Summary of the Thesis in Mathematics by Valentina Monaco

Complete Surfaces of Constant Gaussian Curvature in Euclidean Space $\mathbb{R}^3$.

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SUMMARY

The present thesis deals with complete smooth surfaces of constant Gaussian curvature K, embedded in the Euclidean space $\mathbb{R}^3$. We will treate separately and in the following order the cases $K = 0$; $K > 0$; and $K < 0$. We will show that if the Gaussian curvature is indentically zero, the surfaces are union of parallel lines - i.e. cylinders - (Chapter 2); that every complete and connected regular surface of positve and constant Gaussian Curvature, is a round sphere and, in particular, that compact surfaces exist only in the positive case (Chapter 3). Finally we will prove that complete surfaces do not exist in the negative case - Hilbert's theorem - (Chapter 4).

In particular this thesis is structured as follows:

0.1 Chapter 1

In the introduction (Chapter 1), we will treat some basic concepts.
First we will define the regular surfaces, the first fundamental form, and state the Theorema Egregium.

Definition 1 A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathcal{X} : U \longrightarrow V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathcal{X}$ is differentiable. This means that if we write

$$\mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v)), \qquad (u, v) \in U,$$

   the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivates of all orders in U.

2. $\mathcal{X}$ is a homeomorphism. Since $\mathcal{X}$ is continuous by condition 1, this means that $\mathcal{X}$ has an inverse $\mathcal{X}^{-1} : V \cap S \longrightarrow U$ which is continuous; that is, $\mathcal{X}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

3. (The regularity condition). For each $q \in U$, the differential $d\mathcal{X}_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is one to one.

The natural inner product of $\mathbb{R}^3 \supset S$ induces in each tangent plane $T_p(S)$ of a regular surface S an inner product, to be denoted by $< \ , \ >_p$:
If $w_1, w_2 \in T_p(S) \subset \mathbb{R}^3$, then $< w_1, w_2 >_p$ is equal to the inner product of w_1

and w_2 as vectors in $\mathbb{R}^3$. To this inner product, which is a symmetric bilinear form (i.e., $< w_1, w_2 >=< w_2, w_1 >$ and $< w_1, w_2 >$ is linear both in w_1 and w_2), there corresponds a quadratic form $I_p : T_p(S) \longrightarrow \mathbb{R}$ given by

$$I_p(w) =< w, w >_p= |w|^2 \geq 0. \tag{1}$$

Definition 2 The quadratic form I_p on $T_p(S)$, defined by Eq.(1), is called the *first fundamental form* of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

Theorem 1 (THEOREMA EGREGIUM, GAUSS 1828) *The Gaussian curvature K of a surface is invariant by local isometries.*

Remark 1 *The Gaussian Curvature depends only on the first fundamental form. Cf. [2, page 231].*

Since a diffeomorphism $\varphi : S \longrightarrow \bar{S}$ is an isometry if and only if the differential $d\varphi$ preserves the first fundamental form I

$$I_p(w) = I_{\varphi(p)}(d\varphi_p(w))$$

for all $w \in T_p(S)$ (cf. [2, page 218]); we can state that there exists two parametrizations, $\mathcal{X} : U \longrightarrow S$ and $\bar{\mathcal{X}} : U \longrightarrow \bar{S}$ such that $E = \bar{E}$,
$F = \bar{F}$, $G = \bar{G}$ in U, if and only if the map $\varphi = \bar{\mathcal{X}} \circ \mathcal{X}^{-1} : \mathcal{X}(U) \longrightarrow \bar{S}$ is a local isometry.
Cf. [2, page220].

Then we will study complete surfaces showing Hopf Rinow's theorem:

Definition 3 A regular surfaces S is said to be *geodedically complete* when for every point $p \in S$, any parametrized geodesic $\gamma : [0, \epsilon) \longrightarrow S$ of S, starting from $p = \gamma(0)$, may be extended into a parametrized geodesic $\bar{\gamma} : \mathbb{R} \longrightarrow S$, defined on the entire line $\mathbb{R}$.
In other words, S is complete when for every $p \in S$ the mapping
$exp_p : T_p(S) \longrightarrow S$ is defined for every $v \in T_p(S)$.

Theorem 2 (Hopf-Rinow) *: Let S be a complete surface. Given two points $p, q \in S$, there exists a minimal geodesic[1] joining p to q.*

[1] We say that a parametrized geodesic joining two points is minimal if its length is smaller than or equal to that of any parametrized piecewise regular curve joining these two points.

Finally we will treat some topological concepts proving the following results:

Proposition 1 *A closed surface $S \subset \mathbb{R}^3$ is complete.*

Proposition 2 *A closed surfaces $S \subset \mathbb{R}^3$ is orientable. Cf. [6].*

0.2 Chapter 2

In Chapter 2 our aim is to prove the following theorem:

THEOREM: *Let $S \subset \mathbb{R}^3$ a complete surface of zero Gaussian curvature. Then S is a cylinder or a plane.*

Definition 4 A *cylinder* is a particular ruled surface S. It is a union of parallel lines. Hence through each point $p \in S$ there passes a unique line $R(p)$ (the generator through p) which satisfies the condition that if $q \neq p$ then the lines $R(p)$ and $R(q)$ are parallel or equal.

Example 1 ***A non-complete surface with*** $K \equiv 0$.

Let us consider the open triangle ABC and add to each side a cylindrical surface, with generators parallel to the given side. (Fig. 1).

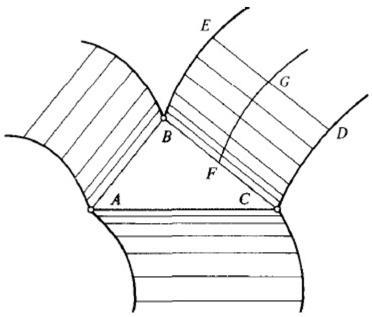

Figure 1: A non-complete surface with $K \equiv 0$.

It is possible to make this construction in such a way that the resulting surface is a regular surface.

For istance, to ensure regularity along the open segment BC, it suffices that the section FG of the cylindrical band $BCDE$ by a plane normal to BC is a curve of the form

$$exp\left(\frac{1}{x^2}\right)$$

Let us observe that the vertices A, B, C of the triangle and the edges BE, CD, etc., of the cylindrical band do not belong to S.

Before proving the theorem we will deal with the study of ruled surfaces, in particular the developable ones:

Definition 5 A (differentiable) *one parameter family of (straight) lines* $\{\alpha(t), w(t)\}$ is a correspondence that assigns to each $t \in I$ a point $\alpha(t) \in \mathbb{R}^3$ and a vector $w(t) \in \mathbb{R}^3, w(t) \neq 0$, so that both $\alpha(t)$ and $w(t)$ depend differentiably on t.
For each $t \in I$, the line L_t which passes through $\alpha(t)$ and is parallel to $w(t)$ is called *the line of the family at t.*
Given a one parameter family of lines $\{\alpha(t), w(t)\}$, the parametrized surface

$$\mathcal{X}(t, v) = \alpha(t) + vw(t), \qquad t \in I, \qquad v \in \mathbb{R},$$

is called the *ruled surface* generated by the family $\{\alpha(t), w(t)\}$. The lines L_t are called the *rulings*, and the curve $\alpha(t)$ is called a *directrix* of the surface $\mathcal{X}$. It should be noticed that we also allow $\mathcal{X}$ to have singular points, that is, points (t, v) where $\mathcal{X}_t \wedge \mathcal{X}_v = 0$.

Among the ruled surfaces, the deveolpables play a distinguished role. Let us start with an arbitrary ruled surface

$$\mathcal{X}(t, v) = \alpha(t) + vw(t) \tag{2}$$

generated by the family $\{\alpha(t), w(t)\}$ with $|w(t)| \equiv 1$. The surface (2.1) is said to be *developable* if

$$(w, w', \alpha') \equiv 0. \tag{3}$$

To find a geometric interpretation for condition (3), we shall compute the Gaussian curvature of a developable surface; and by a simple computation we obtain that the Gaussian curvature K of a developable (regular) surface is identically zero.

To prove the main theorem, recalling that $K(S) \equiv 0$, (and stated $U = S \backslash P$ where U is the set of parabolic points and P is the set of planar ones) we will need a series of results:

Proposition 3 *The unique asymptotic line that passes through a parabolic point $p \in U$ of a surface S of Gaussian curvature $K \equiv 0$ is an (open) segment of a (straight) line in S.*

Proposition 4 *Let r be a maximal asymptotic line passing through a parabolic point $p \in U \subset S$ of a surface S of Gaussian curvature $K \equiv 0$ and let $P \subset S$ be the set of planar points of S. Then $r \cap P = \emptyset$.*

Proposition 5 (Massey) *Let $p \in Bd(U) \subset S$ be a point of the boundary of the set U of parabolic points of a surface S of Gaussian curvature $K \equiv 0$.*
Then through p there passes a unique open segment of line $C(p) \subset S$. Furthemore, $C(p) \subset Bd(U)$; that is, the boundary of U is formed by segments of lines.

We will end the Chapter with the proof of the theorem.

0.3 Chapter 3

In Chapter 3 our aim is to prove the following theorem:

> **THEOREM:** *Let S be a complete, connected, regular surface of constant positive Gaussian curvature K.*
> *Then S is a sphere.*

Before proving this theorem we will introduce the Bonnet's theorem, which states that a complete surfaces of positive Gaussian curvature must be compact.

Theorem 3 (Bonnet) *Let the Gaussian curvature K of a complete surface S satisfy the condition*

$$K \geq \delta > 0.$$

Then S is compact and the diameter ρ of S satisfies the inequality

$$\rho \leq \frac{\pi}{\sqrt{\delta}}.$$

Then we will concentrate on compact surfaces, proving the following result:

Proposition 6 *A regular compact surface $S \subset \mathbb{R}^3$ has at least one elliptic point.*
Thus, if S is compact and K is constant, K must be positive on S.

Remark 2 *The compact, connected surfaces of $\mathbb{R}^3$ for which the Gaussian curvature $K > 0$ are called ovaloids.*

For this reason and having regarded Proposition 6 and Bonnet's theorem; we will show the following equivalent theorem:

THEOREM 1: *Let S be an ovaloid of constant Gaussian curvature. Then S is a sphere.*

Corollary (of the theorem 1): *The sphere is rigid in the following sense. Let $\varphi : \Sigma \longrightarrow S$ be an isometry of a sphere $\Sigma \subset \mathbb{R}^3$ onto a regular surface $S = \varphi(\Sigma) \subset \mathbb{R}^3$. Then S is a sphere.*

Intuitively, this means that it is not possible to deform a sphere made of a flexible but inelastic material.

Remark 3 *It should be noticed that there are surfaces homeomorphic to a sphere which are not rigid. An example is given in Figure 2. We replace the plane region P of the surface S in Fig. 2 by a ''bump'' inwards so that the resulting surface S' is still regular. The surface S'' formed with the ''symmetric bump'' is isometric to S', but there is no linear orthogonal transformation that takes S' into S''. Thus S' is not rigid.*

Figure 2: Surfaces that are homeomorphic to a sphere and that are not rigid.

Finally, to prove the main theorem, we will need the following lemma:

Lemma 1 *Let S be a regular surface and $p \in S$ a point of S satisfying the following condition:*

1. $K(p) > 0$*; that is, the Gaussian curvature in p is positive.*
2. *p is simultaneously a point of local maximum for the function k_1 and a point of local minimum for the function k_2 ($k_1 \geq k_2$).*

Then p is an umbilical point of S.

Since in the proof of the Theorem 1 the assumption that $K = k_1 k_2$ is constant is used only to guarantee that k_2 is a decreasing function of k_1. The same conclusion follows if we assume that the Mean curvature H is constant. This allows us to state

Theorem 1a. *Let S be an ovaloid with Mean curvature H constant. Then S is a sphere.*

The proof is entirely analougous to that of Theorem 1.
Actually, the argument applies whenever $k_2 = f(k_1)$, where f is a decreasing function of k_1.
More precisely we have

Theorem 1b. *Let S be an ovaloid. If there exists a relation $k_2 = f(k_1)$ in S, where f is a decreasing function of $k_1, k_1 \geq k_2$, then S is a sphere.*
Cf. [2, page 322].

0.4 chapeter 4

In this Chapter our aim is to prove the following **Hilbert's theorem** which states that, a complete surface in $\mathbb{R}^3$, of constant negative curvature does not exist.

> **THEOREM:** *A complete geometric surface S of constant negative Gaussian curvature cannot be isometrically immersed in $\mathbb{R}^3$.*

Definition 6 A *geometric surface* is an abstract surface S together with the choice of an inner product $<,>_p$ at each $T_p(S)$, $p \in S$, which varies differentiably with p in the following sense.
For some (and hence all) parametrization $\mathcal{X} : U \longrightarrow S$ around p, the functions

$$E(u,v) =< \mathcal{X}_u, \mathcal{X}_u >, \ F(u,v) =< \mathcal{X}_u, \mathcal{X}_v >, \ G(u,v) =< \mathcal{X}_v, \mathcal{X}_v >,$$

are differentiable function in U.

Definition 7 An *Abstract surface* (differentiable manifold of dimension 2) is a set S toghether with a family of one-to-one maps: $\mathcal{X}_\alpha : U_\alpha \longrightarrow S$ of open sets $U_\alpha \subset \mathbb{R}^2$ into S such that:

1. $\bigcup_\alpha \mathcal{X}_\alpha(U_\alpha) = S$.
2. For each pair α, β with $\mathcal{X}_\alpha(U_\alpha) \cap \mathcal{X}_\beta(U_\beta) = W \neq \emptyset$, we have that $\mathcal{X}_\alpha^{-1}(W), \mathcal{X}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^2$, and $\mathcal{X}_\beta^{-1} \circ \mathcal{X}_\alpha$, $\mathcal{X}_\alpha^{-1} \circ \mathcal{X}_\beta$ are differentiable maps.

An example of abstract surface, that is olso a complete surface of constant negative Gaussian curvature, is the Hyperbolic plane $\mathcal{H}$.

Let $S = \mathbb{R}^2$ be a plane with coordinates (u,v) and define an inner product at each point $q = (u,v) \in \mathbb{R}^2$ by setting

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = E = 1; \qquad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = F = 0; \qquad \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = G = e^{2u}.$$

$\mathbb{R}^2$ with this inner product is a geometric surface $\mathcal{H}$ called the *hyperbolic plane*. The geometry of $\mathcal{H}$ is different from the usual geometry of $\mathbb{R}^2$. For istance the Gaussian curvature of $\mathcal{H}$ is $K \equiv -1$.

Actually the geometry of $\mathcal{H}$ is an exact model for the non-euclidean geometry of Lobachewski, in which all the axioms of Euclid, except the axiom of parallels are assumed.

To prove the main theorem we shall start with some observations. By multiplying the inner product by a constant factor, we may assume that the Gaussian curvature $K \equiv -1$. Moreover, since $exp_p : T_p(S) \longrightarrow S$ is a local diffeomorphism, cf. [2, page 367], it induces an inner product in $T_p(S)$. We choose the unique inner product that makes exp_p a local isometry. Denote by S' the geometric surface $T_p(S)$ with this inner product. If $i : S' \longrightarrow \mathbb{R}^3$ is an immersion, then $\varphi = i \circ exp_p : S' \longrightarrow \mathbb{R}^3$ is an isometric immersion. Thus we are reduced to proving that there exists no isometric immersion $\varphi : S' \longrightarrow \mathbb{R}^3$ of a plane S' with an inner product such that $K \equiv -1$.

Then we will need a series of results:

Lemma 2 *The area of $\varphi(S')$ is infinite.*

Lemma 3 *For each $p \in \varphi(S')$ there is a parametrization $\mathcal{X} : U \subset \mathbb{R}^2 \longrightarrow \varphi(S')$, $p \in \mathcal{X}(U)$, such that the coordinate curves of $\mathcal{X}$ are the asymptotic curves of $\mathcal{X}(U) = V'$ and form a Tchebishef net (we shall express this by saying that the asymptotic curves of V' form a Tchebishef net).*

Lemma 4 *Let $V' \subset \varphi(S')$ be a coordinate neighborhood of $\varphi(S')$ such that the coordinate curves are the asymptotic curves in V'. Then the area A of any quadrilater formed by the coordinate curves is smaller than 2π.*

So far the considerations have been local. We shall now define a map $\mathcal{X} : \mathbb{R}^2 \longrightarrow \varphi(S')$ that is a parametrization of the entire $\varphi(S')$. The map $\mathcal{X}$ is defined as follows. First we have to fix a point $O \in \varphi(S')$ and choose orientations on the asymptotic curves passing through O.

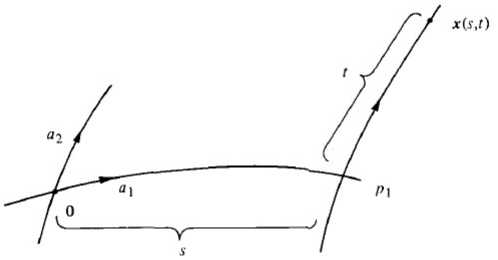

Figure 3: $\mathcal{X} : \mathbb{R}^2 \longrightarrow \varphi(S')$.

Then we have to make a define choice of one of these asymptotic curves, to be called a_1, and denote the other one by a_2. For each $(s,t) \in \mathbb{R}^2$, lay off on a_1 a length equal to s starting from O. Let p' the point thus obtained. Through p' there pass two asymptotic curves, one of which is a_1. Then we have to choose the other one and give it the orientation obtained by the continuous extension, along a_1, of the orientation of a_2. Finally, over this oriented asymptotic curve, we have to lay off a length equal to t starting from p'. The point so obtained is $\mathcal{X}(s,t)$ (Fig.3).

After showing, through a series of lemmas that $\mathcal{X}$ is a parametrization of the entire $\varphi(S')$, we will end the Chapter with the proof of the theorem and with an example of a regular suface of constant, negative Gaussian curvature, which is not-complete. That is the Pseudosphere.

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