AM5 2010: Tracce delle lezioni- 10

DUE FORMULE DI RAPPRESENTAZIONE

La convoluzione con nuclei G_λ dá origine a importanti formule di rappresentazione. Cominciamo con una formula relativa all'equazione di Poisson. Sia

$$N \geq 3, \qquad c_N := N(N-2) \int\limits_{\mathbf{R}^N} \frac{dy}{(1+|y|^2)^{\frac{N+2}{2}}}, \qquad \mathcal{N} := \frac{G_{N-2}}{c_N}$$

Proposizione. $-\Delta\,(\varphi * \mathcal{N}) = \varphi \qquad \text{in } \mathbf{R}^N \qquad \forall\, \varphi \in C_0^\infty(\mathbf{R}^N)\,.$

Tale formula si basa sulla **formula di integrazione per parti**

$$\int\limits_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\, v \;=\; -\int\limits_{\mathbf{R}^N} u\, \frac{\partial v}{\partial x_j} \qquad\qquad \forall u \in C^\infty, \quad \forall v \in C_0^\infty(\mathbf{R}^N)$$

che é a sua volta conseguenza del Teorema Fondamentale del Calcolo. Ad esempio,

$$\int\limits_{\mathbf{R}^N} \frac{\partial(u\,v)}{\partial x_1} \;=\; \int\limits_{\mathbf{R}^{N-1}} \left(\int\limits_{-\infty}^{+\infty} \frac{\partial(u\,v)}{\partial x_1} \right) dx_2 \ldots dx_N \;=0$$

Prova della Proposizione. É $\qquad \Delta\,(\varphi * G_{N-2})(x) = \int\limits_{\mathbf{R}^N} \frac{(\Delta\,\varphi)(x-y)}{|y|^{N-2}}\, dy$

$$= \lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \frac{\Delta_y[\varphi(x-y)]}{(\epsilon^2+|y|^2)^{\frac{N-2}{2}}}\, dy \;=\; \lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \varphi(x-y)\, \Delta_y\, \frac{1}{(\epsilon^2+|y|^2)^{\frac{N-2}{2}}}\, dy$$

$$=\; \lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \varphi(x-y) \sum_{j=1}^N \frac{\partial}{\partial y_j}[-(N-2)\, \frac{y_j}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\;]\, dy \;=$$

$$\lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \varphi(x-y)\, \sum_{j=1}^N [N(N-2) \frac{y_j^2}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}} \;-(N-2)\frac{1}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\;]\, dy \;=$$

$$\lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \varphi(x-y)\; \frac{-N(N-2)\epsilon^2}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}}\, dy \;=\; \lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \varphi(x-\epsilon\xi)\; \frac{-N(N-2)}{(1+|\xi|^2)^{\frac{N+2}{2}}}\, d\xi \;=$$

$$=\; -\varphi(x) N(N-2) \int\limits_{\mathbf{R}^N} \frac{d\xi}{(1+|\xi|^2)^{\frac{N+2}{2}}}$$

Corollario. Sia $N \geq 3$. Allora, per ogni $\varphi \in C_0^\infty(\mathbf{R}^N)$ si ha

$$\varphi(x) = -(\Delta\varphi) * \mathcal{N} = -\frac{1}{c_N} \int\limits_{\mathbf{R}^N} \frac{(\Delta\,\varphi)(x-y)}{|y|^{N-2}}\, dy$$

Un'altra formula di rappresentazione. Sia $\varphi \in C_0^\infty(\mathbf{R}^N),\ N \geq 3$. Allora

$$\varphi(x) \ = \ \frac{N-2}{c_N} \int\limits_{\mathbf{R}^N} \frac{< (\nabla\varphi)(x-y)\ ,\ y >}{|y|^N}\ dy \qquad (*)$$

Prova . 1) Dal Corollario: $\varphi(x) = -\frac{1}{c_N} \sum_j \int\limits_{\mathbf{R}^N} \frac{\partial^2\varphi}{\partial y_j^2}(x-y)\frac{1}{|y|^{N-2}}\, dy =$

$$\frac{2-N}{c_N} \sum_j \int\limits_{\mathbf{R}^N} \frac{\partial}{\partial y_j}\varphi(x-y)\frac{y_j}{|y|^N} dy = \frac{N-2}{c_N} \int\limits_{\mathbf{R}^N} < (\nabla\varphi)(x-y), \frac{y}{|y|^N} > dy$$

Giustificazione della integrazione per parti: $\int\limits_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\frac{1}{|x|^{N-2}} dx =$

$$\lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\frac{1}{(\epsilon^2+|x|^2)^{\frac{N-2}{2}}} dx = -(2-N) \lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} u \frac{x_j}{(\epsilon^2+|x|^2)^{\frac{N}{2}}} dx = (N-2) \int\limits_{\mathbf{R}^N} u \frac{x_j}{|x|^N} dx$$

2) Determinazione diretta :

$$\int\limits_{\mathbf{R}^N} \frac{< (\nabla\varphi)(x-y)\ ,\ y >}{|y|^N}\ dy \ = \lim_{\epsilon\to 0} \sum_{j=1}^N \int\limits_{\mathbf{R}^N} -\frac{\partial}{\partial y_j}\,[\varphi(x-y)] \quad \frac{y_j}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\ dy \ =$$

$$\lim_{\epsilon\to 0} \sum_{j=1}^N \int\limits_{\mathbf{R}^N} \varphi(x-y)\ \frac{\partial}{\partial y_j}\ \frac{y_j}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\ dy \ = \ N \lim_{\epsilon\to 0} \epsilon^2 \int\limits_{\mathbf{R}^N} \left[\frac{\varphi(x-y)}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}}\right] dy \ =$$

$$= \ N \lim_{\epsilon\to 0} \int\limits_{\mathbf{R}^N} \left[\frac{\varphi(x-\epsilon z))}{(1+|z|^2)^{\frac{N+2}{2}}}\right] dz \ = \ N\varphi(x) \int\limits_{\mathbf{R}^N} \frac{dz}{(1+|z|^2)^{\frac{N+2}{2}}} = \frac{c_N}{N-2}\varphi(x)$$

3). Un calcolo alternativo basato sul Teorema della divergenza: se Ω é aperto limitato e a frontiera liscia e $X \in C^1(\overline{\Omega}, \mathbf{R}^N)$, allora

$$\int\limits_{\Omega} div\ X\ dy = \int\limits_{\partial\Omega} < X, \nu > \ d\sigma$$

ove $\nu(y)$ indica la normale esterna ad Ω in $y \in \partial\Omega$. Preso $X(y) := \varphi(x-y)\frac{y}{|y|^N}$,

$$div\ X(y) = < \nabla_y\varphi(x-y), \frac{y}{|y|^N} > \ + \varphi\ div\ \frac{y}{|y|^N} \ = \ - < (\nabla\varphi)(x-y), \frac{y}{|y|^N} >$$

perché $div\, \frac{y}{|y|^N} = div\, \nabla\left(-\frac{1}{(N-2)|y|^{N-2}}\right) = \Delta\left(-\frac{1}{(N-2)|y|^{N-2}}\right) = 0 \quad \forall y \neq 0$. Inoltre, se $\varphi \equiv 0$ fuori della palla B_r, prenderemo $\Omega := \{\epsilon < |y| < r\}$, risultando quindi $\nu(y) = -\frac{y}{|y|}$ se $|y| = \epsilon$. Indicata con ω_N l'area della sfera unitaria in $\mathbf{R}^N$, si trova:

$$\int\limits_{\mathbf{R}^N} \frac{<(\nabla\varphi)(x-y)\,,\, y>}{|y|^N}\, dy = \lim_{\epsilon\to 0} \int\limits_{|y|\geq\epsilon} \frac{(\nabla\varphi)(x-y)\,,\, y>}{|y|^N}\, dy =$$

$$= \lim_{\epsilon\to 0} \int\limits_{|y|=\epsilon} \frac{\varphi(x-y)}{|y|^{N-1}}\, d\sigma = \lim_{\epsilon\to 0} \int\limits_{|y|=\epsilon} \frac{\varphi(x-y)}{\epsilon^{N-1}}\, d\sigma = \omega_N \varphi(x) \qquad (**)$$

NOTA. *Da* $(*)$ *e* $(**)$ *si deduce che* $\omega_N = \frac{c_N}{N-2} = N \int\limits_{\mathbf{R}^N} \frac{dz}{(1+|z|^2)^{\frac{N+2}{2}}}$

DISEGUAGLIANZA HLS: DUE CASI IMPORTANTI

$$\lambda = N-2, \quad \frac{N}{2} > p > 1 \quad \Rightarrow \quad \frac{1}{s} = \frac{N-2p}{Np} \quad \left(= \frac{N-2}{2N} \quad \text{se} \quad p = \frac{2N}{N+2}\right) \quad \Rightarrow$$

$$\|G_{N-2} * f\|_{\frac{Np}{N-2p}} \leq c(N)\ \|f\|_p \qquad \forall f \in L^p(\mathbf{R}^N)$$

$$\lambda = N-1, \quad N > p > 1, \quad \Rightarrow \quad \frac{1}{s} = \frac{N-p}{Np} \quad \Rightarrow$$

$$\|G_{N-1} * f\|_{\frac{Np}{N-p}} \leq c(N,p)\ \|f\|_p \qquad \forall f \in L^p(\mathbf{R}^N)$$

La diseguaglianza $\|G_{N-2} * f\|_{\frac{Np}{N-2p}} \leq c(N)\,\|f\|_p \quad \forall f \in L^p(\mathbf{R}^N)$.
Dice che l'operatore lineare $f \to f * \mathcal{N}$ che 'risolve' l'equazione di Poisson con dato $f \in C_0^\infty$ si estende in modo continuo da $L^p(\mathbf{R}^N), p \in (1, \frac{N}{2})$ a $L^{\frac{Np}{N-p}}$.
In particolare, se $\varphi_n \in C_0^\infty(\mathbf{R}^N)$, $\varphi_n \to_n f$ in L^p e quindi $\varphi_n * \mathcal{N} \to_n f * \mathcal{N}$ in $L^{\frac{Np}{N-p}}$, da $-\Delta(\varphi_n * \mathcal{N}) = \varphi_n$, segue, integrando per parti,

$$-\int\limits_{\mathbf{R}^N} (\varphi_n * \mathcal{N})\, \Delta\varphi = \int\limits_{\mathbf{R}^N} \varphi_n \varphi \qquad \forall \varphi \in C_0^\infty$$

e quindi, passando al limite

$$-\int\limits_{\mathbf{R}^N} (f * \mathcal{N})\, \Delta\varphi = \int\limits_{\mathbf{R}^N} f\, \varphi \qquad \forall \varphi \in C_0^\infty$$

Quanto trovato si esprime dicendo che

$f * \mathcal{N}$ é soluzione dell'equazione $-\Delta u = f$ in $\mathbf{R}^N$ in senso 'debole'.

La diseguaglianza $\|G_{N-1} * f\|_{\frac{Np}{N-p}} \ \le\ c(N,p)\,\|f\|_p \quad \forall f \in L^p(\mathbf{R}^N)$.
Fornisce una semplice dimostrazione di una fondamentale diseguaglianza:

LA DISEGUAGLIANZA DI SOBOLEV

$$\forall p \in (1,N), \exists c = c(N,p): \qquad \left(\int_{\mathbf{R}^N} |u|^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}} \le c \int_{\mathbf{R}^N} |\nabla u|^p \qquad \forall u \in C_0^\infty(\mathbf{R}^N)$$

Prova della diseguaglianza di Sobolev. $\qquad u \in C_0^\infty(\mathbf{R}^N) \ \Rightarrow$

$$|u(x)| \ \le\ c \int_{\mathbf{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\, dy \ =\ c\ (|\nabla u|\ * G_{N-1})\,(x) \quad \forall x \in \mathbf{R}^N \quad \Rightarrow$$

$$\|u\|_{\frac{Np}{N-p}} \ \le\ c\,\|G_{N-1} * |\nabla u|\|_{\frac{Np}{N-p}} \quad \le \quad c \quad \|\nabla u\|_p$$

NOTA. Sobolev vale anche per ogni $N \ge 2$ e $p = 1$. Si puó in effetti dedurre facilmente dal caso $p = 1$, che a sua volta segue dalla diseguaglianza elementare

$$|u(x_1,\dots,x_N)|^N \ \le\ \int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_1}(t,x_2,\dots,x_N)dt| \times \ldots \times \int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_N}(x_1,x_2,\dots,t)dt|$$

che implica $\quad \left(\int_{\mathbf{R}^N} |u(x)|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \ \le$

$$\left[\int_{\mathbf{R}^N} \left[\left(\int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_1}(t,x_2,\dots,x_N)dt| \right)^{\frac{1}{N-1}} \times \ldots \times \left(\int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_N}(x_1,x_2,\dots,t)dt| \right)^{\frac{1}{N-1}} \right] dx \right]^{\frac{N-1}{N}}$$

Sfruttando la speciale forma dell'integrando a secondo membro, si puó provare, sempicemente iterando Holder, che

$$\left[\int_{\mathbf{R}^N} \left[\left(\int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_1}(t,x_2,\dots,x_N)dt| \right)^{\frac{1}{N-1}} \times \ldots \times \left(\int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_N}(x_1,x_2,\dots,t)dt| \right)^{\frac{1}{N-1}} \right] dx \right]^{\frac{N-1}{N}}$$

$$\le \left(\int_{\mathbf{R}^{N-1}} \int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_1}(t,x_2,\dots,x_N)dt|dx_2\dots dx_N \right)^{\frac{1}{N}} \times \ldots \times$$

$$\left(\int_{\mathbf{R}^{N-1}} \int_{-\infty}^{+\infty} |\tfrac{\partial u}{\partial x_N}(x_1,x_2,\dots,t)dt|dx_1\dots dx_{N-1} \right)^{\frac{1}{N}} = \|\tfrac{\partial u}{\partial x_1}\|_1^{\frac{1}{N}} \times\ldots\times \|\tfrac{\partial u}{\partial x_N}\|_1^{\frac{1}{N}} \le \sum_{j=1}^N \|\tfrac{\partial u}{\partial x_j}\|_1.$$

Diseguaglianza di POINCARÉ. Sia $1 < p < N$, $\Omega \subset \mathbf{R}^N$ aperto limitato.

Allora $\exists\, c = c(\Omega) > 0$: $\int\limits_\Omega |\nabla u|^p \ \geq\ c \int\limits_\Omega |u|^p \quad \forall u \in C_0^\infty(\Omega)$

Infatti, da $\frac{p}{N} + \frac{N-p}{N} = 1$, usando Holder e quindi Sobolev, segue

$$\int\limits_{\mathbf{R}^N} |u|^p \ \leq \left(\int\limits_{\mathbf{R}^N} |u|^{\frac{Np}{N-p}}\right)^{\frac{N-p}{N}} vol(\Omega)^{\frac{p}{N}} \ \leq\ M(\Omega) \int\limits_{\mathbf{R}^N} |\nabla u|^p \quad \forall u \in C_0^\infty(\Omega)$$

Poincaré non vale in $\mathbf{R}^N$: $\inf\limits_{u \in C_0^\infty(\mathbf{R}^N),\, u \neq 0} \frac{\int_{\mathbf{R}^N} |\nabla u|^p}{\int_{\mathbf{R}^N} |u|^p} = 0$

Se $u_\epsilon(x) := u(\epsilon x)$, é $\int\limits_{\mathbf{R}^N} |u_\epsilon|^p = \epsilon^{-N} \int\limits_{\mathbf{R}^N} |u|^p$, $\int\limits_{\mathbf{R}^N} |\nabla u_\epsilon|^p = \epsilon^{p-N} \int\limits_{\mathbf{R}^N} |\nabla u|^p$

e quindi $\frac{\int_{\mathbf{R}^N} |\nabla_\epsilon u|^p}{\int_{\mathbf{R}^N} |u_\epsilon|^p} = \epsilon^p \frac{\int_{\mathbf{R}^N} |\nabla u|^p}{\int_{\mathbf{R}^N} |u|^p} \to_\epsilon 0$ Allo stesso modo

si vede che $\lambda_1(\Omega) := \inf\limits_{u \in C_0^\infty(\Omega),\, u \neq 0} \frac{\int_{\mathbf{R}^N} |\nabla u|^2}{\int_{\mathbf{R}^N} |u|^2} < \frac{\int_{\mathbf{R}^N} |\nabla u|^2}{\int_{\mathbf{R}^N} |u|^2} \quad \forall u \in C_0^\infty(\Omega)$

[$l'\inf$ non é realizzato in $C_0^\infty(\Omega)$: $\forall u \in C_0^\infty(\Omega)$ $\exists \epsilon < 1$: $u_\epsilon \in C_0^\infty(\Omega)$]

Diseguaglianze di MORREY. Sia $p > N$.

(i) $\forall\, R > 0\ \exists\ c = c(N,p,R)$: $\|u\|_\infty \ \leq\ c \left(\int\limits_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(B_R)$

(ii) $\exists c = c(p,N)$: $|u(x) - u(y)| \ \leq\ c|x-y|^{\frac{p-N}{p}} \left(\int\limits_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(\mathbf{R}^N)$

Prova di (i). $u \in C_0^\infty(B_R)$, $x \in B_R \ \Rightarrow\ |u(x)| \ \leq\ c \int\limits_{\mathbf{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\, dy \ \leq$

$$c \left(\int\limits_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int\limits_{B_R} \frac{1}{|x-y|^{q(N-1)}}\, dy\right)^{\frac{1}{q}} \ \leq\ c \left(\int\limits_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int\limits_{B_{2R}} \frac{dz}{|z|^{q(N-1)}}\right)^{\frac{1}{q}}$$

perché $p > N \ \Rightarrow\ q(N-1) \ <\ N$.

Prova di (ii) Sia $Q_r := \{x : \ |x_i| \le r \ \forall i\}$ (cubo di lato $2r$ centrato nell'origine). Fissato $\overline{x}$, sia $\overline{u} = \frac{1}{2^N r^N} \int\limits_{Q_r+\overline{x}} u$ la media di u su $Q := Q_r + \overline{x}$. Per ogni $x \in Q$ risulta

$$|\overline{u} - u(x)| = |\frac{1}{(2r)^N} \int\limits_Q [u(y) - u(x)]\, dy| \le \int\limits_Q \left[\frac{|y-x|}{(2r)^N} \int_0^1 |\nabla u(ty + (1-t)x)|\, dt \right] dy$$

$$\le \frac{\sqrt{N}}{(2r)^{N-1}} \int\limits_0^1 \left(\int\limits_{(1-t)x+tQ} \frac{|\nabla u(z)|}{t^N}\, dz \right) dt \le \frac{\sqrt{N}}{(2r)^{N-1}} \left(\int\limits_Q |\nabla u|^p \right)^{\frac{1}{p}} \int_0^1 vol(tQ)^{1-\frac{1}{p}} \frac{dt}{t^N} =$$

$$\sqrt{N}\,(2r)^{1-\frac{N}{p}} \left(\int\limits_{Q_{2r}+\overline{x}} |\nabla u|^p \right)^{\frac{1}{p}} \int_0^1 t^{-\frac{N}{p}}\, dt = c(N,p)\, r^{1-\frac{N}{p}} \left(\int\limits_{Q_{2r}+\overline{x}} |\nabla u|^p \right)^{\frac{1}{p}}$$

Dunque, fissati x, y e posto $r = |x-y|$, $\overline{x} = \frac{x+y}{2}$, per cui $x, y \in Q_r + \overline{x}$, si ha

$$|u(x) - u(y)| \le 2c(N,p)\, r^{1-\frac{N}{p}} \left(\int\limits_{Q_{2r}+\overline{x}} |\nabla u|^p \right)^{\frac{1}{p}} = 2c(N,p)|x-y|^{1-\frac{N}{p}} \left(\int\limits_{\mathbf{R}^N} |\nabla u|^p \right)^{\frac{1}{p}}$$

Morrey (i) non vale in $\mathbf{R}^N$. Se $u_\epsilon(x) := u(\epsilon x)$, é

$$\int\limits_{\mathbf{R}^N} |\nabla u_\epsilon|^p = \epsilon^{p-N} \int\limits_{\mathbf{R}^N} |\nabla u|^p \quad \text{mentre} \quad \|u_\epsilon\|_\infty = \|u\|_\infty$$

Il Teorema di compattezza di RELLICH.

Sia $u_n \in C_0^\infty(B_R)$, con $\sup_n \left(\int\limits_{\mathbf{R}^N} |\nabla u_n|^p \right)^{\frac{1}{p}} < +\infty$. Allora

(i) se $< p < N$, u_n ha una sottosuccessione convergente in $L^r(B_R)\ \forall r < \frac{Np}{N-p}$.

(ii) se $p = N$, u_n ha una sottosuccessione convergente in $L^r(B_R)\ \ \forall r$.

(iii) se $p > N$, u_n ha una sottosuccessione uniformemente convergente in B_R

Prova. (i) Sia $1 \le r < \frac{Np}{N-p}$. Da Holder e quindi Sobolev segue che

$$\sup_n \left(\int\limits_{B_R} |u_n|^r \right)^{\frac{1}{r}} \le c(R) \sup_n \left(\int\limits_{\mathbf{R}^N} |\nabla u_n|^p \right)^{\frac{1}{p}} < +\infty$$

Poi, la diseguaglianza di interpolazione con $\alpha \in (0,1],\ \alpha + (1-\alpha)\frac{N-p}{Np} = \frac{1}{r}$ dá

$$\left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|^r\, dx\right)^{\frac{1}{r}} \leq$$

$$\left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|\right)^{\alpha} \left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|^{\frac{Np}{N-p}}\right)^{\frac{(1-\alpha)(N-p)}{Np}}$$

Il secondo fattore, grazie a Sobolev, resta, nelle nostre ipotesi, limitato e

$$\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|\, dx \leq \int_{\mathbf{R}^N} \left(\int_0^1 | < \nabla u_n(x+th), h > |dt\right)\, dx$$

$$\leq vol(B_{2R})^{1-\frac{1}{p}}\, |h| \int_0^1 \left(\int_{\mathbf{R}^N} |\nabla u_n(x+th)|^p\, dx\right)^{\frac{1}{p}} dt \leq c|h| \sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n(x)|^p\, dx\right)^{\frac{1}{p}}$$

La compattezza di u_n in $L^r(\mathbf{R}^N)$ segue quindi da Frechet-Kolmogoroff.

(ii) In tal caso $\sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^r\right)^{\frac{1}{r}} < +\infty \quad \forall r$, e quindi, come in (i), otteniamo la compattezza di u_n in ogni L^r.

(iii) La (i) nel Teorema di Morrey dice $\sup_n \|u_n\|_\infty < +\infty$ mentre la (ii) assicura la equicontinuitá delle u_n. La conclusione segue quindi dal Teorema di Ascoli-Arzelá.

Nota. Rellich non vale in tutto $\mathbf{R}^N$ né fino all'esponente limite $p^* := \frac{Np}{N-p}$.

(i) Se $f \in C_0^\infty(\mathbf{R}^N)$, $f \neq 0$, $h \in \mathbf{R}^N, h \neq 0$, $f_n(x) := f(x+nh)$, allora $\|\nabla f_n\|_2 \equiv \|\nabla f\|_2$, ma f_n non ha estratte convergenti in alcun L^p

(i) Se $f \in C_0^\infty(B_1)$, $f \neq 0$, $\epsilon_n \to_n 0$, $f_n(x) := \epsilon_n^{\frac{N-2}{2}} f(\frac{x}{\epsilon_n})$ allora

$$\|\nabla f_n\|_2 \equiv \|\nabla f\|_2, \quad \|f_n\|_{\frac{2N}{N-2}} \equiv \|f_n\|_{\frac{2N}{N-2}}$$

e quindi f_n non ha estratte convergenti in $L^{\frac{2N}{N-2}}$ (mentre converge a zero in L^p per $1 \leq p < \frac{2N}{N-2}$).