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Thesis in Mathematics

A generalization of the continued fraction algorithm: The Jacobi-Perron Algorithm

SUMMARY

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Summary

Lagrange established that: *"The expansion of a real number as a simple continued fraction is periodic if and only if it is the real root of a quadratic equation in one variable with rational coefficients"* .
This leads to a characterization of quadratic irrationalities and also, as we will see, to the fundamental unit of any given order in a real quadratic field. After this result mathematicians had abandoned all hope to expect any further information about the arithmetic properties of higher degree algebraic irrationals from the "naive" world of simple continued fractions. This question, however, continued to attract the creative imagination of restless mathematicians like Jacobi who tried to generalize this to cubic irrationals by means of a mixed continued fraction for pairs of real numbers.
The Jacobi Algorithm, in the homogeneous form, essentially proceeds in the following manner: it starts with a triple of three real numbers (u_0, v_0, w_0) and it generates new triples of numbers by the recursion formulas:

$$u_{i+1} = v_i - l_i u_i$$
$$v_{i+1} = w_i - m_i u_i$$
$$w_{i+1} = u_i,$$

for $i = 0, 1, \ldots;\ l_0 = l;\ m_0 = m;\ l, m \in \mathbb{N}$.
But we will use the non-homogeneous form and that is

$$\frac{v_{i+1}}{u_{i+1}} = \frac{\dfrac{w_i}{u_i} - m_i}{\dfrac{v_i}{u_i} - l_i}, \quad \frac{w_{i+1}}{u_{i+1}} = \frac{1}{\dfrac{v_i}{u_i} - l_i}.$$

Now, assuming

$$a_1^{(i)} = \frac{v_i}{u_i},\ a_2^{(i)} = \frac{w_i}{u_i},\ b_1^{(i)} = l_i,\ b_2^{(i)} = m_i,\ u_0 = 1,\ i \geq 0,$$

the Jacobi's Algorithm becomes

$$a_1^{(i+1)} = \frac{a_2^{(i)} - b_2^{(i)}}{a_1^{(i)} - b_1^{(i)}}, \quad a_2^{(i+1)} = \frac{1}{a_1^{(i)} - b_1^{(i)}}$$

and since, for binary expansions, one starts only with the initial values u_0, v_0, this takes the form

$$a^{(i+1)} = \frac{1}{a^{(i)} - b^{(i)}},$$

so that

$$a^{(i)} = b^{(i)} + \frac{1}{a^{(i+1)}}$$

which is the Euclidean algorithm if $b^{(i)} = [a^{(i)}]$.

He showed that if the algorithm, in the non-homogeneous form, of two real numbers becomes periodic then such numbers belong to an algebraic number field of degree $n \leq 3$ but was unable to determine if, taking a set of algebraic irrationals, its algorithm becomes periodic or not. However, he presented some examples of irrationality with periodic development:

$$1, \sqrt[3]{2}, \sqrt[3]{4}, \quad 1, \sqrt[3]{3}, \sqrt[3]{9}, \quad 1, \sqrt[3]{5}, \sqrt[3]{25}.$$

Furthermore his algorithm would have solved the problem of calculating a system of fundamental units in algebraic number fields of degree $n = 3$. But he either did not know it, or was not concerned about units; so did Perron who generalized Jacobi's Algorithm for any real algebraic number field of degree $n \geq 3$. In fact, he published an article [14] in which he revised the Jacobi Algorithm raising it from its isolated cubism to the general n-th dimension. One of his main merits is the study of the convergence of the algorithm in analogy to the convergence of a continued fraction, but generalized to the n-th dimension as we will see later. Much space of Perron's immortal paper is devoted to the characteristic equation (i.e. the equation of degree $n + 1$ whose roots generate the algebraic number field to which the initial n-tuple

of numbers $(a_1^{(0)}, \dots, a_n^{(0)})$ belongs). In fact, the reducibility of this equation is a crucial point for the study of the periodicity. Perron also proved that the algorithm does not becomes periodic if the components of the initial n-tuple are linearly dependent, but was not able to give sufficient conditions for the periodicity. The advantage of periodic algorithms is the fact that they can be applied, as we will see, to calculate units in the corresponding algebraic number fields. In honor of these two great mathematicians this algorithm was called the *Jacobi-Perron Algorithm*, abbreviated JPA.
In this thesis the generalization of a continued fraction in the sense of the JPA is considered. It is organized as follows:

in chapter [1] we recall the standard definitions, fix the notions of basis on theory of the continued fractions and observe that such fractions provide much insight into the nature of numbers. In fact, we will prove that:

Theorem 0.0.1. *A rational number can be expressed as a finite simple continued fraction in just two ways, one with an even and the other with an odd number of convergents. In one form the last partial quotient is 1, in the other it is greater than 1.*

Theorem 0.0.2. *Every irrational number can be represented as an infinite simple continued fraction and the representation, or expansion, is unique.*

We will also discuss the periodic properties of continued fractions and we will investigate the expansion of quadratic irrationals characterising which have infinite periodic continued fraction expansions. In particular we will present the principal result of the chapter, the proof of Lagrange's theorem:

Theorem 0.0.3 (Lagrange)**.** *Let α be real irrational. The continued fraction of $\alpha = [a_0, a_1, \dots, a_{n-1}, b_n]$ is periodic if and only if α is quadratic. Moreover α is reduced if and only if its continued fraction is purely periodic.*

Thus this theorem, in addition to giving a characterization of quadratic irrationalities, states that the numbers represented by purely periodic continued fractions are quadratic irrationals of a particular kind, and we will study how these numbers can be distinguished from other quadratic irrationals. Finally particular interest is devoted to the continued fraction for Euler's number. This will be accomplished in several stages. First of all we will consider the function

$$f(c,x)=\sum_{n\geq 0}\frac{1}{c(c+1)\cdots(c+n-1)}\frac{x^n}{n!}$$

and will show that

$$\frac{z}{c}\frac{f(c+1,z^2)}{f(c,z^2)}=\frac{1}{\dfrac{c}{z}+\dfrac{z}{c+1}\dfrac{f(c+2,z^2)}{f(c+1,z^2)}}=[0,\frac{c}{z},\frac{c+1}{z},\ldots,\frac{c+n}{z},\alpha_{n+2}]$$

where

$$\alpha_{n+2}=\frac{c+n+1}{z}\frac{f(c+n+1,z^2)}{f(c+n+2,z^2)}.$$

In this way, we obtain the *Lambert continued fraction* namely, the continued fraction of

$$e^{1/y}+e^{-1/y}.$$

This is followed by the expansion of the continued fraction for

$$\frac{e^{1/y}-e^{-1/y}}{e^{1/y}+e^{-1/y}}\qquad\forall\ y\in\mathbb{Z}_{\geq 1},$$

and we will obtain

$$\frac{e^{1/y}-e^{-1/y}}{e^{1/y}+e^{-1/y}}=[0,y,3y,5y,\ldots]\qquad\forall\ y\in\mathbb{Z}_{\geq 1}.$$

In particular, for $y=2$:

$$\frac{e-1}{e+1}=[0,2,6,10,\ldots].$$

The chapter will end with the expansion of the continued fraction for e:

$$e=[2,1,2,1,1,4,1,1,6,1,1,\ldots].$$

in chapter [2] we discuss the connection between the expansion of quadratic irrationals in simple continued fraction and the units in a real quadratic field. The main result of this question is expressed in the following theorem:

Theorem 0.0.4. *Let $\alpha \in \mathbb{Q}(\sqrt{d})$ be reduced, such that $D(\alpha) = D$ where D is the discriminant of $\mathbb{Q}(\sqrt{d})$. Suppose also that k is a period of its continued fraction, $v = gcd(q_{k-1}, p_{k-1} - q_{k-2}, p_{k-2})$ and $u = p_{k-1} + q_{k-2}$. Then*

$$w = \frac{u + \sqrt{D}v}{2}$$

is a unit > 1, and every unit > 1 is of that type. Moreover, we have that $N(w) = (-1)^k$.

in chapter [3] we present the main object of this thesis that is: the JPA. We will first define the JPA and state the main theorems and properties of the JPA needed for further investigation.

Definition 0.1. Let $T : E^{n-1} \longrightarrow E^{n-1}$ be a trasformation.
Let $f : E^{n-1} \longrightarrow E^{n-1}$ be any vector function such that

$$f(a^{(k)}) = b^{(k)} = (b_1^{(k)}, b_2^{(k)}, \dots, b_{n-1}^{(k)})$$

where $a_1^{(k)} \neq b_1^{(k)}$. Then f is called a *T-function* or a *function associated with T* if

$$T(a^{(k)}) = (a_1^{(k)} - b_1^{(k)})^{-1}(a_2^{(k)} - b_2^{(k)}, \dots, a_{n-1}^{(k)} - b_{n-1}^{(k)}, 1) \qquad \text{for } k \geq 0.$$

Definition 0.2. A sequence $\{a^{(k)}\}_{k \in \mathbb{N}}$ of vectors in E^{n-1} is called a *Jacobi -Perron Algorithm of the vector $a^{(0)}$* if there exists a T-transformation of E_{n-1} into E_{n-1}, defined as above, such that

$$T(a^{(k)}) = a^{(k+1)} \qquad \forall k.$$

Hence a JPA is a sequence of vectors obtained from an initial vector subsequently applying a T-transformation that allows to find each vector from the previous. We will also show that for $n = 2$ the JPA becomes the continued fraction algorithm if we choose as function associated with T the function f defined in the following way:

$$f(a^{(k)}) = [a^{(k)}]$$

where $[a^{(k)}]$ is the integral part of $a^{(k)}$. Finally we will study the convergence of the JPA in analogy with the convergence of continued fractions and will introduce, for this, the concept of P-boundedness for a T-function:

Definition 0.3. Let $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)}) \in E^{n-1}$. The JPA of $a^{(0)}$ is said to be *convergent*, if

$$\lim_{v\to\infty} \frac{A_i^{(v)}}{A_0^{(v)}} = a_i^{(0)} \qquad \text{for } i = 1, \ldots, n-1.$$

Definition 0.4. A T-function f such that $f(a^{(k)}) = b^{(k)} = (b_1^{(k)}, b_2^{(k)}, \ldots, b_{n-1}^{(k)})$ is said to be *P-bounded*, if

$$0 < \frac{1}{b_{n-1}^{(k)}} \leq C, \quad 0 < \frac{b_i^{(k)}}{b_{n-1}^{(k)}} \leq C \qquad \text{for } i = 1, \ldots, n-1 \text{ and } k \geq 0$$

where C is a real constant, independent of k.

We close this chapter with proving a theorem which lead us to the connection between the convergence of the JPA and P-boundedness for T-function:

Theorem 0.0.5. *Let $a^{(0)} \in E^{n-1}$. The JPA of $a^{(0)}$ is convergent if its T-function f is P-bounded.*

in chapter [4] we study the periodicity of the JPA. For this we first define when the JPA of a vector $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)}) \in E^{n-1}$ is periodic:

Definition 0.5. Let $a^{(0)} \in E^{n-1}$ and let $T : E^{n-1} \longrightarrow E^{n-1}$ a trasformation and its associated T-function f be fixed. The JPA of $a^{(0)}$ is called *periodic* if $\exists\ l \in Z_{\geq 0}, m \in Z_{\geq 1}$ with $l = minL, m = minM$ such that

$$T^{m+v} = T^{v} \qquad \forall\ v \geq l.$$

To study the periodicity of the JPA, we will focus our attention on a particular T-function, the integer part function. This T-function occupies a very significant place in the theory of the JPA. For example we have already observed that for $n = 2$, the JPA with the T-function $f(a_1^{(k)}) = [a_1^{(k)}]$ becomes the Euclidean Algorithm and yields the expansion of any real number by simple continued fraction.
Its main advantage rests with the fact that the $b^{(k)}$ are rational vectors. Besides, this T-function was originally used by Jacobi and Perron in the first definition of the algorithm, and only after the algorithm has been generalized by introducing the concept of T-function. Furthermore, that the T-function $f(a^{(k)}) = [a^{(k)}]$ plays a fundamental role in the study of the periodicity of the JPA is stated in the following theorem:

Theorem 0.0.6. *Let*

$$F(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_{n-1} x - d$$

a P-polynomial of first order. Then $F(x)$ has the following properties:

(i) $F(x)$ has one and only one real root $w \in \left(1, \dfrac{1}{n + k_1 + \cdots + k_{n-2}}\right)$;

(ii) The JPA with the T-function $f(a^{(k)}) = [a^{(k)}]$ of the vector

$$a^{(0)} = (w + k_1, w^2 + k_1 w + k_2, w^3 + k_1 w^2 + k_2 w + k_3, \ldots, w^{n-1} + k_1 w^{n-2} + \cdots + k_{n-1})$$

is purely periodic; the lenght m of the primitive period is

$$m = \begin{cases} n & \text{if } d > 1 \\ 1 & \text{if } d = 1 \end{cases}$$

and the primitive period has the structure

$$\begin{cases} b^{(0)} = (k_1, \ldots, k_{n-1}) \\ b^{(i)} = (k_1, \ldots, k_{n-1-i}, k'_{n-i}, k'_{n-i}, \ldots, k'_{n-i}) & \text{for } i = 1, \ldots, n-2 \\ b^{(n-1)} = (k'_1, \ldots, k'_{n-1}) \\ k'_j = d^{-1} k_j \qquad \text{for } j = 1, \ldots, n-1 \end{cases}$$

for $d > 1$; for $d = 1$, its structure is $b^{(0)} = (k_1, \ldots, k_{n-1})$;

(iii) $F(x)$ is irreducible over the field of rationals.

In the last section we will give some particular examples of periodic JPA with the integer part function f. Those we will enounce are classic cases, which were analyzed by Jacobi, Perron, Bachman, Hasse and later by Leon Bernstein. We will prove:

Theorem 0.0.7. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that*

$$\alpha = \sqrt[n]{D^n + d}$$

where $d, D \in \mathbb{N}$ and $d \mid D$. Let's further assume

$$\begin{aligned} D &\geq (e-2)(n-1)d, & n \leq 4; \\ D &\geq (n-2)d, & n > 4. \end{aligned}$$

Then the JPA with the T-function $f(a^{(k)}) = [a^{(k)}]$ of the vector $a^{(0)}$ with the s-th component defined as follows

$$a_s^{(0)} = \sum_{i=0}^{s} \binom{n-s+i-1}{i} D^i \alpha^{s-i} \qquad \text{for } s = 1, \ldots, n-1$$

is purely periodic; the length of the period is

$$m = \begin{cases} n & \text{if } d \neq 1 \\ 1 & \text{if } d = 1 \end{cases}$$

and if $d > 1$ the period has the structure

$$\begin{cases} b^{(0)} = \left(\binom{n}{1}D, \binom{n}{2}D^2, \dots, \binom{n}{n-1}D^{n-1}\right) \\ \vdots \\ b^{(s)} = \left(\binom{n}{1}D, \binom{n}{2}D^2, \dots, \binom{n}{n-s-1}D^{n-s-1}, \right. \\ \qquad \left. \binom{n}{n-s}tD^{n-s-1}, \dots, \binom{n}{n-1}tD^{n-2}\right) \\ \vdots \\ b^{(n-1)} = \left(\binom{n}{1}t, (\binom{n}{2}tD, (\binom{n}{3}tD^2, \dots, \binom{n}{n-1}tD^{n-2}\right) \\ t = d^{-1}D \end{cases}$$

for $s = 1, \dots, n-2$. If $d = 1$, its form is $b^{(0)} = \left(\binom{n}{1}D, \binom{n}{2}D^2, \dots, \binom{n}{n-1}D^{n-1}\right)$. Moreover the JPA is ideally convergente for $D \geq \frac{e}{2}(n-1)d$.

Many classic researches on the periodicity of the JPA with f such that $f(a^{(k)}) = [a^{(k)}]$ were conducted on the vector $a^{(0)}$ to components in formula of the form

$$a^{(0)} = (\alpha, \alpha^2, \dots, \alpha^{n-1})$$

because it is natural that this is the right approach to the elements

$$\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$$

which create a basis for $\mathbb{Q}(\alpha)$. First of all O. Perron [14] proved that if the algorithm for a given vector $a^{(0)} = (a_1^{(0)}, \dots, a_{n-1}^{(0)})$ becomes periodic, then the $a_i^{(0)}$ belong to an algebraic number field of degree $\leq n$ and they also form a basis for this field if the grade is n. We will analyze and prove this fact in the next chapter. Solving the inverse problem is more complicated. It is not known yet whether each base $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ of an algebraic number field of grade n admits periodic JPA development. We will solve this question for an infinite class of special bases for fields of each degree.

in chapter [5] we examine the connection between the JPA of a vector $a^{(0)} = (a_1^{(0)}, \ldots, a_{n-1}^{(0)})$ and the units of the field $\mathbb{Q}(a_1^{(0)}, \ldots, a_{n-1}^{(0)})$ in analogy with what has been observed in chapter [1] for euclidean algorithm. First of all we will define and build the characteristic equation associated to any periodic JPA. It will allow us to obtain one of the most important properties of a periodic JPA which is expressed in

Theorem 0.0.8. *If the JPA, associated with some T-function, of the vector $a^{(0)} = (a_1^{(0)}, \ldots, a_{n-1}^{(0)})$ is periodic then the components $a_i^{(0)}$ for $i = 1, \ldots, n-1$ are all algebraic numbers of degree $\leq n$.*

The question whether or not this algorithm becomes periodic for any $a^{(0)} \in E^{n-1}$ is unsolved and seems very difficult. The advantage of a periodic JPA is the fact that one or more units of the field $\mathbb{Q}(a_1^{(0)}, \ldots, a_{n-1}^{(0)})$ can be calculated. In fact, we will prove the main result of this in:

Theorem 0.0.9. *Let $a^{(0)} = a^{(0)}(w) \in E_{n-1}$ be a vector with algebraic components of degree $\leq n$. Suppose also that the JPA of $a^{(0)}$ is periodic and that the associated T-function is such that the components of the vectors $b^{(v)}$, for $v = 0, 1, \ldots\ldots,$ are all integers. Then*

$$e = \prod_{i=l}^{l+m-1} a_{n-1}^{(i)}$$

is a unit of the field $\mathbb{Q}(w)$, where l denotes the length of the pre-period and m denotes the length of the period of this periodic JPA.

Furthermore, we will determinate explicitly units of some algebraic number fields based on the periodicity of the JPA of various vectors $a^{(0)}$. Finally, we will conclude this thesis with a word about the fundamentality of the units:

$$\epsilon = \frac{(\alpha - D)^n}{\alpha^n - D^n}$$

in the field $\mathbb{Q}(\alpha)$.

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