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Zeros of the Riemann Zeta-function on the critical line

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February 4, 2012

Layout of the thesis

This thesis is basicly intended as an exposure of fundamental results concerning the so-called *non-trivial zeros* of the Riemann zeta-function $\zeta(s)$: these zeros are strictly connected with the central problem of analytic number theory, i.e. the *Riemann hypothesis*. The starting point is the epoch-making work of Bernhard Riemann, dated 1859 [1]: it was the only paper of the German mathematician about number theory and, taking cue from Euler's relation

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left[1 - p^{-s}\right]^{-1} ,$$

valid for $\text{Re}(s) = \sigma > 1$, showed a much more profound and deep relation between the complex function $\zeta(s)$ and the prime numbers distribution.

The first chapter of this thesis exposes the main features of the zeta-function. In particular, in Section 1.1 we review the analytic continuation of $\zeta(s)$ as a meromorphic function in the whole complex plane, whit a single simple pole at $s = 1$: this was the vital jump which, thanks to Riemann, allowed to study $\zeta(s)$ in the half plane where the Euler product expansion is not valid. Of course in this region, $\sigma \leq 1$, the zeta-function cannot be expressed as a series and this makes life quite difficult; fortunately, Riemann's work included a functional equation for $\zeta(s)$ which, showing a symmetry relatively the *critical line* $\text{Re}(s) = \sigma = 1/2$, has become the starting point for investigating the behaviour of the function for $\sigma \leq 1$. The functional equation is

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{1}$$

and is derived in Chapter 1. In addition, the other conjectures contained in Riemann's paper are exposed in this Section: all but one were proven by Hadamard and von Mangoldt.

Section 1.2 is dedicated to introducing the delicate question of the zeros of $\zeta(s)$: from the functional equation are easily derivable the "trivial zeros", which occur at all negative even integers $s = -2, -4, -6, \ldots$, while the other (non-trivial) zeros are all located in the "critical strip" $\{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$: Riemann conjectured that $\zeta(s)$ has infinitely many zeros in the critical strip, a conjecture proved by Hadamard.

Section 1.3 summarizes, with modern terminology, the original paper written by Riemann, whose title can be translated as "On the Number of Prime Numbers less than a Given Quantity", indicating the main intention of Riemann, that is the achievement of an explicit formula for the prime counting function $\pi(x)$, defined as

$$\pi(x) = \sum_{p \leq x} 1 = \#\{p\,\mathrm{prime} \mid p \leq x\} .$$

This explicit formula involves the zeta-function and, in particular, its non-trivial zeros.

The end of the chapter, Section 1.4, is about the famous Riemann hypothesis (RH), stating that all the non-trivial zeros lies on the critical line $\mathrm{Re}(s) = 1/2$. This conjecture is the only one of the five contained in the Riemann's paper which remains unproved, nevertheless it is taken as hypothesis for thousands of theorems (supporting the term "hypothesis" in place of "conjecture"). Last but not least, RH is strictly connected with primes distribution: some consequence of RH involving primes will be pointed out, the most important being the link with the *prime number theorem* and the magnitude of the error for the Gauss' estimate for the prime counting function $\pi(x)$, an error that would become the smallest possible (meaning a somewhat "random" behaviour of prime numbers), in formulas

$$\pi(x) = \mathrm{Li}(x) + \mathcal{O}(\sqrt{x}\log x) ,$$

where $\mathrm{Li}(x)$ is the logarithmic integral

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \,.$$

At the end of the chapter, different reasons to believe that RH is true will be discussed (most of the mathematicians think so) [6], together with some reasons for doubting of RH [11], for the sake of completeness.

After this introductory chapter, the thesis is divided in two main parts. The first part is outlined in Chapter 2 and is pertaining the "computational" aspect of locating the non-trivial zeros of $\zeta(s)$. There's no doubt that a strong reason for believing in RH is an impressive numerical evidence: in 2004 Gourdon [26] claimed he was able to compute the first 10^{13} non-trivial zeros and all of them lie on the critical line or, in other words, RH turns out to be true for the first 10^{13} zeros. Calculations which are made the present day of course involve a massive use of computers, but the underlying theoretical principles date back to the beginning of XIX century: the pioneer of the field was Gram [15], who managed to calculate the first 15 zeros on the critical line. This was done using the Euler-Maclaurin summation method, which is described in Section 2.1, together with its application to estimate $\Gamma(s)$ and $\zeta(s)$: the numerical estimations performed in this way are vital for Gram's strategy, as explained in the next section of the chapter. In Section 2.2 indeed we start with the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \tag{2}$$

that is entire and zero only corresponding to the non-trivial zeros of $\zeta(s)$; moreover, starting from the proof of the functional equation (1) in Chapter 1, it's simple to show that $\xi(s)$ is real valued on the critical line, so wherever $\xi(1/2+it)$ changes sign we must observe a zero. It's standard notation to write

$$\zeta\left(\frac{1}{2}+it\right) = Z(t)\,e^{-i\vartheta(t)} = Z(t)\cos\vartheta(t) + Z(t)\sin\vartheta(t)\ ,$$

where the so-called Riemann-Siegel theta function is

$$\vartheta(t) = \operatorname{Im} \log \Gamma \left(\frac{1}{4} + \frac{1}{2} it \right) - \frac{t}{2} \log \pi \ .$$

Now Gram performed a smart reasoning, discussed in detail in Section 2.2, concerning the behaviour of the real and imaginary part of $\zeta(1/2 + it)$, in order to prove the existence of 10 zeros on the line segment from 1/2 to $1/2 + it$; subsequently it is shown how the Gram points, defined as the sequence of real numbers g_n satisfying $\vartheta(g_n) = n\pi, \ g_n \geq 10 \ (n = 0, 1, 2, \dots)$, plays an important role in locating the zeros of $\xi(s)$. Gram's technique becomes quite vain when trying to evaluate a larger number of roots. A first improvement is due to Backlund [**?**], who compared the changes of sign of $Z(t)$ in a certain range $0 < t < T$ with the number of zeros on the corresponding limited portion of critical strip, namely $N(T)$: Backlund proved that all the $\xi(s) = 0$ roots in the range $0 < \operatorname{Im}(s) < 200$ are on the critical line and are simple zeros. The end of the Section is dedicated to some remarks about the so-called "Gram's law", which indicates the typical behavior of the zeros of $Z(t)$ in connection with the zeros of $\vartheta(t)$.

In spite of the relevance of Gram and Backlund's works, the most important contribution to the computation of the $\xi(s) = 0$ roots belongs to Siegel, exposed in Section 2.3. Siegel was the first mathematician who fully understood Riemann's Nachlass (i.e. his posthumous notes) in which he found what drove Riemann to state his famous conjecture: Siegel published a paper in 1932 [20] explaining the results concerning a formula that he found in Riemann's private notes. Section 2.3 is dedicated to describe this results, one on an asymptotic formula for $Z(t)$ and another about a new way of representing $\zeta(s)$ in terms of definite integrals, which were fundamental to develop a new powerful method for computing the zeros of $\xi(s)$. Besides, Siegel's discovery pushed back the widely diffused opinion of that period that Riemann's conjecture about the zeros on the critical line was the result of mere intuitions not supported by any solid mathematical justification, giving credit to the great understanding and calculation ability that Riemann possessed respect to the zeta-function. Riemann-Siegel asymptotic formula is a very efficient tool used to compute $\zeta(1/2+it)$ for large t values, which is the range where Euler-Maclaurin summation formula is completely unworkable: especially with the advent of computers, this formula has played a leading role in checking RH for even larger values of $\mathrm{Im}(s)$, integrated with specific algorithms like the Odlyzko-Schönhage algorithm [25]. The formula is

$$Z(t) = 2\sum_{n=1}^{N} n^{-1/2}\cos\left[\vartheta(t) - t\log n\right] + R(t)\ ,$$

where $N = [\sqrt{t/2\pi}]$ and the remainder term $R(t)$ has the following asymptotic expansion

$$R(t) \sim (-1)^{N-1}\left(\frac{t}{2\pi}\right)^{-1/4}\left[\sum_{k\geq 0} C_k \left(\frac{t}{2\pi}\right)^{-k/2}\right]\ ,$$

and the coefficient C_k are computable recursively starting from the first one C_0.

The last section of the chapter, Section 2.4, is devoted to some considerations concerning the Riemann-Siegel formula, which was in possess of Riemann himself, and the possible birth of his famous and still unsolved conjecture.

The second part of the thesis, embodied by Chapter 3, pertains the estimations of the portion of $\xi(s)$ zeros which lies on the critical line.

Section 3.1 deals with Hardy's Theorem [27]: in 1914 Hardy proved that there are infinitely many roots of $\xi(s) = 0$ on the critical line or, equivalently, *there exist infinitely many real numbers* γ *such that* $\zeta(1/2+i\gamma) = 0$. The main strategy in proving this theorem is to use the inverse of the Mellin transform relationship that Riemann used to establish the functional equation and perform complex integrations on suitable paths, together with the estimate $\int_1^T \zeta(1/2+it)dt = T + \mathcal{O}(T^{1/2})$. The section continues with the explanation of the two other important contributions concerning the fraction of roots lying on the critical line. The first result belongs to Hardy and Littlewood [28] and was an improvement of the previous theorem because it states that the number of zeros on the line segment $1/2$ to $1/2 + iT$ (indicated by $N_0(T)$) is at least CT, for some positive constant C and sufficiently large T, that i

$$N_0(T) > CT \ , \ \ \forall T \geq T_0 \, , \exists T_0 > 0 \, , \exists C > 0 \ .$$

This was further improved by Selberg [29] who proved

$$N_0(T) > CT \log T \ , \ \ \forall T \geq T_0 \, , \exists T_0 > 0 \, , \exists C > 0 \ .$$

This brings to mind one of the Riemann's conjectures contained in his memoir, subsequently proved by von-Mangoldt:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T) \ ,$$

where $N(T)$ stands for the numbers of zeros of $\zeta(s)$ in the region $\{s \in \mathbb{C} \,|\, 0 < \mathrm{Re}(s) < 1, \, 0 < t \leq T\}$[1]. Comparing these two expressions, we understand that, roughly speaking, Selberg was the first to prove that a positive fraction of non-trivial zeros of $\zeta(s)$ lies on the critical line.

Section 3.2 explains the ideas behind the Levinson's work [30], i.e.

$$N_0(T + U) - N_0(T) > C(N(T + U) - N(T)) \ ,$$

with $U = TL^{-10}$, $L = \log(T/2\pi)$ and the value of C, unlike Selberg, has been determined by Levinson with $C = 1/3$. In other words, Levinson was able to prove that more than one third of the zeros of $\xi(s)$ lie on the critical line, a very impressive result that, up to now, represents one of the most important theoretical results in favor of the RH.

[1]Using this notation, RH simply becomes $N_0(T) = N(T) \, , \forall T > 0$.

Notations

The following are the standard notations used in analytic number theory. In the whole thesis we will use s to indicate the general complex variable, avoiding to write $s \in \mathbb{C}$ every time. Moreover, we will use $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$, that is $s = \sigma + it$.

The greek letter ρ will indicate a non-trivial zero of $\zeta(s)$:

$$\rho \in \{s \in \mathbb{C} \,|\, \zeta(s) = 0, \mathrm{Re}(s) \in (0,1)\} \ ,$$

and, in order to distinguish between the non-trivial zero ρ and the generic s, we will indicate the real and imaginary part as $\beta = \mathrm{Re}(\rho)$ and $\gamma = \mathrm{Im}(\rho)$ respectively, or $\rho = \beta + i\gamma$.

Every series with infinite terms starting from the natural n_0, usually written as

$$\sum_{n=n_0}^{\infty} a(n) \ ,$$

will be here indicated in the more compact way

$$\sum_{n \geq n_0} a(n) \ .$$

The logarithmic integral $\mathrm{li}(x)$ here, unlike some authors, indicates the Cauchy principal value of the integral

$$\mathrm{li}(x) = \lim_{\epsilon \to 0} \left[\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t} \right]$$

while the notation $\mathrm{Li}(x)$ indicates the well-behaved (in the sense that no Cauchy principal value is needed) integral

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \mathrm{li}(x) - \mathrm{li}(2) \ .$$

The term “region” here means a nonempty connected open set.

Contents

Introduction

In 1859 Bernhard Riemann wrote a short paper titled “Über die Anzahl der Primzahlen unter einer gegebenen Grösse”, which can be translated as “On the Number of Prime Numbers less than a Given Quantity”. As the title suggests, it deals with prime numbers and, in particular, with the prime counting function

$$\pi(x) = \sum_{p \leq x} 1 = \# \{p \,\text{prime} \,|\, p \leq x\} \ .$$

It was the only paper written by Riemann on number theory but it is considered, together with the Dirichlet’s theorem on the primes in arithmetic progression, the starting point of modern analytic number theory. Riemann’s aim was to provide an explicit formula for $\pi(x)$; before him, Gauss already tried to find such a formula but he was only able to prove that the function $\pi(x)$ is well approximated by the logarithmic integral

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \ .$$

Gauss’s estimate was motivated by the observation made by Euler about the divergence of the series

$$S = \sum_p \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \ .$$

In Euler’s terminology, $S = \log(\log \infty)$, which was a consequence of the Euler’s product formula for the harmonic series,

$$\sum_{n \geq 1} \frac{1}{n} = \prod_p \frac{1}{1 - p^{-1}} \ ,$$

so that

$$\log \sum_{n\geq 1} \frac{1}{n} = -\sum_p \log\left(1-p^{-1}\right) = \sum_p \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \dots\right)$$

and the right hand side is the sum of S plus convergent series. Now the harmonic series diverges like $\log n$ for $n \to \infty$, so that S must diverge like the log of it, from which $S = \log(\log\infty)$. Probably, what pushed Gauss to use the logarithmic integral to estimate $\pi(x)$ is an adaptation of Euler's ideas about the divergence of the series S to the case $p \leq x$, conjecturing that

$$\sum_{p\leq x} \frac{1}{p} \sim \log(\log x)$$

even for finite x, so that

$$\log(\log x) = \int_1^{\log x} \frac{dt}{t} = \int_e^x \frac{dy}{y\log y} \ ,$$

which can be interpreted saying that "the integral of $1/y$ with the measure $dy/\log y$ suggests that the density of primes less than y is about $1/\log y$". This is what could have driven Gauss towards the estimate

$$\pi(x) \sim \mathrm{Li}(x) \ . \tag{3}$$

Riemann was intentioned to find an explicit formula for the prime counting function, not only an estimate like Gauss did. In order to do that, he certainly based his work on the excellent approximation (3) but, at the same time, he made use of a generalization of the Euler's product formula, introducing the most important function in analytic number theory, the Riemann zeta-function [2] $\zeta(s)$, defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \ ,$$

for $s = \sigma + it \in \mathbb{C}$. The zeta-function converges absolutely for $\mathrm{Re}(s) > 1$, in which case we can generalize the Euler's product formula to

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \ , \qquad \sigma > 1 \ .$$

[2]Here we are ignoring the Dirichlet L-series which are a generalization of the Riemann zeta-function.

The function $\zeta(s)$ plays a fundamental role inside Riemann's paper: "some" of its zeros appear in the explicit formula that connects $\pi(x)$ to $\mathrm{Li}(x)$, they are the basic ingredient of the error term in Gauss's approximation (3). The zeros we are talking about are the so-called *non-trivial zeros* of $\zeta(s)$, in contrast with the trivial zeros of $\zeta(s)$ which happens for $s = -2, -4, -6, \ldots$: the non-trivial zeros, usually indicated with $\rho = \beta + i\gamma$, are infinite in number and they have $\mathrm{Re}(\rho) = \beta \in (0, 1)$, the region $0 < \mathrm{Re}(s) < 1$ is called *critical strip* and the most important open problem in number theory, the *Riemann Hypothesis* (RH), states that each non-trivial zero has $\beta = \frac{1}{2}$, that is they are all located along the *critical line*, $\mathrm{Re}(s) = \frac{1}{2}$. RH was first conjectured by Riemann in his paper, where he wrote that "it is probable" that all non-trivial zeros have real part equal to $\frac{1}{2}$.

This thesis is intended as an exposition of the ideas contained in Riemann's paper and as a description of some of the most important developments in the study of the zeta-function until today.

In particular, the first chapter contains the Riemann's analytic continuation of $\zeta(s)$ to a meromorphic function with a single simple pole at $s = 1$ with residue 1, the conjectures by Riemann proved (with the exception of RH) some years later by von Mangoldt and Hadamard, the description of the ideas behind the explicit formula for $\pi(x)$ obtained by Riemann and a list of consequences of RH, like the error term in the prime number theorem.

The second chapter investigates the computational aspects behind the RH: as a matter of facts, the most impressive evidence in favor of RH arises from the computation of the non-trivial zeros which lie on the critical line without any exception up to now (the actual number of non-trivial zeros verifying RH is more than 10^{13}). The efforts to locate the zeros of $\zeta(s)$ inside the critical strip date back to Riemann himself (as Siegel found out, studying Riemann's private papers, almost a century after the publication of 1859's article in which no sign of computation were present). Until Siegel made light on the very deep knowledge that Riemann possessed of $\zeta(s)$ and of the its behavior (zeros localization included), the first known computation concerning the non-trivial zeros of $\zeta(s)$ is the one of Gram, who used the *Euler-Maclaurin summation method* to verify that the first fifteen non-trivial zeros have real part $\frac{1}{2}$, as explained in the first part of the second chapter. The second part of Chapter 2 deals with the *Riemann-Siegel formula*, named after the studies of Riemann's unpublished notes made by Siegel, which revealed a powerful method for finding non-trivial zeros already known to Riemann but inexplicably not included by him in his paper. Riemann-Siegel formula allows to perform calculations for large values of $\text{Im}(s)$ inside the critical strip and it is the theoretical basis of every modern computer algorithm for computing the non-trivial zeros of $\zeta(s)$ and, at the same time, this formula is used in different proofs of theorems concerning the zeta-function. The second chapter ends with a section containing some considerations about the possible role that the Riemann-Siegel formula may have had in the birth of the RH.

The third and last chapter of this thesis describes some of the most important theorems about the displacement of non-trivial zeros inside the critical strip: even if each of these theorems is far from proving the RH, still they are fundamental in shedding light on important questions about the zeta-function. The first theorem exposed is due to Hardy and states that there are infinite non-trivial zeros lying on the critical line. The second is a much stronger theorem, by Levinson, that collocates more than one third of non-trivial zeros on the critical line.

Chapter 1

The Riemann's paper: a first introduction to the zeta-function $\zeta(s)$

If I were to awaken after having slept for a thousand years,

my first question would be:

Has the Riemann hypothesis been proven?

D. Hilbert

1.1 Definition as formal series and analytic continuation

The Riemann zeta-function is defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \ , \tag{1.1}$$

where $s = \sigma + it \in \mathbb{C}$. For $\sigma > 1$ the series converges absolutely, defining a holomorphic function and we can use the Euler product (which is a direct consequence of the fundamental theorem of arithmetic) in order to exploit a first connection between $\zeta(s)$ and prime numbers p:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \ , \qquad \sigma > 1 \ , \tag{1.2}$$

and since every factor in (1.2) is different from zero, we may conclude that $\zeta(s) \neq 0$ in the half plane $\sigma > 1$.

It's straightforward to show that $\zeta(s)$ can be extended to a meromorphic function with a single simple pole at $s=1$ in the extended region $\sigma > 0$: starting from the expression (1.1), which makes sense if $\sigma > 1$, we can write

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \sum_{n\geq 1} n \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right] = s \sum_{n\geq 1} n \int_n^{n+1} \frac{dx}{x^{s+1}} \ .$$

But if $x \in (n, n+1)$ then $n = [x]$, the integer part of x (i.e. the largest integer less or equal to x) and

$$\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} dx = s \left[\int_1^\infty \frac{1}{x^s} dx - \int_1^\infty \frac{\{x\}}{x^{s+1}} dx \right] \ .$$

So

$$\zeta(s) = \frac{s}{s+1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx \ , \tag{1.3}$$

where $\{x\} = x - [x]$ is the fractional part of x: (1.3) shows that $\zeta(s)$ is meromorphic for $\sigma > 0$ (the integral converges absolutely for $\sigma > 0$) with a pole in $s=1$ with residue $\text{Res}_{s=1}\zeta(s) = 1$.

In his famous eight paged work [1] Riemann showed that it is possible to analytically continue $\zeta(s)$ over the whole complex plane (excluding the simple pole in $s=1$ with residue 1). To see this, we take a look at the elegant method[1] used by Riemann to prove the fundamental functional equation for $\zeta(s)$:

Theorem 1.1.1. *The function $\zeta(s)$ satisfies the equation*

$$\pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \pi^{-\frac{1}{2}(1-s)} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \ . \tag{1.4}$$

[1] There are different methods for proving this functional equation for $\zeta(s)$, Riemann itself has proved it in two distinct ways. The proof exposed here shall return useful in the future sections.

Proof: By definition, if $\sigma > 0$

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty e^{-t}\, t^{\frac{1}{2}s-1} dt \ ;$$

substituting $t = n^2\pi x$ and summing over $n \in \mathbb{N}$ for $\sigma > 1$ we get

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \sum_{n\geq 1}\int_0^\infty e^{-n^2\pi x}\, x^{\frac{1}{2}s-1} dx \ .$$

We note that going back to $t = n^2\pi s$, the integral becomes

$$\int_0^\infty e^{-t}\left(\frac{t}{n^2\pi}\right)^{\frac{1}{2}s-1}\frac{dt}{n^2\pi} = \frac{\Gamma(s/2)}{(n^2\pi)^{s/2}} \ ,$$

and the sum

$$\sum_{n\geq 1}\int_0^\infty e^{-n^2\pi x}\, x^{\frac{1}{2}s-1} dx = \sum_{n\geq 1}\frac{\Gamma(s/2)}{(n^2\pi)^{s/2}}$$

converges for $\sigma > 1$. This legitimates the inversion, for $\sigma > 1$:

$$\sum_{n\geq 1}\int_0^\infty e^{-n^2\pi x}\, x^{\frac{1}{2}s-1} dx = \int_0^\infty x^{\frac{1}{2}s-1}\sum_{n\geq 1} e^{-n^2\pi x} dx = \int_0^\infty x^{\frac{1}{2}s-1}\psi(x) dx \ ,$$

where $\psi(x) = \sum_{n\geq 1} e^{-n^2\pi x}$. The integral can be split as

$$\int_1^\infty x^{\frac{1}{2}s-1}\psi(x)dx + \int_0^1 x^{\frac{1}{2}s-1}\psi(x)dx = \int_1^\infty \left(x^{\frac{1}{2}s-1}\psi(x) + x^{-\frac{1}{2}s-1}\psi(1/x)\right) dx \ .$$

Introducing

$$\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-n^2\pi x} = 1 + 2\psi(x) \ ,$$

one can prove (see Corollary A.1.2 in Appendix A) that the function $\vartheta(x)$ satisfies

$$\vartheta(x) = \frac{1}{\sqrt{x}}\,\vartheta(1/x) \ ,$$

which gives

$$\psi(1/x) = -\frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x}\,\psi(x) \ ,$$

yielding to

$$\int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-1}\right)\psi(x)dx + \int_1^\infty \left(-\frac{1}{2} + \frac{\sqrt{x}}{2}\right) x^{-\frac{1}{2}s-1} dx \ .$$

The second integral can be computed explicitly giving $1/[s(s-1)]$ (corresponding to the poles of $\zeta(s)$ in $s=1$ and of $\Gamma(s/2)$ for $s=0$). Eventually we managed to have the relation

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{1}{2}s} + x^{\frac{1}{2}(1-s)}\right)\psi(x)\frac{dx}{x} \ . \tag{1.5}$$

Now the integral in (1.5) converges for all s because $\psi(x) = \mathcal{O}(e^{-\pi x})$, which is a consequence of the observation that $\psi(x) = \sum_{n\geq 1} e^{-n^2\pi x} \sim e^{-\pi x}$ for large x: in fact, since $n^2 \geq n$ for $n \geq 1$, we get

$$\psi(x) \leq \sum_{n\geq 1}\left(e^{-\pi x}\right)^n = \frac{e^{-\pi x}}{1-e^{-\pi x}} = \frac{1}{e^{\pi x}-1} < \frac{2}{e^{\pi x}} = \mathcal{O}(e^{-\pi x}) \ .$$

Hence the integral represents an entire function, and the whole second member exhibits the symmetry $s \longleftrightarrow 1-s$, proving the functional equation (1.4). □

Riemann, in his paper, also proved the functional equation for $\zeta(s)$ using an alternative method, based on Cauchy's theorem for complex integrals on closed curves. To be precise, Riemann's second proof (see Appendix A.2) gives the following functional equation:

$$\zeta(s) = \chi(s)\zeta(1-s) \ , \qquad \chi(s) = 2^s\pi^{1-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \ , \tag{1.6}$$

The equivalence between equations (1.4) and (1.6) is evident once we utilize the following two properties of the Γ function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s} \ , \qquad \sqrt{\pi}\,\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2) \ .$$

1.2 Trivial and non-trivial zeros: the function $\xi(s)$

Defining the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \ , \tag{1.7}$$

Theorem 1.1.1 states that $\xi(s)$ is an entire function and we may rewrite equation (1.4) in a more compact way:

$$\xi(s) = \xi(1-s) \ . \tag{1.8}$$

Equation (1.8) shows a symmetry for $\xi(s)$ respect to the critical line $\sigma = \mathrm{Re}(s) = 1/2$ and it is very useful to deduce properties related with $\zeta(s)$ in the region $\sigma < 0$ starting from the properties for $\sigma > 1$, where the Euler product holds.

The function $\xi(s)$ is useful because it is entire: in fact, the terms $s(s-1)$ cancel the pole of $\zeta(s)$ in $s=1$ and the pole of $\Gamma(s/2)$ in $s=0$ (or, in a simpler way, one can just observe that multiplying the right hand side of (1.5) for $s(s-1)$, the resulting function has no poles left).

Starting from this observation, we split the zeros of $\zeta(s)$ in two classes: the *trivial zeros* and the *non-trivial zeros*. Trivial zeros occur for the values $s = -2n\,, n \in \mathbb{N}$; it is easy to derive their presence noticing that the left hand side of the functional equation (1.4) involves the simple poles of $\Gamma(s/2)$. That is, for $s/2 = -1, -2, -3, \ldots$, these poles must cancel out with the zeros of $\zeta(s)$. Every other zero of $\zeta(s)$ lies in the critical strip[2] $0 \leq \sigma \leq 1$, because from the Euler product (1.2), valid for $\sigma > 1$, we have already emphasized that in this region $\zeta(s) \neq 0$: the zeros of $\zeta(s)$ lying on the critical strip are called non-trivial and their exact location is the major problem in analytic number theory.

Looking in particular at (1.7), we note that $\xi(s)$ is zero only in correspondence of the non-trivial zeros of $\zeta(s)$. The fact that $\xi(s) = 0$ only within the critical strip can be seen in another way: in the half plane $\sigma > 1$, the factors in $\xi(s)$ other than $\zeta(s)$ have no zeros, so $\xi(s) \neq 0$ in this region. But (1.8) implies that the same happens for $\sigma < 0$, so every possible zero of $\xi(s)$ lies in the critical strip.

From now on, we will indicate the generic non-trivial zero with $\rho = \beta + i\gamma$. From the definition (1.1) follows forthwith that $\overline{\zeta(s)} = \zeta(\overline{s})$ and combining this fact with equation

[2]Some authors use the term "critical strip" to refer to the region $0 < \sigma < 1$, but concerning the zeros of $\xi(s)$ the substance is the same since it can be proved (as for the first proof of the Prime Number Theorem) that $\xi(s) \neq 0$ on the line $\sigma = 1$ and therefore the same holds on the line $\sigma = 0$ because of (1.8).

(1.8) it is plain that if ρ is a non-trivial zero for $\zeta(s)$, than also $\overline{\rho}$, $1-\rho$ and $1-\overline{\rho}$ are.

The connection between $\xi(s)$ and the non-trivial zeros can be put in a much more significant form. Riemann's belief was that every analytic (meromorphic) function can be defined by its singularities and boundary values. This idea drove Riemann in trying to make explicit the dependance of $\xi(s)$ from the non-trivial zeros ρ of $\zeta(s)$ in the following way: suppose

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) . \tag{1.9}$$

The rough reasoning behind this formula could be the following: $\log \xi(s)$ has singularities whenever a zero of ξ occurs, so it has the same singularities as

$$\sum_{\rho} \log \left(1 - \frac{s}{\rho}\right) . \tag{1.10}$$

Now, if the sum (1.10) converges and if no problems arise concerning boundary conditions (i.e. if the sum is well behaved at ∞ as $\log \xi(s)$), then (1.10) and $\log \xi(s)$ should differ by a constant; setting $s = 0$ we get $\log \xi(0)$ for the value of that constant, hence exponentiating we arrive at (1.9). One could argue problems for the determination of the logarithmic branches of the terms in (1.10): but for a fixes values of s, this problem eventually disappears for sufficiently large ρ values, then in (1.10) we have only a finite number of terms involving a possible undefined multiple of $2\pi i$ and, once exponentiated, they disappear in (1.9).

The main problem in connecting the sum (1.10) with the formula (1.9) is the convergence. We know, in fact, that this kind of reasoning works only in the case of finite product. In the case of infinite product we must, in general, introduce factors in order to provide convergence[3] (see [2], §5). Riemann, aware of that, conjectured the following:

[3]To be precise, concerning the case under consideration, these factors are not necessary, but this fact becomes clear only after noticing that sum (1.10) converges absolutely because converges absolutely the series $\sum_{\mathrm{Im}\rho>0} \frac{1}{\rho(1-\rho)} \sim \int^{\infty} \frac{1}{2\pi T^2} \log \frac{T}{2\pi} dT < \infty$, using the asymptotic zero-density $d\left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}\right)$, see Theorem 1.2.4. For more details, see [4], §1.

Theorem 1.2.1. *The function $\xi(s)$ has the product representation*

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \,, \tag{1.11}$$

where A and B are constants and the sum runs over all the non-trivial zeros of $\zeta(s)$.

Theorem 1.2.1 is a first example of the profound relationship between primes, appearing in the Euler product of $\zeta(s)$, and the non-trivial zeros ρ[4]. This theorem has been proven by Hadamard in 1893 and the proof makes use of the theory on integral functions of finite order, developed by Hadamard himself. An integral function F is of finite order if there exists a positive real number a such that, as $|z| \to \infty$,

$$F(z) = \mathcal{O}(e^{|z|^a}) \,.$$

We set $\operatorname{ord} F = \inf\{a \,|\, F(z) = \mathcal{O}(e^{|z|^a})\}$, the order of the function F. In our case, the integral function $\xi(s)$ is of order 1: in fact, if $|s|$ is large, we have (see [3], §12)

$$\xi(s) \sim e^{C|s|\log|s|} \,, \tag{1.12}$$

for some constant C (being $\Gamma(s/2)$ the main term contributing in (1.12) and using the Stirling's approximation $\log(\Gamma(z)) = (z - 1/2)\log z - z + \mathcal{O}(1)$), so for every $\epsilon > 0$

$$\xi(s) < e^{|s|^{1+\epsilon}}$$

and the order of $\xi(s)$ is exactly 1. Theorem 1.2.1 is just a specific case of a more general and deep theorem by Hadamard:

Theorem 1.2.2. *Every entire function $f(z)$ of order 1 with zeros $\{z_1, z_2, z_3, \dots\}$ (counted with multiplicity) has the product representation*

$$f(z) = e^{A+Bz} \prod_{n\geq 1} \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \,, \tag{1.13}$$

for some constants A and B. Moreover, if $r_n = |z_n|$, then

[4]This relationship will become more evident once the explicit formulas for the prime counting function $\pi(x)$ and the von Mangoldt $\psi(x)$ will appear.

1. $\sum_{n\geq 1} r_n^{-1-\epsilon} < \infty$ *for every* $\epsilon > 0$,

2. *if* $\sum_{n\geq 1} r_n^{-1} < \infty$ *then* $|f(z)| < \exp(C|z|)$ *for* $|z| \to \infty$.

Theorem 1.2.2 has another fundamental implication for the zeros of $\zeta(s)$. Riemann conjectured that $\zeta(s)$ has *infinitely many zeros in the critical strip.* Hadamard was able to prove it using Theorem 1.2.2. If $\xi(s)$ has a finite number of zeros, than the sum $\sum_\rho \frac{1}{\rho}$ converges; but, in this case, we should have $|\xi(s)| < \exp(C|s|)$, in contradiction with what we stated in equation (1.12) (again, this happens because we cannot state that $|\Gamma(s)| < \exp(C|s|)$, everyone can easily convince himself using the Stirling asymptotic approximation for Γ). So, using the observation made before on the symmetrical arrangement of the non-trivial zeros of $\zeta(s)$, we can finally claim:

Theorem 1.2.3. *The function $\zeta(s)$ has infinite many zeros in the critical strip, symmetrically disposed with respect to the real axis and to the critical line $\sigma = 1/2$.*

The two constants, A and B, in Theorem 1.2.1 can be calculated. Setting $s = 0$ we see that $e^A = \xi(0) = \xi(1)$, because of the relation $\xi(s) = \xi(1-s)$. Recalling the definition

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \tag{1.14}$$

and using $\lim_{s\to 1}(s-1)\zeta(s) = 1$, we arrive at

$$e^A = \frac{1}{2\sqrt{\pi}}\Gamma(1/2) = \frac{1}{2}\ .$$

The calculation of B is not so easy. In order to do that, we perform the logarithmic derivative of (1.14):

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} + \frac{\zeta'(s)}{\zeta(s)}\ . \tag{1.15}$$

At the same time, doing the logarithmic derivative of $\xi(s)$ in (1.11) brings

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)\ . \tag{1.16}$$

The complete calculation of B can be found in [3], §12, and the final result is

$$B = -\frac{1}{2}\gamma - 1 + \frac{1}{2}\log 4\pi \ ,$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) \ .$$

We have pointed out equations (1.15) and (1.16) for a more important reason: combining the two forms of the logarithmic derivative of $\xi(s)$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \ . \tag{1.17}$$

This equation for the logarithmic derivative of $\zeta(s)$ is a key tool for the non-elementary proof of the Prime Number Theorem (see the next two Sections) and for the calculation of zero-free regions for $\zeta(s)$. But the importance of equation (1.17) goes beyond this: indeed it is the starting point that von Mangoldt used to prove other two conjectures (now theorems) contained in Riemann's paper:

Theorem 1.2.4. *The number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the critical strip with $0 < \gamma \leq T$ is asymptotically equal to*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T) \ . \tag{1.18}$$

We recall that for every natural number n, the von Mangoldt function is defined as $\Lambda(n) = \log p$ if n is a power of a prime p, $\Lambda(n) = 0$ otherwise.

Theorem 1.2.5. *Define $\psi(x) = \sum_{n\leq x}\Lambda(n)$ and $\psi_0(x) = \psi(x) - \frac{1}{2}\Lambda(x)$ (so, $\psi_0(x)$ and $\psi(x)$ differ only when $x = n \in \mathbb{N}$). Then the following explicit formula holds:*

$$\psi_0(x) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - x^{-2}\right) \ , \tag{1.19}$$

where, in order to provide absolute convergence of the sum over the zeros $\rho = \beta + i\gamma$ on the critical strip, we intend

$$\sum_{\rho} \frac{x^\rho}{\rho} = \lim_{T \to +\infty} \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \, .$$

Theorem 1.2.5 can be shown to be equivalent to the Prime Number Theorem.

Up to now, every conjecture contained in Riemann's paper has been proven to be true, except the one that we are going to discuss from Section 1.4.

1.3 On the Number of Prime Numbers less than a Given Quantity

The title of the current Section is the English translation of the Riemann's paper one,"Über die Anzahl der Primzahlen unter einer Gegebenen Grösse". As the title suggests, the main purpose of the paper was to find an explicit formula for the prime counting function

$$\pi(x) = \sum_{p \leq x} 1 = \#\{p \text{ prime} \,|\, p \leq x\} \, .$$

In order to do that, he used the function $\zeta(s)$, the complex extension of the real function

$$\zeta(x) = \sum_{n \geq 1} \frac{1}{n^x} \, , \qquad x > 1 \, ,$$

already used by Euler for his brilliant proof of the infiniteness of prime numbers, alternative to the Euclid's one.

Taking the logarithm of (1.1) and using the Taylor series of $\log(1-x)$, we write

$$\log \zeta(s) = \sum_{p} \sum_{n \geq 1} \frac{1}{n \, p^{ns}} \, , \qquad \mathrm{Re}(s) > 1 \, . \tag{1.20}$$

Introducing the function $J(x)$ for $x \geq 0$ which is 0 for $x = 0$, jumps of 1 at every prime number p, then jumps of $1/2$ at every prime square p^2, then a jumps of $1/3$ at every prime cube p^3 and so on[5], we rewrite (1.20) using Stieltjes integrals as

$$\log \zeta(s) = \int_0^\infty x^{-s}\, dJ(x) = s \int_0^\infty x^{-s-1} J(x)\, dx\ , \qquad \mathrm{Re}(s) > 1 \tag{1.21}$$

where the last equivalence follows once integrating by parts (at $x = 0$ the term $x^{-s}J(x)$ is null because $J(x) \equiv 0$ for $0 \le x < 2$ and $\lim_{x\to\infty} x^{-s}J(x) = 0$ for $\mathrm{Re}(s) > 1$).

Now the basic steps to achieve the Riemann's formula for $\pi(x)$ are the following (for accurate details, see [4], §1):

1. Use the Fourier inversion formula to reverse (1.21) into

$$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s)\, x^s \frac{ds}{s} = \lim_{T\to\infty} \int_{a-iT}^{a+iT} \log \zeta(s)\, x^s \frac{ds}{s}\ , \qquad a > 1\ ; \tag{1.22}$$

2. Integrate by parts to obtain

$$J(x) = -\frac{1}{2\pi i \log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log \zeta(s)}{s}\right] x^s ds\ , \qquad a > 1\ , \tag{1.23}$$

   which is valid because

$$\lim_{T\to\infty} \frac{\log \zeta(a \pm iT)}{(a \pm iT)} x^{a \pm iT} = 0$$

   since

$$|\log \zeta(a \pm iT)| = \left| \sum_n \sum_p \frac{1}{n} p^{-n(a\pm iT)} \right| \le \sum_n \sum_p \frac{1}{n} p^{-na} = \log \zeta(a)$$

   is a constant.

3. Looking back at the two forms of $\xi(s)$

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = e^{A+Bz}\prod_{n\ge 1}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\ ,$$

[5]For the sake of clarity, we calculate $J(x)$ for $0 \le x \le 5$, keeping in mind that, as usual, in Stieltjes integration theory we assign to the discontinuous function $J(x)$ at each point of discontinuity x_0 the left-right average value $J(x_0) = \frac{1}{2}(J(x_0)^- + J(x_0)^+)$. So $J(x) = 0$ for $0 \le x < 2$, $J(x) = 1/2$ at $x = 2$, $J(x) = 1$ for $2 < x < 3$, $J(x) = 1 + 1/2 = 3/2$ at $x = 3$, $J(x) = 2$ for $3 < x < 4$, $J(x) = 2 + 1/4 = 9/4$ at $x = 4$, $J(x) = 2 + 1/2 = 7/2$ for $4 < x < 5$ and $J(x) = 3$ at $x = 5$.

using the property $z\Gamma(z) = \Gamma(z+1)$ and taking the log of both sides we obtain

$$\log\zeta(s) = \frac{s}{2}\log\pi - \log(s-1) - \log\Gamma\left(\frac{s}{2}+1\right) + A + Bs + \sum_{\rho}\left[\frac{s}{\rho} + \log\left(1-\frac{s}{\rho}\right)\right] \quad (1.24)$$

and we insert this expression in (1.23) to express $J(x)$ as a sum of different terms.

The main term of $J(x)$ comes from the term $\log(s-1)$ in (1.24) and, for $x > 1$, gives rise to the logarithmic integral

$$\mathrm{li}(x) = \lim_{\epsilon\to 0}\left[\int_0^{1-\epsilon}\frac{dt}{\log t} + \int_{1+\epsilon}^{x}\frac{dt}{\log t}\right]$$

as the Cauchy principal value of the divergent integral $\int_0^\infty \frac{dt}{\log t}$. Computing the other terms we obtain the complete formula for $J(x)$:

$$J(x) = \mathrm{li}(x) - \sum_{\rho}\mathrm{li}(x^{\rho}) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}\ , \quad (1.25)$$

where the last term is an indefinite integral which converges for $x > 1$ (which is of course our case, being 2 the smallest prime number).

The second term involves the non-trivial zeros of $\zeta(s)$: this is an outstanding fact which relates primes and the zeta-function in a much profound way than the Euler expansion of $\zeta(s)$. We will see that the distribution of non-trivial zeros on the critical line influences, somehow, the distribution of prime numbers among the natural numbers. Anyway, this series is only conditionally convergent and, in order to sum up correctly the terms, it should be intended as

$$\lim_{T\to\infty}\sum_{0<\gamma\leq T}\left[\mathrm{li}(x^{\rho}) + \mathrm{li}(x^{1-\rho})\right]\ ,$$

and (1.25) remains valid, as proved by von Mangoldt. We will return on these terms involving the non-trivial zeros later.

Turning back to the Riemann's original aim, looking at the definition of $J(x)$ quite handily emerges a relation with $\pi(x)$ given by the formula

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots + \frac{1}{n}\pi(x^{1/n}) + \ldots \quad (1.26)$$

the sum being finite because, for any given x, once $x^{1/n} < 2$ for a certain n (that is when $n > \log_2 x$) we have $\pi(x^{1/m}) = 0$ for every $m \geq n$. Relation (1.26) can be inverted using Möbius inversion formula:

$$\pi(x) = J(x) - \frac{1}{2}\pi(x^{1/2}) - \frac{1}{3}J(x^{1/3}) + \cdots + \frac{\mu(n)}{n}J(x^{1/n}) + \ldots \tag{1.27}$$

the sum being finite as before. Formula (1.25) and the relation (1.27) gives the desired explicit formula for $\pi(x)$. If we temporarily assume $J(x) \sim \mathrm{li}(x)$, then we have a first improvement on the Gauss conjecture $\pi(x) \sim \mathrm{li}(x)$, i.e.

$$\pi(x) \sim \mathrm{li}(x) + \sum_{n=2}^{N} \frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) \ , \tag{1.28}$$

where N is such that $x^{1/(N+1)} < 2$. Looking at (1.25), the error in (1.28) contains different factors: the major contribution to it[6] is of the form

$$\pi(x) - \sum_{n=1}^{N} \frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) = \mathcal{O}\left(\sum_{n=1}^{N}\sum_{\rho} \mathrm{li}(x^{\rho/n})\right) \ . \tag{1.29}$$

We have already emphasized that series of that kind are only conditionally convergent: it is therefore quite surprising that, as Lehmer found out [5], the error (1.29) is relatively small. Integrating repeatedly by parts, indeed, one obtain a series expansion for li(x)

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \cdots = \frac{x}{\log x}\sum_{k\geq 0} \frac{k!}{\log^k x} \ ;$$

so

$$|\mathrm{li}(x)^{\rho}| = \mathcal{O}\left(\left|\frac{x^{\rho}}{\log x^{\rho}}\right|\right) = \mathcal{O}\left(\frac{x^{\beta}}{|\rho|\log x}\right) \ ,$$

hence many terms grows at least as fast as $x^{1/2}/\log x \sim 2\,\mathrm{li}(x^{1/2}) > \mathrm{li}(x^{1/3})$ and are thus comparable with the term $-\frac{1}{2}\mathrm{li}(x^{1/2})$ and more significant than any other term in (1.28).A more profound understanding of this unexpected good behaviour of the correction terms to $\pi(x)$ could be a carried by a deeper understanding of the distribution of the non-trivial zeros of $\zeta(s)$.

[6]Despite of the remarkable explicit formula he had found, Riemann was not able to estimate the size of the contribution of the terms in (1.25) to the calculation of $\pi(x)$.

1.4 The Riemann Hypothesis and its consequences

The failure of the Riemann hypothesis would create

havoc in the distribution of prime numbers.

E. Bombieri

Riemann's paper, despite of its shortness, has been a major breakthrough in Number Theory: Riemann Hypothesis (RH) is the only conjecture contained in it which remains undecided so far. Technically speaking, RH is a conjecture like many other; still, it fully deserves the designation "hypothesis" because many theorems take it as a starting point, i.e. they are of the form: "suppose that the Riemann Hypothesis is true, then...". This fact is indicative of how much RH is trusted to be true by a large part of the modern mathematics community. Then, proving RH would automatically prove thousands of theorems.

RH, together with its extension to the case of Dirichlet L-functions (the so-called Extended Riemann Hypothesis, ERH), is considered by many mathematicians as the most important open problem in mathematics: it has been chosen by Hilbert as one of his famous 23 problems suggested during the International Congress of Mathematicians (8 August 1900, Paris) and is also one of the Clay Mathematics Institute Millennium Prize Problems [6]. RH concerns the arrangement of the $\xi(s)$ roots on the critical strip and has different important implications in pure mathematics[7], so it deserves this relevance because it is not just a peculiarity of a holomorphic function, it is much more.

The Riemann conjecture can be stated as follows:

The Riemann conjecture: *If* $\xi(s) = 0$ *for* $s = \frac{1}{2} + i\alpha$, *then* $\mathrm{Re}(\alpha) = 0$. *(Equivalently, if* $\zeta(\rho) = 0$ *with* $\beta = \mathrm{Re}(\rho) \in (0, 1)$, *then* $\mathrm{Re}(\rho) = 1/2$.*)*

[7] In the case of cryptography, it is not important if RH is true or not, but it could matter if the mathematics surrounding the eventual solution reveals quicker ways to factorize numbers. In particular, it will only matter if it reveals much quicker ways to factorize numbers.

Riemann himself, after computing some of the first non-trivial zeros of $\zeta(s)$ (see in next Chapter the Riemann-Siegel formula), was conscious that the arrangement of the such zeros on the critical line could be a peculiarity which holds for every of them. Quoting Riemann: "It is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation."

In the next Chapter we will see some explicit methods for locating the zeros of $\xi(s)$, but of course this cannot be the method to prove RH even with the help of computers[8] (although it could be a method for disproving RH, by just finding a root for $\xi(s)$ which does not lie on the critical line). In order to prove RH, the results exposed in Chapter 3 are much more significant, because the try to estimate a positive fraction of non-trivial zeros of $\zeta(s)$ satisfying RH, in the hope that this fraction one day will become 100%, even if this of course does not imply RH because we are dealing with infinite amount of zeros.

In this Section we want to mention that, besides the analytical aspects of the zeros of the function $\xi(s)$, the RH has several consequences in number theory and various statements equivalent to RH can be formulated, some of them involving only aspect of "elementary number theory" such as primes, the prime counting function $\pi(x)$, the von Mangoldt function $\psi(x)$ or the Möbius function $\mu(n)$. Some "elementary" consequences of RH are listed in the next:

[8] Few people knows that one of the main reason which pushed Alan Turing in developing the first examples of computers was the calculation of the non-trivial zeros of $\zeta(s)$.

Theorem 1.4.1. *Assuming RH, the following estimates holds:*

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}(\sqrt{x}\log x) \ ; \tag{1.30}$$

$$\psi(x) = x + \mathcal{O}(\sqrt{x}\log^2 x) \ ; \tag{1.31}$$

$$p_n = \mathrm{li}^{-1}(n) + \mathcal{O}(\sqrt{n}\log^{5/2} n) \ ; \tag{1.32}$$

$$\prod_{p\leq x}\left(1-\tfrac{1}{p}\right) = \tfrac{e^{-\gamma}}{\log x} + \mathcal{O}(x^{-1/2}) \ , \tag{1.33}$$

where $\psi(x) = \sum_{n\leq x}\Lambda(n)$ is the von Mangoldt function, p_n is the nth prime number and γ the Euler constant.

The bound for $\pi(x)$ in (1.30), for example, is maybe the most common way to express the RH in a way which does not involve complex analysis and which takes into account directly the primes (through the prime number counting function $\pi(x)$); in fact, condition (1.30) is not just necessary for RH but also sufficient, i.e.

$$\left[\xi\left(\frac{1}{2}+i\alpha\right) \Rightarrow \alpha\in\mathbb{R}\right] \Longleftrightarrow \left[\pi(x) = \mathrm{li}(x) + \mathcal{O}(\sqrt{x}\log x)\right] \ .$$

The error in estimating $\pi(x)$ with $\mathrm{li}(x)$ in (1.30) is the smallest possible: this fact is very significant, if RH is not true then the distribution of prime numbers among naturals is somehow "less random" than one could expect.

Using the general concept of Dirichlet series, that is $\sum_{n\geq 1} a_n/n^s$, where $a(n) : \mathbb{N} \longrightarrow \mathbb{C}$ is an arithmetic function and the fact that the convolution of two such series satisfies the convolution property

$$\sum_{n,m=1}^{\infty}\frac{a(n)b(m)}{(nm)^s} = \sum_{k=1}^{\infty}\frac{(a*b)(k)}{k^s} \ ,$$

with $(a*b)(k)$ indicating the convolution product of the two arithmetic functions $a(n)$ and $b(m)$,

$$(a*b)(k) = \sum_{d|n} a(d)b\left(\frac{n}{d}\right) \ ,$$

one finds the relation

$$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^n} \ , \qquad \mathrm{Re}(s) > 1 \tag{1.34}$$

being $\mu(n)$ the Dirichlet inverse of the identity function 1 (meaning $1(n) = 1$ for all n). If RH holds, then relation (1.34) is valid for $\mathrm{Re}(s) > 1/2$. An equivalent of RH (see [23]) is that

$$\left| \sum_{x\leq x} \mu(n) \right| < x^{\frac{1}{2}+\epsilon} \ , \quad \forall \epsilon > 0 \ . \tag{1.35}$$

The tighter bound

$$\left| \sum_{x\leq x} \mu(n) \right| < x^{\frac{1}{2}}$$

is the famous Mertens conjecture, disproved by Odlyzko and te Riele [7].

The last elementary statement equivalent to RH involves another arithmetic function, the Liouville function

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ represents the number of (non necessary distinct) prime factors in n (multiple factors being counted multiply). A first connection between $\zeta(s)$ and $\lambda(n)$ comes from the identity

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\lambda(n)}{n^s} \ , \quad \mathrm{Re}(s) > 1 \ .$$

Theorem 1.4.2. *RH holds if and only if, for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{\lambda(1) + \lambda(2) + \cdots + \lambda(n)}{n^{\frac{1}{2}+\epsilon}} = 0 \ . \tag{1.36}$$

In other words, Theorem 1.4.2 states that RH is equivalent to the statement that a natural number n has equal probability of having an odd or even number of distinct prime factors (counted with multiplicity).

Another type of consequences of RH concerns the bounds, for $t = \mathrm{Im}(s) \to \infty$, on the value of $\zeta(s)$ on the critical line (the so-called Lindelöf Hypothesis, LH [24]),

$$\zeta(1/2 + it) = \mathcal{O}(t^{\epsilon}) \ , \quad \forall \epsilon > 0$$

and on the line $\mathrm{Re}(s) = 1$,

$$\zeta(1+it) = \mathcal{O}(\log\log t) \ , \qquad \frac{1}{\zeta(1+it)} = \mathcal{O}(\log\log t) \ .$$

Beyond the numerical evidence coming from explicit calculation of the roots of $\xi(s)$ and the "primes randomness" implicit in formula (1.30), other motivations support the idea that RH is true. The most convincing fact is the proof of the *Weil conjecture* by Deligne [8], a theorem pertaining algebraic geometry but which can be viewed as the analogous of RH in the case of local zeta-functions attached on algebraic varieties over finite fields: these functions are rational, satisfy a functional equation and their zeros are displaced according to an accurate pattern which is similar to the case of the non-trivial zeros of $\zeta(s)$ which should lie, inside the critical strip, only on the critical line.

Although most of the mathematicians are quite optimistic on the truth of RH, some doubts still persist. For example, the numerical evidence of RH is not an overwhelming evidence, because for example a famous conjecture in number theory was that the Gauss estimate $\pi(x) \sim \mathrm{li}(x)$ was an upper bound, in the sense that for all x

$$\pi(x) < \mathrm{li}(x) \ . \tag{1.37}$$

The numerical evidence for (1.37) was impressive, nonetheless Littlewood [31] showed that the inequality (1.37) is broken infinitely many times. The search for the lowest number x_S such that $\pi(x_S) > \mathrm{li}(x_S)$ gave birth to the concept of Skewes number, in honour of Skewes who made (assuming RH!) the first estimate [9]

$$x_S = 10^{10^{10^{34}}} \ ,$$

later improved (without assuming RH) in [10]

$$x_S = 10^{10^{10^{3}}} \ .$$

Enormous number like those make mathematicians very prudent about the truth of RH just based on numerical evidence. Other reasonable doubts about RH can be found in [11].

Chapter 2

Numerical calculation of zeros on the critical line

In this Chapter we will discuss some techniques involving the computation of non-trivial zeros for $\zeta(s)$. Today efficient algorithms for computers have been able to prove that more than 10^{13} of such zeros lie on the critical line [26], no exception has been found up to now. Of course explicit computation cannot prove the validity of RH (at most a possible non-trivial zero could disprove RH, even if most of the mathematicians do not consider this case very likely), nevertheless it can support the belief in the truth of RH and the relative efforts in proving it; moreover, the possibility of an outcome of interesting patterns of the zeros arrangement on the critical line may be very useful in view of a deeper understanding of the nature of the non-trivial zeros of $\zeta(s)$.

2.1 The Euler-Maclaurin summation

Euler-Maclaurin summation formula is an important method used in number theory. Basically, it is a tool which enables us to estimate a finite sum of the kind

$$\sum_{k=A}^{B} f(k)$$

as the integral $\int_A^B f(t)dt$ plus an error involving a sum of derivatives $f^{(m)}(x)$ of increasing order m for $x = A$ and $x = B$. Setting $x_0 = A$, $x_1 = A+1$,..., $x_{n-1} = B-1$, $x_n = B$, then the "trapezoidal rule" gives

$$\int_A^B f(t)dt \sim \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2} = \frac{1}{2}f(A) + f(A+1) + \cdots + f(B-1) + \frac{1}{2}f(B)\ .$$

So

$$\sum_{k=A}^{B} f(k) \sim \int_A^B f(t)dt + \frac{1}{2}f(A) + \frac{1}{2}f(B)\ . \tag{2.1}$$

To evaluate the error in estimate (2.1) we observe that, if $[x]$ indicates the step function, the Stieltjes integral

$$\int_a^B f(t)d([x]) = \frac{1}{2}f(A) + f(A+1) + \cdots + f(B-1) + \frac{1}{2}f(B)$$

allows to say that if we add

$$-\int_A^B f(t)dt + \int_A^B f([t])dt = \int_A^B f(t)d([t]-t)$$

to the right hand side of (2.1), we recover the identity

$$\sum_{k=A}^{B} f(k) = \int_A^B f(t)dt + \frac{1}{2}f(A) + \frac{1}{2}f(B) + \int_A^B f(t)d([t]-t)\ . \tag{2.2}$$

The Euler-Maclaurin approach consists of repeated integrations by parts of the last integral in (2.2) to reach a better estimate of the initial approximation (2.1). Instead of the measure $d([t]-t)$ it is convenient to choose the symmetrical measure $d([t]-t+\frac{1}{2})$ and integration by parts gives

$$\sum_{k=A}^{B} f(k) = \int_A^B f(t)dt + \frac{1}{2}f(A) + \frac{1}{2}f(B) - \int_A^B ([t] - t + \frac{1}{2})\, df(t)$$
$$= \int_A^B f(t)dt + \frac{1}{2}f(A) + \frac{1}{2}f(B) + \int_A^B (t - [t] - \frac{1}{2}) f'(t)\, dt \ , \qquad (2.3)$$

where, in the last equality, the Stieltjes measure $df(t)$ becomes $f'(t)\,dt$ only if $f \in C^1([A, B])$. In view of subsequent integration of the last integral in (2.3) , we must exploit *Bernoulli numbers* and *Bernoulli polynomials*. Bernoulli numbers b_n are defined as the rational coefficients in the Taylor series

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} b_n \frac{x^n}{n!} \ ,$$

or, equivalently,

$$b_n = \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) \bigg|_{x=0} \ .$$

Except $b_1 = -1/2$, all odd Bernoulli numbers vanish: $b_{2n+1} = 0$ for all naturals n. The first even Bernoulli numbers are

$$b_0 = 1,\ b_2 = \frac{1}{6},\ b_4 = -\frac{1}{30},\ b_6 = \frac{1}{42},\ \ldots$$

The nth Bernoulli polynomials $B_n(x)$ is defined[1] as the unique polynomial of degree n satisfying

$$\int_t^{t+1} B_n(x)\, dx = t^n \ . \qquad (2.4)$$

[1]Another possible definition is similar to the one of Bernoulli numbers, that is through the generating function

$$F(y, x) = \frac{y e^{yx}}{e^y - 1} = \sum_{n \geq 0} B_n(x) \frac{y^n}{n!} \ .$$

Setting $x = 0$, we instantly see that $B_n(0) = b_n$. Setting instead $x = 1$,

$$F(y, 1) = \frac{y e^y}{e^y - 1} = \frac{y}{1 - e^{-y}} = \frac{-y}{e^{-y} - 1} = F(-y, 0) \ ,$$

so that $B_n(1) = (-1)^n B_n(0) = (-1)^n b_n$.

The recursive relation

$$B_n'(x) = nB_{n-1}(x) \tag{2.5}$$

follows once derived relation (2.4) to get $B_n(t+1) - B_n(t) = n\,t^{n-1}$ and, consequently, $\int_t^{t+1} B_n'(x)/n\,dx = t^{n-1} = \int_t^{t+1} B_{n-1}(x)\,dx$. The first Bernoulli polynomials are

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6},\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ldots$$

It will be handy, in our case, to deal with the so-called *periodic Bernoulli polynomial* $\bar{B}_n(x) = B_n(\{x\})$, where $\{x\} = x - [x]$; if $x = 0$ in the equation $B_n(x+1) - B_n(x) = nx^{x-1}$ we get $B_n(1) = B_n(0)$, so $\bar{B}_n(x)$ is periodic with period 1 and continuous in $\mathbb{R}$.

Turning back to the last integral in (2.3)

$$\begin{aligned}
\int_A^B (t - [t] - \frac{1}{2}) f'(t)\,dt &= \sum_{n=A}^{B-1} \int_0^1 \left(y - \frac{1}{2}\right) f'(n+y)\,dy = \sum_{n=A}^{B-1} \int_0^1 B_1(y) f'(n+y)\,dy \\
&= \sum_{n=A}^{B-1} \left[\frac{1}{2} B_2(y) f'(n+y) \Big|_0^1 - \frac{1}{2} \int_0^1 B_2(y) f''(n+y)\,dy \right] \\
&= -\frac{1}{2} B_2(0) f'(A) + \frac{1}{2} B_2(1) f'(A+1) - \frac{1}{2} B_2(0) f'(A+1) \\
&\quad + \frac{1}{2} B_2(1) f'(A+2) - \cdots + \frac{1}{2} B_2(1) f'(B) - \frac{1}{2} \int_A^B B_2(x - [x]) f''(x)\,dx \\
&= \frac{1}{2} B_2(0) f'(x) \Big|_A^B - \frac{1}{2} \int_A^B \bar{B}_2(x) f''(x)\,dx\ .
\end{aligned}$$

Again, the second term involving the integral of $\bar{B}_2(x) f''(x)$ can be integrated by part as before:

$$\int_A^B (t - [t] - \frac{1}{2}) f'(t)\,dt = \frac{1}{2} B_2(0) f'(x) \Big|_A^B - \frac{1}{6} B_3(0) f''(x) \Big|_A^B + \frac{1}{6} \int_A^B \bar{B}_3(x) f'''(x)\,dx\ .$$

Going on as before, the nth step applied to the original sum gives

$$\begin{aligned}
\sum_{k=A}^{B} f(k) = \int_A^B f(t)dt &+ \frac{1}{2} f(A) + \frac{1}{2} f(B) + \frac{1}{2} B_2(0) f'(x) \Big|_A^B \\
&- \frac{1}{6} B_3(0) f''(x) \Big|_A^B + \cdots + (-1)^n \frac{1}{n!} B_n(0) f^{(n-1)}(x) \Big|_A^B \\
&+ (-1)^{n+1} \frac{1}{n!} \int_A^B \bar{B}_n(x) f^{(n)}(x)\,dx\ .
\end{aligned} \tag{2.6}$$

Euler-Maclaurin summation method is useful if we are able to control the error term, i.e. the last integral in (2.6). To this final cause, we use the following:

Lemma 2.1.1. *If k is an odd natural number, then the sign of $\bar{B}_k(x)$ in the interval $(0, \frac{1}{2})$ is opposite to the sign in the interval $(\frac{1}{2}, 1)$, with $\bar{B}_k(x) = 0$ for $x = 0, \frac{1}{2}, 1$.*

The proof of Lemma 2.1.1 can be found in [4], §6. Roughly speaking, we could say that $\bar{B}_k(x)$ oscillates in the interval $[0, 1]$ if k is odd. Hence, if $f^{(n)}(x)$ is monotone in the interval $[A, B]$, becomes easy to estimate the error in (2.6).

Example: suppose we intend to calculate

$$S = \sum_{k=10}^{100} \log k \ ;$$

we try to estimate S with Euler-Maclaurin summation until the error term contains $\bar{B}_3(x)$, that is

$$S = \int_{10}^{100} \log t \, dt + \frac{1}{2}(\log 100 + \log 10) + \frac{1}{2} B_2(0) \left. \frac{1}{x} \right|_{10}^{100} - \frac{1}{6} B_3(0) \left. \left(-\frac{1}{x^2} \right) \right|_{10}^{100} + R \ , \quad (2.7)$$

where

$$R = \frac{1}{6} \int_{10}^{100} \bar{B}_3(x) \, \frac{2}{x^3} \, dx = \frac{1}{3} \int_{10}^{100} \frac{x^3 - \frac{3}{2}x^2 + \frac{1}{2}x}{x^3} \, dx \ .$$

But $\bar{B}_3(x)$ is positive on $(0, \frac{1}{2})$ and negative on $(\frac{1}{2}, 1)$, while $\frac{2}{x^3}$ decreases; hence R is made by the sum of integrals (the first between 10 and 10.5, the second between 10.5 and 11 and so forth) of oscillating sign, the greater in absolute value being the first one (which turns out to be positive), so we have found the bound

$$|R| \leq \frac{1}{3} \int_{10}^{10.5} \frac{x^3 - \frac{3}{2}x^2 + \frac{1}{2}x}{x^3} \, dx \simeq 0,14 \ ,$$

and comparing with the explicit computation of the other terms in (2.7) which gives approximately 349,84, we see that the relative error is less than $\frac{0.14}{349{,}84} \simeq 4 \cdot 10^{-4}$ (of course the absolute error depends on the sum we are going to estimate and could be large at will, but here we emphasize the efficiency of Euler-Maclaurin method looking at the smallness of the relative error already at the third order).

Going back to the general case, using Lemma 2.1.1 (in particular, the fact that $B_{2h+1}(0) = 0$ for every natural h) we can state the

Euler-Maclaurin summation formula: If $f(x) \in C^{2m}([A, B])$, then

$$\sum_{k=A}^{B} f(k) = \int_A^B f(t)dt + \frac{1}{2}f(A) + \frac{1}{2}f(B) + \frac{1}{2}B_2(0)f'(x)\Big|_A^B + \frac{1}{4!}B_4(0)f'''(x)\Big|_A^B + \cdots + \frac{1}{(2m)!}B_{2m}(0)f^{(2m-1)}(x)\Big|_A^B + R_{2m} \,, \qquad (2.8)$$

with the remainder term R_{2m} given by two possible expressions:

$$R_{2m} = -\frac{1}{(2m)!}\int_A^B \bar{B}_{2m}(x)f^{(2m)}(x)\,dx \,,$$

or, if $f(x) \in C^{2m+1}([A, B])$,

$$R_{2m} = \frac{1}{(2m+1)!}\int_A^B \bar{B}_{2m+1}(x)f^{(2m+1)}(x)\,dx \,,$$

where, as before,l $\bar{B}_k(x) = B_k(x - [x])$.

Observation: It is not the spirit of Euler-Maclaurin method, but theoretically the finite sum on the second member of equation (2.8) could become a series (of course with no remainder term), i.e.

$$\sum_{k=A}^{B} f(k) = \int_A^B f(t)dt + \frac{1}{2}f(A) + \frac{1}{2}f(B) + \sum_{m\geq 1} \frac{1}{(2m)!}B_{2m}(0)f^{(2m-1)}(x)\Big|_A^B \,.$$

Depending on $f(x)$ and the interval $[A, B]$ we choose, in general the approximation of $S = \sum_{k=A}^{B} f(k)$ through Euler-Maclaurin formula gets better if we do not stop the computation after the first few steps, but we must keep in mind that this behaviour does not last indefinitely. In fact, reminding the Euler formula for computing $\zeta(s)$ for even positive integers,

$$\zeta(2m) = (-1)^{m+1}\frac{b_{2m}\,(2\pi)^{2m}}{2\cdot(2m)!} \,,$$

and noticing that for large m values $\zeta(2m) \sim 1$ and that $B_n(0) = b_n$ is the nth Bernoulli number, we find

$$|B_{2m}(0)| \sim \frac{2 \cdot (2m)!}{(2\pi)^{2m}} \ ,$$

so that the remainder term in (2.8), as the order of computation $2m$ increases, despite of an initial reduction, from a certain point gets bigger. Hence there exists a limit of accuracy for the approximation of a sum S using Euler-Maclaurin formula.

Example: recovering the previous case, $S = \sum_{k=10}^{100} \log k$, if we continue the process beyond the third order where we stopped before, we will find an error term (at the order $2m$ of computation) of absolute value bounded by

$$\begin{aligned}
|R_{2m}| =& \frac{1}{(2m+1)!} \left| \int_{10}^{100} \bar{B}_{2m+1}(x) \, \frac{(2m)!}{x^{2m+1}} \, dx \right| \leq \frac{1}{(2m+1)\, 10^{2m+1}} \left| \int_0^{1/2} B_{2m+1}(x) \, dx \right| \\
=& \frac{1}{(2m+1)\, 10^{2m+1}} \left| \int_0^{1/2} \frac{B'_{2m+2}(x)}{2m+2} \, dx \right| \leq \frac{\left| B_{2m+2}\left(\frac{1}{2}\right) - b_{2m+2} \right|}{(2m+1)^2 \, 10^{2m+1}} \\
=& \frac{\left| B_{2m+2}\left(\frac{1}{2}\right) + b_{2m+2} - \ 2b_{2m+2} \right|}{(2m+1)^2 \, 10^{2m+1}} = \frac{|2^{-2m-1} b_{2m+2} - \ 2b_{2m+2}|}{(2m+1)^2 \, 10^{2m+1}} \leq \frac{2\, |b_{2m+2}|}{(2m+1)^2 \, 10^{2m+1}} \ ,
\end{aligned}$$

where the identity $B_n(2x) = 2^{n-1}[B_n(x) + B_n(x+1/2)]$ has been used (see [4], §6). So $|R_{2m}|$ is reasonably small until, for large m, we get

$$|R_{2m}| \lesssim \frac{(2m+2)!}{(2m+1)^2 (20\pi)^{2m+1}}$$

which is no longer a meaningful estimate because it grows without bound with m.

Here follow two applications of Euler-Maclaurin method, the first one relative to $\Gamma(s)$ and the second one to $\zeta(s)$. In the next Section the importance of these two estimates for locating the $\xi(s)$ roots through Euler-Maclaurin will become clear.

2.1.1 Computation of $\log\Gamma(s)$ using Euler-Maclaurin formula

The Γ function can be defined in the whole complex plane except the non-positive integers through the Euler relation:

$$\Gamma(s) = \frac{1}{s}\prod_{n\geq 1}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1} = \lim_{n\to\infty}\frac{n!\,(n+1)^s}{s(s+1)\dots(s+n)}\ ,$$

or simply

$$\Gamma(s) = \lim_{n\to\infty}\frac{n!\,n^s}{s(s+1)\dots(s+n)}\ . \tag{2.9}$$

For our final purpose, we need to evaluate $\mathrm{Im}[\log\Gamma(s)]$, so we exploit Euler-Maclaurin to estimate $\log\Gamma(s)$ from (2.9):

$$\begin{aligned}
\log\Gamma(s) &= \lim_{n\to\infty}\left\{s\log n + \sum_{k=1}^{n}\log k - \sum_{k=0}^{n}\log(s+k)\right\}\\
&= \lim_{n\to\infty}\left\{s\log n + \int_1^n \log x\,dx + \frac{1}{2}\log n + \int_1^n \frac{\bar{B}_1(x)}{x}\,dx\right.\\
&\quad\left. - \int_0^n \log(s+x)\,dx + \frac{1}{2}\left[\log n + \log(s+n)\right] + \int_0^n \frac{\bar{B}_1(x)}{s+x}\,dx\right\}\\
&= \left(s-\frac{1}{2}\right)\log s + C - \int_0^\infty \frac{\bar{B}_1(x)}{s+x}\,dx + \lim_{n\to\infty}\left\{\left(s+n+\frac{1}{2}\right)\log\left(\frac{n}{n+s}\right)\right\}\ ,
\end{aligned}$$

where

$$C = 1 + \int_1^\infty \frac{\bar{B}_1(x)}{x}\,dx$$

is a constant that will be evaluated immediately after having noticed that the last term involving the limit reduces to $-s$, so that

$$\Gamma(s) = s^{s-\frac{1}{2}}e^{-s}e^{C}F(s)\ , \tag{2.10}$$

where

$$F(S) = \exp\left[-\int_0^\infty \frac{\bar{B}_1(x)}{s+x}\,dx\right]\ .$$

To compute A we remind the Legendre duplication formula,

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s)\ ,$$

which using (2.10) gives

$$e^C = \sqrt{2\pi e}\left(\frac{s}{s+\frac{1}{2}}\right)^s \frac{F(2s)}{F(s)F\left(s+\frac{1}{2}\right)} .$$

Taking the limit $s \to \infty$ we obtain $e^C = \sqrt{2\pi}$, being $\lim_{s\to\infty} F(s) = 1$. Hence

$$\begin{aligned}\log\Gamma(s) &= \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi - \int_0^\infty \frac{\bar{B}_1(x)}{s+x}\,dx \\ &= \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \sum_{k=1}^{m} \frac{b_{2k}}{2k(s+x)^{2k-1}} + R_{2m}\ ,\end{aligned} \tag{2.11}$$

with

$$R_{2m} = -\int_0^\infty \frac{\bar{B}_{2m}(x)}{2m(s+x)^{2m}}\,dx = -\int_0^\infty \frac{\bar{B}_{2m+1}(x)}{(2m+1)(s+x)^{2m+1}}\,dx .$$

Letting $m \to \infty$, (2.11) becomes the so-called *Stirling series*.

We are going to make use of the Stirling series in the next Section for complex s values, in particular we will be concerned with the case $s = \frac{1}{2} + it$. In the "slit plane" (i.e. the complex plane without the non-positive real axis $\{s \le 0\}$) all terms in (2.11) are holomorphic functions of s. But unlike the example in the previous Section, to estimate $|R_{2m}|$ we cannot apply an alternating series method because we are dealing with complex quantities. A method for estimating the remainder term of the Stirling series is due to Stieltjes [12] and gives, for $s = re^{i\theta}$,

$$|R_{2m}| \le \left(\frac{1}{\cos(\theta/2)}\right)^{2m+2} \left|\frac{b_{2m+2}}{(2m+2)(2m+1)s^{2m+1}}\right| . \tag{2.12}$$

Looking at (2.11), we see that the estimate (2.12) entails that the error in the Stirling series is at most $\cos(\theta/2)^{-2m-2}$ times the order of magnitude of the first term omitted. In our case, on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, we will have bounded errors because $\cos(\theta/2) \ge \sqrt{2}/2$.

2.1.2 Computation of $\zeta(s)$ using Euler-Maclaurin formula

For estimating $\zeta(s) = \sum_{n\geq 1} n^{-s}$, the Euler-Maclaurin method is not directly feasible because the remainder term is not small; in fact we find for $\mathrm{Re}(s) > 1$

$$\begin{aligned}\zeta(s) &= \int_1^\infty x^{-s}dx + \frac{1}{2}\, x^{-s}\Big|_1^\infty - \frac{b_2 s x^{-s}}{2}\Big|_1^\infty + \cdots + R_{2m} \\ &= \frac{1}{s-1} + \frac{1}{2} + \frac{b_2 s}{2} + \cdots + \frac{b_{2m}s(s+1)\dots(s+2m-2)}{(2m)!} + R_{2m} \ ,\end{aligned}$$

where

$$\begin{aligned}R_{2m} &= -\frac{s(s+1)\dots(s+2m-1)}{(2m)!}\int_1^\infty \bar{B}_{2m}(x)x^{-s-2m}dx \\ &= -\frac{s(s+1)\dots(s+2m)}{(2m+1)!}\int_1^\infty \bar{B}_{2m+1}(x)x^{-s-2m-1}dx \ .\end{aligned}$$

Trying to estimate the remainder term using the alternating series technique, we get

$$\left|\int_1^\infty \bar{B}_{2m+1}(x)x^{-s-2m-1}dx\right| \leq \left|\int_1^{3/2} \bar{B}_{2m+1}(x)x^{-s-2m-1}dx\right| \leq \frac{b_{2m+2}}{m+1} \ ;$$

so, for large $|s|$ values, we may have an error too big to consider valid the $\zeta(s)$ estimate performed through a course use of Euler-Maclaurin formula.

The idea is then to split $\zeta(s)$ into a finite sum $\sum_{n=1}^{N-1} n^{-s}$, which will be summed directly, plus a “tail” $\sum_{n=N}^{\infty} n^{-s}$ on which Euler-Maclaurin method will be applied. The choice of the natural N depends on $|s|$ as we are going to see in a while. In formulas, if $\mathrm{Re}(s) > 1$,

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \sum_{n=N}^{\infty} n^{-s} &= \sum_{n=1}^{N-1} n^{-s} + \frac{N^{-s}}{s-1} + \frac{1}{2}N^{-s} \\ &+ \frac{b_2 s}{2}N^{-s-1} + \cdots + \frac{b_{2m}s(s+1)\dots(s+2m-2)}{(2m)!}N^{-s-2m-1} + R_{2m} \ ,\end{aligned} \tag{2.13}$$

with the remainder term now given by the following two equivalent expressions:

$$\begin{aligned}R_{2m} &= -\frac{s(s+1)\dots(s+2m-1)}{(2m)!}\int_N^\infty \bar{B}_{2m}(x)x^{-s-2m}dx \\ &= -\frac{s(s+1)\dots(s+2m)}{(2m+1)!}\int_N^\infty \bar{B}_{2m+1}(x)x^{-s-2m-1}dx \ .\end{aligned} \tag{2.14}$$

Backlund [13] was the first to show that

$$|R_{2m}| \leq \left| \frac{s(s+1)\dots(s+2m+1)\, b_{2m+2} N^{-\sigma-2m-1}}{(2m+2)!(\sigma+2m+1)} \right| ,$$

so, if N is chosen of the same size of $|s|$, the following estimate holds:

$$|R_{2m}| \leq \left| \frac{(s+2m+1)\, b_{2m+2}}{(\sigma+2m+1)} \right| \mathcal{O}(1) .$$

Equation (2.13) was originally valid only for $\mathrm{Re}(s) > 1$, providing hence an absolute convergence for $\zeta(s)$ written as the standard series; yet, (2.13) holds whenever R_{2m} is well-defined, that is whenever the integral converges. Looking at (2.13) it is therefore evident that formula (2.13) remains valid in the halfplane $\mathrm{Re}(s) > -2m$, which can be viewed as an alternative proof of the analytical continuation of $\zeta(s)$ throughout $\mathbb{C}\backslash\{1\}$.

Concerning the roots of $\xi(s)$ on the critical line, we focus our attention to complex numbers with real part equal to 1/2. If $s = \frac{1}{2} + it$ with a large t value, we have seen that it is mandatory to choose a sufficient large N in order to have a remainder terms in (2.13) not too big. Unfortunately, at the same time we have to deal with the finite sum in (2.13), consisting of 2^{-s}, 3^{-s} and so on forth until $(N-1)^{-s}$: each of them is of the form $n^{-s} = n^{-1/2} e^{-it\log n} = n^{-1/2}[\cos(t\log n) - i\sin(t\log n)]$, thereby involving the computation of a square root, a logarithm, a sine and a cosine, not at all a fast computation. This is the reason why this method cannot be used to calculate $\zeta\left(\frac{1}{2}+it\right)$ for large values of t: to search for the zeros of $\zeta\left(\frac{1}{2}+it\right)$ in a wider range we will need the Riemann-Siegel formula of Section 2.3.

2.2 A first method for locating zeros on the critical line

In order to use Euler-Maclaurin summation for finding a zero on the critical line, we start from the following:

Lemma 2.2.1. *The function* $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$ *is real-valued for* $s = 1/2+it$, $t \in \mathbb{R}$.

Proof: We start from equation (1.5) and we rewrite it using the definition (1.7) of $\xi(s)$:

$$\xi(s) = \frac{1}{2} + \frac{s(s-1)}{2}\int_1^\infty \left(x^{\frac{1}{2}s} + x^{\frac{1}{2}(1-s)}\right)\psi(x)\frac{dx}{x} \ ,$$

and integrating by parts

$$\begin{aligned}
\xi(s) =& \frac{1}{2} + \frac{s(s-1)}{2}\int_1^\infty \frac{d}{dx}\left[\psi(x)\left(\frac{2x^{\frac{1}{2}s}}{s} + \frac{2x^{\frac{1}{2}(1-s)}}{1-s}\right)\right]dx \\
&+ \frac{s(1-s)}{2}\int_1^\infty \psi'(x)\left(\frac{2x^{\frac{1}{2}s}}{s} + \frac{2x^{\frac{1}{2}(1-s)}}{1-s}\right)dx \\
=& \frac{1}{2} + \frac{s(1-s)}{2}\psi(1)\left(\frac{2}{s} + \frac{2}{1-s}\right) + \int_1^\infty \psi'(x)\left[(1-s)x^{\frac{1}{2}s} + sx^{\frac{1}{2}(1-s)}\right]dx \\
=& \frac{1}{2} + \psi(1) + \int_1^\infty \psi'(x)\,x^{\frac{3}{2}}\left[(1-s)x^{\frac{1}{2}(s-1)-1} + sx^{-\frac{1}{2}s-1}\right]dx \\
=& \frac{1}{2} + \psi(1) + \int_1^\infty \frac{d}{dx}\left[\psi'(x)\,x^{\frac{3}{2}}\left(-2x^{\frac{1}{2}(s-1)} - 2x^{-\frac{1}{2}s}\right)\right]dx \\
&- \int_1^\infty \frac{d}{dx}\left[\psi'(x)\,x^{\frac{3}{2}}\right]\left(-2x^{\frac{1}{2}(s-1)} - 2x^{-\frac{1}{2}s}\right)dx \\
=& \frac{1}{2} + \psi(1) - \psi'(1)[-2-2] + 2\int_1^\infty \frac{d}{dx}\left[\psi'(x)\,x^{\frac{3}{2}}\right]\left(x^{\frac{1}{2}(s-1)} + x^{-\frac{1}{2}s}\right)dx \ .
\end{aligned}$$

We have seen in the proof of the functional equation for $\zeta(s)$ that the function $\psi(x)$ satisfies

$$\psi(1/x) = -\frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x}\,\psi(x) \ ;$$

setting $x = 1$ into the derivative of both side, we find

$$\frac{1}{2} + \psi(1) + 4\psi'(1) = 0 \ .$$

Thence

$$\xi(s) = 4\int_1^\infty \frac{d}{dx}\left[\psi'(x)\,x^{\frac{3}{2}}\right]x^{-\frac{1}{4}}\cosh\left[\left(\frac{s}{2} - \frac{1}{4}\right)\log x\right]dx \ .$$

Setting $s = \frac{1}{2} + it$ with real t, it is possible to express $\xi(s)$ on the critical line as

$$\xi\left(\frac{1}{2} + it\right) = 4\int_1^\infty \frac{d}{dx}\left[\psi'(x)\,x^{\frac{3}{2}}\right]x^{-\frac{1}{4}}\cos\left(\frac{t}{2}\log x\right)dx \ , \quad t \in \mathbb{R} \ , \qquad (2.15)$$

which proves that $\xi(s)$ is real-valued on the critical line. □

Before Siegel found out the brilliant method that Riemann possessed to calculate the roots for $\xi(s)$ (and which probably drove Riemann to believe that "it is very probable that all roots are real"), the first known attempt to locate the first roots of $\xi(s)$ belongs to Gram [15], who calculated the first 15 roots, all in the range $0 \leq \mathrm{Im}(s) \leq 50$, which turn out to lie on the critical strip, showing that no other root in the same range exist on the critical strip, hence verifying RH in this range.

Expressing $\xi(s)$ on the critical line through (2.15) makes evident that it is real-valued and continuous, so *every time it changes sign a zero must occur.* Of course, paying attention to eventual zeros of $\zeta(s)$ on the critical line corresponding to a minimum or maximum of $\xi(s)$, in order to verify RH for a certain range $0 \leq t \leq T$, the possibility of roots ρ for $\xi(s)$ occurring with real part $\mathrm{Re}(\rho) = \beta \neq 1/2$ must be excluded in some way. We will return on this point up ahead.

To determine the sign of $\xi\left(\frac{1}{2} + it\right)$ we observe that it is possible to write

$$\begin{aligned} \xi\left(\frac{1}{2}+it\right) =& \frac{1}{2}\left[\frac{1}{2}+it\right]\left[-\frac{1}{2}+it\right]\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\pi^{-\frac{1}{4}-\frac{it}{2}}\zeta\left(\frac{1}{2}+it\right) \\ =& -\frac{1}{2}\exp\left[\mathrm{Re}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right]\pi^{-\frac{1}{4}}\left(t^2+\frac{1}{4}\right)Z(t)\ , \end{aligned} \tag{2.16}$$

where

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right) = \exp\left[i\,\mathrm{Im}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - i\,\frac{\log\pi}{2}\,t\right]\zeta\left(\frac{1}{2}+it\right)$$

is called the *Riemann-Siegel Zeta-function*[2]. In equation (2.16), the factor multiplying $Z(t)$ is real and negative; we also know that $\xi\left(\frac{1}{2}+it\right)$ is real-valued, so also $Z(t) \in \mathbb{R}$ for every real t and the sign of $Z(t)$ is opposite to the sign of $\xi\left(\frac{1}{2}+it\right)$. Then, the research for roots

[2]Some authors are used to call $Z(t)$ simply Riemann-Siegel function, but this is the same name which other authors use to call $\vartheta(t)$. To avoid ambiguities, in this thesis we will explicitly refer to Riemann-Siegel Zeta-function for $Z(t)$ and to Riemann-Siegel theta-function for $\vartheta(t)$.

ρ of $\xi\left(\frac{1}{2}+it\right)$ via a study of its sign must pass through the computation, described in the previous Section, of $\vartheta(t)$ (and hence of the imaginary part of $\log\Gamma$) and of $\zeta\left(\frac{1}{2}+it\right)$.

We start with the computation of $\vartheta(t)$ through Euler-Maclaurin method and we will suppose $t>0$:

$$\begin{aligned}\vartheta(t) =& \operatorname{Im}\log\Gamma\left(\frac{it}{2}+\frac{1}{4}\right)-\frac{\log\pi}{2}t=\operatorname{Im}\left\{\left(\frac{it}{2}-\frac{1}{4}\right)\log\left(\frac{it}{2}+\frac{1}{4}\right)-\left(\frac{it}{2}+\frac{1}{4}\right)\right.\\ &\left.+\frac{1}{2}\log 2\pi+\frac{1}{12\left(\frac{it}{2}+\frac{1}{4}\right)}-\frac{1}{360\left(\frac{it}{2}+\frac{1}{4}\right)^3}+\dots\right\}-\frac{\log\pi}{2}t\\ =&\frac{t}{2}\operatorname{Re}\log\left(\frac{it}{2}+\frac{1}{4}\right)-\frac{1}{4}\operatorname{Im}\log\left(\frac{it}{2}+\frac{1}{4}\right)-\frac{t}{2}-\frac{-\frac{t}{2}}{12\left(\frac{t^2}{4}+\frac{1}{16}\right)}\\ &-\frac{\operatorname{Im}\left(-\frac{it}{2}+\frac{1}{4}\right)^3}{360\left(\frac{t^2}{4}+\frac{1}{16}\right)^3}+\dots+-\frac{\log\pi}{2}t\\ =&\frac{t}{2}\log\frac{t}{2}+\frac{t}{4}\log\left(1+\frac{1}{4t^2}\right)-\frac{1}{4}\left[\frac{\pi}{2}-\arctan\left(\frac{1}{2t}\right)\right]-\frac{t}{2}-\frac{1}{6t\left(1+\frac{1}{4t^2}\right)}\\ &-\frac{1}{45t^3\left(1+\frac{1}{4t^2}\right)^3}+\frac{1}{60t^5\left(1+\frac{1}{4t^2}\right)^3}+\dots-\frac{\log\pi}{2}t\,.\end{aligned}$$

Utilizing the Taylor series of $\log(z)$ and $\arctan(z)$, we finally arrive at

$$\vartheta(t)=\frac{t}{2}\log\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\frac{1}{48t}+\frac{7}{5760t^3}+\mathcal{O}(t^{-5})\,. \tag{2.17}$$

However, using Stirling series to evaluate the error committed, it is possible to show (see [4]) that a valid approximation for (2.17) is simply

$$\vartheta(t)\sim\frac{t}{2}\log\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\frac{1}{48t}\,. \tag{2.18}$$

Equation (2.18) is then sufficient to compute the roots of $\xi(s)$ on the critical line.

Example: suppose we want to compute $\xi(\frac{1}{2}+18i)$, we focus first on $\vartheta(18)$, then on $\zeta(\frac{1}{2}+18i)$ and finally on $Z(18)$. Using (2.18) we find

$$\vartheta(18)\sim 9\log\frac{9}{\pi}-9-\frac{\pi}{8}+\frac{1}{48\cdot 9}\sim 0.080911\,,$$

and following the previous Section for what concerns the computation of $\zeta(\frac{1}{2}+it)$ (or simply looking at Haselgrove's tables [14], see Figure 2.1):

$$\zeta\left(\frac{1}{2}+18i\right) \sim 2.329 - i\,0.189\ ,$$

leading to

$$Z(18) \sim e^{0.080911}(2.329 - i\,0.189) = 2.337 + i\,0.000\ ,$$

which is, in our three decimal approximation, real-valued as expected, a further evidence of how good were the approximations we have used. Since $Z(18)$ is positive, then $\xi(\frac{1}{2}+18i)$ must be negative. It is quite easy to prove that $\xi(x) > 0$ on the real axis[3], so that a change of sign must occur for $\xi(\frac{1}{2}+it)$ in the range $0 \leq t \leq 18$, proving that at least one root $\rho = \frac{1}{2}+i\gamma$ must exist (equivalently, $\xi(\frac{1}{2}+i\gamma) = 0$).

Looking over the Haselgrove's tables for $\zeta(\frac{1}{2}+it)$ in the range $0 \leq t \leq 50$, Figures 2.1 and 2.2, we notice first of all that $\mathrm{Re}\,\zeta(\frac{1}{2}+it)$ is mostly positive, whereas $\mathrm{Im}\,\zeta(\frac{1}{2}+it)$ appears to oscillate quite regularly: 21 changes of sign for $\mathrm{Im}\,\zeta(\frac{1}{2}+it)$ (consequently zeros for $\mathrm{Im}\,\zeta(\frac{1}{2}+it)$) in the range $0 \leq t \leq 50$. Now, a root ρ for $\xi(\frac{1}{2}+it)$ occurs if both the real and the imaginary part of $\zeta(\frac{1}{2}+it)$ are zero, so if $0 \leq t \leq 50$ we must check if any of the 21 changes of sign for $\mathrm{Im}\,\zeta(\frac{1}{2}+it)$ correspond to a zero of $\mathrm{Re}\,\zeta(\frac{1}{2}+it)$: from Figures 2.1 and 2.2 11 of the 21 possible roots of ξ are promptly excluded because, in correspondence to them, $\mathrm{Re}\,\zeta(\frac{1}{2}+it)$ is far from being zero, so just 10 zeros of $\mathrm{Im}\,\zeta(\frac{1}{2}+it)$ seem to be potential zeros for $\mathrm{Re}\,\zeta(\frac{1}{2}+it)$ too.

Writing

$$\zeta\left(\frac{1}{2}+it\right) = e^{-i\vartheta(t)}Z(t) = Z(t)\cos\vartheta(t) - iZ(t)\sin\vartheta(t)\ ,$$

[3]Starting from the definition $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$, if $s = x > 1$ then $x(x-1) > 0$, $\Gamma\left(\frac{1}{2}x\right) > 0$ and of course $\zeta(x) > 0$, so that $\xi(x) > 0$. But from the functional equation we know that $\xi(x) = x(1-x)$, hence also $\xi(x) > 0$ for every $x < 0$. In the range $0 < x < 1$, the extension (1.3) for $\zeta(s)$ prove that $\zeta(x) < 0$ together with $x(x-1) < 0$, while $\Gamma\left(\frac{1}{2}x\right)$ remains positive, so $\xi(x) > 0$ also for $0 < x < 1$. The computation of $\xi(0)$ and $\xi(1)$ is a simple exercise involving the residues of Γ and ζ.

Figure 2.1: Haselgrove's table for $\zeta(\frac{1}{2}+it)$ in the range $0 \le t \le 26.8$.

t	$\zeta(\frac{1}{2}+it)$	t	$\zeta(\frac{1}{2}+it)$	t	$\zeta(\frac{1}{2}+it)$
0.0	-1.46	9.0	$+1.45+0.19i$	18.0	$+2.33-0.19i$
0.2	$-1.18-0.67i$	9.2	$+1.48+0.14i$	18.2	$+2.27-0.43i$
0.4	$-0.68-0.94i$	9.4	$+1.51+0.08i$	18.4	$+2.17-0.66i$
0.6	$-0.28-0.94i$	9.6	$+1.53+0.02i$	18.6	$+2.02-0.86i$
0.8	$-0.02-0.84i$	9.8	$+1.54-0.04i$	18.8	$+1.84-1.03i$
1.0	$+0.14-0.72i$	10.0	$+1.54-0.12i$	19.0	$+1.62-1.16i$
1.2	$+0.25-0.62i$	10.2	$+1.54-0.19i$	19.2	$+1.38-1.24i$
1.4	$+0.32-0.52i$	10.4	$+1.53-0.26i$	19.4	$+1.13-1.28i$
1.6	$+0.37-0.44i$	10.6	$+1.50-0.34i$	19.6	$+0.88-1.26i$
1.8	$+0.41-0.37i$	10.8	$+1.47-0.42i$	19.8	$+0.65-1.18i$
2.0	$+0.44-0.31i$	11.0	$+1.42-0.49i$	20.0	$+0.43-1.06i$
2.2	$+0.46-0.26i$	11.2	$+1.36-0.56i$	20.2	$+0.25-0.90i$
2.4	$+0.48-0.21i$	11.4	$+1.29-0.62i$	20.4	$+0.11-0.70i$
2.6	$+0.50-0.16i$	11.6	$+1.21-0.67i$	20.6	$+0.02-0.48i$
2.8	$+0.52-0.12i$	11.8	$+1.12-0.71i$	20.8	$-0.02-0.25i$
3.0	$+0.53-0.08i$	12.0	$+1.02-0.75i$	21.0	$-0.01-0.02i$
3.2	$+0.55-0.04i$	12.2	$+0.91-0.76i$	21.2	$+0.06+0.19i$
3.4	$+0.56-0.01i$	12.4	$+0.79-0.76i$	21.4	$+0.18+0.38i$
3.6	$+0.58+0.03i$	12.6	$+0.68-0.75i$	21.6	$+0.34+0.52i$
3.8	$+0.59+0.06i$	12.8	$+0.56-0.71i$	21.8	$+0.52+0.62i$
4.0	$+0.61+0.09i$	13.0	$+0.44-0.66i$	22.0	$+0.72+0.67i$
4.2	$+0.62+0.12i$	13.2	$+0.33-0.58i$	22.2	$+0.92+0.66i$
4.4	$+0.64+0.15i$	13.4	$+0.23-0.49i$	22.4	$+1.11+0.60i$
4.6	$+0.66+0.18i$	13.6	$+0.15-0.38i$	22.6	$+1.26+0.49i$
4.8	$+0.68+0.21i$	13.8	$+0.07-0.25i$	22.8	$+1.38+0.34i$
5.0	$+0.70+0.23i$	14.0	$+0.02-0.10i$	23.0	$+1.45+0.16i$
5.2	$+0.73+0.26i$	14.2	$-0.01+0.05i$	23.2	$+1.46-0.03i$
5.4	$+0.75+0.28i$	14.4	$-0.01+0.21i$	23.4	$+1.41-0.21i$
5.6	$+0.78+0.30i$	14.6	$+0.01+0.38i$	23.6	$+1.30-0.38i$
5.8	$+0.81+0.32i$	14.8	$+0.07+0.55i$	23.8	$+1.14-0.50i$
6.0	$+0.84+0.34i$	15.0	$+0.15+0.70i$	24.0	$+0.95-0.58i$
6.2	$+0.87+0.36i$	15.2	$+0.26+0.85i$	24.2	$+0.73-0.60i$
6.4	$+0.91+0.37i$	15.4	$+0.39+0.98i$	24.4	$+0.51-0.55i$
6.6	$+0.94+0.38i$	15.6	$+0.56+1.09i$	24.6	$+0.30-0.43i$
6.8	$+0.98+0.39i$	15.8	$+0.74+1.17i$	24.8	$+0.13-0.25i$
7.0	$+1.02+0.40i$	16.0	$+0.94+1.22i$	25.0	$\pm 0.00-0.01i$
7.2	$+1.06+0.40i$	16.2	$+1.15+1.23i$	25.2	$-0.05+0.26i$
7.4	$+1.11+0.40i$	16.4	$+1.36+1.20i$	25.4	$-0.04+0.55i$
7.6	$+1.15+0.39i$	16.6	$+1.57+1.14i$	25.6	$+0.06+0.85i$
7.8	$+1.20+0.38i$	16.8	$+1.77+1.04i$	25.8	$+0.25+1.11i$
8.0	$+1.24+0.36i$	17.0	$+1.95+0.90i$	26.0	$+0.50+1.34i$
8.2	$+1.29+0.34i$	17.2	$+2.10+0.72i$	26.2	$+0.82+1.49i$
8.4	$+1.33+0.31i$	17.4	$+2.22+0.52i$	26.4	$+1.17+1.56i$
8.6	$+1.37+0.28i$	17.6	$+2.30+0.29i$	26.6	$+1.55+1.54i$
8.8	$+1.41+0.24i$	17.8	$+2.34+0.06i$	26.8	$+1.92+1.42i$

Figure 2.2: Haselgrove's table for $\zeta(\frac{1}{2}+it)$ in the range $27.0 \leq t \leq 50.0$.

t	$\zeta(\frac{1}{2}+it)$	t	$\zeta(\frac{1}{2}+it)$	t	$\zeta(\frac{1}{2}+it)$
27.0	$+2.25+1.21i$	35.0	$+2.60+1.11i$	43.0	$+0.44-0.31i$
27.2	$+2.53+0.91i$	35.2	$+2.84+0.67i$	43.2	$+0.16-0.16i$
27.4	$+2.73+0.55i$	35.4	$+2.94+0.17i$	43.4	$-0.07+0.11i$
27.6	$+2.83+0.15i$	35.6	$+2.89-0.33i$	43.6	$-0.20+0.50i$
27.8	$+2.83-0.27i$	35.8	$+2.70-0.80i$	43.8	$-0.18+0.94i$
28.0	$+2.72-0.68i$	36.0	$+2.38-1.19i$	44.0	$+0.01+1.40i$
28.2	$+2.52-1.05i$	36.2	$+1.97-1.46i$	44.2	$+0.37+1.80i$
28.4	$+2.23-1.35i$	36.4	$+1.50-1.59i$	44.4	$+0.87+2.08i$
28.6	$+1.87-1.57i$	36.6	$+1.03-1.57i$	44.6	$+1.47+2.19i$
28.8	$+1.48-1.69i$	36.8	$+0.60-1.40i$	44.8	$+2.11+2.10i$
29.0	$+1.09-1.70i$	37.0	$+0.26-1.12i$	45.0	$+2.71+1.80i$
29.2	$+0.71-1.61i$	37.2	$+0.04-0.76i$	45.2	$+3.21+1.31i$
29.4	$+0.38-1.43i$	37.4	$-0.05-0.36i$	45.4	$+3.54+0.69i$
29.6	$+0.13-1.18i$	37.6	$+0.01+0.03i$	45.6	$+3.66-0.03i$
29.8	$-0.04-0.89i$	37.8	$+0.19+0.36i$	45.8	$+3.56-0.74i$
30.0	$-0.12-0.58i$	38.0	$+0.46+0.59i$	46.0	$+3.24-1.39i$
30.2	$-0.11-0.29i$	38.2	$+0.80+0.71i$	46.2	$+2.75-1.90i$
30.4	$-0.02-0.03i$	38.4	$+1.14+0.69i$	46.4	$+2.14-2.22i$
30.6	$+0.14+0.17i$	38.6	$+1.44+0.55i$	46.6	$+1.49-2.33i$
30.8	$+0.33+0.30i$	38.8	$+1.67+0.31i$	46.8	$+0.86-2.24i$
31.0	$+0.52+0.34i$	39.0	$+1.79\pm0.00i$	47.0	$+0.33-1.97i$
31.2	$+0.70+0.31i$	39.2	$+1.78-0.33i$	47.2	$-0.06-1.57i$
31.4	$+0.84+0.22i$	39.4	$+1.66-0.64i$	47.4	$-0.27-1.11i$
31.6	$+0.92+0.09i$	39.6	$+1.43-0.88i$	47.6	$-0.31-0.66i$
31.8	$+0.92-0.06i$	39.8	$+1.12-1.02i$	47.8	$-0.21-0.28i$
32.0	$+0.84-0.20i$	40.0	$+0.79-1.04i$	48.0	$-0.01-0.01i$
32.2	$+0.71-0.29i$	40.2	$+0.48-0.95i$	48.2	$+0.24+0.14i$
32.4	$+0.52-0.32i$	40.4	$+0.22-0.75i$	48.4	$+0.47+0.15i$
32.6	$+0.31-0.27i$	40.6	$+0.05-0.48i$	48.6	$+0.64+0.07i$
32.8	$+0.11-0.14i$	40.8	$-0.02-0.18i$	48.8	$+0.71-0.06i$
33.0	$-0.05+0.08i$	41.0	$+0.03+0.12i$	49.0	$+0.67-0.20i$
33.2	$-0.13+0.36i$	41.2	$+0.18+0.36i$	49.2	$+0.53-0.29i$
33.4	$-0.13+0.69i$	41.4	$+0.40+0.51i$	49.4	$+0.34-0.29i$
33.6	$-0.02+1.03i$	41.6	$+0.64+0.57i$	49.6	$+0.14-0.18i$
33.8	$+0.20+1.35i$	41.8	$+0.87+0.52i$	49.8	$-0.02+0.03i$
34.0	$+0.52+1.60i$	42.0	$+1.04+0.37i$	50.0	$-0.08+0.33i$
34.2	$+0.92+1.75i$	42.2	$+1.12+0.18i$		
34.4	$+1.37+1.79i$	42.4	$+1.08-0.04i$		
34.6	$+1.83+1.69i$	42.6	$+0.95-0.22i$		
34.8	$+2.25+1.46i$	42.8	$+0.72-0.32i$		

it is evident that a zero for $\operatorname{Im}\zeta(\frac{1}{2}+it) = -Z(t)\sin\vartheta(t)$ corresponds to a zero of either $Z(t)$ or $\sin\vartheta(t)$. In the matter of our case, along the segment from $\frac{1}{2}$ to $\frac{1}{2}+50i$, near any of the 10 potential roots of $\xi(s)$, $\vartheta(t)$ is far from being a multiple of π, consequently $Z(t)$ must be zero: this proves that if $0 \leq t \leq 50$ then $\zeta(\frac{1}{2}+it)$ possesses exactly 10 roots.

A look over Haselgrove's tables in Figure 2.1 suggests that the first zero of $\zeta(\frac{1}{2}+it)$ lies between $\frac{1}{2}+i14.0$ and $\frac{1}{2}+i14.2$: a more precise localization of such zero can be easily performed through linear interpolation, just like Gram [15] did when he computed the first 15 non-trivial zeros of $\zeta(s)$.

What can we say about the zeros of $\zeta(s)$ on the critical strip for $0 \leq \operatorname{Im}(s) \leq 50$? We have found 10 zeros lying on the segment $S = \{\frac{1}{2}+it \,|\, 0 \leq t \leq 50\}$, but other non-trivial zeros could exist located outside S. To verify that the first 15 zeros on the critical line satisfy RH (the 15th zero computed by Gram was approximately[4] $\rho_{15} = \frac{1}{2}+i65$, so RH holds in this range if no other zeros of $\xi(s)$ can be found with $0 \leq t \leq 65$), Gram [15] used a method based on the Taylor series of $\log\xi\left(\frac{1}{2}+it\right)$ (see [4]) which become soon unfeasible a $\operatorname{Im}(s)$ grows. Backlund [16] developed a much more workable method for verifying RH once given the first n roots of $\xi(\frac{1}{2}+it)$.

The starting point of Backlund's method is the application of the Cauchy's argument principle to the analytic function $\xi(s)$: if ∂R indicates the counterclockwise oriented boundary of $R = \{s \in \mathbb{C} \,|\, -\epsilon \leq \operatorname{Re}(s) \leq 1+\epsilon,\ 0 \leq \operatorname{Im}(s) \leq T\}$ for fixed $\epsilon > 0$, $T > 0$, then

$$N(T) = \frac{1}{2\pi i}\int_{\partial R}\frac{\xi'(s)}{\xi(s)}\,ds = \operatorname{Im}\left\{\frac{1}{2\pi}\int_{\partial R}\frac{\xi'(s)}{\xi(s)}\,ds\right\}$$

is the number of zeros of $\xi(s)$ inside R, each zero counted as many times as its multiplicity[5],

[4] The first 10 zeros have been computed by Gram with good accuracy, while the other 5, from the 11th to the 15th, were only roughly located, as Gram himself admitted. A better localization of these roots can however be deduced from Haselgrove's tables.

[5] We have supposes implicitly that no zero occurs with imaginary part equal to T, otherwise the integra-

where the last equality results from the observation that $N(T) \in \mathbb{N}$. Now, since $\xi(x) \neq 0$ for every real x and as a consequence of the functional equation $\xi(s) = \xi(1-s)$, we can simply consider

$$N(T) = 2\,\mathrm{Im}\left\{\frac{1}{2\pi}\int_L \frac{\xi'(s)}{\xi(s)}\,ds\right\} = \mathrm{Im}\left\{\frac{1}{\pi}\int_L \frac{\xi'(s)}{\xi(s)}\,ds\right\} ,$$

where L is the union of the vertical segment from $1+\epsilon$ to $1+\epsilon+iT$ and the horizontal segment from $1+\epsilon+iT$ to $\frac{1}{2}+iT$. Using the definition (1.7), we write

$$\begin{aligned} N(T) &= \frac{1}{\pi}\,\mathrm{Im}\left\{\int_L \frac{d}{ds}\log\left[\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\right]ds + \int_L \frac{d}{ds}\log\left[s(s-1)\right]ds + \int_L \frac{\zeta'(s)}{\zeta(s)}\,ds\right\} \\ &= \frac{1}{\pi}\vartheta(T) + 1 + \frac{1}{\pi}\,\mathrm{Im}\left\{\frac{1}{\pi}\int_L \frac{\zeta'(s)}{\zeta(s)}\,ds\right\} , \qquad (2.19) \end{aligned}$$

where the last equality follows once noticed that

- the first integrand has $\pi^{-1}\,\mathrm{Im}\log\left[\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\right]$ as antiderivative which, evaluated for $s=1+\epsilon$ and $s=\frac{1}{2}+iT$ and using the definition of the Riemann-Siegel theta -function $\vartheta(t)$, gives exactly $\pi^{-1}\vartheta(T)$;

- the second integral gives π^{-1} times the argument of $\log\left[s(s-1)\right]$: when $s=1+\epsilon$ this argument is zero and, when $s=\frac{1}{2}+iT$, we find $\pi^{-1}\,\mathrm{Im}\log\left(-T^2-\frac{1}{4}\right) = 1$.

Backlund idea is based on the fact if $\mathrm{Re}\,\zeta(s) \neq 0$ on C, then $\zeta(C)$ is a curve contained in the halfplane $\mathrm{Re}(s) > 0$ and the third integral in (2.19) is π^{-1} times the argument of $\log\zeta(s)$ which, for $s \in C$, cannot exceed $\pi/2$ in absolute value, so that the last term in (2.19) cannot axceed $\frac{1}{2}$ in absolute value. As a direct consequence, formula (2.19) states that $N(T)$ is the natural number nearest to $\pi^{-1}\vartheta(T)+1$. The proof that $\mathrm{Re}\zeta(s)$ is never zero on C, together with an example of application of Backlund ideas, can be found in [4]; we conclude here just mentioning that Backund was able to prove that $N(200) = 79$ and at the same time, locating 79 changes of sign for $Z(t)$ in the range $0 \leq t \leq 200$, he managed to prove that RH is true for $|\mathrm{Im}(s)| \leq 200$.

tion over ∂R could not be performed.

2.2.1 Some considerations on the "Gram's law"

Starting from the definition of the Riemann-Siegel theta-function,

$$\vartheta(t) = \operatorname{Im} \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{\log \pi}{2} t \ ,$$

we take the derivative

$$\vartheta'(t) = \operatorname{Im} \frac{i}{2} \frac{\Gamma'\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)} - \frac{1}{2} \log \pi \ .$$

Using the main term from the Stirling series for $\Gamma(s) \sim s \log s$, we may say that

$$\frac{d}{ds} \log \Gamma(s) \sim \log s$$

so that, for $t > 0$,

$$\vartheta'(t) \sim \frac{1}{2} \log \left| \frac{1}{4} + \frac{it}{2} \right| - \frac{1}{2} \log \pi \sim \frac{1}{2} \log \frac{t}{2\pi} > 0 \ .$$

Reminding the equation $\zeta(\frac{1}{2} + it) = Z(t) \cos \vartheta(t) - iZ(t) \sin \vartheta(t)$, since $\vartheta(t)$ is an increasing function of t, we conclude that the zeros of $\cos \vartheta(t)$ and $\sin \vartheta(t)$ alternates. The Haselgrove's tables show a tendency of $\operatorname{Re}\zeta(\frac{1}{2} + it) = Z(t) \cos \vartheta(t)$ to be positive, in particular if $Z(t) \cos \vartheta(t)$ is positive in correspondence to two consecutive zeros of $\sin \vartheta(t)$ then at least one zero of $Z(t)$ (hence a zero of $\zeta(\frac{1}{2} + it)$) must occur.

From Figures 2.1 and 2.2 we see that, in the range $10 \leq t \leq 50$, the zeros of $\operatorname{Im} \zeta(\frac{1}{2} + it)$ are alternately zeros of $Z(t)$ (and therefore zeros of $\operatorname{Re} \zeta(\frac{1}{2} + it)$ as well) and zeros of $\sin \vartheta(t)$: Gram thought, from arguments similar to the one exposed before, that this peculiar pattern could hold for larger t values until, from a certain point on, the tendency of $\operatorname{Re} \zeta(\frac{1}{2} + it)$ disappear in favour of a substantial equilibrium between positive and negative values for $\operatorname{Re} \zeta(\frac{1}{2} + it)$. While the first conjecture was substantially correct, the second one has been disproved by Titchmarsh [17].

The zeros of $\sin \vartheta(t)$ are called *Gram points*: the nth Gram point, g_n, is such that $\vartheta(g_n) = n\pi$. So $\operatorname{Im} \zeta(\frac{1}{2} + ig_n) = 0$ but $\operatorname{Re} \zeta(\frac{1}{2} + ig_n) \neq 0$. Since $\vartheta(t)$ is an increasing function of t, it is not difficult to locate approximately Gram points just looking at Haselgrove's tables and, possibly, performing a linear interpolation (using the fact that $\vartheta'(t) \sim \frac{1}{2}\log\frac{t}{2\pi}$). The alternation of a zero of $Z(t)$ and a zero of $\sin\vartheta(t)$ is strictly related with the persistent positivity of $\operatorname{Re}\zeta(\frac{1}{2}+it)$: as long as it lasts, between $\frac{1}{2}+ig_{n-1}$ and $\frac{1}{2}+ig_n$ there exists at least one zero ρ of $\xi(\frac{1}{2}+it)$. This alternation is often referred as Gram's law and it could be summarized as

$$\operatorname{Re}\zeta\left(\frac{1}{2}+ig_n\right) > 0$$

or, equivalently,

$$(-1)^n Z(g_n) > 0 \ .$$

Gram's law is not a real "law" in the mathematical sense because it is not always true for all n: the first Gram point at which Gram's law fails is $g_{126} \sim 282.455$. However, "Gram's law" has been a very useful relation which helped to locate roots on the critical line, due to the not excessive computing effort needed to locate Gram points, for t values greater that 50, where the direct Euler-Maclaurin computation of $\zeta(\frac{1}{2}+it)$ becomes too demanding. In particular, the turning point of large-scale computation of roots for $\xi(\frac{1}{2}+it)$ was the Riemann-Siegel formula used to evaluate $Z(g_n)$ and to find the points g'_n, close to g_n such that $(-1)^n Z(g'_n) > 0$ whenever the Gram's law fails giving $(-1)^n Z(g_n) < 0$ (which happens very rarely, as was clear from the large-scale computations performed by Lehmer [18], [19], which also proved RH to be true in the range $0 \leq \operatorname{Im}(s) \leq g_{25000}$, all 25000 zeros being simple).

2.3 The Riemann-Siegel formula

The paper of Carl Siegel [20], appeared in 1932, is a milestone in analytic number theory, for two different reasons: first, it disclosed to the mathematical community the Riemann's deep insights about zeta-function which were not at all evident from his brief paper (including

Riemann's "feeling" that all non-trivial zeros for $\zeta(s)$ could have real part equal to $\frac{1}{2}$) and second, it provided a very powerful formula for computing $Z(t)$ for large t values, the *Riemann-Siegel formula*. On this formula are based all modern computer algorithms for the calculation of the ξ roots. Last but not least, the Riemann-Siegel formula plays a fundamental role in theoretical number theory because it is used in different proofs of theorems concerning the zeta-function[6].

The original idea of Riemann, emerged from the studies made by Siegel on Riemann's unpublished papers in the Göttingen library, are a variation of the integrals used to prove the functional equation for $\zeta(s)$ (see Theorem A.2.2 in Appendix A.2). First we remind the identity (A.3):

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz \ , \tag{2.20}$$

where C represents the path along the positive real axis from $+\infty$ toward the origin, then making a counterclockwise circle of radius $r < 2\pi$ around the origin and moving right back to $+\infty$. We want now to split in two the zeta-function defined as formal series, something similar to what was done in Section 2.1.2 when we needed to apply the Euler-Maclaurin formula to $\zeta(s)$:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + R_N \ ,$$

where now the tail of the series, $R_N = \sum_{n=N+1}^{\infty} n^{-s}$, is going to be evaluated through complex integrals. In order to split off $\zeta(s)$ as in equation (2.20), one can use, for the integrand function, the tail of the geometric series $\sum e^{-nx}$:

$$\frac{e^{-Mz}}{e^z - 1} = \sum_{n=M+1}^{\infty} e^{-nz} \ ,$$

so that we can write

$$\zeta(s) = \sum_{n=1}^{M} \frac{1}{n^s} - \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{e^{-Mz}(-z)^{s-1}}{e^z - 1}\, dz \ . \tag{2.21}$$

[6]In Chapter 3 two theorems are proven making use of the Riemann-Siegel formula.

At the same time, this splitting comes out if, instead of the contour C in (2.20), we perform the integration along the path C_N defined as in Appendix A.2, where it is proved that

$$\frac{1}{2\pi i}\int_{C_N-C}\frac{(-z)^{s-1}}{e^z-1}\,dz = 2\sum_{m=1}^{N}(2m\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right) \ ,$$

and consequently

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_{C_N}\frac{(-z)^{s-1}}{e^z-1}\,dz + 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{N}n^{-(1-s)} \ . \tag{2.22}$$

Now, the integral in equation (2.21), for $\mathrm{Re}(s) > 1$, tends to zero when $M \to \infty$ because it is the tail of the converging series which defines $\zeta(s)$; besides, the integral in (2.22) tends to zero for $\mathrm{Re}(s) < 0$ in the limit $N \to \infty$, as shown in Appendix A.2. Yet, none of the two integrals can be neglected for large values of M or N. For $0 < \mathrm{Re}(s) < 1$, remarking that the function $e^{-Mz}(-z)^{s-1}/(e^z-1)$ possesses the same poles with the same residues as $(-z)^{s-1}/(e^z-1)$ in the region having C_N as boundary, the idea is to apply simultaneously the methods that yielded to (2.21) and (2.22):

$$\zeta(s) = \sum_{n=1}^{M}\frac{1}{n^s} + 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{N}n^{-(1-s)} - \frac{\Gamma(1-s)}{2\pi i}\int_{C_N}\frac{e^{-Mz}(-z)^{s-1}}{e^z-1}\,dz \ .$$

Multiplying both members by $\frac{1}{2}s(s-1)\Gamma(s/2)\,\pi^{-s/2}$ and using $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$ together with $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$, we find the corresponding equation for $\xi(s)$:

$$\begin{aligned}\xi(s) = {} & (s-1)\Gamma\left(\frac{s}{2}+1\right)\pi^{-s/2}\sum_{n=1}^{M}\frac{1}{n^s} + (-s)\,\Gamma\left(\frac{3}{2}-\frac{s}{2}\right)\pi^{-(1-s)/2}\sum_{n=1}^{N}\frac{1}{n^{1-s}} \\ & + \frac{1}{4\pi i\,(2\pi)^{s-1}\sin(\pi s/2)}(-s)\,\Gamma\left(\frac{3}{2}-\frac{s}{2}\right)\pi^{-(1-s)/2}\int_{C_N}\frac{e^{-Mz}(-z)^{s-1}}{e^z-1}\,dz \ .\end{aligned}$$

Equation (2.23) is valid for all s but we are interested in the case $s = \frac{1}{2} + it$ and, noticing the symmetry $s \longleftrightarrow (1-s)$, it is natural to consider $N = M$, arriving at

$$\begin{aligned}\xi\left(\frac{1}{2}+it\right) = {} & f(t)\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}} + f(-t)\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}-it}} \\ & + \frac{f(-t)}{2i(2\pi)^{\frac{1}{2}+it}\sin\left[\frac{1}{2}\pi(\frac{1}{2}+it)\right]}\int_{C_N}\frac{-(-z)^{-\frac{1}{2}+it}e^{-Nz}}{e^z-1}\,dz \ .\end{aligned} \tag{2.23}$$

where, for $s = \frac{1}{2}+it$, we define $f(t) = (s-1)\Gamma(s/2+1)\pi^{-s/2} = (-\frac{1}{2}+it)\Gamma(\frac{5}{4}+\frac{it}{2})\pi^{-(1/2+it)/2}$. From equation (2.16) we know that $\xi(\frac{1}{2}+it) = g(t)Z(t)$ where

$$g(t) = -\frac{1}{2}\left(t^2+\frac{1}{4}\right)\exp\left[\operatorname{Re}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right]\pi^{-\frac{1}{4}}\ ;$$

on the critical line, $s = \frac{1}{2}+it$, we have

$$\begin{aligned} g(t) &= \exp\left[\log\Gamma\left(\frac{s}{2}\right)\right]\pi^{-\frac{1}{4}}\,\frac{s(s-1)}{2}\,\exp\left[-i\operatorname{Im}\log\Gamma\left(\frac{s}{2}\right)\right] \\ &= \frac{s}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{1}{4}}(s-1)\,e^{-i\vartheta(t)}\pi^{-\frac{1}{2}it} = f(t)e^{-i\vartheta(t)}\ , \end{aligned} \tag{2.24}$$

hence $f(t) = g(t)e^{i\vartheta(t)}$. Equation (2.24) exhibits the symmetry $s \longleftrightarrow (1-s)$, as a consequence of that we find the relation $g(t) = g(-t)$: hence, referring to $Z(t) = \xi(\frac{1}{2}+it)/g(t)$, the g-terms disappear. Moreover, using $\vartheta(-t) = -\vartheta(t)$ and $2i\sin(\pi s/2) = e^{-i\pi s/2}(e^{i\pi s}-1)$ $= e^{-i\pi/4}e^{\pi t/2}(e^{i\pi/2}e^{-\pi t}-1) = -e^{-i\pi/4}e^{\pi t/2}(1-ie^{-\pi t})$, we finally arrive at

$$Z(t) = 2\sum_{n=1}^{N}\frac{\cos[\vartheta(t)-t\log n]}{\sqrt{n}} + \frac{e^{-i\vartheta(t)}e^{i\pi/4}e^{-\pi t/2}}{\sqrt{2\pi}(2\pi)^{it}(1-ie^{-\pi t})}\int_{C_N}\frac{(-z)^{-\frac{1}{2}+it}e^{-Nz}}{e^z-1}\,dz\ . \tag{2.25}$$

The asymptotic series in (2.25) does not converge under the limit $N \to \infty$, anyhow each term is smaller that the previous one, so that one could ask if the truncated series

$$Z(t) \sim 2\sum_{n=1}^{N}\frac{\cos[\vartheta(t)-t\log n]}{\sqrt{n}}$$

is a useful approximation or not: to answer this question, we need to investigate the remainder term

$$r_N(t) = \frac{e^{-i\vartheta(t)}e^{i\pi/4}e^{-\pi t/2}}{\sqrt{2\pi}\,(2\pi)^{it}(1-ie^{-\pi t})}\int_{C_N}\frac{(-z)^{-\frac{1}{2}+it}e^{-Nz}}{e^z-1}\,dz\ . \tag{2.26}$$

Riemann-Siegel formula deals with the numerical evaluation (2.26).

Consider the integrand of (2.26),

$$I_N(z) = \frac{(-z)^{-\frac{1}{2}+it}e^{-Nz}}{e^z-1}\ ;$$

the modulus of the denominator, $|e^z-1|$, is bounded from below if the integration path C_N does not pass on a zero of e^z-1 (i.e. $\pm 2m\pi i$, $m = 0, 1, 2, \ldots$), in which case we may set $|e^z - 1|^{-1} \leq A$ for some positive constant A. Then the search for large values for the modulus of the integrand of (2.26) coincides with the search of large values for

$$\chi(z) = \mathrm{Re}\left[\left(-\frac{1}{2} + it\right)\log(-z) - Nz\right] ,$$

where

$$\left|(-z)^{-\frac{1}{2}+it}e^{-Nz}\right| = e^{\chi(z)} .$$

The function $\chi(z)$, being the real part of a holomorphic function over C_N, is harmonic and consequently does not possess any maximum in the region having C_N as boundary (maximum principle). Nevertheless, χ does have a saddle point $\alpha = (-\frac{1}{2}+it)/N$; expanding in series near α,

$$\begin{aligned}\chi(z) =& \mathrm{Re}\left\{\left(-\frac{1}{2} + it\right)\log(-\alpha) + \left(-\frac{1}{2} + it\right)\log\left(1 + \frac{z-\alpha}{\alpha}\right) - N\alpha - N(z-\alpha)\right\}\\ =& \,\mathrm{const} + \mathrm{Re}\left\{\left(-\frac{1}{2} + it\right)\left[\frac{z-\alpha}{\alpha} - \frac{1}{2}\left(\frac{z-\alpha}{\alpha}\right)^2 + \dots\right] - N(z-\alpha)\right\}\\ =& \,\mathrm{const} - \frac{1}{2}\mathrm{Re}\left\{\frac{N^2(z-\alpha)^2}{-\frac{1}{2}+it}\right\} + \text{terms with higher powers of (z} - \alpha) .\end{aligned}$$

To find a maximum for $\chi(z)$, we note that the quantity $(z-\alpha)^2/(-\frac{1}{2}+it)$ is real and positive if z lies along the line L defined by $\mathrm{Im}\log(z-\alpha) = \frac{1}{2}\mathrm{Im}\log(-\frac{1}{2}+it)$, so $|I_N(z)|$ has a maximum for $z = \alpha$ (this is an application of the so-called *method of steepest descent*[7], first developed by Van der Corput [21]). Thus along L, the integral's main contribution comes from a finite segment of L centered around α, allowing to use more handful methods of approximation.

Riemann-Siegel formula is very useful where the Euler-Maclaurin method loses its efficacy, i.e. for large t values: if t is large, then $\alpha = (-\frac{1}{2}+it)/N$ lies near the imaginary axis.

[7]The path chosen is important, because α is a saddle point and, for example, if we arrive at this point in the perpendicular direction respect to L, then we are reaching a local minimum along this path. The method of steepest descent works if we find an integration contour where the exponent $\chi(z)$ of our numerator is real and positive.

Remembering that the path C_N crosses the imaginary axis between $2\pi Ni$ and $2\pi(N+1)i$, we must have $\alpha \sim 2\pi Ni$, so that $N^2 \sim (-\frac{1}{2}+it)/2\pi i \sim t/2\pi$: this is the reason why, in the Riemann-Siegel formula, one usually assumes

$$N = \left[\sqrt{\frac{t}{2\pi}}\right] ,$$

the largest integer less than $(t/2\pi)^{1/2}$ (note that, with this choice, we can simply indicate $r_N(t)$ as $r(t)$). As a consequence, $\sqrt{t} \leq (N+1)\sqrt{2\pi}$, hence $\alpha \sim it\sqrt{2\pi/t} = i\sqrt{2\pi t}$ lies approximately between $2\pi Ni$ and $2\pi(N+1)i$ as desired. The direction of L is such that $\operatorname{Im}\log(z-\alpha) = \frac{1}{2}\operatorname{Im}\log(-\frac{1}{2}+it) \sim \frac{1}{2}(\pi/2) = \pi/4$. Summarizing: if $a = i\sqrt{2\pi t}$ is the approximate saddle point for $\chi(z)$, the integrand $I_N(Z)$ of (2.26) along C_N can be estimated as the integral of the same function $I_N(Z)$ along a segment, containing the point a, on the line L (of slope 1, directed from upper right to lower left), and the latter integral can be approximated by local methods. Consider the segment L_1 lying on L which extends from $a+\frac{1}{2}e^{i\pi/4}|a|$ down to $a-\frac{1}{2}e^{i\pi/4}|a|$ (i.e. L_1 is the intersection of L and the disk of radius $\frac{1}{2}|a|$, centered in a): the length of L_1 is bounded by the fact that we want to use a power series in $(z-a)$ for the integrand $I_N(z)$, and the radius of convergence will turn out to be $|a|$. It is non difficult to prove that the remainder $r(t)$ of (2.26), originally calculated from an integral over C_N, can be approximated as the same integral over L_1. To prove that, one considers the path $L_{TOT} = L_0+L_1+L_2+L_3$, where L_0 is the remaining portion of L from $a+\frac{1}{2}e^{i\pi/4}|a|$ up to ∞, L_2 is the vertical line from $a-\frac{1}{2}e^{i\pi/4}|a|$ to the ordinate $\operatorname{Im}(z) = -(2N+1)\pi$ and L_3 goes horizontally from this intersection point right towards ∞: the integral over L_{TOT} has the same poles for $I_N(z)$, so that after proving that the integrals over $L_{0,2,3}$ give minor contribution to the remainder term (see [4], §7), we get

$$r(t) \sim \frac{e^{-i\vartheta(t)}e^{i\pi/4}e^{-\pi t/2}}{\sqrt{2\pi}\,(2\pi)^{it}(1-ie^{-\pi t})}\int_{L_1}\frac{(-z)^{-\frac{1}{2}+it}e^{-Nz}}{e^z-1}\,dz . \tag{2.27}$$

Now[8] (2.27) can be evaluated through local approximations.

[8]In [4], §7, it is shown that the error in approximation (2.27) is bounded by $e^{-t/11}$ for $t \geq 100$.

We are supposing that the denominator $e^z - 1$ can be ignored and that the saddle point method can be applied to the numerator $\exp[\chi(z)]$: matter of course, the numerator should be expanded in powers of $(z-\alpha)$. Instead, a first order approximation is based on the expansion in powers of $(z-a)$: doing that we neglect the dependence of $\alpha = (-\frac{1}{2}+it)/N$ from N and the small real part in α. Similarly to what was done when the saddle point method was utilized for $\chi(z)$, the numerator of $I_N(z)$ now becomes

$$\exp\left\{\left(-\frac{1}{2}+it\right)\log(-a)+\left(-\frac{1}{2}+it\right)\log\left(1+\frac{z-a}{a}\right)-Na-N(z-a)\right\}$$
$$=(-a)^{-\frac{1}{2}+it}e^{-Na}\exp\left\{\left[\left(-\frac{1}{2}+it\right)-N\right]\frac{1}{a}(z-a)-\left(-\frac{1}{2}+it\right)\frac{1}{2a^2}(z-a)^2+\dots\right\}\ .$$

Now the coefficient of $(z-a)$ is approximately, for sufficiently large t values, equal to $it/a \sim \sqrt{t/2\pi}-N = p$ and, remembering the choice $N=[\sqrt{t/2\pi}]$, then p simply represents the fractional part $\{\sqrt{t/2\pi}\}$. The coefficient of the term $(z-a)^2$ is $-it((2a^2) \sim i/(4\pi)$, while the coefficients of the generic higher order term $(z-a)^n$ are approximately $\pm it/[n(i\sqrt{2\pi t})^n]$ which become small for large t values. From these observations, we rewrite the numerator of $I_N(z)$ as

$$(-a)^{-\frac{1}{2}+it}e^{-Na}e^{p(z-a)}e^{i(z-a)^2/(4\pi)}g(z-a)\ ,$$

so that

$$g(z-a)=\exp\left\{\left(-\frac{1}{2}+it\right)\log\left(1+\frac{z-a}{a}\right)-(N+p)(z-a)-i\frac{(z-a)^2}{4\pi}\right\}$$

is the exponential of a power series in $(z-a)$ with small coefficients for large t and it can be expanded as

$$g(z-a)=\sum_{n\geq 0}b_n(z-a)^n\ ,$$

with radius of convergence $|a|$ because $z=0$ is the only singularity for $g(z-a)$, the constant coefficient b_0 is equal to 1 (if $z=a$ in the numerator, we must obtain $(-a)^{-1/2+it}e^{-Na}$) and the others goes to zero in the limit $t\to\infty$. Finally,

$$r(t)\sim\frac{e^{-i\vartheta(t)}e^{i\pi/4}e^{-\pi t/2}(-a)^{-\frac{1}{2}+it}e^{-Na}}{\sqrt{2\pi}\,(2\pi)^{it}(1-ie^{-\pi t})}\int_{L_1}\frac{e^{i(z-a)^2/(4\pi)}e^{p(z-a)}\sum_{n\geq 0}b_n(z-a)^n}{e^z-1}\,dz\ . \quad (2.28)$$

On the segment L_1, which crosses a with slope 1, we have $(z-a) = |z-a|e^{\pm i\pi/4}$ and $\exp[i(z-a)/(4\pi)]$ is real valued on L_1, with a maximum equal to 1 for $z = a$ and a fast decreasing away from a (at the end points of L_1, call them $z_{1,2} = a \pm \frac{1}{2}e^{i\pi/4}|a|$, the value is $e^{-t/8}$): so integral (2.28) is concentrated near $z = a$, where the only term surviving in the series expansion of $g(z-a)$ is $b_0 = 1$. Consequently, equation (2.28) assumes the simpler form:

$$r(t) \sim \frac{e^{-i\vartheta(t)}e^{i\pi/4}e^{-\pi t/2}(-a)^{-\frac{1}{2}+it}e^{-Na}}{\sqrt{2\pi}\,(2\pi)^{it}(1-ie^{-\pi t})}\int_{L_1}\frac{e^{i(z-a)^2/(4\pi)}e^{p(z-a)}}{e^z-1}\,dz\ , \tag{2.29}$$

which was estimated numerically by Riemann himself, as Siegel found out studying his unpublished papers.

First, we take the change of variable $z = u + 2\pi iN$ (and consequently $z - a = u + 2\pi iN - i\sqrt{2\pi t} = u - 2\pi ip$, with $p = \{\sqrt{t/(2\pi)}\}$ as before) and the right hand side of (2.29) becomes

$$\left(\frac{t}{2\pi}\right)^{-\frac{1}{4}}\left(\frac{t}{2\pi}\right)^{\frac{it}{2}}\frac{\exp[-i\vartheta(t) - i\pi(N+p)^2 - i\pi N^2 - 2\pi ip^2]}{(1-ie^{-\pi t})(-2\pi i)}\int_{\Gamma_1}\frac{e^{iu^2/(4\pi)}e^{2pu}}{e^u-1}\,du\ ,$$

where Γ_1 is the line segment of midpoint $2\pi ip$, length $\sqrt{2\pi t}$ and slope 1. Reminding the approximation for the Siegel theta-function,

$$\vartheta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \mathcal{O}(t^{-3})\ ,$$

we note that, for large t values, the quantity

$$E(t) = \frac{\exp\{i[(t/2)\log(t/2\pi) - (t/2) - (\pi/8) - \vartheta(t)]\}}{(1-ie^{-\pi t})}$$

is very near to 1; now, $(N+p)^2 = t/(2\pi)$ and $e^{-i\pi N^2} = (-1)^{N^2} = (-1)^N$, so

$$r(t) \sim \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} e^{i\pi/8}(-1)^{N-1}e^{-2\pi ip^2}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{e^{iu^2/(4\pi)}e^{2pu}}{e^u-1}\,du\ . \tag{2.30}$$

Riemann managed to prove the relation

$$\Psi(p) = e^{i\pi/8}e^{-2\pi ip^2}\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{iu^2/(4\pi)}e^{2pu}}{e^u-1}\,du = \frac{\cos\left[2\pi(p^2 - p - \frac{1}{16})\right]}{\cos(2\pi p)}\ ,$$

where[9] Γ is the line of slope 1, directed from upper right to lower left and crossing the imaginary axis between 0 and $2\pi i$, so that if in (2.32) we approximate Γ_1 with Γ, we finally get *the first term of the Riemann-Siegel formula*:

$$r(t) \sim (-1)^{N-1} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} \frac{\cos\left[2\pi(p^2 - p - \frac{1}{16})\right]}{\cos(2\pi p)} . \tag{2.31}$$

The approximation (2.31) can be easily computed for a given t.

Up to now, the major approximations used to deduce the first order approximation for $r(t)$, equation (2.31), are valid for large t values and can be summarized as follows:

1. $\sum_{n\geq 0} b_n(z-a)^n \sim b_0 = 1$
2. $E(t) = \exp\{i[(t/2)\log(t/2\pi) - (t/2) - (\pi/8) - \vartheta(t)]\}(1 - ie^{-\pi t})^{-1} \sim 1;$
3. $\int_{\Gamma_1} \sim \int_{\Gamma}$.

Higher order approximations are derived [4] using higher order terms in the series $g(z-a) = \sum_{n\geq 0} b_n(z-a)^n$: if we call $\omega = \sqrt{2\pi/t}$, then ω is small for large t and the coefficient b_n turns out to be a polynomial in ω of maximum degree n and minimum degree $[n/3]$. The result of this higher order expansion, up to the Jth term, is

$$r(t) \sim (-1)^{N-1} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} E(t)[b_0c_0 + b_1c_1 + \cdots + b_Jc_J] , \tag{2.32}$$

where

$$c_n = e^{i\pi/8} e^{-2\pi i p^2} \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{iu^2/(4\pi)} e^{2pu}}{e^u - 1} (u - 2\pi i p)^n \, du .$$

Each integral c_n is a linear combination of the derivatives $\Psi^{(k)}(p)$ with numerical coefficients (non depending on t). In particular, the integrals c_n satisfy the relation

$$\sum_{n\geq 0} \frac{(2y)^n c_n}{n!} = e^{2\pi i y^2} \sum_{k\geq 0} \frac{\Psi^{(k)}(p) y^k}{k!} , \tag{2.33}$$

[9]The apparent poles at $p = \frac{1}{4} + k\frac{\pi}{2}$ are not a problem, see [4], §7.

so that, in order to compute c_n, if suffices to equate the coefficient of the powers y^n; however, in view of the computation of $r(t)$, the important property expressed by the relation (2.33) is that each c_n can be expressed as a linear combination of the derivatives $\Psi^{(k)}(p)$, $k \le n$, with coefficients independent of t. The successive steps involve a multiplication of both members of (2.33) by $\sum_{m=0}^{J} m!\, b_m (2y)^{-m}$ and then express the result as polynomials in ω, using the recurrence relation

$$b_{n+1} = \frac{(2n+1)\pi i b_n - b_{n-2}}{4(n+1)\pi^2\omega^{-1}} \ .$$

In view locating the zeros of $Z(t)$, the first four terms of the final asymptotic series for $r(t)$ are more than sufficient and one obtains, at last, the following:

Theorem 2.3.1. *The remainder $r(t)$ in the formula*

$$Z(t) = 2 \sum_{n \le \sqrt{t/(2\pi)}} \frac{\cos\left[\vartheta(t) - t \log n\right]}{\sqrt{n}} + r(t) \tag{2.34}$$

is approximately

$$r(t) \sim (-1)^{N-1} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} \left[A_0 + A_1 \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}} + A_2 \left(\frac{t}{2\pi}\right)^{-1} + A_3 \left(\frac{t}{2\pi}\right)^{-\frac{3}{2}} + A_4 \left(\frac{t}{2\pi}\right)^{-2}\right] \tag{2.35}$$

where $N = [\sqrt{t/(2\pi)}]$, $p = \{\sqrt{t/(2\pi)}\}$ and

$$\begin{aligned}
A_0 &= \Psi(p) = \frac{\cos\left[2\pi(p^2 - p - \frac{1}{16})\right]}{\cos(2\pi p)} \ , \\
A_1 &= -\frac{1}{2^5\, 3\pi^2}\Psi^{(3)}(p) \ , \\
A_2 &= \frac{1}{2^6\pi^2}\Psi^{(2)}(p) + \frac{1}{2^{11}\, 3^2\pi^4}\Psi^{(6)}(p) \ , \\
A_3 &= -\frac{1}{2^6\pi^2}\Psi^{(1)}(p) - \frac{1}{2^8\, 3\cdot 5\pi^2}\Psi^{(5)}(p) - \frac{1}{2^{16}\, 3^4\pi^6}\Psi^{(5)}(p) \ , \\
A_4 &= \frac{1}{2^7\pi^2}\Psi(p) + \frac{1}{2^{13}\, 3\pi^4}\Psi^{(4)}(p) + \frac{1}{2^{17}\, 3^2\cdot 5\pi^6}\Psi^{(8)}(p) + \frac{1}{2^{23}\, 3^5\pi^8}\Psi^{(12)}(p) \ .
\end{aligned}$$

We want to emphasize, one more time, that the Riemann-Siegel formula is the result of different types of approximations, as the truncation of integrals contours (estimated to be of order $e^{-t/11}$, [4]) and mainly of the asymptotic series. However, as for the Euler-Maclaurin formula, the computation of the error terms leads to the general rule of estimating the magnitude of the error committed in computing $Z(t)$ to be less than the first term omitted when truncating the series. Also, the real improvement respect to the Euler-Maclaurin method is evident in the case of large t values: for example, if we want to compute $Z(1000)$, the Riemann-Siegel formula needs the evaluation of $[\sqrt{1000/(2\pi)}] = 12$ terms in the main sum and the error committed (see [4]) is much smaller than the A_4 term which is below $2 \cdot 10^{-8}$, while on the other side Euler-Maclaurin method, to achieve the same accuracy, would require the evaluation of hundreds of terms and consequent overly demanding computational efforts. Hence, explicit computation through the Riemann-Siegel formula suggest that the error committed is very small and $Z(t)$ can be evaluated with great accuracy over a large range of t values[10]. Anyway, the interesting fact is that, until now, the great accuracy in computing $Z(t)$ with Riemann-Siegel is basically no more than a conjecture: no theoretical proofs of this good behavior of the Riemann-Siegel error exist, every estimate made up to now is far from justifying the smallness of such error, even when computed with just the first two terms, A_0 and A_1, in (2.35). In this direction, Siegel himself was able to prove that for every m, if A_m is the first term omitted, there exist t_0 and C such that, for every $t > t_0$, the error term is less than $C(2\pi/t)^{-(2m+1)/4}$; in this sense, the Riemann-Siegel formula is an asymptotic expansion of $Z(t)$ and every term is significant if we intend the the last term in the truncated series, say A_{m-1}, to be less then $C(2\pi/t)^{-(2m-1)/4}$ for t sufficiently large[11].

We conclude this Section mentioning some computational developments since the advent of the Riemann-Siegel formula. The major progresses are, of course, a direct consequence

[10]Even when $t = 14$, for example, the A_4 term is smaller than 10^{-4}; for larget t, the A_4 term becomes rapidly smaller.

[11]Siegel gave explicitly the value t_0 for different m, but this t_0 are very large and unhelpful for practical calculation of $Z(t)$.

of the computers making the scene. We already mentioned the works by Lehmer ([18], [19]),where was proven that every root of $\xi(s)$ was a also zero for $\zeta\left(\frac{1}{2}+it\right)$ for $|t| \leq g_{25000}$: this was done using the Riemann-Siegel formula for locating the zeros of $Z(t)$ but, in order to prove that RH was satisfied in the range $0 \leq t \leq g_{25000}$, was mandatory to verify that the number of these zeros was equal to $N(g_{25000})$, the total number of roots for $\xi(s)$ with $0 \leq \operatorname{Im}(s) \leq g_{25000}$. In this view, the work of Turing [22] was essential: Turing developed a method for computing $N(T)$ based only on the behavior of $\zeta\left(\frac{1}{2}+it\right)$. So, the only informations needed are directly derived from $\zeta(s)$ on the critical line, which is what one naturally look at when trying to locate the changes of sign for $Z(t)$. In particular, Turing's method uses a theorem by Littlewood applied to those Gram points g_n which do not satisfy Gram's law $(-1)^n Z(g_n) > 0$: if $(-1)^n Z(g_n) < 0$ and if we find a point $g_n + h_n$, close to g_n, such that the Gram's law is satisfied (i.e. $(-1)^n Z(g_n + h_n) > 0$), Turing showed that if $h_m = 0$ and if the values of h_n for n near m are small, then $N(g_m) = m + 1$. Now, if $S(T) = N(T) - \pi^{-1}\vartheta(t) - 1$ is the error in the approximation $N(T) \sim \pi^{-1}\vartheta(t) + 1$, then in order to prove $S(g_m) = 0$ one must just prove that $|S(g_m)| < 2$, because $S(g_m)$ is an even integer.

After Lehmer, various works on the computation of the $\xi(s)$ roots came out, all of them using mostly the same underlying ideas and greater computer processing power. Ever since, the the major improvement for determining the roots of $\zeta\left(\frac{1}{2}+it\right)$ has been the Odlyzko-Schönhage algorithm [25], which makes use of the Fast Fourier Transform to convert sums like $\sum_{n=1}^{N} k^{-in}$ (this kind of sums are the most demanding part of the calculations performed through the Riemann-Siegel formula) into rational functions. Also Gourdon's work [26], in which the first 10^{13} zeros of $\xi(s)$ were computed, is based on the Odlyzko-Schönhage algorithm.

2.4 Connections between the Riemann-Siegel formula and the zeros of $\zeta\left(\frac{1}{2}+it\right)$

In Section 2.2, the approximate value for $Z(18)$ has been evaluated through the Euler-Maclaurin summation method: the final result was $Z(18) \sim 2.337$. We would like to compute now $Z(18)$ using the Riemann-Siegel formula. First note that $t = 18$ implies $N = [\sqrt{18/(2\pi)}] = [1.692569] = 1$, so that the asymptotic series must stop with the first term $2\cos\vartheta(t)$. Concerning the error estimates, we begin using the first approximation for it:

$$Z(18) \sim 2\cos\vartheta(18) + (-1)^{1-1}\left(\frac{18}{2\pi}\right)^{-\frac{1}{4}}\Psi(0.692569) \sim 1.993457 + 0.346197 \sim 2.339654\ .$$

Hence, even with the first order approximation (and with a t not so large to render the asymptotic series more precise) the value of $Z(18)$ coincides up to two decimal places with the one obtained (with greater effort and utilizing the Haselgrove's tables) computed with Euler-Maclaurin. Computing the higher order corrections to the remainder term (the A_1, A_2... terms, involving the derivatives of $\Psi(p)$[12]), we achieve a better numerical estimate of $Z(18)$. Including the terms up to A_4, we obtain $Z(18) \sim 2.336796$ which is a very precise approximation, especially once compared with the one based on Euler-Maclaurin. Generally speaking, locating the $Z(t)$ roots is computationally much simpler with Riemann-Siegel than Euler-Maclaurin, even for the first zeros where the Euler-Maclaurin formula is still workable: for larger t, the number of terms one need to evaluate utilizing Euler-Maclaurin is orders of magnitude bigger than the number of terms in the case of Riemann-Siegel formula, because for large t the main sum itself, $Z(t) \sim \sum_{n\leq[\sqrt{t/(2\pi)}]} n^{-1/2}\cos\left[\vartheta(t) - t\log n\right]$, provides a very good approximation for $Z(t)$.

[12]There exist other Heselgrove's tables with the coefficients A_n in powers of $1-2p$, because being $0 < p < 1$ it is natural to expand in powers of the symmetric variable $p' = p - 1/2$.

After having discussed the main aspects of the Riemann-Siegel formula, it is natural to ask if this method of computation for $Z(t)$, developed by Riemann, was the reason why he thought it was "probable" that all the $\xi(s)$ roots have real part equal to $\frac{1}{2}$. Perhaps, as Edwards observed [4], Riemann performed $Z(t)$ computations through his formula and noticed that "in general" this line of work leads from a zero $\tilde{t}$ of the first term, $2\cos\vartheta(t)$, directly to a zero t' of $Z(t)$, not far from $\tilde{t}$. In the previous example, with $t = 18$, we had $\tilde{t} = 1.993457$ and the first order correction added 0.346197 to reach the corresponding zero t' for $Z(t)$ (higher order corrections slightly modified the value t'). So, if this one-to-one correspondence between a zero of $2\cos\vartheta(t)$ and of $Z(t)$ holds over, one may think that in the range $0 < t \leq T$ one finds approximately the same total number of zeros for $Z(t)$ and for $2\cos\vartheta(t)$, the latter being $\sim \pi^{-1}\vartheta(t)$. But, see equation (2.19), the total number $N(T)$ of $\xi(s)$ roots with $0 \leq t \leq T$ is

$$N(T) = \frac{1}{\pi}\vartheta(t) + 1 + S(T) \ ,$$

where

$$S(T) = \frac{1}{\pi}\,\mathrm{Im}\left\{\int_L \frac{\zeta'(s)}{\zeta(s)}\,ds\right\} \ ,$$

with L going from $1+\epsilon$ to $1+\epsilon+iT$ and then to $\frac{1}{2}+iT$. So, $\pi^{-1}\vartheta(t)$ is also an estimate for the total number of non-trivial zeros for $\zeta(s)$: this could be viewed as a "rough" version of the RH and a possible mathematical translation of the Riemann's words about his conjecture.

Still, we have used the term "in general" to describe this supposed one-to-one connection between zeros of $2\cos\vartheta(t)$ and of $Z(t)$. This is motivated by the assumption that the consequence of a small change in t is mainly reflected, in relation to $Z(t)$, in a shift of the first term $2\cos\vartheta(t)$; that is, the other terms in $Z(t)$ are subject to minor variations. Unfortunately, this assumption is not a general rule and it fails in correspondence of some crushing failure of the Gram's law, like the one between g_{6708} and g_{6709}: in this interval, the zero $\tilde{t}$ of $2\cos\vartheta(t)$ is associated to a value $Z(\tilde{t}) \sim -2$. If one tries to move t back to g_{6708}, where $2\cos\vartheta(t) = 2$, the other terms in the Riemann-Siegel formula continue to keep $Z(t) < 0$ (one finds $Z(g_{6708}) \sim -\frac{1}{2}$). The search a nearby zero for $Z(t)$ relies upon continuing move left of g_{6708}, consequently decreasing the value of $2\cos\vartheta(t)$ from the maximum $+2$, with the hope that the other terms will contemporaneously increase enough to give $Z(t) = 0$. Of course, in this case the supposed one-to-one correspondence between the zeros of $2\cos\vartheta(t)$ and of $Z(t)$ fails. From this point of view, if that was what really pushed Riemann to formulate RH in terms of a "probable" rule, the failure of this kind of reasoning runs parallel to the failure of the Gram's law; anyway, Gram's law was motivated by computations made through the Euler-Maclaurin formula, each of them requiring the evaluation of hundreds of terms, while the Riemann-Siegel formula requires, for the same t range (say, $100 \leq t \leq 1000$), less than twenty terms and, in general, the main term dominates. However, we now know the so-called *Lehmer's phenomenon*[13] [5] and the fact that $\zeta\left(\frac{1}{2} + it\right)$ (and as a consequence $Z(t)$) is not a bounded function [24]: this two facts were obviously unknown to Riemann and, as a result, the reasoning outlined above is totally inappropriate to reach a proof of RH.

[13] Lehmer's phenomenon is the fact that, quite rarely, two consecutive zeros of $Z(t)$ are very close or, equivalently, that the graph of $Z(t)$ barely crosses the t-axis.

Chapter 3

Estimates for the zeros of $\zeta(s)$ on the critical line

Chapter 2 dealt with the efforts to locate numerically the non-trivial zeros of $\zeta(s)$ inside the critical strip: as already remarked, these methods were fundamental to give a strong numerical evidence in favor of the RH [26], even if, in a future day, a possible counterexample could in principle come to light thanks to the same numerical computations. The only certain thing is that numerical computation of the $\xi(s)$ roots is not the strategy to catch a proof of the RH.

An important step in the direction of a possible proof of RH has been made in 1914 by Hardy [27], proving that infinite zeros of $\xi(s)$ lie on the critical line, the so-called *Hardy's theorem*. Seven years after, in 1921, a paper by Hardy and Littlewood [28] appeared improving Hardy's theorem: not only infinite non-trivial zeros of *Zetas* have real part $\frac{1}{2}$, but if we indicate with $N_0(T)$ the number of them with imaginary part between 0 and T, then there exists a constant A and a positive T_0 such that $N_0(T) > AT$ if $T > T_0$. Still, comparing this result with the number of $\xi(s)$ roots with $0 \leq \text{Im}(s) \leq T$ provided by the von Mangoldt's

Theorem 1.2.4,

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T)\ ,$$

the factor $\log(T/2\pi)$ makes the fraction $N_0(T)/N(T)$ tend to zero in the limit $T \to \infty$. Eventually, in 1942, Selberg [29] managed to prove that a positive fraction of non-trivial zeros of $\zeta(s)$ lie on the critical line:

$$N_0(T) > KT\log T\ ,$$

for some positive constant K, assuming $T > T_0$ for some T_0. In 1974 Levinson [30] was able to prove the lower bound $K = \frac{1}{3}$, so that more than one third of the $\xi(s)$ roots are located on the critical line: until today, this result has not been significantly improved and different authors believe that this way to proceed will improbably suffices to prove the limit

$$\lim_{T\to\infty} N_0(T)/N(T) = 1\ ,$$

which anyway does not imply RH [23]. Even if none of these results will lead us to the final proof (or disproof) of the RH, no doubt they are very important for the study of $\zeta(s)$ and they substantially contribute to the belief that RH is in fact true.

3.1 Hardy's theorem

Theorem 3.1.1. *There exist infinite zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\gamma = \frac{1}{2}$.*

Proof: There are different proofs of this fundamental theorem, see for example [33]. A proof of Hardy's theorem widely diffused among the texts on the subject is based on the fact that, since $|Z(t)| = |\zeta\left(\frac{1}{2}+it\right)|$, then $\zeta\left(\frac{1}{2}+it\right)$ has a zero $\frac{1}{2}+i\gamma$ of odd multiplicity if and only if $Z(t)$ changes sign for $t=\gamma$; but if $Z(t)$ changes sign at least one time in the interval $T \le t \le 2T$ we must have

$$\left|\int_T^{2T} Z(t)\,dt\right| < \int_T^{2T} |Z(t)|\,dt\ .$$

The previous inequality can be proved in different ways; for example it is possible to give the two estimates [33]

$$\left|\int_T^{2T} Z(t)\,dt\right| = \mathcal{O}(T^{\frac{7}{8}})\ , \qquad \int_T^{2T} |Z(t)|\,dt > CT\ ,$$

with C positive constant, which prove the theorem in the limit $T \to \infty$. Instead we will discuss a proof given by Titchmarsh [17] which makes use of the Riemann-Siegel formula. Concerning the proof of Hardy's theorem, it suffices to use a first order simplified version of the Riemann-Siegel formula (2.34),

$$Z(t) = 2 \sum_{n \le \sqrt{t/(2\pi)}} \frac{\cos\left[\vartheta(t) - t\log n\right]}{\sqrt{n}} + \mathcal{O}(t^{-\frac{1}{4}})\ .$$

We have shown in the previous Chapter that, for large t, the Siegel theta-function and its first derivatives are approximately

$$\vartheta(t) \sim \frac{t}{2}\log t\ , \quad \vartheta'(t) \sim \frac{1}{2}\log\frac{t}{2\pi}\ , \quad \vartheta''(t) \sim \frac{1}{2t}\ ,$$

hence the theta-function $\vartheta(t)$ is monotonically increasing and the equation $\vartheta(t) = \nu\pi$ has only one solution, $t_\nu \sim 2\pi\nu/\log\nu$. The function $Z(t)$ evaluated for $t = t_\nu$ becomes $Z(t_\nu) = (-1)^\nu 2g(t_\nu) + \mathcal{O}(t_\nu^{-\frac{1}{4}})$ with

$$g(t_\nu) = \sum_{n \le \sqrt{t_\nu/(2\pi)}} \frac{\cos\left(t_\nu \log n\right)}{\sqrt{n}} = 1 + \frac{\cos\left(t_\nu \log 2\right)}{\sqrt{2}} + \dots\ ;$$

roughly speaking, the quantity $g(t_\nu)$ is equal to 1 plus a series of terms with oscillatory sign and decreasing absolute value and, consequently, it is very likely that $g(t_\nu) > 0$ and that $Z(t_\nu)$ changes sign in the interval $(t_\nu, t_\nu + 1)$. The key of the proof is to show that

$$\sum_{\nu=M+1}^{N} Z(t_{2\nu}) \sim 2N\ , \qquad \sum_{\nu=M+1}^{N} Z(t_{2\nu+1}) \sim -2N\ ,$$

so that, letting $N \to \infty$, we conclude that $Z(t)$ changes sign infinite times (i.e. $Z(t)$ has infinite zeros of odd multiplicity) because $Z(t_{2\nu})$ is positive infinite times and $Z(t_{2\nu+1})$ is negative infinite times. Now consider the sum

$$\sum_{\nu=M+1}^{N} g(t_{2\nu}) = \sum_{\nu=M+1}^{N} \left(\sum_{n\leq\sqrt{t_{2\nu}/(2\pi)}} \frac{\cos(t_{2\nu}\log n)}{\sqrt{n}} \right)$$
$$= N - M + \sum_{2\leq n\leq\sqrt{t_{2\nu}/(2\pi)}} \frac{1}{\sqrt{n}} \sum_{\tau\leq t_{2\nu}\leq t_{2N}} \cos(t_{2\nu}\log n)\ , \qquad (3.1)$$

where $\tau = \max\{2\pi n^2,\, t_{2M+2}\}$. Indicating

$$\phi(\nu) = \frac{t_{2\nu}\log n}{2\pi}\ ,$$

the inner sum in (3.1) is simply

$$\sum_{\tau\leq t_{2\nu}\leq t_{2N}} \cos(2\pi\phi(\nu))\ ;$$

Noticing that, since $\vartheta(t_{2\nu}) = 2\nu\pi$, then

$$\vartheta'(t_{2\nu})\,\frac{dt_{2\nu}}{d\nu} = 2\pi\ ,$$

and we find

$$\phi'(\nu) = \frac{\log n}{2\pi}\,\frac{dt_{2\nu}}{d\nu} = \frac{\log n}{\vartheta'(t_{2\nu})} > 0\ .$$

At the same time, for large ν,

$$\phi''(\nu) = -\log n\,\frac{\vartheta''(t_{2\nu})}{[\vartheta'(t_{2\nu})]^2}\,\frac{dt_{2\nu}}{d\nu} < -\frac{8\pi\log n}{t_{2\nu}\log^3 t_{2\nu}} < -A\,\frac{\log n}{t_{2N}\log^3 t_{2N}}\ ,$$

for a certain constant A. Now it is possible to prove ([33], §5) that if $f(x)\in C^2([a,b])$ and $\lambda \leq -f''(x) \leq h\lambda$ in the interval $[a,b]$ with $b\geq a+1$, then

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = \mathcal{O}[(b-a)\sqrt{\lambda}] + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\ ;$$

as a consequence, the inner sum in (3.1) becomes

$$\sum_{\tau\leq t_{2\nu}\leq t_{2N}} \cos(t_{2\nu}\log n) = \frac{e^{2\pi i\phi(\nu)} + e^{-2\pi i\phi(\nu)}}{2}$$
$$= \mathcal{O}\left(t_{2N}\,\frac{\log^{1/2} n}{\sqrt{t_{2N}}\log^{3/2} t_{2N}}\right) + \mathcal{O}\left(\frac{\sqrt{t_{2N}}\log^{3/2} t_{2N}}{\log^{1/2} n}\right)$$
$$= \mathcal{O}(\sqrt{t_{2N}}\log^{3/2} t_{2N})\ .$$

Then

$$\sum_{2\leq n\leq\sqrt{t_{2N}/(2\pi)}} \frac{1}{\sqrt{n}} \sum_{\tau\leq t_{2\nu}\leq t_{2N}} \cos\left(t_{2\nu}\log n\right) = \mathcal{O}(t_{2N}^{3/4}\log^{3/2} t_{2N}) = \mathcal{O}(N^{3/4}\log^{3/2} N) \ ,$$

so that we have provided the following equation:

$$\sum_{\nu=M+1}^{N} Z(t_{2\nu}) = 2N + \mathcal{O}(N^{3/4}\log^{3/2} N) \sim 2N \ . \tag{3.2}$$

In a similar manner, one can prove that

$$\sum_{\nu=M+1}^{N} Z(t_{2\nu+1}) \sim -2N \ . \tag{3.3}$$

Combining (3.2) and (3.3) and the theorem is proven. □

3.2 A positive fraction of $\zeta(s)$ zeros lies on the critical line

In this Section $N(T)$ indicates the number of $\xi(s)$ roots with $0 \leq \mathrm{Im}(s) \leq T$ and $N_0(T)$ are the number of such roots located on the critical line (hence, $N(T) = N_0(T)$ is a concise reformulation of RH).

Theorem 3.2.1.

$$N_0\left(T + \frac{T}{\log^{10}(T/2\pi)}\right) - N_0(T) > C\left(T + \frac{T}{\log^{10}(T/2\pi)}\right) - N(T) \ , \tag{3.4}$$

with $C > \frac{1}{3}$.

The complete proof of Theorem 3.2.1 is very long and full of technical difficulties [30]; here we will restrict the attention on the basic ideas behind it. The functional equation for $\zeta(s)$, equation (1.4), can be written as

$$h(s)\zeta(s) = h(1-s)\zeta(1-s) \ , \tag{3.5}$$

with $h(s) = \pi^{-s/2}\Gamma(s/2)$. Setting

$$h(s) = e^{f(s)} ,$$

if $|\arg(s)| \leq \pi - \delta$ one can apply the Stirling's approximation and if $|\mathrm{Im}\log(s/2\pi)| < \pi$ we arrive at

$$f(s) = \frac{s-1}{2}\log\frac{s}{2\pi} - \frac{s}{2} + \frac{1}{2}\log 2 + \mathcal{O}(|s|^{-1}) , \tag{3.6}$$

that once differentiated gives

$$f'(s) = \frac{h'(s)}{h(s)} = \frac{1}{2}\log\frac{s}{2\pi} + \mathcal{O}(|s|^{-1}) .$$

Taking the derivatives of both sides in (3.5) we obtain

$$\begin{aligned} f'(s)h(s)\zeta(s) + h(s)\zeta'(s) &= -f'(1-s)h(1-s)\zeta(1-s) - h(1-s)\zeta'(1-s) \\ &= -f'(1-s)h(s)\zeta(s) - h(1-s)\zeta'(1-s) , \end{aligned}$$

so that

$$[f'(s) + f'(1-s)]h(s)\zeta(s) = -h(s)\zeta'(s) - h(1-s)\zeta'(1-s). \tag{3.7}$$

On the critical line, $s = \frac{1}{2} + it$, the right hand side of (3.7) becomes

$$-h\left(\frac{1}{2} + it\right)\zeta'\left(\frac{1}{2} + it\right) + \text{c.c.} ,$$

hence $\zeta\left(\frac{1}{2} + it\right) = 0$ whenever

$$\arg\left\{h\left(\frac{1}{2} + it\right)\zeta'\left(\frac{1}{2} + it\right)\right\} = \frac{\pi}{2} + k\pi .$$

Comparing the second version of the functional equation for $\zeta(s)$, (1.6), with the definition of $h(s)$ in (3.5), the function $\chi(s) = h(1-s)/h(s)$ appears in equation (3.7) as follows:

$$\zeta'(s) = -\chi(s)\{[f'(s) + f'(1-s)]\zeta(1-s) + \zeta'(1-s)\} .$$

Consequently, $\zeta\left(\frac{1}{2} + it\right) = 0$ whenever

$$\arg\left\{h\left(\frac{1}{2} + it\right)\left\{\left[f'\left(\frac{1}{2} + it\right) + f'\left(\frac{1}{2} - it\right)\right]\zeta\left(\frac{1}{2} + it\right) + \zeta'\left(\frac{1}{2} + it\right)\right\}\right\} = \frac{\pi}{2} + k\pi . \tag{3.8}$$

Now $\arg\{h(s) = \exp[f(s)]\}$ can be obtained by (3.6), so that the evaluation left is the one of the argument of

$$G(s) = \zeta(s) + \frac{\zeta'(s)}{f'(s) + f'(1-s)} \ , \tag{3.9}$$

for $s = \frac{1}{2}+it$. Roughly speaking, the central idea of Levinson's proof is that if $\arg\{G(\frac{1}{2}+it)\}$ does not change then the change in argument in (3.8) depends exclusively on $h(\frac{1}{2}+it)$; we note that $\arg\{h(\frac{1}{2}+it)\} = \operatorname{Im}\log\Gamma(\frac{1}{4}+\frac{it}{2}) - \frac{t}{2}\log\pi = \vartheta(t)$, so that $\zeta\left(\frac{1}{2}+it\right)$ reaches a new zero whenever $\vartheta(t)$ increases of π, which is nothing but the Gram's law. But, from (3.6),

$$\vartheta(t) = \operatorname{Im} f(s) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} + \mathcal{O}(\log T)$$

and the fact that $\zeta\left(\frac{1}{2}+it\right)$ gets a zero every time $\vartheta(t)$ increases of π means that the number $N_0(T)$ of zeros on the critical line in the range $0 \le t \le T$ is the essentially comparable, due to the von Mangoldt Theorem 1.2.4, with the number $N(T)$ of zeros on the critical strip in the same range

$$N_0(T) = \frac{T}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \mathcal{O}(\log T) = N(T) \ ,$$

So the essence of Levinson's proof is to prove that $\arg\{G(\frac{1}{2}+it)\}$ changes very slowly in t and consequently $N_0(T)$, even if different from $N(T)$ in this case, is anyway estimated as a significant fraction of $N(T)$ (about $\frac{1}{3}$). The following steps of used by Levinson in his proof are very technical and complicated, but the main strategy of the whole proof is the one exposed before. Now, if $N_G(R)$ indicates the number of zeros of G inside the rectangle $R = \{s \in \mathbb{C} \,|\, \frac{1}{2} \le \operatorname{Re}(s) \le 3,\ T \le \operatorname{Im}(s) \le T+U\}$, then by the principle argument

$$N_G(R) = \frac{1}{2\pi i}\int_{\partial R}\frac{G'(s)}{G(s)}\,ds = \frac{1}{2\pi}\,\Delta_R \arg\{G(s)\}$$

It involves standard but long calculations (including the use of the Jensen's formula, [2], [23]) to show that the contribution to $N_G(R)$ of the integration along the three sides of R not lying on the critical line can be estimated as $\mathcal{O}(\log T)$, so that

$$\arg\{G(\frac{1}{2}+it)\}\Big|_T^{T+U} = -2\pi N_G(R) + \mathcal{O}(\log T).$$

At this point, a direct computation of $N_G(R)$ is not possible, so that Levinson used a theorem by Littlewood: if $\Psi(s)$ is an integral function (hence, $\Psi(s)G(s)$ has the same zeros as $G(s)$) and for every $0 < a < \frac{1}{2}$

$$2\pi\left(\frac{1}{2}-a\right)N_G(R) \leq \frac{U}{2}\log\left[\frac{1}{U}\int_T^{T+U}|\Psi(a+it)G(a+it)|^2\,dt\right]+\mathcal{O}(UL^{-1})\,, \qquad (3.10)$$

where $L = \log(T/2\pi)$. The quality of this kind of estimate for $N_0(T)/N(T)$ strictly relies upon which function $\Psi(s)$ is chosen; the one chosen by Levinson is not the optimal choice [23] but, at the same time, renders the calculations much easier. The estimation of the integral in (3.10) is obtained through the Riemann-Siegel formula and, after long calculations, Levinson's proof is completed showing that for $a = \frac{1}{2}+\mathcal{O}(L^{-1})$ the integral in (3.10) is less than $C'U$ for a certain positive constant C', from which depends the value of the constant C in (3.4).

Conclusions

The present thesis deals with the Riemann zeta-function $\zeta(s)$ and it is intended ad an exposure of some of the most relevant results obtained until today. The zeta-function plays a fundamental role in number theory and one of the most important open question in mathematics is the Riemann Hypothesis (RH), which states that all non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$. The zeta-function appeared for the first time in 1859 on a Riemann's paper originally devoted to the explicit formula connecting the prime counting function, $\pi(x)$, with the logarithmic integral $\mathrm{Li}(x)$; nevertheless, the paper contained other outstanding results, like the analytical continuation of $\zeta(s)$ through the whole complex plane, and deep conjectures, each of them involving the non-trivial zeros of $\zeta(s)$. All these conjectures, thanks to von Mangoldt and Hadamard, afterwards became theorem except, as said before, the conjecture about the displacement of non-trivial zeros along the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

In Chapter 1 we have seen the ideas inside Riemann's paper and the basic properties of $\zeta(s)$; also, some consequences of RH are discussed, together with considerations about the possibility of RH to be true or false.

Chapter 2 is dedicated to the computational aspects of the RH: we walked along an historical path from the first calculations of non-trivial zeros by Gram, who made use of the Euler-Maclaurin summation method, to the much more powerful Riemann-Siegel formula, which is the foundation of every modern algorithm to locate these zeros (and, at the same time, is a formula widely used in many proofs concerning $\zeta(s)$). Chapter 2 also includes considerations on the so-called "Gram's law" and on the possible origin of RH by Riemann.

Ultimately, Chapter 3 discussed two prominent theorems, by Hardy and Levinson respectively, which prove that infinite non-trivial zeros of $\zeta(s)$ satisfy RH (Hardy) and that, in particular, this is true for more than one third of them (Levinson).

Appendix A

A.1 Poisson summation formula and functional equation for $\vartheta(x)$

A very powerful tool in analytic number theory is the so-called *Poisson summation formula*: it is a consequence of the theory of Fourier series and here we will not consider it in the most general case. However, as long as we are concerned, the hypothesis of the following theorem are sufficient to deal with a large part of number theory objects, like the function $\vartheta(x)$ used to prove the functional equation for $\xi(s)$.

Theorem A.1.1. *Let $f, \hat{f} \in L^1(\mathbb{R})$ (where $\hat{f}(x) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i n t} dt$ is the Fourier transform of f) and assume that both f and $\hat{f}$ have bounded variation on $\mathbb{R}$. Then*

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \ , \tag{A.1}$$

each series converging absolutely.

Proof: The function $g(x) = \sum_{m \in \mathbb{Z}} f(x+m)$ is periodic of period 1 and possesses an absolutely convergent Fourier series expansion

$$g(x) = \sum_{n \in \mathbb{Z}} c_g(n) \, e^{2\pi i n x} \ ,$$

with coefficients

$$c_g(n) = \int_0^1 g(t)e^{-2\pi i n t}dt = \int_0^1 \sum_{m\in\mathbb{Z}} f(t+m)e^{-2\pi i n t}dt = \int_{-\infty}^{+\infty} f(t)e^{-2\pi i n t}dt = \hat{f}(n) \ ,$$

the third equality holding for bounded variation of f. Now (A.1) derives setting $x = 0$. □

Corollary A.1.2. *The function* $\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-n^2\pi x}$ *satisfies*

$$\vartheta(x) = \frac{1}{\sqrt{x}}\,\vartheta(1/x) \ . \tag{A.2}$$

Proof: Consider the function $f(x) = e^{-\alpha x^2}$ for a fixed $\alpha > 0$: f satisfies the hypothesis of Theorem A.1.1, so

$$\vartheta(\alpha/\pi) = \sum_{m\in\mathbb{Z}} e^{-\alpha m^2} = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\alpha t^2} e^{-2\pi i n t} dt \ .$$

Now

$$\int_{-\infty}^{+\infty} e^{-\alpha t^2} e^{-2\pi i n t}dt = 2\int_0^{+\infty} e^{-\alpha t^2}\cos 2\pi n t\ dt = \frac{2}{\sqrt{\alpha}}\int_0^{+\infty} e^{-x^2}\cos\frac{2\pi n x}{\sqrt{\alpha}}\ dx = \frac{2}{\sqrt{\alpha}}\,F\left(\frac{\pi n}{\sqrt{\alpha}}\right),$$

where

$$F(y) = \int_0^{+\infty} e^{-x^2}\cos 2yx\ dx \ ;$$

but $F(y)$ satisfies the differential equation $F'(y) + 2yF(y) = 0$ and, using $\int_0^{+\infty} e^{-x^2} = \sqrt{\pi}/2$, we conclude that

$$F(y) = \frac{\sqrt{\pi}}{2}\, e^{-y^2} \ .$$

Hence

$$\int_{-\infty}^{+\infty} e^{-\alpha t^2} e^{-2\pi i n t}dt = \sqrt{\frac{\pi}{\alpha}} e^{-\pi^2 x^2/\alpha}$$

and taking $\alpha = \pi x$ we obtain (A.2). □

A.2 Riemann's proof of the functional equation for $\zeta(s)$

The starting point is the definition of $\zeta(s)$ as complex integral which appeared in Riemann's paper[1].

Lemma A.2.1. *Let C represent, in the complex plane, the contour which runs just above the positive real axis from ∞ directed leftward to the origin, make a counterclockwise circle of radius $r < 2\pi$ around the origin and move right to ∞ running just below the positive real axis. Then*

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz \ , \tag{A.3}$$

where $(-z)^{s-1} = \exp[(s-1)\log(-z)]$ is defined on the complement of the non-negative real axis, with $|\,\mathrm{Im}\log(-z)| < \pi$.

Proof: The integral is convergent and, choosing $r < 2\pi$, we are sure that no multiple of $2\pi i$ is enclosed by C; then Cauchy's theorem assures that the value of the integral does not depend on how small the radius r is, so that we can take the limit $r \to 0$ and the integral along C reduces to the sum of the two integral

$$\int_\infty^0 \frac{x^{s-1}e^{-\pi i(s-1)}}{e^x - 1}\, dx + \int_0^\infty \frac{x^{s-1}e^{\pi i(s-1)}}{e^x - 1}\, dx = 2i\sin[(s-1)\pi]\zeta(s)\Gamma(s) \ ,$$

where, when moving upward the positive real axis, $\arg(-z) = -\pi$ so that $(-z)^{s-1} = x^{s-1}e^{-\pi i(s-1)}$ and, when moving back downward the positive real axis $(-z)^{s-1} = x^{s-1}e^{\pi i(s-1)}$ because now $\arg(-z) = \pi$ and where we have used the relation

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \ .$$

[1] Riemann, throughout his paper, used the "factorial function"

$$\Pi(s) = \int_0^\infty x^s e^{-s} dx = s\Gamma(s) = \Gamma(s+1)$$

instead of $\Gamma(s)$, as can be seen in [4] where all computations are made using $\Pi(s)$ as Riemann did. Here we follow the contemporary vogue in number theory which replaces the old-fashioned Π with Γ.

The Lemma is proved just remembering that $\Gamma(s)\Gamma(1-s) = \pi/\sin s\pi$. □

The subsequent step taken by Riemann was the proof of the following functional equation for $\zeta(s)$, which enables to continue it analytically over the whole complex plane, excluding the simple pole at $s = 1$.

Theorem A.2.2. *The function $\zeta(s)$ satisfies the following functional equation:*

$$\zeta(s) = 2^s \pi^{1-s} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \; . \tag{A.4}$$

Proof: First of all we define, on the complex plane, the path C_N which runs over the positive real axis from $+\infty$ to $(2N+1)\pi$, then describes counterclockwise the square of corners $(2N+1)\pi(i \pm 1)$ and $(2N+1)\pi(-i \pm 1)$ and finally, from $(2N+1)\pi$, returns to $+\infty$ following the positive real axis. Making use of the path C of the previous Lemma, it is evident that the path $C_N - C$ has winding number equals to 1 about the points $\pm 2m\pi i$, with $m = 1, \ldots, N$, which are simple points for the function $(-z)^{s-1}/(e^z - 1)$ with residues $(\mp 2m\pi i)^{s-1}$. Then, using the residue theorem,

$$\frac{1}{2\pi i} \int_{C_N - C} \frac{(-z)^{s-1}}{e^z - 1}\, dz = \sum_{m=1}^{N} [(-2m\pi i)^{s-1} + (2m\pi i)^{s-1}] = 2 \sum_{m=1}^{N} (2m\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \; ,$$

where we have used the simplification

$$i^{s-1} + (-i)^{s-1} = \frac{1}{i}\,[e^{s \log i} - e^{s \log(-i)}] = \frac{1}{i}\,[e^{s\pi i/2} - e^{-s\pi i/2}] = 2 \sin\left(\frac{\pi s}{2}\right) \; .$$

Separating C_N as the sum of the square S and the path C'_N from $+\infty$ to $(2N+1)\pi$ and its opposite C''_N from $(2N+1)\pi$ to $+\infty$, then $|e^z - 1|^{-1}$ is bounded on S by a constant A independent of N and $|(-z)^{s-1}| = A' N^{\sigma - 1}$ for some constant A', so that

$$\left| \int_S \frac{(-z)^{s-1}}{e^z - 1}\, dz \right| \leq A'' N^{\sigma} \; ,$$

for some positive constant A''. Hence, if $\sigma < 0$, the integral over S will tend to zero as $N \to \infty$; since the same will happen to the integrals over C'_N and C''_N, in the limit $N \to \infty$

the integral over $C_N - C$ will tend to the integral over C which, using (A.4), turns out to be equal to $\zeta(s)/\Gamma(1-s)$. For $N \to \infty$, the sum $\sum_{m=1}^{N}$ converges to $\zeta(1-s)$, which is well defined for $\sigma < 0$ and the identity (A.4) is obtained for $\sigma < 0$. But two meromorphic functions which are equal on a region are identical, thus (A.4) is valid for every s. $\square$

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