Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di Analisi 3

A.A. 2009-2010 - Docente: Prof. P. Esposito

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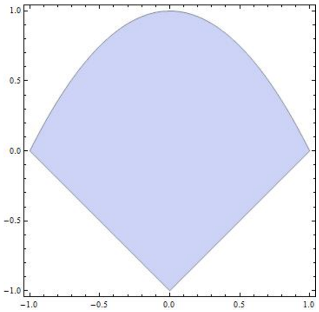
Soluzioni del tutorato numero 9 (10/12 Maggio 2010)
Integrali con Fubini e con cambio di varibile

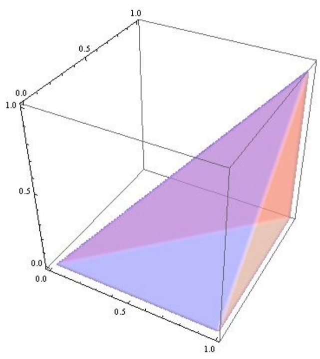
I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
http://www.lifedreamers.it/liuck

1. $A = \left\{(x,y) \in \mathbb{R}^2 : |x| - 1 \leq y \leq 1 - x^2\right\} = \left\{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1, |x| - 1 \leq y \leq 1 - x^2\right\}.$



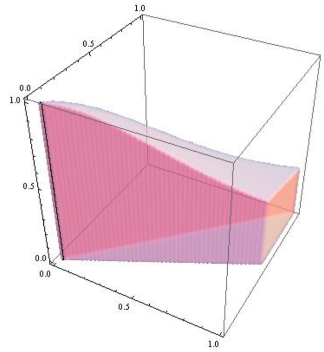
$$\int_A |x| \cos(\pi y) dx dy = \int_{-1}^{1} |x| dx \int_{|x|-1}^{1-x^2} \cos(\pi y) dy = \int_{-1}^{1} |x| dx \left[\frac{\sin(\pi y)}{\pi}\right]_{|x|-1}^{1-x^2} =$$
$$= \frac{1}{\pi}\int_{-1}^{1} |x| dx \left(\sin\left(\pi\left(1-x^2\right)\right) - \sin(\pi(|x|-1))\right) = \frac{2}{\pi}\int_0^1 x\left(\sin\left(\pi\left(1-x^2\right)\right) - \right.$$
$$\left. - \sin(\pi(x-1))\right) dx = \frac{2}{\pi}\int_0^1 x \sin\left(\pi\left(1-x^2\right)\right) dx - \frac{2}{\pi}\int_0^1 x \sin(\pi(x-1)) dx \overset{t=\pi\left(x^2-1\right)}{=}$$
$$= \frac{2}{\pi}\int_{-\pi}^{0} \frac{\sin(-t)}{2\pi} dt - \frac{2}{\pi}\left(\left[\frac{-x\cos(\pi(x-1))}{\pi}\right]_0^1 + \int_0^1 \frac{\cos(\pi(x-1))}{\pi} dx\right) =$$
$$= \frac{2}{\pi}\left[\frac{\cos(-t)}{2\pi}\right]_{-\pi}^{0} - \frac{2}{\pi}\left(-\frac{1}{\pi} + \left[\frac{\sin(\pi(x-1))}{\pi^2}\right]_0^1\right) = \frac{2}{\pi^2} - \frac{2}{\pi}\left(-\frac{1}{\pi}\right) = \frac{4}{\pi^2}.$$

2. $B = \left\{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq y\right\}.$



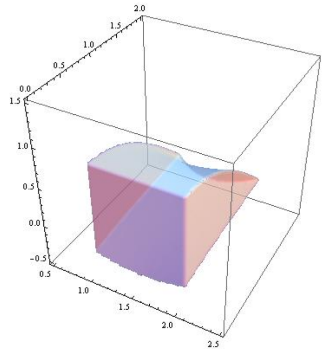
$$\int_B x^4y^2e^{xyz}dxdydz = \int_0^1 x^4dx\int_0^x y^2dy\int_0^y e^{xyz}dz = \int_0^1 x^4dx\int_0^x y^2dy\left[\frac{e^{xyz}}{xy}\right]_0^y =$$
$$= \int_0^1 x^3dx\int_0^x y\left(e^{xy^2}-1\right)dy = \int_0^1 x^3dx\left[\frac{e^{xy^2}}{2x}-\frac{y^2}{2}\right]_0^x = \int_0^1 x^3\left(\frac{e^{x^3}}{2x}-\frac{x^2}{2}-\frac{1}{2}\right)dx =$$
$$= \int_0^1\left(\frac{x^2e^{x^3}}{2}-\frac{x^5}{2}-\frac{x^2}{2}\right)dx = \left[\frac{e^{x^3}}{6}-\frac{x^6}{12}-\frac{x^3}{6}\right]_0^1 = \frac{e}{6}-\frac{1}{12}-\frac{1}{6}-\frac{1}{6} = \frac{e}{6}-\frac{5}{12}.$$

3. $C = \left\{(x,y,z)\in\mathbb{R}^3 : x \le 2y \le 2x \le 2, 0 \le x^2z+y^2z+z \le 1\right\} =$
$= \left\{(x,y,z)\in\mathbb{R}^3 : 0 \le x \le 1, \frac{x}{2} \le y \le x, 0 \le z \le \frac{1}{x^2+y^2+1}\right\}$



$$\int_C \left(x^2+y^2+1\right)\arctan x\,dxdydz = \int_0^1 \arctan x\,dx\int_{\frac{x}{2}}^x \left(x^2+y^2+1\right)dy\int_0^{\frac{1}{x^2+y^2+1}} dz =$$
$$= \int_0^1 \arctan x\,dx\int_{\frac{x}{2}}^x \left(x^2+y^2+1\right)\frac{1}{x^2+y^2+1}dy = \int_0^1 \arctan x[y]_{\frac{x}{2}}^{x} = \int_0^1 \frac{x\arctan x}{2}dx =$$
$$= \left[\frac{x^2\arctan x}{4}\right]_0^1 - \int_0^1 \frac{x^2}{4\left(x^2+1\right)}dx = \frac{\pi}{16}+\frac{1}{4}\int_0^1\left(\frac{1}{x^2+1}-1\right)dx = \frac{\pi}{16}+\frac{1}{4}[\arctan x - x]_0^1 =$$
$$= \frac{\pi}{16}+\frac{\pi}{16}-\frac{1}{4} = \frac{\pi}{8}-\frac{1}{4}.$$

4. $D = \left\{(x,y,z)\in\mathbb{R}^3 : -\frac{\sin^3 x+\sin^4 x}{(\pi+2y)^2} \le z \le \cos(y\sin x), 0 \le y\sin x \le \frac{\pi}{2}, \frac{\pi}{3} \le x \le \frac{2}{3}\pi\right\} =$
$= \left\{(x,y,z)\in\mathbb{R}^3 : -\frac{\sin^3 x+\sin^4 x}{(\pi+2y)^2} \le z \le \cos(y\sin x), 0 \le y \le \frac{\pi}{2\sin x}, \frac{\pi}{3} \le x \le \frac{2}{3}\pi\right\}$



$$Vol(D) = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} dx \int_0^{\frac{\pi}{2\sin x}} dy \int_{-\frac{\sin^3 x + \sin^4 x}{(\pi+2y)^2}}^{\cos(y\sin x)} dz = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} dx \int_0^{\frac{\pi}{2\sin x}} \left(\cos(y\sin x) + \frac{\sin^3 x + \sin^4 x}{(\pi+2y)^2}\right) dy =$$

$$= \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} dx \left[\frac{\sin(y\sin x)}{\sin x} - \frac{\sin^3 x + \sin^4 x}{2(\pi+2y)}\right]_0^{\frac{\pi}{2\sin x}} = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \left(\frac{1}{\sin x} - \frac{\sin^3 x + \sin^4 x}{2\pi + \frac{2\pi}{\sin x}} + \frac{\sin^3 x + \sin^4 x}{2\pi}\right) dx =$$

$$= \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \left(\frac{1}{\sin x} - \frac{\sin^4 x}{2\pi} + \frac{\sin^3 x + \sin^4 x}{2\pi}\right) dx = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \left(\frac{1}{\sin x} + \frac{\sin^3 x}{2\pi}\right) dx =$$

$$= \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \left(\frac{\sin x}{1-\cos^2 x} + \frac{\sin x\left(1-\cos^2 x\right)}{2\pi}\right) dx \overset{t=\cos x}{=} -\int_{\frac{1}{2}}^{-\frac{1}{2}} \left(\frac{1}{1-t^2} + \frac{1-t^2}{2\pi}\right) dt =$$

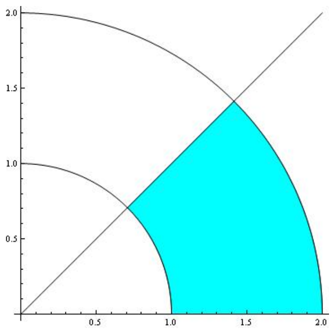
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2(1+t)} + \frac{1}{2(1-t)} + \frac{1}{2\pi} - \frac{t^2}{2\pi}\right) dt = \left[\frac{\log(1+t)}{2} - \frac{\log(1-t)}{2} + \frac{t}{2\pi} - \frac{t^3}{6\pi}\right]_{-\frac{1}{2}}^{\frac{1}{2}} =$$

$$= \frac{\log\left(\frac{3}{2}\right)}{2} - \frac{\log\left(\frac{1}{2}\right)}{2} + \frac{1}{4\pi} - \frac{1}{48\pi} - \frac{\log\left(\frac{1}{2}\right)}{2} + \frac{\log\left(\frac{3}{2}\right)}{2} + \frac{1}{4\pi} - \frac{1}{48\pi} = \log\left(\frac{3}{2}\right) - \log\left(\frac{1}{2}\right) + \frac{11}{24}\pi =$$

$$= \log 3 + \frac{11}{24}\pi.$$

5. $A = \left\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x,\ 1 \le x^2 + y^2 \le 4\right\} \quad f(x,y) = \dfrac{y}{x}$

   Passiamo in coordinate polari cioè poniamo $(x,y) = \Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$.



   Notiamo che $\Phi^{-1}(A) = \left\{(\rho,\theta) \in \mathbb{R}^2 : 1 \le \rho \le 2,\ \theta \le \frac{\pi}{4}\right\}$ e che la matrice Jacobiana di Φ è $J\Phi(\rho,\theta) = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \Longrightarrow |\det J\Phi| = \rho$.

   Per la formula del cambio di variabile abbiamo quindi che

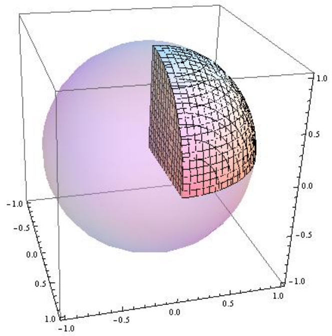
$$\int_A f(x,y)dxdy = \int_{\Phi^{-1}(A)} f(\Phi(\rho,\theta))\ \rho\ d\rho d\theta = \int_{\Phi^{-1}(A)} \frac{\rho\sin\theta}{\rho\cos\theta}\rho\ d\rho d\theta = \int_{\Phi^{-1}(A)} \rho\tan\theta\ d\rho d\theta =$$

$$= \int_1^2 d\rho \int_0^{\frac{\pi}{4}} d\theta \rho\tan\theta = \int_1^2 d\rho\ \rho \int_0^{\frac{\pi}{4}} d\theta\tan\theta = \int_1^2 d\rho\ \rho\left(-\log\cos\theta\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\log 2\int_1^2 \rho\ d\rho =$$

$$= \frac{1}{2}\log 2\ \frac{1}{2}\rho^2\Big|_1^2 = \frac{1}{4}\log 2(4-1) = \frac{3}{4}\log 2.$$

6. $B = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1,\ x,y,z \ge 0\right\} \quad f(x,y,z) = \dfrac{x^2}{4 - x^2 - y^2 - z^2}$

   Passiamo in coordinate sferiche cioè poniamo
   $(x,y,z) = \Phi(\rho,\theta,\varphi) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$.
   Notiamo che $\Phi^{-1}(B) = \left\{(\rho,\theta,\varphi) \in \mathbb{R}^3 : 0 \le \rho \le 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{2}\right\}$ e che la matrice Jacobiana di Φ è $J\varphi(\rho,\theta,\varphi) = \begin{pmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix} \Longrightarrow$



$|\det J\Phi| = \rho^2 \sin\varphi$.
Per la formula del cambio di variabile abbiamo quindi che

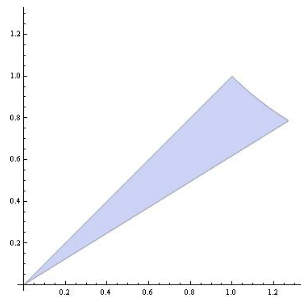
$$\int_B f(x,y,z)\,dxdydz = \int_{\Phi^{-1}(B)} f(\Phi(\rho,\theta,\varphi))\rho^2\sin\varphi\,d\rho d\theta d\varphi = \int_0^1 d\rho \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \frac{\rho^4\cos^2\theta\sin\varphi}{4-\rho^2}$$

$$= \int_0^1 d\rho \frac{\rho^4}{4-\rho^2} \int_0^{\frac{\pi}{2}} d\theta\ \cos^2\theta \int_0^{\frac{\pi}{2}} d\varphi\ \sin\varphi = \int_0^1 d\rho \frac{\rho^4}{4-\rho^2}\int_0^{\frac{\pi}{2}} d\theta\ \cos^2\theta\,(-\cos\varphi)\Big|_0^{\frac{\pi}{2}} =$$

$$= \int_0^1 d\rho \frac{\rho^4}{4-\rho^2}\int_0^{\frac{\pi}{2}} d\theta\ \cos^2\theta = \frac{\pi}{4}\int_0^1 d\rho \frac{\rho^4}{4-\rho^2} d\rho = \frac{\pi}{4}\int_0^1 \left(-4-\rho^4+\frac{4}{2-\rho}+\frac{4}{2+\rho}\right) d\rho =$$

$$\frac{\pi}{2}\left[-4\rho - \frac{\rho^3}{3} - 4\log(2-\rho) + 4\log(2+\rho)\right]_0^1 = \frac{\pi}{4}\left(4\log 3 - \frac{13}{3}\right) = \pi\left(\log 3 - \frac{13}{12}\right)$$

7. $C = \left\{(x,y)\in\mathbb{R}^2 : 0 \le x^2 - y^2 \le xy \le 1,\ x \ge 0\right\}$ $\quad f(x,y) = (x^4 - y^4)e^{xy}$



Poniamo $u = x^2 - y^2$ e $v = xy$ cioè $(u,v) = G(x,y) = (x^2 - y^2, xy)$.
$G(C) = \{(u,v)\in\mathbb{R}^2\ :\ 0 \le u \le v \le 1\} = \{(u,v)\in\mathbb{R}^2\ :\ 0 \le u \le 1,\ u \le v \le 1\}$.
Notiamo che se $(x,y)\in C$ allora $\begin{cases} u = x^2 - y^2 \\ v = xy \end{cases} \implies u^2+4v^2 = (x^2+y^2)^2 \implies x^2+y^2 = \sqrt{u^2+4v^2}$ da cui ricaviamo che $\begin{cases} \sqrt{u^2+4v^2} + u = 2x^2 \\ \sqrt{u^2+4v^2} - u = 2y^2 \end{cases} \implies \begin{cases} x = \sqrt{\frac{1}{2}(\sqrt{u^2+4v^2}+u)} \\ y = \sqrt{\frac{1}{2}(\sqrt{u^2+4v^2}-u)} \end{cases}$
Questo ci dice che G è una applicazione biettiva tra G e $G(B)$ e che $G^{-1}(u,v) = \Phi(u,v) = (\sqrt{\frac{1}{2}(\sqrt{u^2+4v^2}+u)}, \sqrt{\frac{1}{2}(\sqrt{u^2+4v^2}-u)})$. Il cambio di variabili $(u,v) = \Phi(x,y)$ è equivalente a $(x,y) = \Phi(u,v)$. Inoltre $\Phi^{-1}(C) = G(C)$ e $\det JG(x,y) = \det\begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix} = 2(x^2+y^2)$ quindi $\det J\Phi(u,v) = \frac{1}{2(x^2+y^2)}\Big|_{(x,y)=\Phi(u,v)}$.

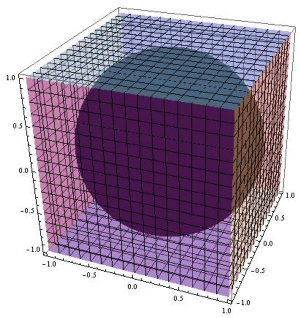
Per il teorema del cambio di variabile abbiamo quindi che

$$\int_C f(x,y)dxdy = \int_{\Phi^{-1}(C)} f(\Phi(x,y))\frac{1}{2(x^2+y^2)}\Big|_{(x,y)=\Phi(u,v)} dudv = \frac{1}{2}\int_{\Phi^{-1}(C)} (x^2-y^2)e^{xy}\Big|_{(x,y)=\Phi(u,v)} dudv =$$

$$= \frac{1}{2}\int_{\Phi^{-1}(C)} ue^v dudv = \frac{1}{2}\int_0^1 du \int_u^1 dv\ ue^v = \frac{1}{2}\int_0^1 ue^v|_u^1 = \frac{1}{2}\int_0^1 du\ eu - ue^u =$$

$$= \frac{1}{2}\left(\frac{e}{2}u^2\Big|_0^1 - ue^u\Big|_0^1 + \int_0^1 e^u\right) = \frac{1}{2}\left(\frac{e}{2} - e + e - 1\right) = \frac{e}{4} - \frac{1}{2}$$

8. $D = \{(x,y,z) \in \mathbb{R}^3 : \max\{|x|,|y|,|z|\} \leq 1 \leq x^2+y^2+z^2\}$ $\quad f(x,y) = x^2y^2z^2$



Notiamo che $D = D_1 \backslash D_2$ dove $D_1 = \{(x,y,z) \in \mathbb{R}^3 : \max\{|x|,|y|,|z|\} \leq 1\}$ e D_2 è la palla unitaria di $\mathbb{R}^3$ $D_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$.
Per il teorema di Fubini abbiamo che

$$\int_{D_1} f(x,y,z)dxdydz = \int_{-1}^1 dx \int_{-1}^1 dy \int_{-1}^1 dz\ x^2y^2z^2 = 8\int_0^1 dx\ x^2 \int_0^1 dy\ y^2 \int_0^1 dz z^2 = \frac{8}{27}$$

Passando alle coordinate sferiche abbiamo invece che

$$\int_{D_2} f(x,y,z)dxdydz = \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^\pi d\varphi\ f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi =$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^\pi d\varphi\ \rho^8\cos^2\theta\sin^2\theta\sin^5\varphi\cos^2\varphi = \int_0^\pi d\varphi \int_0^{2\pi} d\theta \int_0^1 d\rho\ \frac{1}{4}\rho^8\sin^2(2\theta)\sin^5\varphi\cos^2\varphi =$$

$$= \frac{1}{36}\int_0^\pi d\varphi \int_0^{2\pi} d\theta\sin^2(2\theta)\sin^5\varphi\cos^2\varphi = \frac{1}{72}\int_0^\pi d\varphi \int_0^{4\pi} dt\sin^2(t)\sin^5\varphi\cos^2\varphi =$$

$$= \frac{\pi}{36}\int_0^\pi \sin^5\varphi\cos^2\varphi \overset{t=\cos\varphi}{=} \frac{\pi}{36}\int_{-1}^1 (1-t^2)^2t^2 = \frac{\pi}{18}\int_0^1 \left(t^2 - 2t^4 + t^6\right)dt =$$

$$= \frac{\pi}{18}\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{\pi}{18}\,\frac{8}{105} = \frac{4}{945}\pi$$

Pertanto

$$\int_D f(x,y,z)dxdydz = \int_{D_1} f(x,y,z)dxdydz - \int_{D_2} f(x,y,z)dxdydz = \frac{8}{27} - \frac{4}{945}\pi$$

9. $E = \{x \in \mathbb{R}^3 : < Mx, x > \leq 1\}$
Siccome D è simmetirca e definita positiva allora per il teorema spettrale esiste una

matrice ortogonale Q tale che $Q^T M Q = D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$. Consideriamo allora il cambio di variabile $x = \Phi(y) = Qy$. Si noti che $|\det J\Phi| = |\det Q| = 1$ perché Q è ortogonale.
Inoltre $\Phi^{-1}(E) = \{y \in \mathbb{R}^3 \ : < MQy, Qy > \leq 1\}$ ma $< MQy, Qy >=< Q^T MQy, y >=< Dy, y >= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ quindi $\Phi^{-1}(E) = \{y \in \mathbb{R}^3 \ : \ \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \leq 1\}$ è l'ellissoide in $\mathbb{R}^3$ centrato nell'origine avente semiassi di lunghezze $\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \frac{1}{\sqrt{\lambda_3}}$. Per la formula del cambiamento di variabile abbiamo quindi che
$Vol(E) = \int_E 1 \, dx_1 dx_2 dx_3 = \int_{\Phi^{-1}(E)} 1 \, dy_1 dy_2 dy_3$. Consideriamo ora il cambio di variabile $(z_1, z_2, z_3) = \Psi(y_1, y_2, y_3) = (\sqrt{\lambda_1} y_1, \sqrt{\lambda_2} y_2, \sqrt{\lambda_3} y_3)$. Dato che $\Psi^{-1}(\Phi^{-1}(E)) = \{(z_1, z_2, z_3) \ : \ z_1^2 + z_2^2 + z_3^2 \leq 1\} = B_1(0)$ è la palla unitaria di $\mathbb{R}^3$ e $\det J\Psi = \sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3}$ abbiamo

$$Vol(E) = \int_{\Phi^{-1}(E)} 1 \, dy_1 dy_2 dy_3 = \int_{B_1(0)} \sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3} \, dz_1 dz_2 dz_3 = \sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3} Vol(B_1(0)) =$$

$$= \frac{4}{3}\pi \sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3} = \frac{4}{3}\pi \sqrt{\det M}$$