \mid 0 \le \rho \le 1,\; 0 \le \theta \le 2\pi\}$

$|\det J\Phi| = \frac{8}{3\sqrt{3}}\rho \qquad \int_E (x+2)^2 - 4(x^2+y^2) dxdx =$

$$\int_E 4x + 4 - 3x^2 - 4y^2 dxdy = 3\int_E \frac{16}{9} - (x - \frac{2}{3})^2 - \frac{4}{3}y^2 dxdy =$$

$$= 3\int_{\Phi^{-1}(E)} \left(\frac{16}{9} - \frac{16}{9}\rho^2\cos^2\theta - \frac{16}{9}\rho^2\sin^2\theta\right)\frac{8}{3\sqrt{3}}\rho d\rho d\theta =$$

$$= \frac{128}{9\sqrt{3}}\int_0^1 d\rho \int_0^{2\pi} d\theta \rho - \rho^3 = \frac{256}{9\sqrt{3}}\pi \int_0^1 d\rho\ \rho - \rho^3 = \frac{256}{9\sqrt{3}}\pi\left(\frac{1}{2} - \frac{1}{4}\right) =$$

$$= \frac{64}{9\sqrt{3}}\pi$$