The Jacobian ideal of an hypersurface

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One of the main purposes of algebraic geometry is to explore the algebraic structure that lies within geometric objects in order to better understand their properties.

When focusing solely on the geometric aspects of an issue, it is very likely to miss some of the key information that could help to take hold on the problem. On the other hand, using algebra to describe geometry offers a different point of view.

Algebraic geometry uses tools like tensor products, the Hilbert polynomial of a graded module or exact sequences, which are born in the field of commutative or homological algebra, to examine geometric objects like varieties and more generally sheaves and schemes.

The first natural example is that of the zero locus of an homogeneous polynomial f in $\mathbb{C}[x_0, ..., x_n]$. Such space is a variety V of codimension 1 in $\mathbb{P}^n$ whose properties depend on the polynomial defining it. For instance we have that the polynomials in the irreducible factorization of f describe the irreducible components of V.

In this thesis we will focus on the study of singularities of an irreducible hypersurface. The simplest approach to this matter is to consider the partial derivatives of f and see where they vanish at the same time.

The ideal generated by the partial derivatives of a polynomial f is called the **Jacobian ideal** of f.

It is the object of a lot of studies, but at the same time, it is not yet fully understood. In this thesis we will consider some of its aspects, in order to give an idea of its known properties and of how it is related to some open problems.

Due its structure, it is evident that the singular locus Σ of V is the zero locus of the Jacobian ideal J of f. The simplest case to consider is therfore when $\Sigma = \emptyset$ or as well when V is not singular.

Under these conditions J contains a power of the irrilevant ideal $(X_0, ..., X_n)$, so it has maximum depth in $\mathbb{C}[X_0, ..., X_n]$ and it is generated by a regular sequence. This implies that the Koszul complex of the partial derivatives of f is a free minimal resolution for the quotient of $\mathbb{C}[x_0, ..., x_n]$ with respect to J. Such ring is called the **Jacobian ring** of f and will have main relevance in this thesis as it contains a great amount of information regarding the hypersurface V. Furthermore in this case the Jacobian ring R is an Artinian Gorenstein ring, which means that it has finite length and that if we denote with $\mathcal{F}$ a minimal resolutions for R then $\mathcal{F} \cong \mathcal{F}^*$.

The main properties of the Jacobian ring R of a smooth hypersurface are described by a classical theorem due to Macaulay.

Theorem 1 (Macaulay). *Given a regular sequence of homogeneous polynomials $f_0, ..., f_n$ of degree $degf_i = d_i$, then the quotient R of $\mathbb{C}[X_0, ..., X_n]$ satify:*

(i) *Letting ρ be equal to $\sum d_i - (n+1)$, then dim $R_k = 0$ for every $k > \rho$. so R is a module of finite length over $\mathbb{C}[X_0, ..., X_n]$. Furthermore dim $R_\rho = 1$.*

(ii) *For every positive integer $k \leq \rho$, the natural multiplication*

$$R_k \otimes R_{\rho-k} \longrightarrow R_\rho \cong \mathbb{C}$$

is a perfect pairing.

This case has been deeply studied by Griffiths and his school during the 80's because of its relevance in problems connected to the period map, Torelli type theorems and Hodge theory.

In this thesis we provide an elementary proof of the fact that if the polynomial f is sufficiently general, it can be reconstructed from its Jacobian ideal J. This result, due to Carlson and Griffiths, is exactly a Torelli type result and it suggests that J, or equivalently R, contains all the geometric information necessary to individuate the hypersurface $V(f)$.

In particular Griffiths proved that the Hodge decomposition

$$H^n(V, C) = \oplus H^{p,n-p}$$

is fully determinated by R. As a matter of facts he showed that there is an isomorphism

$$H^{p,n-p} \cong R_{a(p)} \quad \forall\, 0 \leq p \leq n,$$

where $a(p) = (n - p + 1)d - (n + 2)$.

In the light of this result, it is evident that the dualities

$$H^{p,n-p} \cong (H^{n-p,p})^* \quad \forall\, 0 \leq p \leq n$$

that come from the Hodge theory can all be obtained from the perfect pairings described in Macaulay's theorem.

Several mathematicians have tried to generalize this simple yet elegant result to broader cases, but only special results have been obtained.

The Jacobian ideal has been studied also by experts of singularities. This is due to the fact that when f is not homogeneous and defines an hypersurface with only isolated singularities, J provides information regarding those points.

This is also true in more general cases when the singular locus of $V(f)$ has positive dimension.

In the beginning, this thesis was intended to be an exploration of some possible generalizations of Macaulay's theorem to singular hypersurfaces. It was meant to be an attempt to extend some of the Griffiths' results we referred to above.

Then the initial project evolved to the study of a natural generalization of the Jacobian ring to the singular case. The idea arose from an article by Choudary and Dimca where they examine some of the homological properties of R.

In that work, the aim of the authors is to analize potential relations between the singularities of an hypersurface V defined by $f \in \mathbb{C}[X_0, ..., X_n]$ and the homology of the Koszul complex associated to the partial derivatives of the polynomial itself. An intersting result arising from this study is

Proposition 2. *Let Σ be the singular locus of the hypersurface V. Then*

$$H_k(K) = 0 \qquad \textit{for } k > \mathit{dim}(\Sigma) + 1.$$

Furthermore Choudary and Dimca introduce the notion of Poicaré series in order to study the structure of the Jacobian ring of V as a graded module. They end up by providing a formula for the dimension of the homogeneous parts of R, when the singular locus Σ of V is a 0-dimensional complete intersection.

Theorem 3. *Let V be an hypersurface of degree d with Jacobian ideal J and Jacobian ring R. Suppose that the singular locus Σ of V is a zero dimensional complete intersection with associated ideal $I = (g_1, ..., g_n)$ and $a_i = \mathit{deg}\ g_i$. Then,*

$$P(R)(t) = \frac{(1-t^{d-1})^{n+1}}{(1-t)^{n+1}} + t^{(n+1)(d-1)-\sum a_i} \frac{(1-t^{a_1})...(1-t^{a_n})}{(1-t)^{n+1}}.$$

This computation shows that differently from the smooth case, the Jacobian ring of a singular hypersurface is not even Artinian.

However this object continues to possess peculiar features. In fact we prove that when Σ has dimension zero, from the degree $\rho = (n+1)(d-2)$ on, the homogeneous parts of R are all isomorphic to $R_{\rho+1}$.

This kind of "stabilization" for the dimensions of the homogeneous parts of the Jacobian ring, inspired us to look after a replacement for R in the case of V singular. We detected the quotient of the **adjoint ideal** A of V with respect to the Jacobian ideal, as such possible substitute. Given the sheaf of ideals $\mathcal{A}$ locally generated by the partial derivatives of f, the adjoint ideal of V is defined as the direct sum

$$A = \bigoplus_{k \geq 0} H^0(\mathbb{P}^n, \mathcal{A}(n)).$$

As a matter of facts, when the hypersurface is smooth, the adjoint ideal coincides with $\mathbb{C}[X_0, ..., X_n]$ itself and so $A/J = R$.

We proved that, when dealing with hypersurfaces whose singular loci are complete intersections, the quotient A/J is a module of finite lenght over $\mathbb{C}[X_0, ..., X_n]$ and it features a symmetry between the dimensions of its homogeneous parts.

Theorem 4. *Let V be an hypersurface in $\mathbb{P}^n$ defined by a reduced polynomial f of degree d. Suppose that V has a zero dimensional singular locus which is a complete intersection. Set J and A respectively the Jacobian and the adjoint ideal of V.*

Then there exist a symmetry between the homogeneous parts of A/J with respect to $\rho = (n+1)(d-2)$, in the sense that for every $k \in \mathbb{Z}$

$$dim_{\mathbb{C}}(A/J)_k = dim_{\mathbb{C}}(A/J)_{\rho-k}.$$

In the proof of Theorem (4) we found a formula for the Poincaré series of the module A/J

$$P(A/J)(t) = \frac{(1-t^{d-1})^{n+1}}{(1-t)^{n+1}} - \frac{(1-t^{(n+1)(d-1)-\sum a_i})(1-t^{a_1})...(1-t^{a_n})}{(1-t)^{n+1}},$$

where d is the degree of the hypersurface V and the a_i are the degrees of the polynomials g_i such that $(g_1, ..., g_n) = A$.

From the equation above we deduced that $(A/J)_\rho = 0$.

Thus, differently from the smooth case, under the hypothesis described above, multiplication in $\mathbb{C}[x_0, ..., x_n]$ cannot induce the perfect pairing

$$(A/J)_k \otimes (A/J)_{\rho-k} \longrightarrow \mathbb{C}$$

we were looking for.

We end our work by outlining possible options for further research.

Open problems:

- It remains unknown whether it is possible to extend Theorem (4) to case of an hypersurface whose singular locus Σ is not in general a complete intesection.

- Even if the perfect pairing in Macaulay's theorem does not extend to the singular case, the symmetry between the homogeneous parts of A/J could be a symptom of the fact that the duality

$$(A/J)_k \cong (A/J)_{\rho-k}$$

  may in effect exist.

- It could be interesting to explore the potential relations between the coefficents of the Poincaré series of A/J and the geometric properties of the hypersurface V (like the genus), then understand what role plays the symmetry of Theorem (4) when Σ is a complete intersection.

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