Soluzioni Tutorato di Statistica 1 del 25/02/2010
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Esercizio 1.
X v.a., $X \sim N(\mu, \sigma^2)$
$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$
$E[e^{tx}] = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$
$= \frac{e^{t\mu}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{tx} e^{-t\mu} e^{(x-\mu)^2/2\sigma^2} dx =$
$= \frac{e^{t\mu}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-1/2\sigma^2\{(x-\mu)^2 - 2\sigma^2 t(x-\mu) + (\sigma^2 t)^2 - (\sigma^2 t)^2}\} dx =$
$= \frac{e^{t\mu + (\sigma^2 t)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-1/2\sigma^2[(x-\mu)-(\sigma^2 t)]^2} dx =$
$= \frac{e^{t\mu + (\sigma^2 t^2)/2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-z^2/2\sigma^2} dx = e^{t\mu + (\sigma^2 t^2)/2}$
Dunque $m(t) = e^{t\mu + (\sigma^2 t^2)/2}$
$E[X] = m'(t)|_{t=0} = e^{t\mu + (\sigma^2 t^2)/2}(\mu + t\sigma^2)|_{t=0} = \mu$
$Var[X] = E[X^2] - E[X]^2$
$E[X^2] = m^{''}(t)|_{t=0}$
$m^{''}(t) = e^{t\mu + (\sigma^2 t^2)/2}(\mu^2 + t\sigma^2\mu + \sigma^2 + t\mu\sigma^2 + (t\sigma^2)^2)|_{t=0} =$
$= \mu^2 + \sigma^2$
$Var[X] = m^{''}(t)|_{t=0} - m'(t)^2_{t=0} = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Esercizio 2.
X v.a., $X \sim Po(\lambda)$.
$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} 1_{\{0,1,2...\}}(x)$
$m(t) = \sum_{x=0}^{+\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{+\infty} \frac{e^{tx}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{+\infty} \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} e^{e^t\lambda} = e^{\lambda(e^t-1)}$
$E[x] = m'(t)|_{t=0}$
$m'(t) = \lambda e^t e^{\lambda(e^t-1)}|_{t=0} = \lambda$
$Var[X] = E[X^2] - E[X]^2$
$m^{''}(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}|_{t=0} = \lambda + \lambda^2$
$Var[X] = m^{''}(t)|_{t=0} - m'(t)^2_{t=0} = \lambda = \lambda^2 - \lambda^2 = \lambda$

Esercizio 3.
X v.a., $X \sim Unif(a, b)$
$f_X(x) = \frac{1}{b-a}, x \in (a, b)$
$m(t) = \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{1}{t(b-a)} \int_a^b te^{tx} dx = \frac{e^{bt} - e^{at}}{t(b-a)}$
$E[x] = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$
$E[x^2] = \int_a^b x^2 \frac{1}{b-a} = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + a^2 + ab}{3}$
$Var[x] = \frac{b^2 + a^2 + ab}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$

Esercizio 4.
X e Y variabili aleatorie con densità congiunta data da:
$f_{X,Y}(x, y) = cx(y - x)e^{-y}$ con $0 \leq x \leq y < +\infty$

1. Per trovare c t.c. $f_{X,Y}(x, y)$ sia una densità basta calcolare

$\int\int f_{X,Y}(x,y)dxdy = 1$, allora
$\int_0^{+\infty}\int_x^{+\infty} cx(y-x)e^{-y}dydx = \int_0^{+\infty} cx\{\int_x^{+\infty} ye^{-y}dy - \int_x^{+\infty} xe^{-y}dy\}dx =$
$\int_0^{+\infty}\{e^{-x}(x+1) - x\int_x^{+\infty} e^{-y}dy\}dx = \int_0^{+\infty} cx\{2xe^{-x}+e^{-x}\}dx =$
$\int_0^{+\infty} cxe^{-x}dx = c$ allora $c = 1$

2. $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$, $0 \le x \le y$
   $f_{Y|X}(y|x) =, \frac{f_{X,Y}(x,y)}{f_X(x)}$, $0 \le x \le y < \infty$
   $f_X(x) = \int_x^{=\infty} x(y-x)e^{-y}dy = xe^{-x}$ allora $f_{Y|X} = \frac{(y-x)e^{-y}}{e^{-x}}$
   $f_Y(y) = \int_0^y x(y-x)e^{-y}dx = \frac{1}{6}y^3e^{-y}$ allora $f_{X|Y}(x|y) = \frac{6x(y-x)}{y^3}$

3. $E[X|Y] = \int_0^y x\frac{6x(y-x)}{y^3}dx = \frac{1}{2}y$
   $E[Y|X] = \int_x^{+\infty} y\frac{(y-x)e^{-y}}{e^{-x}}dy = x+2$

Esercizio 5.

Se X è il numero della prima estratta e Y è il max dei due numeri estratti, allora $X = 1,2,3$ e $Y = 2,3$

1. $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_X(x)$
   $f_{X,Y}(1,2) = 1/6$
   $f_{X,Y}(2,2) = 1/6$
   $f_{X,Y}(3,2) = 0$
   $f_{X,Y}(1,3) = 1/6$
   $f_{X,Y}(2,3) = 1/6$
   $f_{X,Y}(3,3) = 1/3$

2. $P[X=1|Y=3] = \frac{P[X=1,Y=3]}{P[Y=3]} = \frac{1/6}{2/3} = 1/4$

3. $Cov[X,Y] = E[XY] - E[X]E[Y]$
   $E[X] = 1/3 + 2/3 + 1 = 2$
   $E[Y] = 2/3 + 2 = 8/3$
   $E[XY] = \sum xyf_{X,Y}(x,y) = 1/3 + 1/2 + 2/3 + 1 + 3 = 11/2$
   $Cov[XY] = 11/2 - 16/3 = 17/3$