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Tensor products of commutative algebras over a field

Sintesi

Candidato

Claudia Cittadini

Relatore

Prof.ssa Florida Girolami

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Abstract

In this thesis we aim at showing how the tensor product $A \otimes_k B$, where k is a field and A, B are k-algebras, inherits several properties from A and B. In the first chapter, after giving the definitions of the tensor product of modules and of algebras and recalling some properties, we focus our attention on three questions:

1. When every invertible element in $A \otimes B$ is of the form $a \otimes b$, with $a \in U(A)$ and $b \in U(B)$.
2. When the tensor product of k-algebras is local.
3. When the tensor product of extension fields of k is Noetherian.

The second chapter is about the computation of the Krull dimension of tensor products of k-algebras, with a particular interest to a special class of algebras, called *AF-rings*. We show how it is possible to determine precisely the value of the dimension for tensor products of AF-rings and give an estimate for the tensor product of two domains. In the last part of the second chapter we show an analogue of the Seidenberg's inequalities for polynomial rings, that holds for the dimension of the tensor product of a domain with a ring. In the third chapter we seek conditions for a tensor product to have *S-property, strong S-property and catenarity.* Throughout the thesis, we only consider commutative and unitary rings.

0.1 Chapter 1

We first introduce the definition of *tensor product* of A-modules:

Proposition 1. *Let M, N be A-modules. Then there exist an A-module T and an A-bilinear map $g : M \times N \longrightarrow T$ with the following properties:*

1. *If P is an A-module and $f : M \times N \longrightarrow P$ is an A-bilinear map, there exists a unique A-linear map $\psi : T \longrightarrow P$ such that $f = \psi \circ g$.*
2. *If T' is an A-module and g' is a map with the same properties of T and g, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.*

T is called the *tensor product* of M and N and

$$T := \left\{ \sum_{i=1}^{k} (m_i \otimes n_i),\ m_i \in M,\ n_i \in N \right\};$$

furthermore, for every $m_i \in M$, $a \in A$ and $n_i \in N$, we have:

$$(m_1 + m_2) \otimes n_1 = m_1 \otimes n_1 + m_2 \otimes n_1$$

$$m_1 \otimes (n_1 + n_2) = m_1 \otimes n_1 + m_1 \otimes n_2$$

$$(am_1) \otimes n_1 = a(m_1 \otimes n_1) = m_1 \otimes (an_1).$$

Let us give some examples:

Example 1. *1.* $\dfrac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \dfrac{\mathbb{Z}}{n\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{d\mathbb{Z}}$, *where $d = gcd(m, n)$.*

2. Let G be a finite abelian group; then $\mathbb{Q} \otimes_{\mathbb{Z}} G = 0$.

Let $T := B \otimes_A C$, where B and C are two A-algebras; we define a multiplication on T as follows:

$$\left(\sum_i (b_i \otimes c_i) \right) \left(\sum_j (b'_j \otimes c'_j) \right) = \sum_{i,j} (b_i b'_j \otimes c_i c'_j).$$

With such a definition, T is a commutative ring with unit $1 \otimes 1$.

Like most authors, we are interested in particular in tensor products over a field k. In this case, if A and B are k-algebras we have the following commutative diagram:

$$\begin{tikzcd} & A \arrow[dr, "h"] & \\ k \arrow[ur, "f"] \arrow[dr, "g"'] & & A \otimes B \\ & B \arrow[ur, "i"'] & \end{tikzcd}$$

where $h(a) = a \otimes 1$ and $i(b) = 1 \otimes b$ are injective. Hence, it is natural to ask under which conditions $A \otimes_k B$ inherits some properties from A and B. We show, for example, that a tensor product of two k-algebras is not always a domain, even though A and B are domains.

Example 2. *Let $k = \mathbb{Q}$ and $A = B = \mathbb{Q}(i)$, $A \otimes_k B$ is not a domain. Indeed the element $(\frac{1}{2} \otimes 1 - \frac{i}{2} \otimes i)$ is a zero divisor: $(\frac{1}{2} \otimes 1 - \frac{i}{2} \otimes i)(\frac{1}{2} \otimes 1 + \frac{i}{2} \otimes i) = \frac{1}{4} \otimes 1 + \frac{i}{4} \otimes i - \frac{i}{4} \otimes i - \frac{1}{4} \otimes 1 = 0$.*

In [29], Zariski gives several basic properties of tensor products of algebras and in particular he proves:

Proposition 2. *([29], Chap. III, Corollary 1, p. 198)*
If k is an algebraically closed field and L, F are two extension fileds of k, then $L \otimes_k F$ is a domain.

Remark 1. *The condition that k has to be algebraically closed is not necessary: indeed, it suffices to note that, for example, $\mathbb{Q} \otimes_{\mathbb{Q}} K \cong K$, for every field $K \supset \mathbb{Q}$.*

The first question we focus on, is when every invertible element of $A \otimes B$ is of the form $a \otimes b$, with $a \in U(A)$ and $b \in U(B)$. In [24], M. E. Sweedler proves the following theorem:

Theorem 1. *([24], Th. 1.2, p. 260)*
Let k be an algebraically closed field, A, B be k-algebras and k algebraically closed in A and B. Then every invertible element of $A \otimes_k B$ is of the form $a \otimes_k b$, where $a \in U(A)$ and $b \in U(B)$.

Later in [25], he gives conditions that characterize when a tensor product of algebras over a field is local:

Theorem 2. *([25], Th. 1, p. 8)*
If A and B are k-algebras, then the following statements are equivalent:

1. *$A \otimes B$ is local.*

2. (a) *A is local (with maximal ideal M) and B is local (with maximal ideal N).*

   (b) *Either A or B is algebraic over k.*

   (c) *$\frac{A}{M} \otimes \frac{B}{N}$ is local.*

The last part of the first chapter is concerned with the Noetheriannes of a tensor product of two fields. We can easily show an example of a tensor product of two algebras that is Noetherian:

Example 3. *Let R be a ring and A and B be two R-algebras. If A is finitely generated over R and B is a Noetherian ring, then $A \otimes_R B$ is Noetherian.*
Indeed the hypothesis on A leads to assert that $A \otimes_R B$ is a finitely generated algebra over B; since B is Noetherian, then $A \otimes_R B$ is Noetherian.

The question, anyway, is not trivial, because we may find examples of tensor products of two Noetherian rings that are not Noetherian. In particular, we have:

Lemma 1. *([27], Lemma 9, p. 33)*

Let k be a field and F, L, M be extension fields of k, let $\varphi : M \longrightarrow L, \chi : M \longrightarrow F$ be homomorphisms. If there exists a proper ascending chain of fields $k \subset k_1 \subset k_2 \subset \ldots \subset k_n \subset \ldots \subset M$, then $L \otimes_k F$ is not Noetherian.

The problem of determining when a tensor product of algebras is Noetherian is unsolved, at the present, in the general case; in the thesis we give necessary and sufficient conditions for a tensor product of a field with itself to be Noetherian.

Hence, we have:

Theorem 3. *([27], Th. 11, p. 34)*

For an extension field F of k the following statements are equivalent:

1. *$F \otimes_k F$ is Noetherian.*
2. *The ascending chain condition is satisfied by the intermediary fields between k and F.*
3. *F is a finitely generated extension field of k.*

Proof.

- 1) $\Longrightarrow$ 2): The statement follows directly from Lemma 1.
- 2) $\Longrightarrow$ 3): Let K be a field that is maximal among the finitely generated subfields of F. If $K \subset F$, then there exists a in $F \setminus K$ such that $K \subset K(a)$, but this is a contradiction. Hence $K = F$.
- 3) $\Longrightarrow$ 1): Since $F \otimes_k F$ is a finitely generated algebra over F and F is Noetherian, the statement is done.

$\square$

0.2 Chapter 2

The second chapter is about the Krull dimension of tensor products; for a ring R, we denote with $\mathrm{d}(R)$ the (Krull) dimension of R. In particular we consider a special class of commutative algebras, that is the class of *AF-rings*. First, we extend the definition of "transcendence degree over a field"[1], to arbitrary k-algebras. If K is an extension field of k, we denote with $\mathrm{t}(K)$ the transcendence degree of K over k.

Definition 1. *Let R be a k-algebra. The* transcendence degree of R over k *(denoted with $\mathrm{t}(R)$) is defined as follows:*

1. *If R is a domain, then $\mathrm{t}(R) := \mathrm{t}(K)$, where K is the quotient field of R.*
2. *If R is a ring, then $\mathrm{t}(R) := \sup\left\{\mathrm{t}\left(\frac{R}{P}\right),\quad P \in \mathrm{Spec}(R)\right\}$*

We assume throughout this chapter that $\mathrm{t}(R)$ is finite.

Before introducing Wadsworth's class of algebras, we prove an inequality holding for every k-algebra:

Proposition 3. *Let R be a k-algebra, then*

$$\mathrm{ht}(P) + \mathrm{t}\left(\frac{R}{P}\right) \leq \mathrm{t}\left(R_P\right), \tag{1}$$

for every $P \in \mathrm{Spec}(R)$.

Remark 2. *In view of Formula (1), from our assumption that $t(R) < \infty$ it follows that also $d(R) < \infty$.*

Since Grothendieck published [13], only a few authors have studied the prime ideal structure of tensor products of k-algebras (e.g Sharp in [23] (1977), Vámos in [27] (1977), Wadsworth in [28] (1979), Bouchiba et al. in [4] (2002)). All these

[1]For the basic properties of the transcendence degree of a field, see e.g. ([29], Ch. II, § 12)

works deal with dimension of tensor products of some special algebras; at the present the general problem is still unsolved.

In [23], Sharp proved that, if K and L are two extension fields of k, then

$$\mathrm{d}(K \otimes L) = \min\{\mathrm{t}(K), \mathrm{t}(L)\} \tag{2}$$

(actually, this result was already found by Grothendieck ten years earlier). In 1979, A. Wadsworth, for giving a generalization of Sharp's result, introduced a new class of k-algebras, called the *AF-rings*:

Definition 2. *A k-algebra A is an* AF-ring *if it satisfies*

$$\mathrm{ht}\, P + \mathrm{t}\left(\frac{A}{P}\right) = \mathrm{t}(A_P),$$

called the Altitude Formula, *for every prime ideal P in A.*

Examples of AF-rings are zero-dimensional rings and finitely generated k-algebras.

In [28], Wadsworth proved that AF-rings are the most tractable rings for Krull dimension computation; to see this, we show how tensor products behave with them.

Proposition 4. *If $A_1, A_2, ..., A_n$ are AF-rings, then $A_1 \otimes A_2 \otimes ... \otimes A_n$ is an AF-ring.*

Corollary 1. *Let A be an AF-ring. Then $A[X]$ is an AF-ring.*

Proof. Since $A[X] \cong k[X] \otimes A$, the statement follows directly from Proposition 4. □

Now, let us recall two functions, defined in [28], that are useful to express the value of the dimension of tensor products.

Definition 3. *Let R be a k-algebra, $P \in \mathrm{Spec}(R)$, $d, s \in \mathbb{Z}$ with $0 \leq d \leq s$. Then:*

$$\Delta(s, d, P) := \mathrm{ht}\, PR[X_1, ..., X_s] + \min\left(s, d + \mathrm{t}\left(\frac{R}{P}\right)\right);$$

$$D(s, d, R) := \max\{\Delta(s, d, P) \ \mid P \in \mathrm{Spec}(R)\}.$$

Wadsworth found a formula to compute the Krull dimension of a tensor product of two k-algebras, in which one of them is an AF-domain; we have:

Theorem 4. *Let A be an AF-domain, $t := \mathrm{t}(A)$, $d := \mathrm{d}(A)$ and R a k-algebra. Then:*

$$\mathrm{d}(A \otimes R) = D(t, d, R).$$

Later, he proved a fundamental theorem, that shows how a tensor product of AF-domains can be easily expressed in terms of transcendence degrees and dimensions:

Theorem 5. *Let $A_1, A_2, ..., A_n$ be AF-domains and $t_i := \mathrm{t}\,(A_i)$ and $d_i := \mathrm{d}\,(A_i)$. Then:*

$$\mathrm{d}\,(A_1 \otimes A_2 \otimes ... \otimes A_n) = t_1 + t_2 + ... + t_n - \max\{t_i - d_i \quad ; \quad 1 \leq i \leq n\}.$$

Remark 3. *Theorem 5 is a generalization of Formula* (2).

If A_1 and A_2 are domains but not AF-rings, Wadsworth shows in [28] that in general it is no longer possible to express $\mathrm{d}(A_1 \otimes A_2)$ in terms of the above invariants, but we can only obtain upper and lower bounds. We recall from [15]:

Definition 4. *1. Let R be a domain and K be its quotient field. R is said to have* valuative dimension n *($\mathrm{d}_v(R) = n$), if each valuation overring of R has dimension at most n and if there exists a valuation overring of R of dimension n. If such integer does not exist, R is said to have infinite valuative dimension.*

2. Let A be a ring, then

$$\mathrm{d}_v(A) := \sup\left\{ \mathrm{d}_v\left(\frac{A}{P}\right) \quad : \quad P \in \mathrm{Spec}(A)\right\}.$$

Furthermore, we have:

Proposition 5. *Let A be a k-algebra, then:*

$$\mathrm{d}(A) \leq \mathrm{d}_v(A) \leq \mathrm{t}(A).$$

Proof. For the proof we remind to ([15], Proposition 5 of Chapter IV). $\square$

Now, we can show bounds for the dimension of a tensor product of a domain R with itself:

Theorem 6. *([28], Th. 4.1, p. 399)*
Let R be a domain and a k-algebra. Then:

$$\mathrm{d}(R) + \mathrm{t}(R) \leq \mathrm{d}(R \otimes R) \leq \mathrm{d}_v(R) + \mathrm{t}(R).$$

Remark 4. *If $\mathrm{d}(R) = \mathrm{d}_v(R)$, Theorem 6 gives the exact value of $\mathrm{d}(R \otimes R)$. For example, if R is Noetherian or if it is a valuation ring (cfr. [15], Corollary 1, p.67), we have:*

$$\mathrm{d}(R \otimes R) = \mathrm{d}(R) + \mathrm{t}(R).$$

In 1953, Seidenberg in [22] proved one of the most important inequalities holding for a polynomial ring $A[X]$, that is:

$$1 + \mathrm{d}(A) \leq \mathrm{d}(A[X]) \leq 1 + 2\,\mathrm{d}(A).$$

Actually, this formula was extended by Jaffard in [15], who proved:

$$n + \mathrm{d}(A) \leq \mathrm{d}(A[X_1, ..., X_n]) \leq n + (n+1)\,\mathrm{d}(A). \tag{3}$$

In the last part of the chapter we show some properties to have analogues of these inequalities for tensor products of k-algebras; we denote with $A[n]$ the polynominal ring $A[X_1, ..., X_n]$ for any ring A. Bouchiba in [3], proved:

Theorem 7. *Let A and R be k-algebras and assume that A is a domain. Then:*

$$\mathrm{d}(R[\mathrm{t}(A)]) - (\mathrm{t}(A) - \mathrm{d}(A)) \leq \mathrm{d}(A \otimes R) \leq \mathrm{d}(R[\mathrm{t}(A)]).$$

Hence, we have:

Corollary 2. *Let A and R be k-algebras such that A is a domain. Then:*

$$\mathrm{d}(A) + \mathrm{d}(R) \leq \mathrm{d}(A \otimes R) \leq \mathrm{t}(A) + (\mathrm{t}(A) + 1)\,\mathrm{d}(R)$$

Proof. By the Jaffard's inequalities in (3), it follows that

$$\mathrm{d}(R) + \mathrm{t}(A) \leq \mathrm{d}(R[\mathrm{t}(A)]) \leq \mathrm{t}(A) + (\mathrm{t}(A) + 1)\,\mathrm{d}(R)$$

and by Theorem 7 the proof follows immediately. □

0.3 Chapter 3

Finally in the third chapter, we investigate sufficient conditions for a tensor product to inherit the S-property and the catenarian property. Kaplansky in [16], in order to treat Noetherian domains and Prüfer domains in a unified manner, introduced the concept of *S(eidenberg)-domains* and *Strong S(eidenberg)-rings*; these rings have been subsequently studied by S. Malik and J.L. Mott in [17].

Definition 5. *A domain R is said to be an* S-domain *if for every prime ideal P of height 1, $PR[X]$ again has height 1.*

Definition 6. *Let A be a ring. A is said to be a* Strong S-ring *if for every prime ideal Q in A, $\dfrac{A}{Q}$ is an S-domain.*

The class of S-domains is not actually stable with respect to polynomial extensions (see [8], Example p.40), hence we recall from [17] the following definition:

Definition 7. *Let A be a ring. A is said to be a* stably strong S-ring *if $A[X_1, ..., X_n]$ is a strong S-ring, for every $n \geq 0$.*

In particular we have by [17]:

Proposition 6. *Let A be a stably strong S-ring. Then*

$$\mathrm{d}(A[X_1, ..., X_n]) = n + \mathrm{d}(A),$$

for every $n \geq 1$.

Let us recall the definition of a catenarian domain:

Definition 8. *A domain R is said to be a* catenarian domain *if it is locally finite dimensional (LFD), in the sense that every prime ideal has a finite height, and if for every pair of adjacent primes $P \subset Q$ in R, $\mathrm{ht}\, Q = 1 + \mathrm{ht}\, P$ (or, equivalentely, if for every pair of prime ideals $P \subseteq Q$, all the saturated chains between P and Q have the same length).*

The class of catenarian domains, like the class of strong S-rings, is not stable in polynomial extensions. Hence we recall by [7] the following definition:

Definition 9. *A domain R is said to be a* universally catenarian domain *if $R[X_1, ..., X_n]$ is catenarian for every positive n.*

We also show from [7] that if R is a universally catenarian domain, then R is a stably strong S-domain.

One of our aims in this chapter is to extend the definitions we have given, to arbitrary rings. To do this, we recall from [4] the following definition:

Definition 10. *Let A be a ring. A is said to satisfy* MPC *(Minimal Prime Comaximality) if its mimimal prime ideals are pairwise comaximal or, equivalentely, if every maximal ideal in A contains only one minimal prime ideal.*

Following [4], we see now how to extend the definitions of the S-property and catenarity to the MPC context.

Let A be an arbitrary ring and consider the following properties:

- (P_1): $\dfrac{A}{P}$ is an S-domain for every minimal prime ideal P of A.
- (P_2): If $\operatorname{ht} P = 1$ then $\operatorname{ht}(PA[X]) = 1$, for each prime ideal P of A.
- (Q_1): A is LFD and $\operatorname{ht} Q = \operatorname{ht} P + 1$, for all adjacent primes $P \subset Q$ of A.
- (Q_2): $\dfrac{A}{P}$ is a catenarian domain for each minimal prime ideal P of A.

It is easy to prove that if A satisfies MPC, then A satisfies (P_1) (resp. (Q_1)) if and only if it satisfies (P_2) (resp. (Q_2)).

Hence, we can give the following definitions:

Definition 11. *Let A be a ring. A is said to be an* S-ring *if it satisfies MPC and (P_1) (or (P_2)).*

A is said to be a catenarian ring *if it satisfies MPC and (Q_1) (or (Q_2)).*

We are interested in studying the transfer of these properties to tensor products of k-algebras. After showing several examples in which the MPC property is inherited, we can give necessary and sufficient conditions for a tensor product of k-algebras to have S-property.

Theorem 8. *Let A and B be k-algebras such that $A \otimes B$ satisfies MPC. Then $A \otimes B$ is an S-ring if and only if at least one of the following statements holds:*

1. *A and B are S-ring.*
2. *A is an S-ring and $\operatorname{t}\left(\dfrac{A}{p}\right) \geq 1$, for each $p \in \operatorname{Min}(A)$.*

3. B *is an S-ring and* $\mathrm{t}\left(\dfrac{A}{q}\right) \geq 1$*, for each* $q \in \mathrm{Min}(B)$*.*

4. $\mathrm{t}\left(\dfrac{A}{p}\right) \geq 1$ *and* $\mathrm{t}\left(\dfrac{B}{q}\right) \geq 1$*, for each* $p \in \mathrm{Min}(A)$ *and* $q \in \mathrm{Min}(B)$*.*

Seeking conditions for a tensor product to inherit the strong S-property and catenarian property, we focus our attention mainly on $A \otimes B$, where at least one of the algebras is an algebraic extension of k; indeed in [4] it is shown that, moving beyond this hypothesis, the study of the transfer of these properties to tensor products becomes more difficult. In this context, we show some cases in which tensor products inherit the strong S-property and catenarity:

Theorem 9. *Let* K *be an algebraic field extension of* k *and* A *be a* k*-algebra.*

1. *If* A *is a domain and it contains a separable algebraic closure of* k*, then* $K \otimes A$ *is a strong S-ring (resp. stably strong S-ring, catenarian, universally catenarian) if and only if* A *is a strong S-ring (resp. stably strong S-ring, catenarian, universally catenarian).*

2. *If* A *is two-dimensional and* $K \otimes A$ *satisfies MPC, then* $K \otimes A$ *is a strong S-ring (resp. catenarian) if and only if* A *is a strong S-ring (resp. catenarian).*

However, it is still an open problem to determine under which hypotheses $K \otimes A$ inherits the catenarity from A; following [26], we give a positive answer in some special cases. In particular, in the last part, we reformulate the notion of catenarian rings, removing the MPC property and show:

Theorem 10. *Let* A *be a Noetherian ring that is a* k*-algebra and let* K *be an extension field of* k *with* $t := \mathrm{t}(K) < \infty$*. If* A *is universally catenarian, then so is* $K \otimes A$*.*

Corollary 3. *Let K and L be two extension fields of k, such that either* $\mathrm{t}(K) < \infty$ *or* $\mathrm{t}(L) < \infty$*. Then $K \otimes L$ is universally catenarian.*

Proof. Since every field is universally catenarian, the statement follows directly from Theorem 10. $\square$

Theorem 11. *Let A be a universally catenarian ring which is a k-algebra, K be an extension field of k such that $K \otimes A$ is Noetherian. Then $K \otimes A$ is universally catenarian.*

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