Università degli Studi Roma Tre
Facoltà di Scienze Matematiche Fisiche e Naturali

Synthesis

Asymptotic-numerical approximation of rapidly oscillatory solutions of second order differential equations and of their zeros

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Introduction

Throughout this Thesis we use an asymptotic-numerical method to compute solutions and zeros of second-order linear ordinary differential equation as

$$y'' + q(x)y = 0, \tag{1}$$

on a half-line, in the oscillatory case.

Recall that every second-order linear ordinary differential equation like

$$Y'' + a(x)Y' + b(x)Y = 0, \tag{2}$$

can be taken into the form (1) by appropriate changes of indipendent and dependent variables [3], as

$$t(x) = t_0 + t_0' \int_{x_0}^{x} e^{-\int_{x_0}^{\sigma} a(\tau)d\tau} d\sigma, \tag{3}$$

where x_0, t_0, $t_0' \neq 0$, or

$$Y = e^{-\frac{1}{2}\int_{x_0}^{x} a(\tau)d\tau} y, \tag{4}$$

whenever coefficient $a(x)$ has a continuous derivative.

We therefore consider the "canonical" form in equation (1) as the starting point under the appropiate regularity of the coefficients $a(x)$ and $b(x)$ of (2).

Following Boruvka's terminology [3], we refer to the coefficient $q(x)$ in (1) as to *the carrier* of the equation. Below, we apply our method to several cases, i.e.:

$$q(x) = a + \frac{b}{x} + O(x^{-p}) \text{ with } a > 0,\ b \in \mathbf{R},\ p > 1; \tag{5}$$

$$q(x) = cx^m\,[1 + o(1)] \text{ as } x \to +\infty, \text{ with } c, m \in \mathbf{R}^+; \tag{6}$$

$$\begin{aligned} q(x) \;=\; e^{ax} + O(e^{bx}) \;\; & \text{with } x \in (\rho; +\infty), \\ & \rho, a > 0,\ 0 < b < a. \end{aligned} \tag{7}$$

Actually, we can pass from a case to another by simple transformations hence all problems could be treated in a unified way. In fact, from

$$\frac{d^2y}{dx^2} + [e^{ax} + g(x)]\, y = 0, \tag{8}$$

setting $t = e^x$, $y(x) \equiv y(\log t) := Y(t)$ and appling the transformation in (4) we obtain

$$Z'' + \left[\frac{1}{4t^2} + t^{a-2} + \frac{g(\log t)}{t^2}\right] Z = 0. \tag{9}$$

Note that the dominant term in the carrier is t^{a-2}, hence for, e. g., $a = 2$ we are in the case of the asymptotically constant carrier, for $a > 2$ we fall in the case of the asymptotically polynomial carrier. When $0 < a < 2$, equation (1) may not have oscillatory solutions since the condition $q(x) > 0$ does not sufficient to ensure the oscillatory behavior. Indeed, the following Theoreme provides a sufficient condition for the oscillatority of solutions to equation (1)

***Theorem* 1** [6, Theoreme 7.1, p. 362]. Let $q(x)$ be real-valued and continuos for large $x > 0$. Then, if

$$-\infty \le \limsup_{x\to\infty} x^2 q(x) < \frac{1}{4} \qquad \left[\text{ or } \frac{1}{4} < \liminf_{x\to\infty} x^2 q(x) \le +\infty\right], \tag{10}$$

equation (1) is nonoscillatory [or, respectively, oscillatory] as $x \to +\infty$.

It is easy to see that, for every value of $a > 0$ in (9), the carrier there satisfies the condition (10), which implies that the equation is oscillatory.

If we start from equation (1) with the carrier in (6), i.e.,

$$\frac{d^2y}{dt^2} + [t^\alpha + h(t)]\, y = 0, \tag{11}$$

and set $t = e^x$, $y(t) \equiv Y(x)$, and apply transformation (4) with $a(x) \equiv -1$, as before, we obtain

$$Z'' + \left[e^{x(\alpha+2)} + e^{2x} h(e^x) - \frac{1}{4}\right] Z = 0. \tag{12}$$

In this case, $\beta := \alpha + 2 \geq 2 \Leftrightarrow \alpha \geq 0$, hence the leading coefficent of the carrier is e^{β}, that is, we have an asymptotically exponential carrier

In this Thesis, we studied the problem of the asymptotic-numerical the approximation of solutions to second-order linear ordinary differential equations in the *rapidly oscillatory* case. Zeros of the corresponding solutions can also be evaluated by our approach without computing first the solutions themselves.

The basic ingredients of the method are the theory of transformation of second-order differential equations, developed by the group theorist O. Boruvka [2, 3], and the so-called Liouville-Green (or WKB, or WKBJ) asymptotic theory, rigorously founded by F.W.J. Olver, [7].

We have applied the algorithm to three classes of rapidly oscillatory "carrier" $q(x)$ (see (5), (6), (7)), to compute a basis of solutions to equation (1) as well as zeros of solutions, providing convergence results for the first two. In each case, a few examples have been given, showing (in suitable Tables and pictures) efficiency and accurancy of the algorithm.

Boundary-value (BV) problems and Cauchy problems for equation (1) with rapidly oscillatory solutions have been solved by such algorithm. In the Cauchy problem, we compared such solution with that obtained by a very accurate, well established numerical method, the Runge-Kutta-Nystrom method with 12/10 stages. Here some examples were also given, for which Tables and pictures show the accurancy achieved by our algorithm. All examples have been worked out in the environment of MATHEMATICA, since symbolic manipulations were essentials and the ensuing numerical treatment rather simple. The Runge-Kutta-Nystrom code was runned in MATLAB.

1 The underlying theories

O. Boruvka's transformation theory was developed from Kummer's theory starting from an equation written in the "canonical" form

$$y'' + q(x)y = 0, \tag{13}$$

The coefficient q is a continuous functions on the interval $j := (a, b)$, where possibly $a = -\infty$ and $b = +\infty$. Boruvka called the function q the *carrier* of (13), respectively, and introduced the very important notion of *phase* (or *phase function*) of a linear second-order differential equation. A phase function, $\alpha(x)$, is any C^3-solution of the equation $\tan\alpha(x) = \frac{u(x)}{v(x)}$, where (u, v) denotes a basis for equation (13). Moreover, α has the property that $\alpha' \neq 0$ and, by differentiation, it turns out that

$$\alpha'(x) = -\frac{W}{u^2(x) + v^2(x)}, \tag{14}$$

where $W := uv' - u'v$ is the (constant) Wronskian of u, v. It is possible to avoid the explicit knowledge of u and v, and obtain a close-form third-order nonlinear differential equation satisfied by α alone. This task can be accomplished by repeated differentiations and using the fact that $u'' = -qu$, $v'' = -qv$. The resulting equation can be written as

$$\alpha'^2(x) = q(x) - \frac{1}{2}\{\alpha, x\}, \tag{15}$$

where

$$\{\alpha, x\} := \frac{\alpha'''(x)}{\alpha'(x)} - \frac{3}{2}\left(\frac{\alpha''^2(x)}{\alpha'^2(x)}\right)^2$$

denotes the so-called *Schwarzian derivative* of α [3, 8]. The close-form equation in (15) is a special case of the Kummer equation. As a consequence, it can be easily shown that every solution to (15) is a phase function related to the basis $(u(x), v(x))$ given by

$$u(x) := |\alpha'(x)|^{-1/2} \sin \alpha(x), \qquad v(x) := |\alpha'(x)|^{-1/2} \cos \alpha(x), \tag{16}$$

see [2]

Therefore, knowing a single phase function, it is possible to retrieve a basis (and thus all solutions) to equation (13). In addition, zeros of every solution, for instance $u(x)$, can be *directly* obtained through $\alpha(x)$, since $u(x) = 0$ is equivalent to $\alpha(x) = k\pi$, $k \in \mathbf{Z}$. Therefore, the relation

$$|\alpha(x_k) - \alpha(x_{k-1})| = \pi \tag{17}$$

holds, being x_{k-1} and x_k *any two consecutive* zeros of *any given solution* of (13). In fact, once that $\alpha(x)$ has been computed, and one of such two zeros is known, it is possible to evaluate the other from(17). This amounts to solve a nonlinear equation like $\alpha(x) = const.$ Note that, by (14), all phase functions are strictly monotone. The concept of phase plays a central role in the description of the solutions of second-order linear differential equations (and of their properties). The approximation of a certain phase by the procedure described below, enables to compute *globally* all zeros of *any* particular solution. We connected this idea to Olver's Liouville-Green (WKB) asymptotic theory.

In [7], F.W.J. Olver describes a method for obtaining the zeros of a Bessel function and of other solutions of second-order differential equations solving a certain nonlinear differential equation. More precisely, he derived a third-order nonlinear differential equation satisfied by the generic zero, $x = \rho(\alpha)$, of the solution $y(x, \alpha)$,

$$y(x, \alpha) := u(x) \cos \alpha - v(x) \sin \alpha = 0,$$

considering it as a function of the real parameter α. Here, $u(x)$ and $v(x)$ represent any two linearly independent solutions of (1). This third-order equation can be written as

$$\rho'^2 = \frac{1}{q(\rho)} \left(1 - \frac{1}{2} \{\rho, \alpha\} \right), \tag{18}$$

see [7, 8].

It is easily checked that equation (15) can be taken into (18) upon the transformation $\rho = \rho(\alpha)$ and conversely. To evaluate the zeros of cylinder functions, Olver used the iterative scheme

$$\begin{aligned} \rho_0'^2 &= \frac{1}{q(\rho_0)}, \\ \rho_{n+1}'^2 &= \frac{1}{q(\rho_n)}\left(1 - \frac{1}{2}\{\rho_n, \alpha\}\right), \qquad n = 0, 1, 2, \ldots, \end{aligned} \tag{19}$$

suggested by (18). It turns out that $\{\rho, \alpha\} = O(\alpha^{-3})$ as $\alpha \to +\infty$ [7].

We found more convenient to base our approach on equation (15) rather than (18), hence we used instead the iterative scheme

$$\begin{aligned} \alpha_0'^2 &= q(x), \\ \alpha_{n+1}'^2 &= q(x) - \frac{1}{2}\{\alpha_n, x\}, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{20}$$

The sequence $\{\alpha_n'\}$ defined in (20), with $q(x)$ belonging to the classes (5), 6) or 7), converges (in a suitable sense) to a solution, $\alpha'(x)$, of (15),which possesses a certain asymptotic behavior and satisfies the relation (14), (u, v) being a *Liouville-Green* (or WKBJ) *basis*, see [8]. We call this phase function a *Liouville-Green phase* and write $\alpha(x) \equiv \alpha_{LG}(x)$. The convergence of α_n to α_{LG} is usually very fast and thus the method seems to be very competitive, especially to evaluate zeros of rapidly oscillatory solutions, when compared to methods based on the preliminary numerical evaluation of the solution to the original differential equation (1). The Liouville-Green basis is given by the Olver's Theorem [8, Ch. 6, Thm 11.1], which is of fundamental importance and ensure the existance of two linearly independent solutions, holomorphic in a complex domain, of an equation written as

$$\frac{d^2w}{dz} = (f(z) + g(z))\, w,$$

where $f(z)$ and $g(z)$ are holomorphic and $f(z)$ does not vanish.

Introducing the *auxiliary function* ϕ, setting

$$\phi := (\alpha')^2, \quad \phi_n := (\alpha_n')^2, \quad n = 0, 1, 2, \ldots, \tag{21}$$

equation (15) transforms into

$$\phi(x) = q(x) + [\phi, x]\,, \tag{22}$$

where we

$$[\phi, x] := -\frac{1}{4}\frac{\phi''(x)}{\phi(x)} + \frac{5}{16}\left(\frac{\phi'(x)}{\phi(x)}\right)^2,$$

and the iterative scheme in (20) becomes

$$\begin{aligned} \phi_0(x) &= q(x), \\ \phi_{n+1}(x) &= q(x) + [\phi_n, x]\,, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{23}$$

Note that the *nonlinear third-order* differential equation in (15), satisfied by $\alpha'(x)$, has been reduced to the *nonlinear second-order* equation in (22), satisfied by $\phi(x)$. The phase function α can be then recovered from ϕ through a quadrature. In addition, the scheme in (23) has the advantage with respect to that in (20) of avoiding the evaluation of square roots, which would imply some complications when working in the complex plane. Using ϕ_n instead of α_n is also convenient in a view of *symbolic manipulations*. In fact, the algorithm in (23) yields ϕ_n as a rational function of q and its derivatives up to the order $2n$.

From (16) and (17), using to the auxiliary function ϕ, we obtain

$$\begin{aligned} u(x) &= \phi^{-1/4}(x) \sin\left(\int^x \phi^{1/2}(t)dt\right), \\ v(x) &= \phi^{-1/4}(x) \cos\left(\int^x \phi^{1/2}(t)dt\right), \end{aligned} \tag{24}$$

and

$$\int_{x_k}^{x_{k+1}} \phi^{1/2}(t)dt = \pi. \tag{25}$$

If $\alpha(x)$ denotes a Liouville-Green phase, then the relation (25) ensures that the equation

$$\alpha(x_k) - \alpha(x) = \int_x^{x_k} \phi^{1/2}(t)dt = \pi, \tag{26}$$

x_k being *any* fixed zero, has the unique solution $x = x_{k-1}$. The relation (26) actually refers to the Liouville-Green basis (u, v). In practice we solve, rather,

$$\int_x^{x_k} \phi_n^{1/2}(t)dt - \pi = 0, \tag{27}$$

and start from a certain "large" zero, x_h, and compute successively the approximate values of the smaller zeros x_k, $k = h-1, h-2, \ldots$, which will be denoted by $\tilde{x_k}$.

Therefore, in evaluating zeros of solutions to (1), we can identify two main steps: we first obtain an approximation, ϕ_n, to ϕ, according to the algorithm (23) (see Theorem 2 and 3), and then solve equation (27).

When, for istans, $y(x)$ is a rational function, following this procedure, the degree of numerator and denominator of the rational functions involved in $\phi_n(x)$, in general increase exponentially with n, thus implying an *exponential* computational complexity. The convergence of the algorithm is however very fast (see (34),(39)-(40) in Sections 2, 3), thus providing very accurate results in few iterations.

In the examples below, $q(x)$ is given in terms of elementary functions, hence we are able to combine symbolic manipulations with numerical evaluations. We proceed as follows:

- compute ϕ_n symbolically with MATHEMATICA, following the scheme in (23), for $n = 0, 1, 2, 3, 4$;
- evaluate numerically the integral $J(x) := \int_x^{x_k} \phi_n^{1/2}(t)dt$ using Simpson's rule,

$$I_{2,m} = \frac{h}{6}\left[f(x_0) + 2\sum_{r=1}^{m-1} f(x_{2r}) + 4\sum_{s=0}^{m-1} f(x_{2s+1}) + f(x_{2m})\right]$$

, where

$$h = \frac{(b-a)}{m}, \quad m \geq 1,$$

and

$$x_k = a + k\frac{h}{2}, \quad k = 0, 1, \ldots, 2m.$$

In our case, $f(x) = \phi_n^{1/2}(x)$. To improve the accuracy in the evaluation of the integral, we observed experimentally that the value of m should increase with n, when $n = 0, 1, 2$ we choosed $m = 5$, when $n = 3, 4$, we choosed $m = 7$. The integration interval $[a, b]$, in our case is $[x, x_k]$, where x_k represents the "large" zero from which we started;

- find the zero of the function $f_{k,n}(x) := J(x) - \pi$ using the MATHEMATICA's function $FindRoot[f, \{x, x_0\}]$, which searches for a numerical root of f using Newton's method, starting from $x = x_0$.

In the following examples, we find the first 20 zeros of a given solution to equation (1), compare them with the exact values, and display the errors. Moreover, we solved a boundary-value problem and a Cauchy problem using the knowledge of the phase function of equation (1), constructing a basis of solutions, and thus *all* solutions.

2 Asymptotically constant carrier

Assume that the carrier enjoys the following properties: first of all $q(x)$ is a restriction to the real halfline, say $x > \rho$, of a function $q(z)$ holomorphic in the annular sector

$$S_{\rho,\gamma} \equiv \{z : z \in \mathbf{C}, |z| > \rho, |\arg(z)| < \gamma\}, \tag{28}$$

for certain constants ρ, γ, with $\rho \geq 0$ and $0 < \gamma \leq \frac{\pi}{2}$. Moreover, we assume that $q(z)$ possesses the asymptotic structure

$$q(z) = a + Q(z), \quad a > 0 \text{ (constant)}, \tag{29}$$

where

$$Q(z) = O(z^{-p}), \quad p > 1 \text{ (constant)}. \tag{30}$$

Denoting by $S(0,\theta)$ the sector with vertex at the origin and semiangle θ, consider the sequence of "decreasing" sectors

$$S_n = x_n + S(0,\theta), \tag{31}$$

$\{x_n\}$ being the increasing sequence of their vertices. The convergence of ϕ_n to ϕ is given by the following theorem:

Theorem 2 [11, Theorem II.1], [12]. Suppose that equation (1) is given, with $q(x)$ in the class (5) with $b = 0$ and that $S_{\rho',\gamma'}$ is a suitable subsector of $S_{\rho,\gamma}$. Then, there exists $x_0 \in S_{\rho',\gamma'} \cap \mathbf{R}$, which depends, in general, on all parameters entering the problem, such that the function $\phi_n(z)$, given by the complexified scheme (23), "converges" to the function $\phi_{LG}(z)$ defined by

$$\phi_{LG} := \frac{W^2}{(u^2(x) + v^2(x))^2} \equiv \frac{1}{(U(x)V(x))^2}. \tag{32}$$

where

$$\begin{aligned} u(x) &= \frac{U(x) + V(x)}{2}, \\ v(x) &= \frac{U(x) - V(x)}{2i}, \quad x \in S_{\rho,\gamma} \cap \mathbf{R}, \end{aligned} \tag{33}$$

(Liouville-Green basis).
The convergence is intended in the sense that, for any $\beta > a^{-1/2}$,

$$|\phi_n(z) - \phi_{LG}(z)| = a^{-1/2}(\beta a^{1/2})^{-2n} O((|z| - n\beta\sqrt{2})^{-1-p}), \tag{34}$$

$n = 0, 1, 2, \dots,$ for $z \in S_n$, where

$$x_n = x_0 + \frac{n\beta\sqrt{2}}{\sin\theta} \tag{35}$$

(see (31)).

Clearly, the error in (34) is *exponentially small.*

Theorem 2 also holds for the more general class (5) with $b \neq 0$.

3 Asymptotically polynomial carrier

We now consider the carrier in (6). It is required that $q(x)$ be the restriction to the real half-line $x > \rho$ of a function, $q(z)$, analytic in the annular sector $S_{\rho,\gamma}$ as in (28), and that

$$q(z) = cz^m \left[1 + o(1)\right], \qquad z \to \infty \text{ in } S_{\rho,\gamma}. \tag{36}$$

Without any loss of generality, we stipulate that $q(z)$ does not vanish in $S_{\rho,\gamma}$ and thus $q(x) > 0$ for $x > \rho$.

The convergence of the algorithm (23) to ϕ_{LG} is given by the following Theorem, established in [10] (see also [11]).

***Theorem* 3** [10, 11]. Let equation (1) be given with $q(x)$ "asymptotically polynomial", as in (6), (36). Then, there exists $x_0 \in (\rho, +\infty)$, depending on all parameters of the problem, such that the sequence $\phi_n(x)$ in (23) converges to the function $\phi_{LG}(x)$ defined by

$$\phi_{LG} := \frac{W^2\left[u, v\right]}{\left[u^2(x) + v^2(x)\right]^2} \equiv \frac{1}{\left[u^2(x) + v^2(x)\right]^2}, \tag{37}$$

where

$$\begin{aligned} u(x) &:= \mathrm{Re}\left[U_+(x)\right] = \frac{U_+(x) + U_-(x)}{2}, \\ v(x) &:= \mathrm{Im}\left[U_+(x)\right] = \frac{U_+(x) - U_-(x)}{2i} \end{aligned} \tag{38}$$

is a *real Liouville-Green basis* on the real half-line $x > \rho'$, and $U_+(x) = \overline{U_-(x)}$. The convergence is intended in the sense that

$$\left|\phi_{LG}(x) - \phi_n(x)\right| \leq C h_n(x) \left(x - n\sqrt{2}\right)^{-2}, \; x > x_n, \tag{39}$$

where

$$
\begin{aligned}
&h_0(x) := 1, \\
&h_n(x) := C^n \prod_{j=0}^{n-1} \left(x - j\sqrt{2}\right)^{-m}, \qquad n = 1, 2, 3, \ldots,
\end{aligned}
\tag{40}
$$

and

$$
x_n := x_0 + n\frac{\sqrt{2}}{\sin\gamma^*}, \qquad x_0 \geq \rho^*, \quad n = 0, 1, 2, \ldots, \tag{41}
$$

γ^* and ρ^* being the semi-angle and the radius of a suitable annular subsector of $S_{\rho,\gamma}$, and C a suitable constant.

4 Asymptotically exponential carrier

In this section we apply the algorithm (23) when the carrier is "asymptotically exponential" as in (7). In [13] it was established the following

Lemma **4** [13, Lemma 5.1]. Assume that

$$
q(x) = e^{ax} + g(x), \tag{42}
$$

where $a > 0$, $g^{(i)}(x) = O(e^{bx})$, $\forall i \geq 0$, $0 < b < a$, and that ϕ_n, $n \geq 0$, is recursively defined by the scheme (23). Then,

$$
\phi_n = e^{ax} + g(x) + c_n + G_n(x), \tag{43}
$$

where

$$
\begin{array}{rclcrcl}
c_0 & = & 0, & c_n & = & \frac{a^2}{16}, & n \geq 1 \\
G_0 & = & 0, & G_n^{(i)} & = & O(e^{-\omega x}), & n \geq 1,\ \forall i \geq 0,\ \omega := a - b.
\end{array}
\tag{44}
$$

5 Boundary-value problems

The method described in the previous sections, makes it possible to approximate a (Liouville-Green) basis of solutions to equation (1), and thus *any* solution, to a given initial- or boundary-value problem for (1). In fact, knowing *any* phase, $\alpha(x)$, a basis can be constructed by the relations in (16).

In practice, we are able to construct an *approximate* basis $(u_n(x),\, v_n(x))$, through the iterative scheme in (20), namely

$$u_n(x) := |\alpha_n'(x)|^{-1/2}\sin\alpha_n(x), \qquad v_n(x) := |\alpha_n'(x)|^{-1/2}\cos\alpha_n(x). \tag{45}$$

Consider, e.g., the boundary-value problem

$$y'' + q(x)y = 0 \qquad a < x < b, \tag{46}$$

$$y(a) = A, \qquad y(b) = B, \tag{47}$$

where the coefficient $q(x)$ belongs to one of the classes described above. Its solution has necessarily the form

$$y(x) = c_1 u(x) + c_2 v(x), \tag{48}$$

where $(u(x), v(x))$ is a basis of solutions to (1), and the constants c_1, c_2 can be determined as functions of a, b, A, B, solving the linear system

$$\begin{cases} c_1 u(a) + c_2 v(a) = A \\ c_1 u(b) + c_2 v(b) = B. \end{cases} \tag{49}$$

Since the phase function $\alpha(x)$ defines the basis in (16), the solution (48) takes the form

$$y(x) = c_1|\alpha'(x)|^{-1/2}\sin\alpha(x) + c_2|\alpha'(x)|^{-1/2}\cos\alpha(x), \tag{50}$$

and the system (49) has a unique solution if and only if

$$|\alpha'(a)\alpha'(b)|^{-1/2}\begin{vmatrix} \sin\alpha(a) & \cos\alpha(a) \\ \sin\alpha(b) & \cos\alpha(b) \end{vmatrix} = |\alpha'(a)\alpha'(b)|^{-1/2}\sin\left[\alpha(a) - \alpha(b)\right] \neq 0,$$

and hence $\sin\left[\alpha(a) - \alpha(b)\right] \neq 0$, or

$$\alpha(a) - \alpha(b) \neq k\pi,\ \forall k \in \mathbf{Z}. \tag{51}$$

If such condition is satisfied, we obtain, by Cramer's rule

$$\begin{cases} c_1 = \frac{A|\alpha'(a)|^{1/2}\cos\alpha(b) - B|\alpha'(b)|^{1/2}\cos\alpha(a)}{\sin[\alpha(a)-\alpha(b)]} \\ \\ c_2 = \frac{B|\alpha'(b)|^{1/2}\sin\alpha(a) - A|\alpha'(a)|^{1/2}\sin\alpha(b)}{\sin[\alpha(a)-\alpha(b)]}. \end{cases} \tag{52}$$

Consider now the *approximate* solution

$$y_n(x) := c_1(n)u_n(x) + c_2(n)v_n(x),$$

where $u_n(x)$ and $v_n(x)$ are given by (45). The "constants" $c_1(n)$, $c_2(n)$ can also be determined *approximately* as functions of the known data, imposing the boundary conditions, i.e., solving the linear system

$$\begin{cases} c_1(n)\sin\alpha_n(a) + c_2(n)\cos\alpha_n(a) = A|\alpha_n'(a)|^{1/2} \\ c_1(n)\sin\alpha_n(b) + c_2(n)\cos\alpha_n(b) = B|\alpha_n'(b)|^{1/2}, \end{cases} \tag{53}$$

for the unknowns $c_1(n)$ and $c_2(n)$. This system has a unique solution if and only if $\sin[\alpha_n(a) - \alpha_n(b)] \neq 0$ and the result coincide with that in (52), replacing α' with α_n'.

To recover the phase function α_{LG} from the auxiliary function ϕ_{LG} we should choose a branch of the square root, $\sqrt{\phi_{LG}(x)}$ (being $\phi_{LG}(x) = (\alpha_{LG}'(x))^2$), and a value k associated with a point $\overline{x} \in (0, \infty)$, such that $\alpha_{LG}(\overline{x}) = k$,

$$\alpha_{LG}(x) = \int_{\overline{x}}^{x} \phi_{LG}^{1/2}(t)\,dt + k. \tag{54}$$

Therefore, the approximate phase function $\alpha_n(x)$ will be given by

$$\alpha_n(x) = \int_{\overline{x}}^{x} \phi_n^{1/2}(t)\,dt + k, \tag{55}$$

setting $\alpha_n(\overline{x}) = k$ and recalling that $\phi_n(x)$ approximates $\phi_{LG}(x)$, according to Theorems 2 or 3.

We now estimate the error made when the approximate solution, $y_n(x)$, is used instead of $y(x)$, on the interval $[a, b]$. Writing

$$c_1(n) = c_1 + \epsilon_1, \quad c_2(n) = c_2 + \epsilon_2, \tag{56}$$

we obtain promptly

$$\begin{aligned} |y_n(x) - y(x)| &\leq |c_1||u_n(x) - u(x)| + |c_2||v_n(x) - v(x)| \\ &+ |\epsilon_1||u_n(x)| + |\epsilon_2||v_n(x)|. \end{aligned} \tag{57}$$

Hence we can recognize two kinds of error, one due to the fact that we use the approximate basis of solutions $(u_n(x), v_n(x))$ instead of $(u(x), v(x))$, and the other one due to the fact that we use the approximate coefficients $c_1(n)$, $c_2(n)$ instead of c_1 and c_2. Setting

$$E_1 := |c_1||u_n(x) - u(x)| + |c_2||v_n(x) - v(x)|, \tag{58}$$

and choosing the positive branch of the square roots of ϕ_n and ϕ_{LG} we have, after a little algebra,

$$E_1 \leq \left(\frac{|c_1| + |c_2|}{\sqrt{\alpha'_n}\sqrt{\alpha'}} \right) \left(\frac{|\alpha'_n - \alpha'|}{\sqrt{\alpha'_n} + \sqrt{\alpha'}} + \sqrt{\alpha'}|\alpha_n - \alpha| \right).$$

From the relation

$$\alpha'_n - \alpha' = \frac{\phi_n - \phi_{LG}}{\alpha'_n + \alpha'}, \tag{59}$$

being $\alpha'(x) \neq 0 \forall x \in (0, +\infty)$, there exist $n_0 > 0$ and $\eta_n > 0$ such that, for $n > n_0$,

$$\eta_n := \sup_{x \in [a,b]} \frac{H_n(x)}{|\alpha'_n(x) + \alpha'(x)|} < \infty. \tag{60}$$

Here $H_n(x)$ depends on the class of the carriers. When the carrier is asymptotically constant, from Theorem 2 we have

$$H_n(x) := C\, a^{-1/2} (\beta a^{1/2})^{-2n} (|x| - n\beta\sqrt{2})^{-1-p}.$$

When the carrier is asymptotically polynomial, from Theorem 3 we have

$$H_n(x) := C' h_n(x) \left(x - n\sqrt{2} \right)^{-2},$$

where C and C' are two positive constants. It follows that

$$|\alpha'_n - \alpha'| \leq \eta_n, \tag{61}$$

and furthermore

$$|\alpha_n(x) - \alpha(x)| \leq \eta_n(b-a) \tag{62}$$

for all $x \in [a,b]$. From (61) follows, for a and n sufficiently large,

$$(\alpha')^{-1/2} \leq (\alpha'_n - \eta_n)^{-1/2} \quad \text{and} \quad \alpha' \leq \alpha'_n + \eta_n.$$

Therefore the estimate for E_1 becomes, after some algebra,

$$E_1 \leq \sqrt{2}\,(|c_1| + |c_2|) \sup_{x\in[a,b]} \frac{1}{\alpha'_n(x)} \left[\frac{5}{4}\sqrt{\alpha'_n(x)}(b-a) + \frac{2-\sqrt{2}}{\sqrt{\alpha'_n(x)}}\right]\eta_n, \tag{63}$$

for η_n sufficiently small, i.e. for n sufficiently large.

The error ϵ_1 made using $c_1(n)$ instead of c_1 turns out to be

$$\begin{aligned} |\epsilon_1| \quad &\leq \quad \left[3\left(\tfrac{|A|(\alpha'_n(a)+\eta_n)^{1/2}+|B|(\alpha'_n(b)+\eta_n)^{1/2}}{\sin^2[\alpha_n(a)-\alpha_n(b)]}\right)(b-a)\right. \\ &+ \quad \left.\tfrac{|A|(\alpha'_n(a)+\eta_n)^{-1/2}+|B|(\alpha'_n(b)+\eta_n)^{-1/2}}{2|\sin[\alpha_n(a)-\alpha_n(b)]|}\right]\eta_n, \end{aligned} \tag{64}$$

and the same holds for ϵ_2. Hence the global error can be estimated as

$$\begin{aligned} E \quad &\leq \quad (|c_1(n)| + |c_2(n)|)\sup\nolimits_{x\in[a,b]}\left[\tfrac{5\sqrt{2}}{4\sqrt{\alpha'_n(x)}}(b-a) + \tfrac{2\sqrt{2}-2}{(\alpha'_n(x))^{\frac{3}{2}}}\right]\eta_n \\ &+ \quad (2\sqrt{2}+2)\sup\nolimits_{x\in[a,b]}\left[\tfrac{|A|(\alpha'_n(a)+\eta_n)^{-\frac{1}{2}}+|B|(\alpha'_n(b)+\eta_n)^{-\frac{1}{2}}}{2\sqrt{\alpha'_n(x)}|\sin[\alpha_n(a)-\alpha_n(b)]|}\right. \\ &+ \quad 3\left(\tfrac{|A|(\alpha'_n(a)+\eta_n)^{\frac{1}{2}}+|B|(\alpha'_n(b)+\eta_n)^{\frac{1}{2}}}{\sqrt{\alpha'_n(x)}\sin^2[\alpha_n(a)-\alpha_n(b)]}\right)(b-a) \\ &+ \quad \left.\tfrac{5}{4\sqrt{\alpha'_n(x)}}(b-a) + \tfrac{2-\sqrt{2}}{(\alpha'_n(x))^{\frac{3}{2}}}\right]\eta_n. \end{aligned} \tag{65}$$

6 Cauchy problems

Consider the Cauchy problem

$$\begin{cases} y'' + q(x)y = 0, \quad x > a \geq 0, \\ y(a) = A \\ y'(a) = B, \end{cases} \tag{66}$$

where $q(x)$ belongs to one of the classes described above. Again, its general solution can be written as

$$y(x) = c_1 u(x) + c_2 v(x), \tag{67}$$

where $(u(x), v(x))$ is a basis of solutions, and the constants c_1, c_2 can be evaluated solving the linear system

$$\begin{cases} c_1 u(a) + c_2 v(a) = A \\ c_1 u'(a) + c_2 v'(a) = B. \end{cases} \tag{68}$$

From the phase function $\alpha(x)$ we obtain the basis in (16), then the solution will be as in (50). System (68) has a unique solution if and only if the determinant is nonzero, and in fact

$$\begin{vmatrix} (\alpha'(a))^{-1/2} \sin\alpha(a) & (\alpha'(a))^{-1/2} \cos\alpha(a) \\ -\frac{\alpha''(a)}{2(\alpha'(a))^{3/2}} \sin\alpha(a) + \sqrt{\alpha'(a)} \cos\alpha(a) & -\frac{\alpha''(a)}{2(\alpha'(a))^{3/2}} \cos\alpha(a) - \sqrt{\alpha'(a)} \sin\alpha(a) \end{vmatrix} = -1.$$

We obtain (by Cramer's rule) the constants c_1 and c_2,

$$\begin{cases} c_1 = \frac{A\left[\alpha''(a) \cos\alpha(a) + 2\alpha'(a)^2 \sin\alpha(a)\right] + 2B\alpha'(a) \cos\alpha(a)}{2(\alpha'(a))^{3/2}} \\ \\ c_2 = \frac{A\left[-\alpha''(a) \sin\alpha(a) + 2\alpha'(a)^2 \cos\alpha(a)\right] + 2B\alpha'(a) \sin\alpha(a)}{2(\alpha'(a))^{3/2}}. \end{cases} \tag{69}$$

Considering now the *approximate* solution

$$y_n(x) := c_1(n) u_n(x) + c_2(n) v_n(x),$$

with $(u_n(x), v_n(x))$ given by (45), the "constants" $c_1(n)$ and $c_2(n)$ can be determined *approximately* solving the linear system

$$\begin{cases} c_1(n)u_n(a) + c_2(n)v_n(a) = A \\ c_1(n)u_n'(a) + c_2(n)v_n'(a) = B. \end{cases} \tag{70}$$

This system has a unique solution which coincide with that in (69), replacing α' with α_n'.

As in the previous section, we want to estimate the error made when the approximate solution $y_n(x)$ is used instead of $y(x)$. Also in this case we write

$$c_1(n) = c_1 + \epsilon_1, \quad c_2(n) = c_2 + \epsilon_2, \tag{71}$$

and after rather lengthy though elementary calculations we obtain the same estimate in (57). Again, there are two kinds of errors, one due to the fact that we are using the approximate solutions $u_n(x)$, $v_n(x)$ instead of $u(x)$, $v(x)$,

$$E_1' := |c_1||u_n(x) - u(x)| + |c_2||v_n(x) - v(x)|, \tag{72}$$

the other due to the fact that we are using the approximate coefficients $c_1(n)$ and $c_2(n)$ instead of c_1 and c_2.

The estimate for the error E_1' coincides with that in (63), but in this case we impose that

$$\eta_n := \sup_{x\in[a,\tilde{b}]} \frac{H_n(x)}{|\alpha_n'(x) + \alpha'(x)|} < \infty \tag{73}$$

and

$$|\alpha_n(x) - \alpha(x)| \le \left| \int_{\overline{x}}^{x} |\alpha_n'(t) - \alpha'(t)| dt \right| \le \eta_n(\tilde{b} - a), \tag{74}$$

so that

$$E_1' \le \sup_{x\in[a,\tilde{b}]} \frac{|c_1| + |c_2|}{\alpha_n'(x)} \left[\sqrt{\alpha_n'(x)}(\tilde{b} - a) + \frac{1}{2\sqrt{\alpha_n'(x)}} \right] \eta_n, \tag{75}$$

where $\tilde{b} > a$ can be always chosen to satisfy inequality (74).

To estimate ϵ_1 define

$$\Gamma_n := a^{-\frac{1+2n}{2}} K_n \sup_{x\in[a,\tilde{b}]} \left[\left(|x| - n\sqrt{2}\right)^{-1-p} + \frac{\left(|x| - (n+1)\sqrt{2}\right)^{-1-p}}{\sqrt{2}} \right]$$

and

$$\widehat{\Gamma}_n := 2\widehat{K}_n C^{n+1}\left[\prod_{j=0}^{n}(x-j\sqrt{2})^{-m}\right]\left[(x-n\sqrt{2})^{-2}+(x-(n+1)\sqrt{2})^{-2}\right],$$

where $K_n := \max\{K_a,\, K_b\}$, $\widehat{K}_n := \max\left\{\widehat{K}_a,\, \widehat{K}_b\right\}$ and C is a suitable constant. Here K_a, K_b, $\widehat{K}_a$, $\widehat{K}_b$ are four positive constants. When the carrier is asymptotically constant, exist two values x_0, $x_0' > 0$ such that

$$\begin{aligned} \left|\tfrac{q'(x)}{q(x)^{3/2}}\right| &\leq K_a \ \text{ for } a < x_0 \leq x < \tilde{b}, \\ |q(x)|^{-1/2} &\leq K_b \ \text{ for } a < x_0' \leq x < \tilde{b}, \end{aligned} \tag{76}$$

while, when the carrier is asymptotically polynomial, exist other two values $\hat{x}_0$, $\hat{x}_0' > 0$ such that

$$\begin{aligned} \left|\tfrac{q'(x)}{q(x)^{3/2}}\right| &\leq \widehat{K}_a \ \text{ for } a < \hat{x}_0 \leq x < \tilde{b}, \\ |q(x)|^{-1/2} &\leq \widehat{K}_b \ \text{ for } a < \hat{x}_0' \leq x < \tilde{b}. \end{aligned} \tag{77}$$

Hence, when $q(x)$ is asymptotically constant, an estimate for ϵ_1 is

$$\begin{aligned} |\epsilon_1| &\leq \tfrac{|A|}{2(\alpha_n'(a)+\eta_n)^{3/2}}\Gamma_n + \tfrac{|A|\left[\alpha_n''(a)+\Gamma_n+2(\alpha_n'(a)+\eta_n)^2\right]+2|B|(\alpha_n'(a)+\eta_n)}{2(\alpha_n'(a)+\eta_n)^{3/2}}\eta_n(\tilde{b}-a) \\ &+ \tfrac{14|A|(\alpha_n'(a)+\eta_n)^2+10|B|(\alpha_n'(a)+\eta_n)+3|A|(\alpha_n''(a)+\Gamma_n)}{4(\alpha_n'(a)+\eta_n)^{7/2}}\eta_n, \end{aligned} \tag{78}$$

and proceeding similarly we obtain for ϵ_2

$$\begin{aligned} |\epsilon_2| &\leq \tfrac{|A|}{2|\alpha_n'(a)|^{3/2}}\Gamma_n + \tfrac{|A|\left[|\alpha_n''(a)|+2(\alpha_n'(a))^2\right]+2|B||\alpha_n'(a)|}{2|\alpha_n'(a)|^{3/2}}\eta_n(\tilde{b}-a) \\ &+ \tfrac{2|A|(\alpha_n'(a))^2+2|B||\alpha_n'(a)|+3|A||\alpha_n''(a)|}{4|\alpha_n'(a)|^{7/2}}\eta_n. \end{aligned} \tag{79}$$

Therefore, when $q(x)$ is asymptotically constant, the global error can be estimated as

$$
\begin{aligned}
E' \ \leq\ & \sup_{x\in[a,\tilde{b}]} \left\{ \tfrac{|c_1|+|c_2|}{\alpha_n'(x)} \left[\sqrt{\alpha_n'(x)}(\tilde{b}-a) + \tfrac{1}{2\sqrt{\alpha_n'(x)}} \right] \eta_n \right. \\
+\ & \tfrac{2}{\sqrt{\alpha_n'(x)}} \left[\tfrac{|A|}{2(\alpha_n'(a)+\eta_n)^{3/2}} \Gamma_n + \tfrac{|A|\left[\alpha_n''(a)+\Gamma_n+2(\alpha_n'(a)+\eta_n)^2\right]+2|B|(\alpha_n'(a)+\eta_n)}{2(\alpha_n'(a)+\eta_n)^{3/2}} \eta_n(\tilde{b}-a) \right] \\
+\ & \left. \tfrac{1}{\sqrt{\alpha_n'(x)}} \left[\tfrac{16|A|(\alpha_n'(a)+\eta_n)^2+12|B|(\alpha_n'(a)+\eta_n)+6|A|(\alpha_n''(a)+\Gamma_n)}{4(\alpha_n'(a)+\eta_n)^{7/2}} \right] \eta_n \right\}.
\end{aligned}
\tag{80}
$$

When $q(x)$ is asymptotically polynomial, the estimates of ϵ_1, ϵ_2 and E' turn out to be the same, with $\widehat{\Gamma}_n$ replacing Γ_n.

6.1 Comparing with the Runge-Kutta-Nystrom 12/10 method

In order to assess the performance of our method, we compared it with another one very effective, that is the so-called “Runge-Kutta-Nystrom 12/10” algorithm.
This method consist of a numerical integrator for ordinary differential equations of the form

$$y''(x) = f(x, y), \tag{81}$$

subject to initial conditions $y(x_0) = y_0$, $y'(x_0) = dy_0$, which exploits two Runge-Kutta methods of order 10 and 12.
The MATLAB program RKN12(10) implements such methods providing a high-order algorithm with automatically control of the step-size, used in problems where extremely stringent error tolerances are required.

In [4], it is assumed that the Runge-Kutta-Nystrom (RKN) process consists of formulae of orders q and p (with $q > p$) and the algorithm is shown. The conditions that the RKN parameters should satisfy for the process up to order 6 were given in [5], and those for the process up to order 11 was found by El-Mikkawy in 1986.

We considered two examples, both concerning a perturbed generalized

Airy equation. Choosing first the phase function $\alpha(x)$, the carrier $q(x)$ can be estimated, and hence the exact solution is known. Thus, the accurancy of our method and that of the RKN12(10) method can be compared computing the discrepancy between the approximate solution and the exact one.

From such examples it appears that the RKN12(10) method is faster than ours, but recall that our method exploits (in the present examples) also symbolic manipulations which are rather slow. On the other hand, our method works for *arbitrarily rapid* oscillations of solutions, and has an infinite accurancy in those sections where symbolic manipulations are used.

Moreover, in several cases the RKN12(10) method fails completely, for instance, when the step-size falls below the minimum acceptable value $\frac{t_{fin}-t_{in}}{10^{12}}$, see [4]. An example of this occurrence is encountred solving the Cauchy problem for the harmonic oscillator,

$$\begin{cases} y'' + \omega^2 y = 0, \qquad x > 0, \\ y(0) = 0 \\ y'(0) = \omega. \end{cases} \tag{82}$$

Choosing increasing values of ω, the algorithm RKN12(10) at $\omega = 10^5$ had to be stopped, while our method provided a solution with an the error essentially equal to zero, even for values of ω larger than this.

7 Examples

In this section we are concerned with the asymptotic-numerical approximation of zeros of a given solution to equation (1) when the carrier is asymptotically constant, polynomial and exponential. We also approximate the solution of a boundary-value problem. All examples were worked out within the MATHEMATICA environment.

7.1 Computing zeros from the phase function

In this example, we start from an explicit form of a phase function, $\alpha(x)$. Recall that any given phase function uniquely identifies the carrier of (1) throught the relation (15). Choosing for instance $\alpha(x) = x + \frac{1}{x}$, $x > 0$, we obtain

$$q(x) = \frac{x^8 - 4x^6 + 3x^4 - 4x^2 + 1}{x^8 - 2x^6 + x^4}, \tag{83}$$

hence equation (1) becomes

$$y'' + \left(\frac{x^8 - 4x^6 + 3x^4 - 4x^2 + 1}{x^8 - 2x^6 + x^4} \right) y = 0. \tag{84}$$

We want to compute the first 20 zeros of a solution to (84), starting from the $21th$ exact zero, and compare them with the "exact" zeros that can be obtained through MATHEMATICA. Figures 1 and 4 show the approximating function $\phi_n(x)$ of the auxiliary function $\phi(x)$ and the errors (in a logarithmic scale) at each iteration. We can see that the functions ϕ_n, $n = 0, 1, 2, 3, 4$ approximate the function $\phi(x)$, from a certain point on, and that the errors decrease dramatically increasing the values of k and n.

7.2 A perturbed generalized Airy equation

Consider now the case of a perturbed generalized Airy equation, corresponding to the carrier $q(x) = x^2 - \frac{3}{4x^2}$ in (1), i.e.,

$$y'' + \left(x^2 - \frac{3}{4x^2} \right) y = 0 \quad x > 0. \tag{85}$$

In this case the phase function $\alpha(x)$ is known, $\alpha(x) = \frac{x^2}{2}$, and thus the function

$$y(x) = x^{-1/2} \sin\left(\frac{x^2}{2} \right)$$

is a solution to (85) (see (16)), and its zeros are

$$x_k = \sqrt{2k\pi}, \quad k = 1, 2, 3, \ldots$$

To test our method, we evaluate the $21th$ zero from this formula, and then compute the first 20 zeros using our algorithm for $n = 0,\ 1,\ 2,\ 3$. Figures 2-5 show results similar to those in the previous example. Note that knowing $\alpha(x)$ we also know $\phi(x)$, in fact $\phi(x) = (\alpha'(x))^2 = x^2$. This function has also been plotted in Figure 2.

7.3 A perturbed exponential carrier

Consider equation (1) with the carrier $q(x) = e^{2x} - 4$, i.e., equation

$$y'' + (e^{2x} - 4)y = 0. \tag{86}$$

This is a special case of

$$y'' + \left(\lambda^2 e^{2x} - \nu^2\right) y = 0, \tag{87}$$

whose general solution is given by

$$y(z) = C_\nu \left(\lambda e^z\right) \tag{88}$$

where C_ν is the general solution to Bessel differential equation [1].

For $\lambda = 1,\ \nu = 2$, we have thus

$$y(x) = C_2(e^x). \tag{89}$$

Consider in particular the solution

$$y(x) = J_2(e^x),$$

and compute the first 20 zeros of this function using (27), for $k = 1, 2, \ldots, 20$, and $n = 0, 1, 2, 3, 4$. We then compare the results with the first 20 “exact” zeros of $J_2(e^x)$ obtained by MATHEMETICA. In Figures 3, 6 the behavior of $\phi_n(x)$, for $n = 0, 1, 2, 3, 4$, and the errors, in logarithmic scale, $-\frac{1}{\log(x_k - \tilde{x}_k)}$, are shown.

7.4 A boundary-value problem for a perturbed generalized Airy equation

Consider the two-point BV problem for the perturbed generalized Airy equation

$$\begin{cases} y'' + \left(x^2 - \frac{3}{4x^2}\right) y = 0, & a < x < b \\ y(a) = A \\ y(b) = B, \end{cases} \tag{90}$$

whose solution has (necessarily) the form

$$y(x) = c_1|\alpha'(x)|^{-1/2} \sin\alpha(x) + c_2|\alpha'(x)|^{-1/2} \cos\alpha(x)$$

with $\alpha(x) = \frac{x^2}{2}$. Let choose the value of the two constants, for instance, $c_1 = 1$, $c_2 = -1$, and compute the boundary values A, B so that the boundary conditions in (90) are satisfied. Then, we compute $\alpha_n(x)$ by (55), using the Simpson's rule to evaluate the integral, where we chose $\overline{x} \equiv a$ and $k = \alpha(a)$.

According to (45), an approximate solution is

$$y_n(x) = c_1(n)|\alpha_n'(x)|^{-1/2} \sin\alpha_n(x) + c_2(n)|\alpha_n'(x)|^{-1/2} \cos\alpha_n(x),$$

where $c_1(n)$ and $c_2(n)$ have been determinated solving the linear system (53) using the MATHEMATICA's function *LinearSolve*$[\mathbf{M}, b]$. Here the system matrix, $\mathbf{M}$, is the square matrix

$$\mathbf{M} := \begin{pmatrix} |\alpha_n'(a)|^{-1/2} \sin\alpha_n(a) & |\alpha_n'(a)|^{-1/2} \cos\alpha_n(a) \\ |\alpha_n'(b)|^{-1/2} \sin\alpha_n(b) & |\alpha_n'(b)|^{-1/2} \cos\alpha_n(b) \end{pmatrix},$$

and b is the vector

$$b = \begin{pmatrix} A \\ B \end{pmatrix}.$$

This problem has been solved on two intervals, $6 < x < 8$ and $9.5 < x < 11.5$, for $n = 3, 4$. The errors turned out to be of order of 10^{-12} and 10^{-15}, respectively.

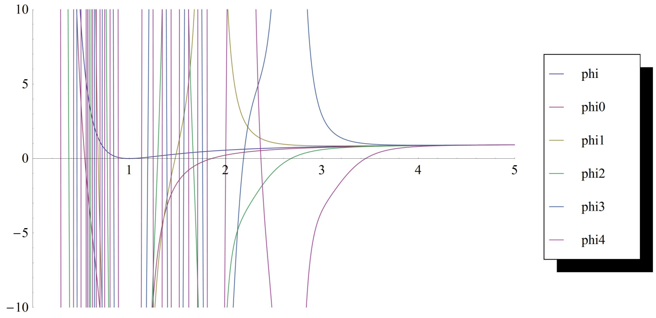

Figure 1: Functions $\phi_n(x)$, approximating $\phi(x) = 1 + \frac{1}{x^4} - \frac{2}{x^2}$, for $n = 0, 1, 2, 3, 4$.

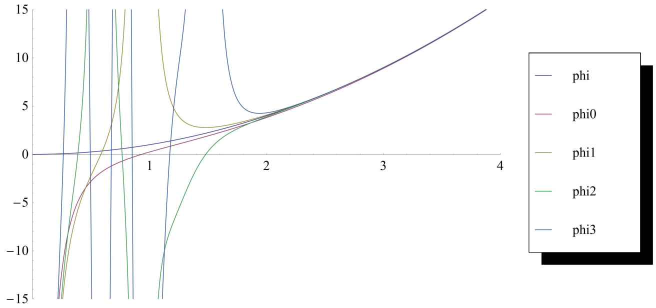

Figure 2: Approximations of $\phi(x) = x^2$ by $\phi_n(x)$ with $n = 0, \ldots, 3$, when $q(x) = x^2 - \frac{3}{4x^2}$.

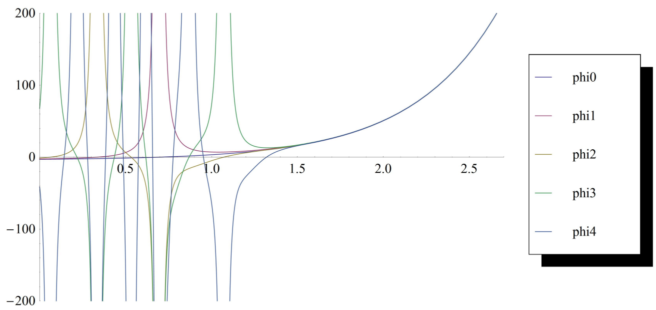

Figure 3: Approximations of $\phi(x)$ by $\phi_n(x)$ with $n = 0, \ldots, 4$, when $q(x) = e^{2x} - 4$.

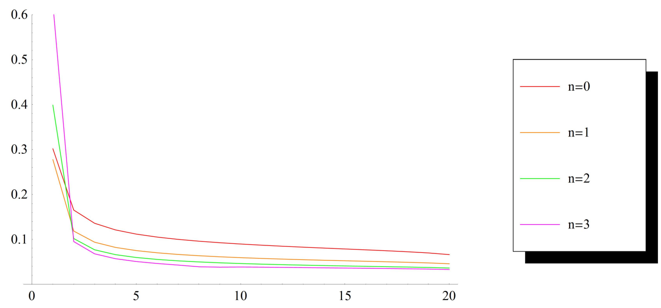

Figure 4: $-\frac{1}{\log(x_k-\tilde{x}_k)}$ for $k=1,2,\ldots,20$, obtained using $\phi_n(x)$ with $n=0,\ldots,3$, when $\alpha(x)=x+\frac{1}{x}$.

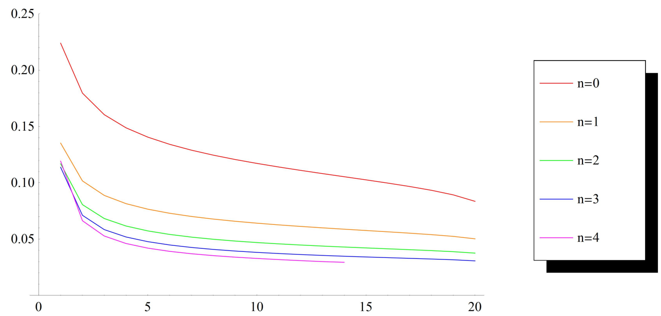

Figure 5: $-\frac{1}{\log(\tilde{x}_k-x_k)}$ for $k=1,\ldots,20$ using $\phi_n(x)$ with $n=0,\ldots,4$, when $q(x)=x^2-\frac{3}{4x^2}$.

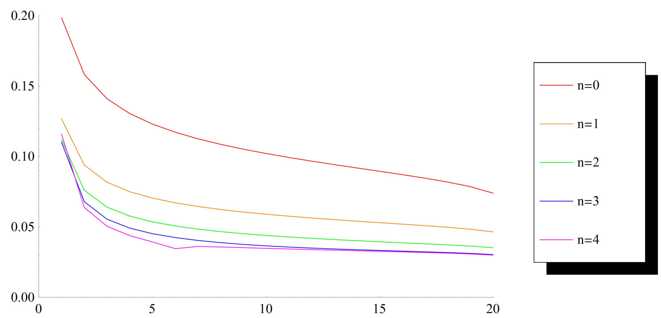

Figure 6: $-\frac{1}{\log(\tilde{x}_k-x_k)}$ for $k=1,\ldots,20$ using $\phi_n(x)$ with $n=0,\ldots,4$, when $q(x)=e^{2x}-4$.

8 Conclusion and future directions

There is no question that the second-order linear ordinary differential equations (see (1)), though so classical and simple, play a fundamental role in the entire body of Mathematics, as well as in a large number of applications. Investigations on them, in a way or another, have been continuing even over the last years. These equations play a role, e.g., in celestial mechanics, quantum mechanical scattering theory, theoretical physics and chemistry, and electronics, as is witnessed by the recent scientific literature. For instance, T.E. Simos and Z.A. Anastassi in [9] consider the numerical integration of the Schrödinger equation, which describes the time variation of the quantum state of a physical system, and the related IVPs with oscillatory solution. Developing methods computationally effective and accurate at the same time, in the *rapidly oscillatory* case, is still an open problem.

In this Thesis, we studied an asymptotic-numerical method capable of computing rapidly oscillatory solutions as well as their zeros with high accuracy. This method rests on the investigations carried out by O. Boruvka and F. W. J. Olver, and has been applied to a number of cases. In each example, the results are shown to be very accurate, with absolute errors possibly of order of 10^{-18}. The algorithm was not very fast, but just because we used symbolic manipulations (carried out with MATHEMATICA) as an essential ingredient. This however ensured a high accuracy. Since the method converges very fast, only few iterations are required, and hence the exponential complexity due to the symbolic part can be limited severely.

Symbolic manipulations could be used because, in our examples, the *carrier*, i.e., the coefficient $q(x)$ in equation (1), was assumed to be explictly known, in terms of elementary functions.

As a future direction, one can conceive to compute the *phase function* $\alpha(x)$ (or the related auxiliary function $\phi(x)$), by purely numerical methods which changes, in some sense, the difficulty of computing *higly oscillatory* solutions into the evaluation of a phase function, which is a *monotonic* function. An efficient solution of such problem, which ultimately may shift the difficulty of handling rapidly oscillatory problems at worst to a stiff problem, is expected to be fast and accurate at the same time.

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