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Pricing of derivatives of financial markets: classic techniques and open problems

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The goal of this work is to give a survey of the mathematical techniques exploited in some advanced economic and financial models with special attention to models for pricing and hedging derivative securities.

Since the early fifties, a growing body of studies have become to apply to economic and financial models those mathematical techniques developed to deal with stochastic phenomena, that is those phenomena of the real world whose dynamics is influenced by some random mechanism. This on the ground that the time evolution of the near totality of important features of economic and financial phenomena is affected by the occurrence of random events. For instance, the prices of commodities and consumption goods, exchange rates, bond returns, stock and derivative prices and so on clearly show, at least partly, an erratic behavior. Hence, probability theory and specifically the theory of stochastic processes has begun to play a crucial role in economic and financial modeling.

All epistemological value of the theory of probability is based on this: that large scale random phenomena in their collective action create strict, non random regularity.
(Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*)

In economics and financial phenomena, randomness stems from the incomplete knowledge of reality, from the lack of information, from complexity, from the fact that causes are diverse, that tiny perturbations may result in large effects. For over a century now, Science has abandoned Laplace's deterministic vision, and has fully accepted the task of deciphering randomness and inventing adequate tools for its description. The surprise is that, after all, randomness has many facets and that there are many levels to uncertainty, but, above all, that a new form of predictability appears, which is no longer deterministic but statistical.

Financial markets offer an ideal testing ground for these statistical ideas. The fact that a large number of participants, with divergent anticipations and conflicting interests, are simultaneously present in these markets, leads to unpredictable behavior. Moreover, financial markets are, sometimes strongly, affected by external news, which are, both in date and in nature, to a large extent unexpected. The statistical approach consists in drawing from past observations some information on the frequency of possible price changes. If one then assumes that these frequencies reflect some intimate mechanism of the markets themselves, then one may hope that these frequencies will remain stable in the course of time. For example, the mechanism underlying the roulette or the game of dice is obviously always the same, and one expects that the frequency of all possible outcomes will be invariant in time, although of course each individual outcome is random. The "bet" that probabilities are stationary is very reasonable in the case of roulette or dice; nevertheless, it is much less justified in the case of financial markets. It is clear, for example, that financial markets do not behave now as they did 30 years ago: many factors contribute to the evolution of the way markets behave (development of derivative markets, world-wide and computer-aided trading, etc.). "Young markets" (such as emergent countries markets) and more mature markets (exchange rate markets, interest rate markets, etc.) behave quite differently. The statistical approach to financial markets is based on the idea that whatever evolution takes place, this happens sufficiently slowly (on the scale of several years) so that the observation of the recent past provides some information which is useful to give a statistical description of a not too far future. However, even this "weak stationarity" hypothesis is sometimes badly in error, in particular in the case of a crisis, which marks a sudden change of market behavior. Hence, the statistical description of financial fluctuations is certainly imperfect. Nevertheless, it is extremely helpful: in practice, the "weak stationarity" hypothesis is in most cases reasonable, at least to describe risks. In other words, the amplitude of the possible price changes is, to a certain extent, probabilistically predictable. It is thus rather important to devise adequate tools, in order to control financial risks when it is possible.

Trying to give a definition of financial markets, we could say that financial markets are all trades of financial tools performed with the goal of transferring assets from individuals in surplus to individuals in deficit and redistributing the risks of economic and financial business. These two features are the primary office of financial markets. Hence, financial markets are the key in allocating economic and financial resources to get profit and reduce risk. In a financial market, demand and supply of capital are to be balanced and investments of money are to be efficient. *Financial tools* are those contracts concerning obligations and rights of financial nature suitably exchanged and negotiated via financial markets to achieve the target of getting profit and reducing risk. In particular, we call financial tools: securities representative of risky capital, government securities negotiable in market of money, mutual funds, general contracts to trade goods, money and even rates of interest, and all possible combinations of the mentioned contracts and securities. *Securities* (often called *assets*) are financial tools, which in exchange of an amount of money (*asset price*) at a specified time give the right to get a profit (*asset dividend*) at a future time. It is customary to distinguish between securities whose profit can be considered safe (*bonds*) and securities whose gain is not certain (*stocks*). However, the price market of both bond and stocks fluctuate continuously in response to the arrival of fresh information. Interesting is the case of *derivatives markets*, which are financial tools whose price depends upon or is derived from one or more assets, called *underlying.* A derivative is merely a contract between two or more parties. Its value is determined by fluctuations in the underlying asset. In our work we are interested primarily in *options.* An option is a contract between two parties, a *holder* and a *writer.* By paying a *prime* to the writer, the holder acquires the right, but not the obligation, to buy from the writer (*call option*), or to sell to the writer (*put option)*, one unit of an underlying asset, represented with S, within a fixed date T, called *maturity* or *expiration date*, at a predetermined price K, called *strike-price.* The writer, upon the payment of the prime, takes the obligation to satisfy the holder's right upon the exercise of the option. In particular, we will deal with *European call option* whose characteristic feature is that they can be exercised only at the maturity. If S_T is the price of the underlying asset at maturity, then the *payoff* of the option, that is the value of the contract at maturity, is given by

$$(S_T - K)^+ \stackrel{\text{def}}{=} \begin{cases} S_T - K & \text{If } S_T - K > 0 \\ 0 & \text{otherwise} \end{cases} .$$

In case $S_T - K > 0$, the holder will exercise the option and make the profit $S_T - K$ by buying the stock for the strike price K and selling it immediately at the market price S_T. Otherwise, the option will be not exercised, since the market price of the asset is less than the strike price.

More generally, we will consider European derivatives defined by their maturity time T and their nonnegative payoff function $h(S_T)$. At time $t < T$ this contract has a value, known as the *derivative price*, which varies with t and the observed stock price S_t. Problems of pricing and hedging derivative securities in an uncertain environment are important to investors ranging from large trading institutions to pension funds. The most common underlying assets include stocks, bonds, commodities, currencies, interest rates and market indexes. As discussed above, the dynamics of asset prices in financial markets is influenced by the occurrence of a large number of random events giving raise to a stochastic perturbation, which, under suitable market condition, is modeled by the celebrate Wiener process, or standard Brownian motion, constituting the basic stochastic diffusion. The *Wiener process* is a real-valued stochastic process starting from the zero state and advancing toward an infinite horizon, with stationary normally distributed independent increments and continuous trajectories. More precisely, writing $(W_t)_{t\geq 0}$ for the Wiener process we have that

1. $W_0 = 0$;
2. the random variables $\left(W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}\right)$ are independent for any finite increasing sequence of times $0 = t_0 < t_1 < \ldots < t_n$;
3. the increment $W_t - W_s$ is normally distributed with null mean and variance $\mathbf{E}\left[(W_t - W_s)^2\right] = t - s$ for all times $0 \leq s < t$.

We denote by $(\Omega, \mathfrak{E}, \mathbf{P}) \equiv \Omega$ the probability space where our Brownian motion is defined and the expectation $\mathbf{E}[\cdot]$ is computed.

The increasing family of σ-field $(\mathfrak{F}_t)_{t\geq 0}$ generated by $(W_t)_{t\geq 0}$, represents the flow of information behind on the stochastic perturbation. Hence, the process $(W_t)_{t\geq 0}$ can be viewed as the quantitative counterpart of the continuous arrival of fresh information in the market buffeting the dynamics of the asset prices.

The independence of increments makes the Wiener process an ideal candidate to define a complete family of independent infinitesimal increments dW_t, which are centered and normally distributed with variance dt and serve as a differential model of noise. The drawback is that the trajectories of $(W_t)_{t\geq 0}$ are not regular enough, in the sense that they are almost surely of no bounded variation. Hence, they cannot be point wise differentiated in the standard way.

Nevertheless, we can define an integral respect to Wiener process. For a fixed finite time T, let $(X_t)_{t\geq 0}$ be a stochastic process adapted to the filtration $(\mathfrak{F}_t)_{t\geq 0}$ generated by the Brownian motion up to time T, such that

$$\mathbf{E}\left[\int_0^T (X_t)^2\, dt\right] < +\infty.$$

Using iterated conditional expectations and the independent increments property of Brownian motion, it is possible to show

$$\mathbf{E}\left[\left(\sum_{k=1}^n X_{t_{k-1}}\left(W_{t_k} - W_{t_{k-1}}\right)\right)^2\right] = \mathbf{E}\left[\sum_{k=1}^n \left(X_{t_{k-1}}\right)^2 (t_k - t_{k-1})\right],$$

for any finite increasing sequence of times $0 = t_0 < t_1 < \ldots < t_n = t \leq T$, which is the basic equation for the construction of stochastic integrals. The *Itô stochastic integral* of $(X_t)_{t\geq 0}$ with respect to the Brownian motion $(W_t)_{t\geq 0}$ is then defined as,

$$\int_0^t X_s dW_s \stackrel{\text{def}}{=} \|\cdot\|_{L^2(\Omega)}\text{-}\lim_{n\to\infty} X_{t_{k-1}}\left(W_{t_k} - W_{t_{k-1}}\right).$$

As a function of time t, this stochastic integral defines a time continuous square integrable process satisfying the *Itô isometry*

$$\mathbf{E}\left[\left(\int_0^t X_s dW_s\right)^2\right] = \mathbf{E}\left[\int_0^t X_s^2 ds\right],$$

and the *martingale property*

$$\mathbf{E}\left[\int_0^t X_u dW_u \mid \mathfrak{F}_s\right] = \int_0^s X_u dW_u \quad \mathbf{P}\text{-a.s}, \ \forall s \leq t.$$

The composition of a sufficiently regular function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with the Wiener process defines a new stochastic process $(Y_t)_{t\geq 0}$, where $Y_t \equiv g(t, W_t)$. The purpose of the chain rule is to compute the differential $dg(W_t)$ or equivalently its integral $g(W_t) - g(W_0)$. A suitable application of Taylor's formula yields

$$g(W_t) - g(W_0) = \int_0^t g'(W_s)\, dW_s + \frac{1}{2}\int_0^t g''(W_s)\, ds,$$

or in the differential shorthand

$$dg(W_t) = g'(W_t)\, dW_t + \frac{1}{2} g''(W_t)\, dt,$$

which is the simplest version of the *Itô formula*. More generally, given an *Itô process* $(X_t)_{t\geq 0}$ satisfying some technical integrability condition, it is possible to define the *Itô formula* for a process $(Y_t)_{t\geq 0}$, where $Y_t \equiv g(t, X_t)$, which is written in the differential shorthand as

$$dg(t, X_t) = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\, d\langle X\rangle_t,$$

where $\langle X\rangle_t = \int_0^t X_s^2 ds$ is the *quadratic variation*of the Itô process $(X_t)_{t\geq 0}$.

The stochastic process which is most commonly exploited for modeling the price of a stock $(S_t)_{t\geq 0}$ is the *geometric Brownian motion*, given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where μ and σ are positive constant parameters known as *drift* and *volatility* of the stock, respectively. Somewhat loosely speaking, in the geometric Brownian motion model the *infinitesimal expected return* and the *infinitesimal expected variance* are given by

$$\mathbf{E}_t[dS_t] = \mu S_t\, dt, \quad \text{and} \quad \mathbf{D}_t^2[dS_t] = \sigma^2 S_t^2\, dt.$$

Hence, it is natural to think of μ as the *instantaneous expected rate of return* and σ as the *instantaneous deviation of the rate of return* of the stock for each unit invested in the stock. Actually, many authors formally write

$$\mathbf{E}_t\left[\frac{dS_t}{S_t}\right] = \mu\, dt, \quad \text{and} \quad \mathbf{D}_t^2\left[\frac{dS_t}{S_t}_t\right] = \sigma^2\, dt.$$

Therefore, the right side of (1) has the natural financial interpretation of a return term plus a risk term. It is worth noting that as stock prices are generated by the agents' demand for the stocks a diffusion model such as (1) should be explained in terms of the interaction of the agents trading in the financial market. Indeed, D. Kreps [36] showed that geometric Brownian motion can be justified as a rational expectation equilibrium in a market with highly rational agents who all believe in this model.

In 1973, Fischer Black and Myron Scholes published a market model, the celebrate *Black & Scholes model*, in which they solved the problem of the pricing of European call or put options having as underlying a stock, under the assumptions of *no arbitrage condition*, meaning that there is not a risk-free way to make a profit in the market, and *continuous hedging condition*, meaning that it is possible to hedge a derivative continuously in time via a portfolio composed by the underlying stock and a free risk asset. The derived analytic formula for the fair price of the option, the so called *Black & Scholes formula*, is actually exploited in the real markets when the assumptions of the Black & Scholes (B&S) model are likely reasonable. In B&S model, it is assumed the existence of a risk free asset, a *bond*, whose price $(B_t)_{t\geq 0}$ is driven by the ordinary differential equation

$$dB_t = rB_t dt,$$

where $r > 0$ is the instantaneous constant interest rate for lending or borrowing money, and a stock, whose price $(S_t)_{t\geq 0}$ is modeled according to a geometric Brownian motion as in (1). The B&S analysis of an European derivative yields an explicit hedging strategy in the underlying stock and risk free bond whose terminal payoff is equal to the payoff $h(S_T)$ of the derivative at maturity, no matter the path the stock price takes. Thus, selling the derivative and holding a dynamically adjusted portfolio according to this strategy covers an investor against all risk of eventual loss, because a loss incurred at the final time from one part of this portfolio will be exactly compensated by a gain in the other part. This *replicating strategy*, as it is known, therefore provides an insurance policy against the risk of being at a loss. It is also called a *dynamic hedging strategy* since it involves continuous trading to hedge the risk. The essential step in the B&S methodology is the construction of this replicating strategy and the argument, based on no arbitrage principle, that the value of the replicating portfolio at any time has to be the fair price of the derivative at that time. If the price of an European call option at time t with underlying $(S_t)_{t\geq 0}$, is denoted by $C(t, S_t)$, then $C(t, x)$ is the solution of the celebrated *B&S partial differential equation (PDE)*

$$\mathcal{L}_{BS} C = 0, \tag{2}$$

where

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right), \tag{3}$$

with the terminal condition $h(x) = (x - K)^+$. Equation (3) has a closed-form solution, which is the *B&S formula*, given by

$$C(t, x) = x\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{4}$$

where

$$\begin{aligned} d_1 &= \frac{\log(x/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

and

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy.$$

This rather simple formula for the price of a call option, in terms of the current stock price x, the time to maturity $T-t$, the strike price K, the stock volatility σ, and the risk free interest rate r, explains the popularity of the model in the financial services industry since the mid-1970s. The fact that $C(t, x)$ given by (4) satisfies Equation (2) with the final condition $h(x) = (x - K)^+$ can easily be checked directly. In this work we study some techniques to solve (2) in more general frameworks than the original B&S one. Due to the crucial importance in financial applications, several different techniques have been developed to solve (2), depending on the model given for the price of the stock and the structure of the terminal condition. Actually, it is widely recognized that the simplicity of the popular B&S model is no longer sufficient to capture modern market phenomena, especially since the 1987 crash. It rests upon a number of assumptions that are limiting factors to obtain a stochastic model as realistic as possible. Among these are continuity of the stocks price process, the ability to hedge continuously without transaction costs, independent Gaussian returns, and constant volatility. We shall focus here on relaxing the last assumption by allowing volatility to vary randomly, for the following reason: volatility tends to fluctuate at a high level for a while, then at a low level for a similar period, then high again, and so on. It "reverts in mean" many times during the life of a derivative contract. In addition, modeling volatility as a stochastic process is motivated a priori by empirical studies of stock price returns in which estimated volatility is observed to exhibit "random" characteristics. In addition, the effects of transaction costs show up, under many models, as uncertainty in the volatility. In this work we present three important models which consider the stock satisfying a modified geometric brownian motion, in which also the volatility is modeled as a stochastic process. The difference between these models is the assignment of the volatility. The *Hull&White model* follows the dynamics

$$\begin{cases} dS_t = \alpha S_t dt + \sigma_t S_t dW_t \\ \sigma_t = f(Y_t) \\ dY_t = bY_t dt + cY_t dZ(t) \end{cases},$$

where $f(y) = \sqrt{y}$, $b < 0$, and $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are uncorrelated Wiener processes. In the *Cox-Ingersoll-Ross model* volatility is a *mean-reverting* process

$$\begin{cases} dX_t = \alpha X_t dt + \sigma_t X_t dW_t \\ \sigma_t = f(Y_t) \\ dY_t = (a + bY_t)\, dt + c\sqrt{Y_t} dZ_t \end{cases},$$

where $f(y) = \sqrt{y}$, a, b, c are constant parameters, and $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are correlated Wiener processes. The last model volatility model presented is the *Log Ornstein-Uhlenbeck* process which is also mean-reverting. The model dynamics is given by

$$\begin{cases} dX_t = \alpha X_t dt + \sigma_t X_t dW_t \\ \sigma_t = f(Y_t) \\ dY_t = (a + bY_t)\, dt + c dZ_t \end{cases},$$

where $f(y) = e^y$, a, b, c are constants and the processes $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are correlated Wiener processes.

In the last part of our work, we are interested in studying a model for the pricing of options whose underlying is driven by the following *bivariate Ornstein-Uhlenbeck process*

$$\begin{cases} dX_t = aX_t dt + Y_t dW_t \\ dY_t = \alpha\,(m - Y_t)\, dt + \beta dZ_t \end{cases} \tag{5}$$

where α is the rate of mean reversion and m is the long-run mean level of Y, a is a constant and $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are uncorrelated Wiener processes. The assumption that the stochastic process $(X_t)_{t\geq 0}$ is driven by an Ornstein-Uhlenbeck process is reasonable when we think of risky-asset price of a commodity underlying the option or when we assume that $(X_t)_{t\geq 0}$ models the logarithm of a stock price, $X_t = \ln S_t$. Nevertheless, assuming stochastic volatility as a mean reverting Ornstein-Uhlenbeck (OU) process does not prevent the possibility that volatility become negative. On the other hand, since OU process is Gaussian, the distribution of Y_T conditional on Y_t is normal with mean

$$\mathbf{E}\left[Y_T\right] = m + \left(Y_t - m\right) e^{-\alpha(T-t)},$$

and variance

$$\mathbf{D}^2\left[Y_T\right] = \frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)$$

Thus, the probability that Y_T becomes negative is given by

$$\mathbf{P}\left(Y_T \leq 0\right) = \Phi\left(-\frac{\mathbf{E}\left[Y_T\right]}{\mathbf{D}^2\left[Y_T\right]}\right).$$

This probability is undeniably small for a wide range of reasonable parameter values. We will deal with the problem of pricing an European call option whose underlying satisfies the bivariate O-U process (5), by exploiting the theory of semigroups. We will study the problem of the existence of the resolvent of the infinitesimal generator of the semigroup associated to the O-U process (5) and give the guideline to find its explicit representation.

Bibliography

[1] Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications.* New York: Wiley.

[2] Ball, C. A. & Roma, A. (1994). "Stochastic Volatility Option Pricing". *The Journal of Financial and Quantitative Analysis* 29, 589-607.

[3] Barlow, M. T. (2002). "A Diffusion Model for Electricity Prices". *Mathematical Finance* 12, 287-298.

[4] Billingsley, P. (1986). *Probability and Measure* (2d ed.). New York: Wiley.

[5] Björk, T. (1998). *Arbitrage Theory in Continuos Time.* Oxford: Oxford University Press.

[6] Black, F. & Scholes, M. (1973). "The Pricing of Options and Corporate Liabilities". *Journal of Political Economy* 81, 637-654.

[7] Cattiaux, P. (1992). "Stochastic Calculus and Degenerate Boundary Value Problems". *Annales de L'Istitut Fourier* 42, 541-624.

[8] Cerrai, S. (2000). "Analytic Semigroups and Degenerate Elliptic Operators with Unbounded Coefficients: A Probabilistic Approach". *Journal of Differential Equations* 166, 151-174.

[9] Chen, Z.Q. & Zhao, Z. (1995). "Diffusion processes and second order elliptic operators with singular coefficients for lower order terms". *Math. Ann.* 302, 323-357

[10] Chung, K. L. (1974). *A Course in Probability Theory* (2d ed.). New York: Academic Press.

[11] Chung, K. L. & Williams, R. (1990). *An Introduction to Stochastic Integration* (2d ed.). Boston: Birkhäuser.

[12] Cox, J. & Huang, C.- F. (1988). "Options Pricing and Its Applications". In S. Bhattacharya & G. Constantinides, *Frontiers of Financial Theory.* Totowa, N.J.: Rowman & Littlefield, 272-288.

[13] Cox, J., Ross, S. & Rubinstein M. (1979). “Option Pricing: A Simplified Approach”. *Journal of Financial Economics* 7, 229-263.

[14] Cox, J. & Rubinstein M. (1985). *Options Markets.* Englewood Cliffs, N.J.: Prentice-Hall.

[15] Da Prato, G. & Lunardi, A. (1995). ”On the Ornstein-Uhlenbeck Operator in Spaces of Continuous Functions”. *Journal of Functional Analysis-* 131, 94-144.

[16] Delbaen, F. & Schachermayer, W. (1993). “A General Version of the Fundamental Theorem of Asset Pricing”. Working Paper. Department of Mathematics, University of Vienna.

[17] Duffie, D. (1996). *Dynamic Asset Pricing Theory* (2d ed.). Princeton: Princeton University Press.

[18] El Karoui, N., Peng, S. & Quenez, M. (1994). “Backward Stochastic Differential Equations in Finance”. Laboratoire de Probabilités, Université de Paris.

[19] Elliott, R. (1982). *Stochastic Calculus and Applications.* New York: Springer-Verlag.

[20] Elliot, R. & Kopp, P.E (1999). *Mathematics of Financial Markets.* New York: Springer-Verlag.

[21] Es-Sarhir, A. & Farkas, B. ”Invariant measures and regularity properties of perturbed Ornstein–Uhlenbeck semigroups”. *J. Differential Equations* 233, 87-104.

[22] Ethier, S. & Kurtz, T. (1986). *Markov Processes: Characterization and Convergence.* New York: Wiley.

[23] Friedman, A. (1964). *Partial Differential Equations of the Parabolic Type.* Englewood Cliffs, N.J.: Prentice-Hall.

[24] Friedman, A. (1975). *Stochastic Differential Equations and Applications*, Vol. I. New York: Academic Press.

[25] Fouque, J.- P., Papanicolaou, G. & Sircar, K. R. (2000). *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge: Cambridge University Press.

[26] Galor, O. & Zeira, J. (1993). ”Income Distribution and Macroeconomics”. *Review of Economic Studies* 60, 35-52.

[27] Gihman, I. & Skorohod, A. (1972). *Stochastic Differential Equations.* Berlin: Springer-Verlag.

[28] J.M Harrison &S.R. Pliska, (1981). "Martingales and Stochastic intefrals in the theory of continuous trading", *Stoch. Proc. & Appl* 11, 215-260.

[29] Ikeda, N. & Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* Amsterdam: North-Holland.

[30] Jarrow, R. (1988). *Finance Theory.* Englewood Cliffs, N.J.: Prentice-Hall.

[31] Jarrow, R. & Rudd, A. (1983). *Option Pricing.* Homewood, Ill.: Richard D. Irwin.

[32] Karatzas, I. & Shreve, S. (1988). *Brownian Motion and Stochastic Calculus.* New York: Springer-Verlag.

[33] Karatzas, I. & Shreve, S. (1995). "Methods of Mathematical Finance". Chapters 1-3, Draft Manuscript, Department of Mathematics, Carnegie-Mellon University, Pittsburgh.

[34] Kopp, P. (1984). *Martingales and Stochastic Integrals.* Cambridge: Cambridge University Press.

[35] Kreps, D. (1981). "Arbitrage and equilibrium in economies with infinitely many commodities". *J. Math. Econ.* 8, 15-35.

[36] Kreps, D. (1982). "Multiperiod Securities and the Efficient Allocation of Risk: A comment on the Black-Scholes Option Pricing Model". *The Economics of Uncertainty and Information* (ed. J. McCall), 203-232. University of Chigago Press.

[37] Malliaris, A. (1982). *Stochastic Methods in Economics and Finance.* Amsterdam: North-Holland.

[38] Merton, R. (1990). *Continuous-Time Finance.* Oxford: Basil Blackwell.

[39] Nicolato, E. & Vernados, E. (2003). "Option Pricing in Stochastic Volatility Models of the Ornstein-Uhlenbeck Type", *Mathematical Finance* 13, 445-466.

[40] Øksendal, B. (1985). *Stochastic Differential Equations.* Berlin: Springer-Verlag.

[41] Protter, P. (1990). *Stochastic Integration and Differential Equations.* New York: Springer-Verlag.

[42] Revuz, D. & Yor, M. (1991). *Continuous Martingales and Brownian Motion.* New York: Springer-Verlag.

[43] Schobel, R. & Zhu, J. (1999). "Stochastic Volatility With an Ornstein–Uhlenbeck Process: An Extension", *European Finance Review* 3, 23-46.

[44] Stein, E.M. & Stein, J.C. (1991)."Stock Price Distributions with Stochastic Volatility: An Analytic Approach", *The Review of Financial Studies* 4, 727-752.

[45] Steele, J.M. (2001). *Stochastic Modelling and Applied Probability.* New York: Springer-Verlag.

[46] Stokey, N. & Lucas, R. (1989). *Recursive Methods in Economic Dynamics (with Ed Prescott).* Cambridge, Massachussetts: Harward University Press.

[47] Stroock, D. & Varadhan, S. R. S. (1979). *Multidimensional Diffusion Processes.* New York: Springer-Verlag.

[48] Wilmott, P., Dewynne, J. & Howison, S. (1993). *Option Pricing: Mathematical Models and Computation.* Oxford: Oxford Financial Press.