Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di Analisi 3

A.A. 2009-2010 - Docente: Prof. P. Esposito

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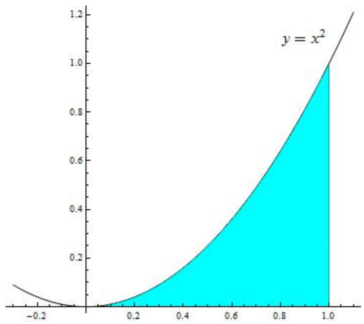
Soluzioni del tutorato numero 8 (5 Maggio 2010)
Integrali

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
http://www.lifedreamers.it/liuck

1. $A = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le x^2\right\}$.

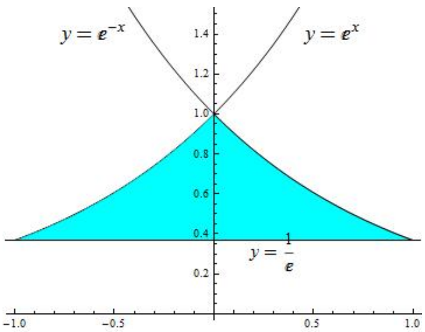


Dato che A è un insieme normale applicando il teorema di Fubini si ottiene che

$$\int_A xye^{x^6}dxdy = \int_0^1 dx \int_0^{x^2} dy \;\; xye^{x^6} = \int_0^1 dx \;\; \frac{1}{2}y^2xe^{x^6}\Big|_0^{x^2} = \frac{1}{2}\int_0^1 dx \; x^5e^{x^6} \overset{(t=x^6)}{=}$$

$$= \frac{1}{12}\int_0^1 e^t dt = \frac{1}{12}e^t\Big|_0^1 = \frac{1}{12}(e-1)$$

2. $B = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{e} \le y \le e^{-|x|}\right\} = \left\{(x,y) \in \mathbb{R}^2 : \; -1 \le x \le 1, \; \frac{1}{e} \le y \le e^{-|x|}\right\}$.



$$\int_B \log y \; dxdy = \int_{-1}^1 dx \int_{\frac{1}{e}}^{e^{-|x|}} dy \;\; \log y = \int_{-1}^1 dx \left(y \log y\Big|_{\frac{1}{e}}^{e^{-|x|}} - \int_{\frac{1}{e}}^{e^{-|x|}} dy \; 1 \right) =$$

$$= \int_{-1}^1 dx - |x|e^{-|x|} + \frac{1}{e} - e^{-|x|} + \frac{1}{e} = \int_{-1}^1 dx \; \frac{2}{e} - |x|e^{-|x|} - e^{-|x|}dx =$$

$$= 2\int_0^1 dx \; \frac{2}{e} - xe^{-x} - e^{-x}dx = \frac{4}{e} - 2\int_0^1 xe^{-x}dx - 2\int_0^1 e^{-x}dx =$$

$$= \frac{4}{e} - 2\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx\right) - 2\int_0^1 e^{-x}dx = \frac{4}{e} + \frac{2}{e} - 4\int_0^1 e^{-x}dx =$$

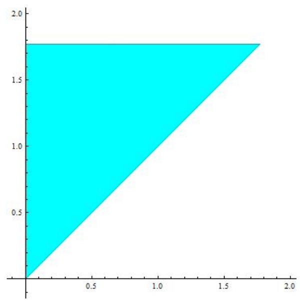
$$= \frac{6}{e} + 4e^{-x}\Big|_0^1 = \frac{10}{e} - 4$$

L'integrale si poteva calcolare anche integrando prima in x e poi in y:
$B = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{e} \leq y \leq 1,\ \log y \leq x \leq -\log y\right\}$

$$\int_B \log y dxdy = \int_{\frac{1}{e}}^1 dy \int_{\log y}^{-\log y} dx \log y = -2\int_{\frac{1}{e}}^1 \log^2 y\ dy = -2\left(y\log^2 y\Big|_{\frac{1}{e}}^1 - 2\int_{\frac{1}{e}}^1 \log y\right) =$$

$$= \frac{2}{e} + 4\int_{\frac{1}{e}}^1 \log y\ dy = \frac{2}{e} + 4\left(y\log y - y\Big|_{\frac{1}{e}}^1\right) = \frac{2}{e} - 4 + \frac{4}{e} + \frac{4}{e} = \frac{10}{e} - 4$$

3. $C = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq \sqrt{\pi}, x \leq y \leq \sqrt{\pi}\} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq y, 0 \leq y \leq \sqrt{\pi}\}$



$$\int_C x^2 \sin\left(y^2\right) dxdy = \int_0^{\sqrt{\pi}} dy \int_0^y dx\ x^2 \sin\left(y^2\right)\ = \int_0^{\sqrt{\pi}} dy\ \ \frac{1}{3}x^3 \sin\left(y^2\right)\Big|_0^y\ =$$

$$= \frac{1}{3}\int_0^{\sqrt{\pi}} dy\ y^3 \sin(y^2) = \frac{1}{6}\int_0^{\pi} t\sin t\ dt = \frac{1}{6}\left(-t\cos t\Big|_0^{\pi} + \int_0^{\pi} \cos t dt\right) = \frac{1}{6}\left(\pi + \sin t\Big|_0^{\pi}\right) = \frac{\pi}{6}$$

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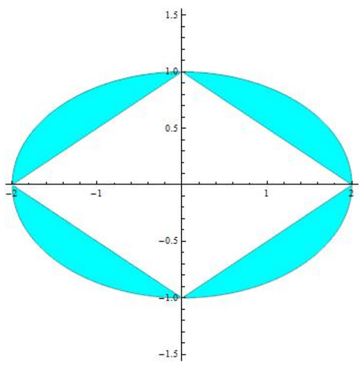
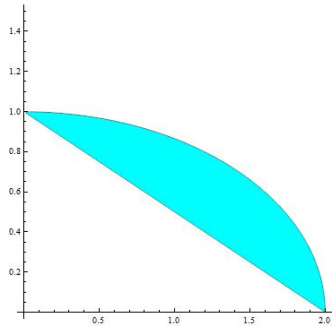
4. Sia $D = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \leq \frac{|x|}{a} + \frac{|y|}{b}\right\}$.
   Chiaramente $Area(D) = 4Area(\tilde{D})$ dove $\tilde{D} = D \cap \{(x,y)\ :\ x \geq 0,\ y \geq 0\}$.





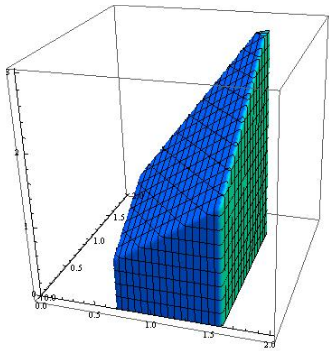
$$Area(\tilde{D}) = \int_{\tilde{D}} 1\ dxdy = \int_0^a dx \int_{b\left(1-\frac{x}{a}\right)}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\ 1 = \int_0^a dx\ b\sqrt{1 - \frac{x^2}{a^2}} - b(1 - \frac{x}{a}) =$$

$$= \int_0^a dx\, b\sqrt{1-\frac{x^2}{a^2}} - ab - \frac{b}{2a}x^2\Big|_0^a = -\frac{1}{2}ab + \int_0^a dx\, b\sqrt{1-\frac{x^2}{a^2}} \overset{x=a\sin t}{=}$$

$$= -\frac{1}{2}ab + ab\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t\, dt = -\frac{1}{2}ab + ab\int_0^{\frac{\pi}{2}} \cos^2 t\, dt = -\frac{1}{2}ab + \frac{\pi}{4}ab$$

Quindi $Area(D) = 4Area(\tilde{D}) = \pi ab - 2ab = (\pi - 2)ab$

5. $E = \{(x,y,z) \in \mathbb{R}^3 : \log 2 \le x \le \log 5, 0 \le y \le x, 0 \le z \le x+y\}$.



$$\int_E \frac{z}{x^2(x+y)\sinh x}dxdydz = \int_{\log 2}^{\log 5} dx \int_0^x dy \int_0^{x+y} dz\, \frac{z}{x^2(x+y)\sinh x} =$$

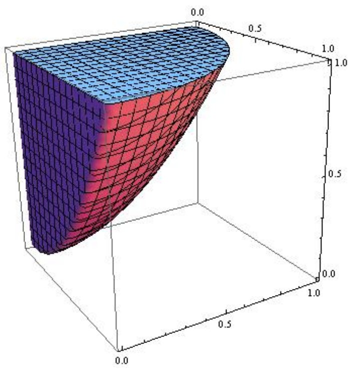
$$= \int_{\log 2}^{\log 5} dx \int_0^x dy\, \frac{z^2}{2x^2(x+y)\sinh x}\Big|_0^{x+y} = \int_{\log 2}^{\log 5} dx \int_0^x dy\, \frac{x+y}{2x^2\sinh x} =$$

$$= \int_{\log 2}^{\log 5} dx\, \frac{xy + \frac{1}{2}y^2}{2x^2\sinh x}\Big|_0^x = \int_{\log 2}^{\log 5} dx\, \frac{x^2 + \frac{1}{2}x^2}{2x^2\sinh x} = \frac{3}{4}\int_{\log 2}^{\log 5} dx\, \frac{1}{\sinh x} = \frac{3}{2}\int_{\log 2}^{\log 5} dx\, \frac{1}{e^x - e^{-x}}$$

$$= \frac{3}{2}\int_{\log 2}^{\log 5} dx\, \frac{e^x}{e^{2x}-1} \overset{(t=e^x)}{=} \int_2^5 \frac{1}{t^2-1}dt = \frac{3}{2}\int_2^5 \frac{1}{2}\frac{1}{t-1} - \frac{1}{2}\frac{1}{t+1}dt =$$

$$= \frac{3}{4}\left(\log(t-1) - \log(t+1)\right)\Big|_2^5 = \frac{3}{4}\log\left(\frac{t-1}{t+1}\right)\Big|_2^5 = \frac{3}{4}\left(\log\frac{2}{3} - \log\frac{1}{3}\right) = \frac{3}{4}\log 2$$

6. Sia $F = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le 1, x^2+y^2 \le z \le 1\}$.



Notiamo che $F = \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \tilde{F}, x^2+y^2 \le z \le 1\right\}$ dove $\tilde{F} = \{(x,y) \,:\, x^2 + y^2 \le 1, x, y \ge 0\}$ quindi per il teorema di Fubini

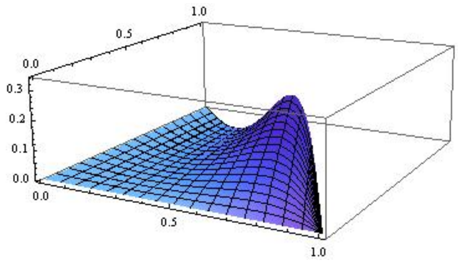
$$\int_F xy\; dxdydz = \int_{\tilde{F}} dxdy \int_{x^2+y^2}^1 dz\;\; xy = \int_{\tilde{F}} dxdy\; xyz\Big|_{x^2+y^2}^1 = \int_{\tilde{F}} dxdy\;\; xy - xy(x^2+y^2) =$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy\;\; xy - x^3y - xy^3 = \int_0^1 dx\; \frac{1}{2}xy^2 - \frac{1}{2}x^3y^2 - \frac{1}{4}xy^4\Big|_0^{\sqrt{1-x^2}} =$$

$$= \int_0^1 \frac{1}{2}x(1-x^2) - \frac{1}{2}x^3(1-x^2) - \frac{1}{4}x(1-x^2)^2 dx = \frac{1}{2}\int_0^1 x - x^3 - x^3 + x^5 - \frac{1}{2}x(1-2x^2-x^4)dx =$$

$$= \frac{1}{2}\int_0^1 \frac{1}{2}x - x^3 + \frac{1}{2}x^5\; dx = \frac{1}{8}x^2 - \frac{1}{8}x^4 + \frac{1}{24}x^6\Big|_0^1 = \frac{1}{8} - \frac{1}{8} + \frac{1}{24} = \frac{1}{24}$$

7. $G = \left\{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1,\; x^5y \le z \le x^5y\, e^{x^2-xy}\right\}$
   Notiamo che $G = \left\{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1,\; 0 \le y \le x,\; x^5y \le z \le x^5y\, e^{x^2-xy}\right\}$



$$Vol(G) = \int_G 1\; dxdydz = \int_0^1 dx \int_0^x dy \int_{x^5y}^{x^5ye^{x^2-xy}} dz = \int_0^1 dx \int_0^x dy\; x^5ye^{x^2-xy} - x^5y =$$

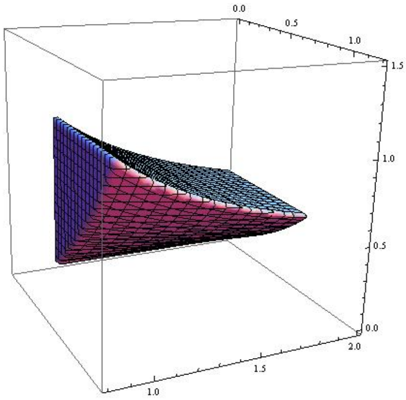
$$= \int_0^1 dx\; x^5e^{x^2} \int_0^x dy\; ye^{-xy} - \int_0^1 dx \int_0^x dy\; x^5y = \int_0^1 dx\; x^5e^{x^2}\left(-\frac{y}{x}e^{-xy}\Big|_0^x + \int_0^x \frac{e^{-xy}}{x}dy\right) +$$

$$- \int_0^1 \frac{1}{2}x^7 dx = \int_0^1 \left(-x^5 - \frac{x^5e^{x^2}}{x^2}e^{-xy}\Big|_0^x\right) dx - \frac{1}{16}x^8\Big|_0^1 = \int_0^1 -x^5 - x^3 + x^3e^{-x^2}\; dx - \frac{1}{16} =$$

$$= -\frac{1}{6}x^6 - \frac{1}{4}x^4\Big|_0^1 + \int_0^x x^3e^{x^2} dx - \frac{1}{16} = -\frac{1}{4} - \frac{1}{6} - \frac{1}{16} + \frac{1}{2}\int_0^1 te^t dt = -\frac{23}{48} + \frac{1}{2}\int_0^1 te^t dt =$$

$$= -\frac{23}{48} + \frac{1}{2}\left(te^t - e^t\Big|_0^1\right) = -\frac{23}{48} + \frac{1}{2} = \frac{1}{48}$$

8. $H = \{(x,y,z) \in \mathbb{R}^3 : 1 \le y \le 2,\; 0 \le x \le \arctan(yz),\; 0 \le yz \le 1\}$.
   $H = \left\{(x,y,z) \in \mathbb{R}^3 : 1 \le y \le 2, 0 \le z \le \frac{1}{y}, 0 \le x \le \arctan(yz)\right\}$



$$\int_H \frac{x\log y}{1+y^2z^2}dxdydz = \int_1^2 dy\int_0^{\frac{1}{y}} dz\int_0^{\arctan(yz)}dx\ \frac{x\log y}{1+y^2z^2} = \int_1^2 dy\int_0^{\frac{1}{y}}dz\frac{1}{2}\frac{\log y\arctan^2(yz)}{1+y^2z^2}$$

$$\overset{(t=\arctan(yz))}{=} \frac{1}{2}\int_1^2 dy\int_0^{\frac{\pi}{4}} dt\frac{t^2\log y}{y} = \frac{1}{2}\int_1^2 dy\ \frac{\log y}{y}\int_0^{\frac{\pi}{4}} t^2dt = \frac{\pi^3}{384}\int_1^2 \frac{\log y}{y}dy =$$

$$= \frac{\pi^3}{768}\log^2 y\Big|_1^2 = \frac{\pi^3}{768}\log^2 2$$