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Recent developments on the Kronecker function rings Theory

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Chapter 1

1.1 b-operation and integral closure

Our work starts from the definition of the *b-operation*, a (semi)star operation on an integral domain D. Here, we start by giving some notation.
Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ be the set of all nonzero D-submodules of K and $\boldsymbol{F}(D)$ the nonzero fractionary ideals of D and, finally, let $\boldsymbol{f}(D)$ be the finitely generated D-submodules of K. Hence:

$$\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D).$$

Definition 1.1.1. If M is a D-module contained in K, *the completion of* M is the D-module

$$\widetilde{M} := \bigcap_{V_\lambda \in S} MV_\lambda.$$

The module M is said to be *complete* if $M = \widetilde{M}$.

Remark 1.1.2. If $\bar{D}$ denotes the integral closure of D in K and if we set $\bar{M} := M\bar{D}$, then $\widetilde{\bar{M}} = \widetilde{M}$, where $\widetilde{\bar{M}}$ is the completion of the $\bar{D}$-module $\bar{M}$.

Proof: By definition, S is the set of all valuation overrings of $\bar{D}$, hence:

$$\widetilde{\bar{M}} = \bigcap_{V_\lambda \in S} \bar{M}V_\lambda = \bigcap_{V_\lambda \in S} MV_\lambda = \widetilde{M}.$$

□

It follows that the class of complete $\bar{D}$-modules coincides with the class of complete D-modules. Whenever K is the quotient field of D, let $\overline{\boldsymbol{F}}(D)$ be the set of all nonzero D-submodules of K and $\boldsymbol{f}(D)$ the set of finitely generated D-submodules of K. Hence, we can introduce a semistar operation on K using the completion, as follows.

Definition 1.1.3. The mapping $b : \overline{\boldsymbol{F}}(D) \longrightarrow \overline{\boldsymbol{F}}(D)$, $E \mapsto E^b$ such that $E^b := \widetilde{E}$ is called *b-operation on D*.

The following result shows in particular that the b-operation is a semistar operation on D.

Proposition 1.1.4. *Let $\bar{D}$ be the integral closure of D in K. Let $M, N, L \in \overline{\boldsymbol{F}}(D)$. The b-operation satisfies the following conditions:*

1. $D^b = \bar{D}$;
2. $M^b \supseteq M$;
3. *If* $M \supseteq N$ *then* $M^b \supseteq N^b$;
4. $(M^b)^b = (M)^b$;
5. $(MN)^b = (M^b N^b)^b$, *where by the product MN of two D-modules M, N we mean the D-module generated by the products mn ($m \in M$, $n \in N$, the product is meant to be inside K);*
6. $(zM)^b = zM^b$, $z \in K$;
7. *If $(MN)^b \subseteq (ML)^b$, and if M is either finite or is the completion of a finite D-module, then $N^b \subseteq L^b$.*

Corollary 1.1.5. *We have*

$$(Dx)^b = \bar{D}x, \;\; x \in K \tag{1.1}$$

$$\bar{D}M^b = M^b. \tag{1.2}$$

In this context, it is important to define the integral dependence of an element in a very general form. For any non-negative integer q, we denote by M^q the D-module, submodule of K, generated by the monomials $m_1 m_2 \ldots m_q$, $m_i \in M$, and M^0 stands for D.

Definition 1.1.6. An element z of K is said to be *integrally dependent on the module $M \subseteq K$* if it satisfies an equation on the form

$$z^q + a_1 z^{q-1} + \ldots + a_q = 0, \quad a_i \in M^i. \tag{1.3}$$

This definition is equivalent to the following one: *z is integrally dependent on M if there exists a finite D-module N, contained in K, such that*

$$zN \subseteq MN. \tag{1.4}$$

Relation (1.3) is a consequence of (1.4), using a basis of N. On the other hand, if (1.3) holds, then (1.4) is satisfied by taking for N the module $M_f^{q-1} + M_f^{q-2} z + \ldots + M_f z^{q-2} + D z^{q-1}$, where M_f is a finite submodule of M such that $a_i \in M_f^i$ for $i = 1, 2, \ldots, q-1$.

By relation (1.4), it follows that the set of elements of K which are integrally dependent on M is itself a D-module. We may call that D-module *the integral closure of M in K.*

Theorem 1.1.7. *The completion M^b of M in K coincides with the integral closure of M in K.*

Remark 1.1.8. By Definition 1.1.6 and Theorem 1.1.7 it follows that the completion of M is independent of the choice of the ring D. Therefore, if M is also a module over another ring D_1, subring of K, (for example, if D_1 is a subring of D) then the completion of M as a D_1-module is the same as the completion of M as a D-module.

Since ideals are special modules, we are interested in *complete ideals in* D, where an ideal I is said to be complete if it is complete as a D-module.

Corollary 1.1.9. *If I is an ideal in D, then the completion I^b of I is a complete ideal in the integral closure $\bar{D}$ of D in K. Furthermore, if S^* denotes the set of all valuations v of the quotient field of $\bar{D}$ which are non-negative on D, then:*

$$I^b = \bigcap_{v \in S^*} I A_v, \tag{1.5}$$

where A_v is the valuation ring associated to the valuation v.

We are interested, now, to the case of *complete ideals in an integrally closed domain D.*

Let D be a domain, integrally closed in K and let I be an ideal in D. By Corollary 1.1.9, we have that the completion I^b of I is a complete ideal in $D = \bar{D}$. Hence, we have:

$$I^b = \bigcap_{v \in \Sigma} I A_v = \bigcap_{v \in \Sigma} (D \cap I A_v).$$

We recall now some notions on *valuation ideals.*

Definition 1.1.10. Let D be an integral domain and K the quotient field of D. An ideal J in D is said to be a *valuation ideal* if it is the intersection of D with an ideal J' of a valuation overring A_v of D. If v is the valuation associated to A_v, we say that J is a valuation ideal associated with v.

Proposition 1.1.11. *If v is a valuation, non-negative on D and J is an ideal of D, then the following properties are equivalent:*

1. *J is a valuation ideal;*
2. *If $a, b \in D,\ a \in J$ and $v(a) \leq v(b)$, then $b \in J$;*
3. *The following condition holds:*

$$JA_v \cap D = J. \tag{1.6}$$

Now, coming back to complete ideals, since $D \cap IA_v$ is a valuation ideal, we see that *every* complete ideal in D is an intersection of valuation ideals. On the other hand, if J is a valuation ideal in D associated to $v \in \Sigma$, then $J = D \cap JA_v$ by condition (1.6), where from $J^b = D \cap J^b \subseteq D \cap JA_v = J$, so that $J^b = J$. This means that every valuation ideal in D is a complete ideal, as every finite or infinite intersection of valuation ideals. Therefore, the class of complete ideals in D coincides with the class of ideals which are intersections of valuation ideals. At any time, we can replace K by the quotient field K_0 of D, just restricting v to K_0. So, we assume that K is the quotient field of D.
If I is a complete ideal, then

$$I = \bigcap_{v \in \Sigma} (D \cap IA_v)$$

represents I as intersection (also infinite) of valuation ideals: this representation can be non-unique and it may be even not the finer one. There may be some representations that are finite intersections. We present a result in the case of Noetherian domains.

Theorem 1.1.12. *Let D be a Noetherian domain, K a field containing D and let $\bar{D}$ be the integral closure of D in the quotient field of D.*
If an ideal I is the completion in K of an ideal J in D (in particular, if $D = \bar{D}$ and I is complete), then I is a finite intersection of valuation ideals of $\bar{D}$ associated with discrete valuations of rank 1.

The proof of this theorem needs the following Lemma on complete D-modules and can be found in the complete thesis in Theorem 1.1.14.

Lemma 1.1.13. *Let K be a field containing D, let $M \in \boldsymbol{f}(D)$ and let $\{m_i\}$ be a finite D-basis of M. For each i denote with D_i the ring generated by the* *quotients m_j/m_i, $j \neq i$, over D and let $\bar{D}_i$ be the integral closure of D_i in K. If M^b is the completion of M in K, then*

$$M^b = \cap_i \bar{D}_i m_i.$$

1.2 Integral closure and properties

In the first section, we introduced the notion of *integral closure* over modules. Now, we want to study more in detail this topic in a different situation: we will consider ideals in rings, not necessarily integral domains. First of all, we remind the definition of integral closure in the case of an ideal of a ring.
Note that the definition of integral dependence has been stated for modules. Since the ideals are very important in Commutative Algebra, we give here the definition of integral dependence for ideals.

Definition 1.2.1. Let R be a ring and let I be an ideal of R. An element $z \in R$ is said to be *integral over I* if there exist an integer $n \geq 1$ and elements $a_i \in I^i$, $i = 1, \ldots, n$, such that:

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

An equation of this type is said to be *an equation of integral dependence of z over I, of degree n.*

Let's give now some examples of the integral closure of an ideal.

Examples 1.2.2.

1. Let D be a ring and let z and w be arbitrary elements in D. Consider the ideal $I = (z^2, w^2)$ of D. Hence $zw \in \bar{I} = \overline{(z^2, w^2)}$, since it satisfies an equation of integral dependence over I: $a_0 + a_1(zw) + (zw)^2 = 0$, where $a_1 = 0 \in I$ and $a_0 = -z^2w^2 \in I^2 = (z^2, w^2)^2$. In the same way, for any integer $0 \leq i \leq d$, $z^i w^{d-i} \in \overline{(z^d, w^d)}$.

2. $\bar{I} \subseteq \sqrt{I}$, since by the equation of integral dependence over I, $z^n \in (a_1, \ldots, a_n) \subseteq I$.

3. Radical and prime ideals are integrally closed.

4. The nilradical of the ring is contained in the integral closure of any ideal of the ring, since for each nilpotent element there exists $n \in \mathbb{N}$ such that $z^n = 0$, that is an equation of integral dependence of z over any ideal I.

5. Intersections of integrally closed ideals are integrally closed. (See [SH-06, Remark 1.1.3 (6)])

6. *Persistence*: If $R \xrightarrow{\varphi} S$ is a ring homomorphism, then $\varphi(\bar{I}) \subseteq \overline{\varphi(I)S}$. In fact, if we apply the homomorphism φ to an equation of integral dependence of an element z over I, we obtain an equation of integral dependence of $\varphi(z)$ over $\varphi(I)S$. (See [SH-06, Remark 1.1.3 (7)])

Talking about integral closure of an ideal, it is important to show some properties of this notion, like the good behaviour under localization.

Proposition 1.2.3. *Let R be a ring and I an ideal in R. For any multiplicatively closed subset T of R, $T^{-1}\bar{I} = \overline{T^{-1}I}$.*
Furthermore, the following statements are equivalent:

1. $I = \bar{I}$;

2. *For every multiplicative part T of R, $T^{-1}I = \overline{T^{-1}I}$;*

3. *For every prime ideal P of R, $I_P = \overline{I_P}$;*

4. *For every maximal ideal M of R, $I_M = \overline{I_M}$.*

We have proved that the completion of an ideal is an ideal, so when the completion coincides with the integral closure, we already know that the integral closure of an ideal is an ideal. Now, we will prove this fact in general.

Proposition 1.2.4. *Let R be a ring and L an ideal in R. Then, the integral closure $\bar{L}$ of L is an integrally closed ideal (in R).*

Chapter 2

2.1 The Kronecker function ring

Definition 2.1.1. Let D_0 be a PID with quotient field K_0 and let K be a finite field extension of K_0. Let D be the integral closure of D_0 in K. Let $D[X]$ be the polynomial domain with coefficients in D.
The classical Kronecker function ring of D is:

$$\begin{aligned}\mathrm{Kr}(D) &:= \left\{ \tfrac{f}{g} \mid f, g \in D[X],\ g \neq 0 \text{ and } \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g) \right\} \\ &= \left\{ \tfrac{f'}{g'} \mid f', g' \in D[X],\ \text{and } \boldsymbol{c}(g') = D \right\}\end{aligned}$$

where $\boldsymbol{c}(g)$ denotes *the content* of a polynomial $g \in D[X]$, that is the ideal of D generated by the coefficients of g.

Note that we are assuming that D is a Dedekind domain, since it is the integral closure of D_0, which is a PID, in a finite field extension K of the quotient field K_0 of D_0 ([G-78, Thm. 37.8 and Thm. 41.1]).
Under these hypothesis, every nonzero ideal in D is invertible, so that for every $0 \neq g \in D[X]$ we can choose a polynomial $u \in K[X]$ such that $(D : \boldsymbol{c}(g)) := \boldsymbol{c}(g)^{-1} = \boldsymbol{c}(u)$. This ring has some basics properties:

Proposition 2.1.2.

1. $\mathrm{Kr}(D)$ *is a domain with identity with quotient field $K(X)$ and, in particular,* $\mathrm{Kr}(D) \cap K = D$
2. $\mathrm{Kr}(D)$ *is a Bézout domain.*
3. *If I is a finitely generated ideal of D, then $I\mathrm{Kr}(D) \cap K = ID$.*

Since we are working with a Dedekind domain, we can immediately see an important result:

Lemma 2.1.3. ***Gauss Lemma:*** *Let $f, g \in D[X]$, where D is an integral domain. If D is a Prüfer domain, then:*

$$\boldsymbol{c}(fg) = \boldsymbol{c}(f)\boldsymbol{c}(g),$$

and conversely.

This theory has been generalized by Krull, who worked in a more general context using integrally closed domains, (not necessarily Dedekind domains). One of the greatest troubles that exists in the generalization of the Kronecker idea is that *Gauss Lemma* works fine for Prüfer (and in particular Dedekind domains), but not in general.
However, in an integral domain, it holds always the inclusion of ideals

$$\boldsymbol{c}(fg) \subseteq \boldsymbol{c}(f)\boldsymbol{c}(g).$$

A very useful result is *the Dedekind-Mertens Lemma.*

Lemma 2.1.4. ***Dedekind-Mertens Lemma:*** *Let $f, g \in D[X]$, where D is an integral domain. Let $m := deg(g)$, then:*

$$\boldsymbol{c}(f)^m\boldsymbol{c}(fg) = \boldsymbol{c}(f)^{m+1}\boldsymbol{c}(g)$$

Lemma 2.1.5. ***Gauss-Krull Lemma.*** *Let $\star$ be an e.a.b. star operation on an integral domain D and let $f, g \in D[X]$ then:*

$$\boldsymbol{c}(fg)^\star = \boldsymbol{c}(f)^\star\boldsymbol{c}(g)^\star.$$

Definition 2.1.6. Let D be an integrally closed domain with quotient field K and let $\star$ be an e.a.b. star operation on D.
The Star-Kronecker function ring of D is given by:

$$\mathrm{Kr}(D, \star) := \left\{ \tfrac{f}{g} \mid f, g \in D[X],\ g \neq 0 \text{ and } \boldsymbol{c}(f)^\star \subseteq \boldsymbol{c}(g)^\star \right\}.$$

It is an integral domain with quotient field $K(X)$ and it is called the $\star$*-Kronecker function ring of D*

Then the following properties hold:

Proposition 2.1.7.

1. $\mathrm{Kr}(D, \star)$ *is a domain with identity with quotient field $K(X)$ and, in particular,* $\mathrm{Kr}(D, \star) \cap K = D^\star = D$

2. $\mathrm{Kr}(D,\star)$ *is a Bézout domain.*

3. *If* I *is a finitely generated ideal of* D*, then* $I\mathrm{Kr}(D,\star) \cap K = I^\star D$*.*

Proof: For the proof, see Proposition 2.1.7 in the complete thesis. $\square$

It is important to show that the two definitions of the Kronecker function ring coincide when we are in a Dedekind domain. Actually, we only need to assume to have a Prüfer domain.

Theorem 2.1.8. *If* D *is a Prüfer domain, then each* $\star$*-operation on* D *is a.b., and any two* $\star$*-operations on* D *are equivalent.*

Proof: For the proof, see Theorem 2.1.8 in the complete thesis. $\square$

Hence, in such a domain, every star operation is equivalent to the identical one and $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D)$.

Corollary 2.1.9. *If* D *admits an e.a.b. star operation, then* D *is integrally closed.*

We give here the proof, since it is rather easy and illuminating.

Proof: Let $F \mapsto F^\star$ be an e.a.b. $\star$-operation and let $\mathrm{Kr}(D,\star)$ be the star-Kronecker function ring of D. We have proved that $\mathrm{Kr}(D,\star) \cap K = D$ and that $\mathrm{Kr}(D,\star)$ is a Bézout domain, so it is integrally closed. Therefore, D is integrally closed too. $\square$

Theorem 2.1.10. *Let* D *be an integrally closed domain with quotient field* K *and let* $\star$ *be an e.a.b.* $\star$*-operation on* D*, with Kronecker function ring* $\mathrm{Kr}(D,\star)$*. If* W *is a valuation overring of* $\mathrm{Kr}(D,\star)$*, then* W *is the trivial extension of* $W \cap K$ *to* $K(X)$*.*

Corollary 2.1.11. *Each Kronecker function ring of an integrally closed domain* D *contains* $\mathrm{Kr}(D,b)$*, the Kronecker function ring of* D *with respect to the* b*-operation.*

Theorem 2.1.12. *Let* D *be an integrally closed domain with quotient field* K*, and let* $\mathrm{Kr}(D,b)$ *be the Kronecker function ring of* D*.*

1. *If V is an integrally closed overring of D, then each Kronecker function ring* $\mathrm{Kr}(V,\star)$ *of V is an overring of* $\mathrm{Kr}(D,b)$ *such that* $\mathrm{Kr}(V,\star)\cap K = V$.

2. *If R is an overring of* $\mathrm{Kr}(D,b)$*, then R is a Kronecker function ring of* $R\cap K$.

Theorems 2.1.10 and 2.1.12 imply that there is a one-to-one correspondence between valuation overrings of an integrally closed domain D and valuation overrings of $\mathrm{Kr}(D,b)$. Since $\mathrm{Kr}(D,b)$ is a Bézout domain, the set of valuation overrings of $\mathrm{Kr}(D,b)$ is in one-to-one correspondence with the set of proper prime ideals of $\mathrm{Kr}(D,b)$. This is a good reasons to consider the Kronecker function ring D: we have reduced the problem of finding all valuation overrings of D to the study of the set of proper ideals of $\mathrm{Kr}(D,b)$.

Since $\mathrm{Kr}(D,b)$ is a Prüfer domain, the dimension of $\mathrm{Kr}(D,b)$ is the same as the valuative dimension of $\mathrm{Kr}(D,b)$ (where the *valuative dimension* of D is the maximum of the Krull dimension of its valuation overrings if finite, otherwise it is infinite). Moreover, a valuation v and each its trivial extension have the same value group and the same rank, so that D and $\mathrm{Kr}(D,b)$ have the same valuative dimension. Therefore, we have proved the following:

Proposition 2.1.13. *Let D be an integrally closed domain with Kronecker function ring* $\mathrm{Kr}(D,b)$*. Then* $\dim_v D = \dim \mathrm{Kr}(D,b)$.

2.2 General setting

The problem of the construction of a Kronecker function ring for general integral domains was investigated independently by Halter-Koch in 2003 and by Fontana-Loper since 2001.

Halter-Koch had an axiomatic approach and used the theory of module systems. He established a connection with Krull's theory and Kronecker function rings and introduced the Kronecker function rings for integral domains with an ideal system which does not verify, necessarily, the cancellation property (e.a.b.).

Fontana-Loper based their work on the theory of semistar operations.
Halter-Koch gave an abstract definition which does not depend on semistar operations or valuation overrings.

Definition 2.2.1. Let K be a field, R a subring of $K(X)$ and $D := R\cap K$. If

(Kr.1) X is a unit in R;

(Kr.2) $f(0) \in fR$ for each $f \in K[X]$;

Then R is called *K-function ring of D*.

Using only these two axioms, he proved that R has similar properties as a Kronecker function ring:

Theorem 2.2.2. *Let R be a K-function ring of $D = R \cap K$, then:*

1. *R is a Bézout domain with quotient field $K(X)$;*

2. *D is integrally closed in K;*

3. *For each polynomial $f := a_0 + a_1X + \cdots + a_nX^n \in K[X]$, we have $(a_0, \ldots, a_n)R = fR$.*

Note that these properties are very close to those described in Proposition 2.1.2.
The next step is to describe Fontana-Loper's approach and to find out the relation with Halter-Koch's K-function rings. Using the semistar operations, we generalize the concept of Kronecker function ring and we can now define the Kronecker function ring for any e.a.b. semistar operation:

Definition 2.2.3. Let D be any integral domain and let $\star$ be any semistar operation. We define *the Kronecker function ring of D with respect to the semistar operation* $\star$ by:

$$\begin{aligned}\mathrm{Kr}(D,\star) := \{f/g \mid f,g \in D[X], g \neq 0 \text{ and there exists} \\ h \in D[X] \smallsetminus \{0\} \text{ with } (\boldsymbol{c}(f)\boldsymbol{c}(h))^\star \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^\star\} \cup \{0\}.\end{aligned}$$

Our next steps are:

- Show that the semistar Kronecker function ring leads to a natural extension of the classical Kronecker function ring;

- Study the connections between the semistar Kronecker function ring $\mathrm{Kr}(D,\star)$ and the axiomatically defined K-function ring;

- Show that $\mathrm{Kr}(D,\star)$ defines a new semistar operation on D.

Definition 2.2.4. Given any semistar operation $\star$ of D, we can associate to $\star$ an (e.)a.b. semistar operation of finite type $\star_a$ of D, called *the (e.)a.b. semistar operation associated to* $\star$, defined as follows for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$F^{\star_a} := \bigcup\{((FH) : H^\star) \mid H \in \boldsymbol{f}(D)\},$$
$$E^{\star_a} := \bigcup\{F^{\star_a} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}.$$

It is clear that $(\star_f)_a = \star_a$. Furthermore, if $\star = \star_f$, then $\star$ is (e.)a.b. if and only if $\star = \star_a$. Hence, if $\star$ is an e.a.b. semistar operation, then $\star_a$ is the unique (e.)a.b. semistar operation of finite type and that is equivalent to $\star$.

Then:

- When $\star = \star_f$, then $\star$ is e.a.b. if and only if $\star = \star_a$.
- $D^{\star_a}$ is integrally closed and contains the integral closure of D.

Remark 2.2.5. Focusing on the star operations, $\star_a$ is expected to be a star operation too and that is because it is defined on the "star closure" of D, or on an integral domain which is "star closed". Precisely, even if $\star$ is a semistar operation, we call *the $\star$-closure of D*:

$$D^{\mathrm{cl}^\star} := \cup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\}.$$

It follows immediately that $D^{\mathrm{cl}^\star}$ is an integrally closed overring of D and D is said $\star$-closed if $D = D^{\mathrm{cl}^\star}$. Let's define now a new (semi)star operation on D, when $D = D^{\mathrm{cl}^\star}$ or more in general a semistar operation on D, denoted by $\mathrm{cl}^\star$: for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$, define

$$F^{\mathrm{cl}^\star} := \cup\{((H^\star : H^\star)F)^\star \mid H \in \boldsymbol{f}(D)\},$$
$$E^{\mathrm{cl}^\star} := \cup\{F^{\mathrm{cl}^\star} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}.$$

Setting $\bar{\star} = \mathrm{cl}^\star$, it is easy to see that $D^{\mathrm{cl}^{\bar{\star}}} = D^{\mathrm{cl}^\star}$, it coincides with $D^{\star_a}$ and that $D^{\mathrm{cl}^\star}$ contains the classical integral closure $\bar{D}$ of D. Furthermore:

$$\star_f \le \mathrm{cl}^\star \le \star_a \text{ and } (\star_f)_a = (\mathrm{cl}^\star)_a = (\star_a)_a = \star_a.$$

We have seen in Section 2 that for a domain D and a semistar operation $\star$ on D, a valuation overring V of D is a *$\star$-valuation overring of* D whenever $F^\star \subseteq FV$ for each $F \in \boldsymbol{f}(D)$. By definition, the $\star$-valuation overrings coincide with the $\star_f$-valuation overrings.

Proposition 2.2.6. *Let D be a domain and let $\star$ be a semistar operation on D.*

1. *The $\star$-valuation overring of D also coincide with the $\star_a$-valuation overrings.*

2. $D^{\mathrm{cl}^\star} = \cap\{V \mid V$ *is a* $\star-$*valuation overring of* $D\}$.

Proof: For the proof, see Proposition 2.3.5 in the complete thesis. $\square$

Now, let's see how the two approaches are related.

Theorem 2.2.7. *Let $\star$ be a semistar operation of an integral domain D with quotient field K. Then:*

1. *V is a $\star$-valuation overring of D if and only if $V(X)$ is a valuation overring of* $\mathrm{Kr}(D,\star)$. *The map $W \mapsto W \cap K$ establishes a bijection between the set of all valuation overrings of* $\mathrm{Kr}(D,\star)$ *and the set of all the $\star$-valuation overrings of D.*

2. $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_f) = \mathrm{Kr}(D,\star_a) =$
   $= \cap\{V \mid V$ *is a* $\star-$*valuation overring of* $D\}$ *is a Bézout domain with quotient field $K(X)$.*

3. *$E^{\star_a} = E\mathrm{Kr}(D,\star) \cap K = \cap\{EV \mid V$ is a $\star-$valuation overring of $D\}$, for each $E \in \overline{\boldsymbol{F}}(D)$.*

4. *$R := \mathrm{Kr}(D,\star)$ is a K-function ring of $R \cap K = D^{\star_a}$, in the sense of Definition 2.2.1.*

Proof: For the proof, see Theorem 2.3.6 in the complete thesis. $\square$

Chapter 3

3.1 Nagata ring

Nagata extended the notion of Kronecker function ring, considering a special ring of rational functions for arbitrary integral domains and, even, arbitrary rings.

Definition 3.1.1. Let D be a ring with identity. We define the *Nagata ring of* D, the ring

$$\mathrm{Na}(D) := D(X) := \left\{ \frac{f}{g} \;\mid\; f, g \in D[X] \text{ and } \boldsymbol{c}(g) = D \right\}$$

Nagata's ring is particularly interesting, because it has some important properties that D itself need not have, even though it maintains a strong relation with the ideal structure of D. For the proof of the following, see Proposition 3.1.2 in the complete thesis.

Proposition 3.1.2. *Let D be an arbitrary ring and let* $\mathrm{Na}(D)$ *be the Nagata ring of D.*

1. *The map $M \mapsto MD(X)$ establishes a 1-1 correspondance between the maximal ideals of D and the maximal ideals of $D(X)$.*

2. *For each ideal I of D,*

$$ID(X) \cap D = I, \qquad \frac{D(X)}{ID(X)} \cong \left(\frac{D}{ID}\right)(X);$$

   I is finitely generated $\Leftrightarrow$ $ID(X)$ is finitely generated.

Among the new properties acquired by $D(X)$ are the following:

3. *The residue field at each maximal ideal of $D(X)$ is infinite;*

4. *An ideal contained in a finite union of ideals is contained in one of them;*

5. *Each finitely generated locally principal ideal is principal and, therefore,* $\mathrm{Pic}(D(X)) = 0$.

In the proof of this proposition, we needed the following Lemma, whose proof can be found in [G-78, Lemma 33.2].

Lemma 3.1.3. *Let Q be an ideal of the ring R, and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over R. Then $QR[\{X_\lambda\}] \cap R = Q$, and if Q is P-primary in R, then $QR[\{X_\lambda\}]$ is $PR[\{X_\lambda\}]$-primary in $R[\{X_\lambda\}]$.*

In general, $\mathrm{Na}(D)$ is not a Bézout domain, but it has the following properties:

Proposition 3.1.4. *Let D be an integral domain. The following properties are equivalent:*

1. *D is a Prüfer domain;*

2. $\mathrm{Na}(D)$ *concides with* $\mathrm{Kr}(D)$*;*

3. $\mathrm{Na}(D)$ *is a Bézout domain;*

4. $\mathrm{Na}(D)$ *is a Prüfer domain;*

5. *Every ideal of* $\mathrm{Na}(D)$ *is extended from D.*

Proof: For the proof, see Proposition 3.1.4 in the complete thesis. □

An easy example is the following:

Example 3.1.5. Let V be a valuation ring of a field K. Let v be the valuation associated to V and Γ be the value group of v. Then the map:

$$\begin{array}{rcl} w : K[X] & \longrightarrow & \Gamma \cup \{\infty\} \\ f := \sum_{i=0}^{n} a_i x^i & \longmapsto & w(f) := \begin{cases} \infty & \text{if } f = 0; \\ \min\{v(a_i) \mid 0 \le i \le n\} & \text{else;} \end{cases} \end{array}$$

defines naturally a valuation on $K(X)$, called *Gaussian extension of the valuation v*, just setting for each $f/g \in K(X)$, $w(f/g) = w(f) - w(g)$.
The valuation ring W of w is such that:

$$W = \mathrm{Kr}(V) = \mathrm{Na}(V) = V(X).$$

Using the semistar operations, we generalize the Nagata ring:

Definition 3.1.6. Given <u>any</u> integral domain D and <u>any</u> semistar operation $\star$ on D, we define *the semistar Nagata ring* as follows:

$$\mathrm{Na}(D,\star) := \left\{ \frac{f}{g} \ \mid \ f,g \in D[X],\ g \neq 0 \text{ and } \boldsymbol{c}(g)^\star = D^\star \right\}.$$

Remarks 3.1.7.

1. Note that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_f)$. So the assumption $\star = \star_f$ is not restrictive in Nagata semistar rings.

2. If $\star = d$ is the identity (semi)star operation of D, then:

$$\mathrm{Na}(D,d) = D(X).$$

Since Kang generalized the Nagata rings with the star operations in the 1980's, we will show some results, that he proved on star Nagata rings, generalized to the semistar setting:

Proposition 3.1.8. *Let $\star$ be a nontrivial semistar operation of an integral domain D. Set:*

$$N(\star) := N_D(\star) := \{h \in D[X] \mid \boldsymbol{c}(h)^\star = D^\star\}.$$

1. *$N(\star) = D(X) \setminus \cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$ is a saturated multiplicatively closed subset of $D[X]$ and $N(\star) = N(\star_f)$.*

2. $\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$.

3. $\mathrm{Na}(D,\star) = D[X]_{N(\star)} = \cap\{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.

4. *$\mathcal{M}(\star_f)$ coincides with the canonical image in $\mathrm{Spec}(D)$ of the maximal spectrum of $\mathrm{Na}(D,\star)$; i.e. $\mathcal{M}(\star_f) = \{M \cap D \mid M \in \mathrm{Max}(\mathrm{Na}(D,\star))\}$.*

From the last point of previous proposition we have:

Corollary 3.1.9. *Let D be an integral domain, then:*

Q is a maximal t-ideal of $D \Leftrightarrow Q = M \cap D$, for some $M \in \mathrm{Max}(\mathrm{Na}(D,v))$.

Let study now the semistar-operation associated to $\mathrm{Na}(D,\star)$, but first we need some notions.

Definition 3.1.10. Let D be an integral domain.

1. If Δ is a nonempty set of prime ideals of D, then the semistar operation $\star_\Delta$ defined on D as follows, for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\star_\Delta} := \cap\{ED_P \mid P \in \Delta\},$$

is called *the spectral semistar operation associated to* Δ.

2. A semistar operation $\star$ of D is called *a spectral semistar operation* if there exists a nonempty subset $\Delta \subseteq \mathrm{Spec}(D)$ such that $\star = \star_\Delta$.

3. We say that $\star$ is *a quasi-spectral semistar operation of* D if, for each nonzero ideal I of D such that $I^\star \cap D \neq D$, there exists a quasi-$\star$-prime P of D such that $I \subseteq P$.

4. A semistar operation of D is said to be *stable* if for all $E, F \in \overline{\boldsymbol{F}}(D)$

$$(E \cap F)^\star = E^\star \cap F^\star.$$

Let's see some properties of this new operation $\star_\Delta$. For the proofs of the following two lemmas, see, respectively, [FL-03, Lemma 2.4] and [FL-03, Lemma 2.5].

Lemma 3.1.11. *Let D be an integral domain and let $\emptyset \neq \Delta \subseteq \mathrm{Spec}(D)$. Then:*

1. *$E^{\star_\Delta} D_P = ED_P$, for each $E \in \overline{\boldsymbol{F}}(D)$ and for each $P \in \Delta$.*

2. *$(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$, for all $E, F \in \overline{\boldsymbol{F}}(D)$.*

3. *$P^{\star_\Delta} \cap D = P$, for each $P \in \Delta$.*

4. *If I is a nonzero integral ideal of D and $I^{\star_\Delta} \cap D \neq D$, then there exists $P \in \Delta$ such that $I \subseteq P$.*

Lemma 3.1.12. *Let $\star$ be a nontrivial semistar operation of an integral domain D. Then:*

1. *$\star$ is spectral if and only if $\star$ is quasi-spectral and stable.*

2. *Assume that $\star = \star_f$, then $\star$ is quasi-spectral and $\mathcal{M}(\star) \neq \emptyset$.*

Theorem 3.1.13. *Let $\star$ be a nontrivial operation and let $E \in \overline{\boldsymbol{F}}(D)$. Set*

$$\tilde{\star} := (\star_f)_{sp} := \star_{\mathcal{M}(\star_f)}.$$

[$\tilde{\star}$ is called the spectral semistar operation associated to $\star$.] Then:

1. $\tilde{\star} \leq \star_f$.

2. $E^{\tilde{\star}} = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$ *[and* $E^{\star_f} = \cap\{E^{\star_f} D_Q \mid Q \in \mathcal{M}(\star_f)\}$*].*

3. $E\mathrm{Na}(D, \star) = \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$*, thus:*
   $E\mathrm{Na}(D, \star) \cap K = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$.

4. $E^{\tilde{\star}} = E\mathrm{Na}(D, \star) \cap K$.

Proposition 3.1.8 (*4*) assures that, whenever we contract a maximal ideal of $\mathrm{Na}(D, \star)$ is to D, we obtain exactly a prime ideal in $\mathcal{M}(\star_f)$. This conclusion can be reversed. Furhermore, the two semistar operations, $\tilde{\star}$ and $\star$, generate the same Nagata ring.

Corollary 3.1.14. *Let $\star, \star_1, \star_2$ be semistar operations of an integral domain D. Then:*

1. $\mathrm{Max}(\mathrm{Na}(D, \star)) = \{QD_Q(X) \cap \mathrm{Na}(D, \star) \mid Q \in \mathcal{M}(\star_f)\}$.

2. $(\tilde{\star})_f = \tilde{\tilde{\star}}$.

3. $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.

4. $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star})$.

5. $\star_1 \leq \star_2 \Rightarrow \mathrm{Na}(D, \star_1) \subseteq \mathrm{Na}(D, \star_2) \Leftrightarrow \widetilde{\star_1} \leq \widetilde{\star_2}$.

Proof: For the proof, see Proposition 3.2.6 in the complete thesis. $\square$

At this point, a natural question is: what kind of relationship exists between $\mathrm{Kr}(D\star)$ and $\mathrm{Na}(D, \star)$?

3.2 Relations between $\mathrm{Na}(D, \star)$, $\mathrm{Kr}(D, \star)$ and the semistar operations.

Let's turn our attention to the question of valuation overrings. We introduced this topic in Section 3, talking about the two definitions of Kronecker function ring in the semistar case.

Proposition 3.2.1. *Let D be a domain and let $\star$ be a semistar operation on D. A valuation overring V of D is a $\tilde{\star}$-valuation overring if and only if V is an overring of D_P for some $P \in \mathcal{M}(\star_f)$.*

An important fact is the following:

Proposition 3.2.2. *Let D be a domain with quotient field K and let $\star$ be a semistar operation on D. Then* $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$.

Proof: See [FL-03, Proposition 4.1] $\square$

An immediate first question to ask is whether the two semistar operations $\tilde{\star}$ and $\star_a$ are the same or very different. Proposition 3.2.1 shows that for a semistar operation $\star$ on a domain D, the $\tilde{\star}$-overrings of D are all the valuation overrings of the localizations of D at the primes in $\mathcal{M}(\star_f)$. We also know from Theorem 2.3.6 that the $\star_a$-valuation overrings, or equivalently the $\star$-valuation overrings, of D correspond exactly to the valuation overrings of the Kronecker function ring $\mathrm{Kr}(D,\star)$. In particular: each $\star_a$-valuation overring is also a $\tilde{\star}$-valuation overring.
It is clear that these two set of valuation domains can be different and, even if they coincide, it may happen that $\tilde{\star} \neq \star_a$. It is possible to make positive statements about the relationship between $\widetilde{(-)}$ and $(-)_a$ under certain conditions. Despite that we limit to stating a result that generalizes the fundamental result that is at the basis of Krull's theory of Kronecker function rings: $\mathrm{Na}(D) = \mathrm{Na}(D,d) = \mathrm{Kr}(D,b) = \mathrm{Kr}(D)$ if and only if D is a Prüfer domain [G-78, Thm. 33.4].

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