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by
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PERIODIC ORBITS CLOSE TO ELLIPTIC TORI for the $(\mathbf{N}+\mathbf{1})$-BODY PROBLEM

SUMMARY

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Towards the end of the ninetheenth century the king of Sweden Oscar II offered a considerable sum of money as a prize to anyone who would have managed to solve the *three-body problem*. The prize was awarded to Henri Poincaré. The great french scientist proved that, in some sense, a general solution to the three-body problem does not exist: indeed it would need a formulation of inconceivable complexity to deal with phenomena as open trajectories, collisions, and caotic motions. In modern language, what Poincaré showed is just that the three-body problem is a *non-integrable* dynamical system.

Even if it is not possible to find a general solution, of course it could be very interesting to study particular ones. One of the main related questions regards the existence of periodic trajectories. In *"Les Méthodes nouvelles de la Mécanique Céleste"* Poincaré wrote:

> *"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable.* [...]
> *Étant données des équations de la forme définie dans le n. 13* [1] *et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut.*
> *D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusu'ici réputée inabordable."*
>
> ([Poi1892], Tome 1, ch. III, a. 36)

This conjecture has been the basis for many research works concerning the problem of the existence of periodic orbits for Hamiltonian dynamical systems. Among them we recall the ones of Poincaré himself, followed by Lyapunov, Birkhoff, Moser, Weinstein, etc. However, the conjecture is still open and we are still very far away from a complete proof.

An intermediate step is the search of periodic orbits in the vicinity of invariant submanifolds. In the thirties, Birkhoff and Lewis ([Bir]-[BL]-[Lew]) established the existence of infinitely many periodic solutions in a neighborhood of an elliptic equilibrium[2] whose linear frequencies are sufficiently *non-resonant*. This result also requires a non-degeneracy *"twist"* condition, involving finitely many Taylor coefficients of the Hamiltonian at the equilibrium, and implying the system to be genuinely non-linear.

[1] The Hamilton's equations for N-body problem.

[2] Actually, Birkhoff and Lewis established the existence of infinitely many periodic solutions close to a non-constant periodic elliptic solution, but the proof is essentially the same for an elliptic equilibrium.

Moreover, if the Hamiltonian is sufficiently smooth, KAM theory ensures, in a neighborhood of the equilibrium small enough, the existence of Lagrangian (maximal) invariant tori filling up a set of positive Lebesgue measure. In [CZ] the existence of infinitely many others periodic orbits (with larger and larger minimal period) accumulating to the KAM torus itself has been proved applying the Birkhoff-Lewis type Theorem of [Mos].

In the last years, an exhaustive perturbation theory for *elliptic* tori has been developed by many authors, see [Mos], [Eli], [Kuk], [Pös89], [Pös96], [Bou], and [XY]; in short, the persistence of elliptic tori is ensured requiring appropriate "non-resonance conditions" among the frequencies and further non-degeneracy conditions. We recall that a lower-dimensional invariant torus is called elliptic, or linearly stable, if the linearized system along the torus, possesses purely imaginary eigenvalues.

Recently, in [BBV], the existence of periodic orbits with larger and larger minimal period clustering to elliptic invariant tori has finally been proved.

One of the main motivation for studying such problem comes from classical topics in Celestial Mechanics such as the many-body problem, which of course has inspired the whole development of KAM theory [3]. Indeed, as we will show in Chapter 2, in the *planetary* case, where one body (the "Star" or the "Sun") has mass much bigger than that of the other ones (the N "planets"), the $(N+1)$-body problem could be seen as a *nearly-integrable Hamiltonian system*, where, under suitable assumptions, the bounded motions of the unperturbed (integrable) system lie on N-dimensional elliptic tori.

The persistence of a majority of such elliptic tori has been shown for the spatial planetary three-body problem in [BCV03], and for the planar planetary $(N+1)$-body problem in [BCV06].

In [BBV] a general Birkhoff-Lewis type result about the existence of periodic solutions accumulating onto elliptic invariant tori is proved. Furthermore, it is shown that such result can be applied to the spatial planetary three-body problem, proving the existence of infinitely many periodic solutions accumulating on the elliptic KAM tori found in [BCV03].

In this thesis we extend the result of [BBV] to the planetary planar $(N+1)$-body problem discussed in [BCV06]. To this end we have first to prove a new general theorem on the existence of periodic orbits close to elliptic tori; in particular we have modified the hypotheses of the theorem in [BBV] requiring the vector of linear and elliptic frequencies to be non-resonant up to a *finite* order. Then, in order to show the applicability of the general theorem to the $(N+1)$-body problem of [BCV06], we need to check the non-resonance condition above. This task will be accomplished through a careful KAM-analysis

[3] For a brief account of ideas and results concerning the application of KAM theory to the N-body problem see [CC]

exploiting (of course with some efforts) the *analytic* properties of the involved functions.

We now proceed to present a detailed summary of the contempt of thesis.

Chapter 1: Periodic orbits close to elliptic tori

We consider a normal form Hamiltonian describing the dynamics in a neighborhood of an elliptic torus, namely

$$\mathcal{H}_*(\mathcal{I}_*,\varphi_*,Z_*,\overline{Z}_*) \;=\; \omega\cdot\mathcal{I}_* \;+\; \Omega Z_*\cdot\overline{Z}_* \;+\sum_{2|k|+|a+\overline{a}|\geq 3} R^*_{k,a,\overline{a}}(\varphi_*)\mathcal{I}_*^k Z_*^a\overline{Z}_*^{\overline{a}}\,, \qquad (1)$$

where $(\mathcal{I}_*,\varphi_*)\in\mathbb{R}^n\times\mathbb{T}^n$ are action-angle variables and $(Z_*,\overline{Z}_*)\in\mathbb{C}^{2m}$ are called the normal (or elliptic) coordinates. The phase space is equipped with the symplectic form $d\mathcal{I}_*\wedge d\varphi_*+\mathrm{i}\,dZ_*\wedge d\overline{Z}_*$. In these coordinates

$$\mathcal{T}\;:=\;\Big\{\mathcal{I}_*=0\,,\varphi_*\in\mathbb{T}^n\,,Z_*=\overline{Z}_*=0\Big\}$$

is the invariant elliptic torus, while

$$\omega\,\in\,\mathbb{R}^n \qquad \text{and} \qquad \Omega\,\in\,\mathbb{R}^m$$

are respectively the *torus* (or linear) frequencies and the *elliptic* (or normal) frequencies.

The *frequency vector* (ω,Ω) is assumed to satisfy the "second order Melnikov non-resonance condition"

$$\begin{gathered}\big|\omega\cdot\ell\,+\,\Omega\cdot h\big|\;\geq\;\frac{\gamma}{1+|\ell|^\tau}\;,\\ \forall\,\ell\in\mathbb{Z}^n\,,\quad\forall\,h\in\mathbb{Z}^m,\,|h|\leq 2\,,\quad(\ell\,,h)\neq(0\,,0),\end{gathered}\qquad (2)$$

for some positive constants $\gamma,\tau\in\mathbb{R}$. This implies that the linear frequency vector ω is rationally independent (actually Diophantine), while *a priori* the whole frequency vector (ω,Ω) could meet some resonance relations.

Expanding the functions $R^*_{k,a,\overline{a}}(\varphi_*)$ in Fourier series as

$$R^*_{k,a,\overline{a}}(\varphi_*)=\sum_{\ell\in\mathbb{Z}^n}R^*_{k,a,\overline{a},\ell}\,e^{\mathrm{i}\ell\cdot\varphi_*}\,,$$

we can define the symmetric "twist" matrix $\mathcal{R}\in\mathrm{Mat}(n\times n,\mathbb{R})$

$$\begin{aligned}\mathcal{R}_{ii'}\;&:=\;\big(1+\delta_{(i,i')}\big)\,R^*_{e_i+e_{i'},0,0,0}\;+\\ &\quad-\sum_{\substack{1\leq j\leq m\\ \ell\in\mathbb{Z}^n}}\frac{1}{\omega\cdot\ell+\Omega_j}\Big(R^*_{e_i,e_j,0,\ell}\,R^*_{e_{i'},0,e_j,-\ell}+R^*_{e_i,0,e_j,-\ell}\,R^*_{e_{i'},e_j,0,\ell}\Big)\,,\end{aligned}\qquad (3)$$

and the matrix $\mathcal{Q}\in\mathrm{Mat}(m\times n,\mathbb{R})$

$$\begin{aligned}\mathcal{Q}_{ji}\;&:=\;R^*_{e_i,e_j,e_j,0}\;+\\ &\quad-\sum_{\substack{1\leq i'\leq n\\ \ell\in\mathbb{Z}^n}}\frac{\ell_{i'}}{\omega\cdot\ell+\Omega_j}\Big(R^*_{e_i,e_j,0,\ell}\,R^*_{e_{i'},0,e_j,-\ell}+R^*_{e_i,0,e_j,-\ell}\,R^*_{e_{i'},e_j,0,\ell}\Big)\;+\\ &\quad-\sum_{\substack{1\leq j'\leq m\\ \ell\in\mathbb{Z}^n}}\frac{1}{\omega\cdot\ell+\Omega_{j'}}\Big(R^*_{0,e_j,e_j+e_{j'},-\ell}\,R^*_{e_i,e_{j'},0,\ell}+R^*_{0,e_j+e_{j'},e_j,\ell}\,R^*_{e_i,0,e_{j'},-\ell}\Big)\,.\end{aligned}\qquad (4)$$

Our general theorem about the existence of periodic orbits close to elliptic invariant tori of Hamiltonian systems says:

Theorem 1. *Given an Hamiltonian of the form* (1)*, let the frequency vector* (ω, Ω) *satisfy the second order Melnikov non-resonance condition* (2) *for some positive constant* $\gamma, \tau \in \mathbb{R}$.
Assume that the "twist" matrix is invertible, i.e.

$$\det(\mathcal{R}) \neq 0 \qquad (\text{ "twist" condition }), \tag{5}$$

and that the frequency vector (ω, Ω) *satisfies the "non-resonance condition up to order* M*":*

$$(\omega, \Omega) \cdot \vec{k} \neq 0 \quad \forall \vec{k} \in \mathbb{Z}^{n+m}, \; 0 < |\vec{k}|_1 \leq M \,, \tag{6}$$

for a suitable constant $M = M\Big(n \,, m \,, \big|\mathcal{Q}\mathcal{R}^{-1}\big|\Big) \in \mathbb{N}$,
Then, $\exists\, \eta_0 > 0$ *such that* $\forall \eta \in (0, \eta_0]$ *there exists an open set of periods* $\Theta_\eta \subset [\frac{1}{\eta^2}\,, +\infty]$ *such that* $\forall\, T \in \Theta_\eta$ *the Hamiltonian system generated by* $\mathcal{H}_*$ *admits at least* n *geometrically distinct* T*-periodic solutions* $\bar{\varrho}_\eta(t)$*. The trajectories of the* $\bar{\varrho}_\eta(t)$*'s are closer and closer to the elliptic torus* $\mathcal{T}$ *as* η *tends to zero. In particular* $\mathcal{T}$ *lies in the closure of the family of periodic orbits* $\bar{\varrho}_\eta$, $\eta \in (0, \eta_0]$.

Theorem 1 consists in a suitable adaptation of Theorem 1.1 of [BBV]. The main modification introduced lies in the non-resonance hypotheses. Apart from the *second order Melnikov condition* (2) (that holds automatically for elliptic KAM tori), we have assumed the *non-resonance condition up to a finite order* (6): the frequency vector (ω, Ω) might as well admit some resonances, provided they happen at a sufficiently large order.

Such condition was not present in the original result of [BBV]. Indeed, the theorem of [BBV] had been thought to be applied to the three-body problem KAM tori of [BCV03]: in such case the low number (m=2) of elliptic directions allows to carry on the proof without further assumptions over the frequency vector. Instead, the elliptic tori arising in the $(N+1)$-body problem we are dealing with (in particular for $N \geq 3$) are N-dimensional; hence the result of [BBV] seems no longer applicable (or rather we are not able to check its hypoteses).

This is the reason why we have modified it *ad hoc*, obtaining Theorem 1.

Sketch of the proof of Theorem 1 : First of all, as we are interested in the region of phase space near the torus $\mathcal{T}$, we introduce a small rescaling parameter $\eta > 0$ measuring the distance from $\mathcal{T}$. Then, since (ω, Ω) satisfy the second order Melnikov non-resonance conditions (2), in view of an averaging procedure, the Hamiltonian $\mathcal{H}_*$ is casted, in a suitable set of coordinates

$(I, \phi, z, \overline{z}) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2m}$, and sufficiently close to the torus $\mathcal{T}$, in a small perturbation of the integrable Hamiltonian

$$H_{\rm int} := \omega \cdot I + \frac{\eta^2}{2} \mathcal{R} I \cdot I + \Omega z \overline{z} + \eta^2 \mathcal{Q} I \cdot z \overline{z} \, .$$

The Hamiltonian system generated by $H_{\rm int}$ possesses the elliptic tori $\mathcal{T}(I_0) := \{I = I_0, \ \phi \in \mathbb{T}^n, \ z = \overline{z} = 0\}$. The torus $\mathcal{T}(I_0)$ supports the linear flow $t \to (I_0, \ \phi_0 + (\omega + \eta^2 \mathcal{R} I_0) t, \ 0, \ 0)$, whereas on the normal space the dynamic is described by $\dot{z} = {\rm i}(\Omega + \eta^2 \mathcal{Q} I_0) z, \ \dot{\overline{z}} = {\rm i}(\Omega + \eta^2 \mathcal{Q} I_0) \overline{z}$.
$\widetilde{\omega} = \omega + \eta^2 \mathcal{R} I_0$ and $\widetilde{\Omega} = \Omega + \eta^2 \mathcal{Q} I_0$ will be called respectively the vector of the "*shifted linear frequencies*" and of the "*shifted elliptic frequencies*".

By the "twist condition" (5) the system generated by $H_{\rm int}$ is properly nonlinear. In particular, such condition ensures that the shifted linear frequencies $\widetilde{\omega}$ vary with the actions I_0. Hence is it always possible to find *completely resonant* frequencies $\widetilde{\omega} \in (2\pi/T)\mathbb{Z}^n$ for some $T = O(\eta^{-2})$ and $I_0 = O(1)$. In such case, $\mathcal{T}(I_0)$ is a completely resonant torus supporting the family of T-periodic motions $\mathcal{P} := \{I(t) = I_0, \ \phi(t) = \phi_0 + \widetilde{\omega} t, \ z(t) = \overline{z}(t) = 0.\}$

Our task is to find periodic solutions for the Hamiltonian system generated by $\mathcal{H}_*$ bifurcating from the ones of $H_{\rm int}$. Nevertheless, in general, the family $\mathcal{P}$ will not persist in its entirety for the complete Hamiltonian system due to resonances among the oscillations.

The key point to continue some periodic solutions of the family $\mathcal{P}$ is to choose properly the "1-dimensional parameter" T (the period) and the actions I_0 : the period T and the "shifted elliptic frequencies" $\widetilde{\Omega}(I_0)$ must satisfy a suitable *non-resonance property.* Through "ergodization" arguments, condition (6) of Theorem 1 makes possible to find an open set of "non resonant" periods T.

After this construction, the proof is based on a *Lyapunov-Schmidt reduction.* First, the non-resonance property over the periods T and the "twist condition" (5) allow to solve the *range equation* by means of the Contraction Mapping Theorem. Then, by a variational argument we find at least n geometrically distinct T-periodic solutions of the *bifurcation equation* given by the previous Lyapunov-Schmidt reduction.

Chapter 2: The planetary planar (N+1)-body problem

In this Chapter we report the result of [BCV06] concerning the existence of quasi-periodic orbits lying on N-dimensional invariant elliptic tori for the planetary planar $(N+1)$-body problem. Roughly, such orbits are the continuations of Keplerian elliptic trajectories obtained neglecting the mutual interactions among the N planets.

First of all, in order to deal with the *planetary* case, we introduce, as customary, a small parameter ε such that, setting the mass of the "Sun" $m_0 := 1$ (that is tantamount to fix a new unity of mass) and denoting by m_i the mass of the i^{th} planet, we have

$$m_i = \varepsilon \, \mu_i \, , \qquad i = 1, \ldots, N \, , \qquad 0 < \varepsilon < 1 \tag{7}$$

for some fixed constants μ_i.

Then, as in the classical approach to the matter, we show that the planetary $(N+1)$-body problem could be viewed as a *nearly-integrable* Hamiltonian system in the perturbative parameter ε : the integrable limit consists just of the N decoupled two-body systems given by the Sun and the i^{th} planet. As anyone knows, for suitable initial data, each unperturbed two-body system admits the Keplerian solutions with the planet revolving around the Sun on a ellipse. Obviously, such solutions in general do not persist when the gravitational interaction among the planets is taken into account. However, the orbital elements of these "osculating" ellipses provide a good set of coordinates to describe the true motions. Indeed, the main results of this chapter will be stated, as usual, in terms of the major semi-axes $a_i \, , \; i = 1, \ldots, N$ of the osculating ellipses. Following [BCV06], we suppose that the a_i's satisfy

$$0 < a_i < \theta \, a_{i+1} \, , \qquad \qquad 1 \leq i \leq N-1 \, , \tag{8}$$

for a suitable constant $0 < \theta < 1$.

Moreover, we focus our attention on a caricature of the outer solar system: we assume that two planets (such as Jupiter and Saturn in the real world) have mass considerably bigger than the other ones; besides, the two big planets are supposed to have an orbit which is internal with respect to the orbits of the small planets (as in the case of the real small planets Uranus, Neptune...). Precisely, we assume that, for some fixed $\bar{\mu}_i$,

$$\begin{aligned} \mu_i &= \bar{\mu}_i && \text{for} \quad i = 1, 2 \, , \\ \mu_i &= \delta \, \bar{\mu}_i && \text{for} \quad i = 3, \ldots, N \, , \qquad 0 < \delta < 1 \, . \end{aligned} \tag{9}$$

Under this assumptions, the existence of N-dimensional elliptic tori for the planar $N+1$-body problem has been proved in [BCV06] for a large set of initial osculating major semi-axes. The proof is based on an appropriate averaging/KAM procedure.

Thus, we have:

Theorem 2. *Fix a compact set $A \subset \mathbb{R}^N$ of osculating major semi-axes satisfying* (8) *for a small θ. There exist two positive constant $\delta^\star$ $\varepsilon^\star = \varepsilon^\star(\delta)$ such that for any $\delta < \delta^\star$ and for any $\varepsilon < \varepsilon^\star(\delta)$, there exists a Cantor set $\mathcal{A}(\varepsilon) \subset A$ such that for any vector of semi-axes $a \in \mathcal{A}(\varepsilon)$ it is possible to find a real-analytic symplectic transformation Φ casting the $(N+1)$-body problem Hamiltonian into the normal form* (1)*. In particular, for $a \in \mathcal{A}(\varepsilon)$, the $(N+1)$-body system possesses N-dimensional elliptic invariant tori foliated by the quasi-periodic (Diophantine) flows $t \to \psi_* + \omega t$ with frequency $\omega = \omega(a)$.*

Moreover the second order Melnikov non-resonance condition (2) *holds true for the frequency vector $\big(\omega(a), \Omega_*(a)\big)$, $a \in \mathcal{A}(\varepsilon)$.*

Finally, the subset $A(\varepsilon) \subset A$ has density close to one as ε tends to 0.

The hypotheses (7) and (9) over the masses of the planets are needed to check that, for δ and ε small enough, the eigenvalues of the averaged quadratic part of the $(N+1)$-body problem Hamiltonian are non vanishing and distinct, that is the main condition needed to apply elliptic KAM theory. Furthermore (and this is the really original idea of [BCV06]), the "outer solar system" model provides particular expressions in the quadratic part of the Hamiltonian, making possible to compute the eigenvalues asymptotically.

Such explicit evaluation, besides allowing to perform elliptic KAM theory as said before, will turn out to be fundamental in the sequel since the eigenvalues of the averaged Hamiltonian are strictly related to the elliptic frequencies of the N-dimensional tori found in Theorem 2. In Chapter 3 the asymptotics we be exploited in order to check the non-resonance condition (6) of Theorem 1 in the particular case of the $(N+1)$-body problem, finally proving the announced result on the existence of periodic orbits accumulating on the elliptic tori of [BCV06].

In short we note here that in Subsection 2.2.3 we state an abstract KAM Theorem from [Pös89], that is of course the main tool used to get the existence of the elliptic tori. Of such theorem, we present a fully detailed version, similar to the one stated in [BBV], enriched with results from the original Pöschel's paper: in particular the KAM normal form describing the dynamics in a neighborhood of the tori is furnished together with estimates concerning the KAM transformation and other various relevant quantities. These informations are indispensable in order to carry on the calculations of the third Chapter.

Another important ingredient for the proof of the next chapter is the appropriate choice of the parameters involved in the KAM procedure.

Chapter 3: Abundance of periodic solutions in the planetary planar $(N+1)$-body problem

In this chapter we prove that, for small parameters δ and ε, the hypotheses of Theorem 1 are fulfilled for the (elliptic) normal form Hamiltonian for the $(N+1)$-body problem provided by Theorem 2, unless to discard a suitable (small) subset of the set of osculating major semi-axes $\mathcal{A}(\varepsilon)$ found above.

First, in Section 3.1, we check the non-degeneracy "twist" condition (5) as already done in [BBV] in the three-body case. At the same time, we provide a uniform bound over the size of the matrices $\mathcal{Q}$ and $\mathcal{R}^{-1}$ in the statement of Theorem 1: since the constant M depends upon them (and grows to ∞ with them) we need to exclude that the norms of the matrices $\mathcal{Q}$ and $\mathcal{R}^{-1}$ tend to ∞ as $\varepsilon \to 0$. In order to do that, we exploit the special form of the symplectic transformation Φ supplied by the KAM Theorem of Póschel to estimate the Tailor-Fourier coefficients $R^*_{k,a,\bar{a},\ell}$ involved in the definitions (3)-(4).

Then, in Section 3.2, we deal with the non-resonance condition (6). After stating some preparatory results on analytic functions of several variables, the non-resonance condition up to order M of Theorem 1 is shown to be met for small δ and ε, and for suitable semi-axes.

The main difficulties to be overcome in the proof are due to the fact that the linear and elliptic frequencies do not vary independently; moreover we don't know such dependence exactly.

Anyway, through a careful handling of the asymptotics found in Chapter 2 and an accurate control of the size of the (small) additive terms introduced by the averaging/KAM procedure, we manage to find a subset of semi-axes $\widetilde{\mathcal{A}}(\delta,\varepsilon) \subset \mathcal{A}(\varepsilon) \subset A$ over which the non-resonance condition (6) holds, provided δ and ε are suitably small. Furthermore, the measure of the set of "discarded" semi-axes $\mathcal{A} \smallsetminus \widetilde{\mathcal{A}}(\delta,\varepsilon)$ is proved to be small with δ. This is certainly the most original part of the thesis. The proof makes use of the fact that the frequency map $a \mapsto \big(\omega(a), \Omega(a)\big)$ is analytic in order to exploit the geometrical properties of (a certain "non-resonant" class of) analytic functions (of several variables). We perform three consecutive steps in which the vector $\big(\omega(a), \Omega_*(a)\big)$ is split as the sum of an appropriate (*Rüßmann non-degenerate*) "dominant part" $D_i(a)$ and a "perturbative part" $P_i(a)$, $i = 0, 1, 2$. Roughly, after proving the non-resonance up to order M of the dominant part over a suitable subset of semi-axes, we will be able to find δ and ε small enough to control the (small) perturbative part so that the whole frequency vector is still non-resonant (up to order M).

At last, recollecting all the results shown so far, in section 3.3 we apply Theorem 1 obtaining our final result:

Theorem 3. *Consider a planetary planar* $(N+1)$*-body system* $(N \geq 3)$ *and let the masses of the planets satisfy* (7) *and* (9).
For every compact set A *of osculating Keplerian major semi-axes, where* (8) *holds for a suitable universal constant* $\theta = \theta_0$ *(depending only on the masses* m_i*), there exists a positive constant* $\tilde{\delta}$ *such that if* $0 < \delta \leq \tilde{\delta}$*, then for sufficiently small* ε*, i.e.* $\forall\, 0 < \varepsilon \leq \tilde{\varepsilon}$*, for some* $\tilde{\varepsilon} = \tilde{\varepsilon}(\delta)$ *(with* $\tilde{\varepsilon}(\delta) \to 0^+$ *as* $\delta \to 0^+$*), the system affords infinitely many periodic solutions, with larger and larger minimal period, clustering to an elliptic KAM torus, provided that the osculating major semi-axes belong to a suitable subset of* A *of density*[4] *closer and closer to* 1 *as* $\delta \to 0$.

[4]The "density" is intended with respect to the Lebesgue measure.

This was the content of the main body of the thesis. In addiction, we provide three appendices in which some useful results are displayed.

In Appendix A we present the proof of the [BBV] averaging Theorem used in the proof of Theorem 1.

In Appendix B a brief review of the Ljusternik-Schnirelman category theory is provided. The variational argument in the first chapter makes use of these results.

In Appendix C we recall in full details a linear algebra lemma from [BCV06] about suitable "perturbations" of simple eigenvalues of real matrices. Such result is needed to find the asymptotics of Chapter 2..

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