Soluzioni del tutorato di Statistica 1 del 27/02/2009
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Esercizio 1.
Vedi esercizio 1, tutorato1 del 09/03/2006

Esercizio 2.
X, Y hanno distribuzione congiunta data da:

$$f_{X,Y}(x,y) = 2 \cdot 1_{(0,y)}(x) 1_{(0,1)}(y)$$

1.

$$E[XY] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X,Y}(x,y) dx dy = \int_0^1 \int_0^y 2xy dx dy = \int_0^1 y^3 dy = \frac{1}{4}$$

$$f_X(x) = \int_{-\infty}^{+\infty} 2 \cdot 1_{(0,y)}(x) 1_{(0,1)}(y) dy = \int_0^1 2 \cdot 1_{(0,y)}(x) dy = 1_{(0,1)}(x) \int_x^1 2 dy = (2-2x) 1_{(0,1)}(x)$$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^1 x(2-2x) dx = \frac{1}{3}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} 2 \cdot 1_{(0,y)}(x) 1_{(0,1)}(y) dx = 1_{(0,1)}(y) \int_0^y 2 dx = 2y \cdot 1_{(0,1)}(y)$$

$$E[Y] = \int_0^1 2y^2 dy = \frac{2}{3}$$

$$Cov[X,Y] = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{3}\frac{2}{3} = \frac{1}{36}$$

2.

$$f_{Y|X} = \frac{f_{X,Y}}{f_X}$$

$$f_{Y|X} = \frac{2 \cdot 1_{(0,y)}(x) 1_{(0,1)}(y)}{(2-2x) 1_{(0,1)}(x)} = \frac{1_{(x,1)}(y)}{(1-x)}$$

per $x \in (0,1)$

Esercizio 3.

$X \sim Exp(\lambda)$. $f_X(x) = \lambda e^{-\lambda x}$ per $x > 0$.

$$E[e^{tX}] = \int_0^{+\infty} \lambda e^{tx} e^{-\lambda x} dx = \lambda \int_0^{+\infty} e^{-x(\lambda - t)} dx = \frac{\lambda}{\lambda - t}$$

$$E[X] = \frac{d}{dt} \left(\frac{\lambda}{\lambda - t} \right) |_{t=0} = \frac{1}{\lambda} = m'(0).$$

$$E[X^2] = m''(0) = \frac{2}{\lambda^2}.$$

$$Var[X] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Esercizio 4.

$$f_{X,Y}(x,y) = e^{-y}(1-e^{-x}) 1_{(0,y)}(x) 1_{(0,\infty)}(y) + e^{-x}(1-e^{-y}) 1_{(0,x)}(y) 1_{(0,\infty)}(x)$$

1. $f_{X,Y}$ è una densità se $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx dy = 1$
   Allora:
$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-y}(1-e^{-x}) 1_{(0,y)}(x) 1_{(0,+\infty)}(y) dx dy +$$
$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x}(1-e^{-y}) 1_{(0,x)}(y) 1_{(0,+\infty)}(x) dy dx = 1$$

2.
$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = 1_{(0,+\infty)}(x)\left\{\int_x^{+\infty} e^{-y}(1-e^{-x}dy) + \int_0^x e^{-x}(1-e^{-y})dy\right\} =$$
$$= xe^{-x}1_{(0,+\infty)}(x)$$

Per simmetria si ha $f_Y(y) = ye^{-y}1_{(0,+\infty)}(y)$

3.
$$E[Y|X=x] = \int_{-\infty}^{+\infty} y f_{Y|X}(y|X=x)dy$$

$$f_{Y|X}(y|X=x) = \frac{e^{-y}(1-e^{-x})1_{(0,y)}(x)1_{(0,\infty)}(y) + e^{-x}(1-e^{-y})1_{(0,x)}(y)1_{(0,\infty)}(x)}{xe^{-x}1_{(0,\infty)}(x)}$$

$$= \frac{e^{-y}(1-e^{-x})1_{(x,\infty)}(y) + e^{-x}(1-e^{-y})1_{(0,x)}(y)}{xe^{-x}}$$

Allora:
$$E[Y|X=x] = \frac{1}{xe^{-x}} \cdot \left\{\int_x^{+\infty} ye^{-y}(1-e^{-x})dy + \int_0^x ye^{-x}(1-e^{-y})\right\} dy$$

4.
$$P(X \leq 2, Y \leq 2) = 2\int_0^2 \int_0^y e^{-y}(1-e^{-x})dxdy$$

5. $E[X] = \int_0^{+\infty} x^2e^{-x}dx = 2$ Per simmetria $E[Y] = 2$.
$E[X^2] = \int_0^{+\infty} x^3e^{-x}dx = 6$
$Var[X] = Var[Y] = 2$

$$E[XY] = 2\int_0^{+\infty} ye^{-y} \int_0^y x(1-e^{-x})dxdy = 5$$

Allora il coefficiente di correlazione è:
$$\rho(X,Y) = \frac{5-4}{\sqrt{4}} = \frac{1}{2}$$

6. Una possibile scelta è considerare X ed Y indipendenti, quindi fare il prodotto delle marginali.

Esercizio 5.
Vedi teorema 2.3 pag.80