PER LA PROSSIMA VOLTA (LUNEDÌ 12/10)

- Principio di induzione. Dimostrare

$$\sum_{k=1}^{n} (2k-1) = n^2$$

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3}$$

- Sup/Inf/Max/min

$$B = \left\{ \frac{3n+2}{n} : n \in \mathbb{N} \right\}$$

$$C = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$$ Studia per n pari e n dispari

- scrivere nella forma $z = a + ib$:

$$\frac{1}{i} \quad , \quad \frac{1-i}{1+i} \quad , \quad 13 \cdot \frac{1+i}{2-3i}$$

Soluzioni

- Principio di induzione

i) $$\sum_{k=1}^{n} (2k-1) = n^2$$

① $P(n=1)$: $$\sum_{k=1}^{1} (2k-1) = (1)^2$$

$$2(1) - 1 = (1)$$

$$1 = 1 \quad \text{vera}$$

② $P(n+1)$: $$\sum_{k=1}^{n+1} (2k-1) = (n+1)^2$$

$$\sum_{k=1}^{n} (2k-1) + (2(n+1)-1) = (n+1)^2$$

$$n^2 + 2n + 2 - 1 = (n+1)^2$$

$$n^2 + 2n + 1 = (n+1)^2$$

$$(n+1)^2 = (n+1)^2 \quad \text{vera}$$

ii) $$\sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3}$$

① $P(n=1)$ $$\sum_{k=1}^{1} (\quad 1)^2 = \frac{1(4\cdot(1)^2-1)}{3}$$

$$[2(1)-1]^2 = \frac{3}{3}$$

$$1 = 1 \quad \text{vera}$$

→

② $P(n+1)$: $\sum_{k=1}^{n+1}(2k-1)^2 = \frac{(n+1)[4(n+1)^2-1]}{3}$

$\sum_{k=1}^{n}(2k-1)^2 + [2(n+1)-1]^2 = \frac{(n+1)}{3}[4(n^2+2n+1)-1]$

$\frac{n}{3}(4n^2-1)+[2n+2-1]^2 = \frac{(n+1)}{3}[4n^2+8n+4-1]$

$\frac{1}{3}(4n^3-n)+(2n+1)^2 = \frac{(n+1)}{3}[4n^2+8n+3]$

$\frac{1}{3}(4n^3-n)+4n^2+4n+1 = \frac{(n+1)}{3}[4n^2+8n+3]$

$\frac{1}{3}[4n^3-n+12n^2+12n+3] = \frac{1}{3}(n+1)(4n^2+8n+3)$

$\frac{1}{3}[4n^3+12n^2+11n+3] = \frac{1}{3}[4n^3+4n^2+8n^2+8n+3n+3]$

$\frac{1}{3}[4n^3+12n^2+11n+3] = \frac{1}{3}[4n^3+12n^2+11n+3]$

vera!

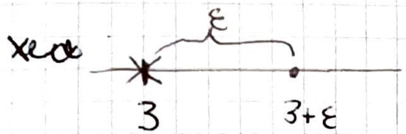
sup/inf/max/min

i) $B = \left\{x = \frac{3n+2}{n} : n \in \mathbb{N}\right\}$ $\qquad \mathbb{N} = \{1, 2, 3 \dots\}$

$x = \frac{3n+2}{n} = 3 + \frac{2}{n}$

$B = \Big\{ \underset{n=1}{5}, \underset{n=2}{4}, \underset{n=3}{3+\frac{2}{3} = \frac{11}{3}}, \underset{n=4}{3+\frac{2}{4} = \frac{14}{4} = \frac{7}{2}}, \dots, 3+\frac{2}{n} \Big\}$

per n grande $\quad x = 3 + \frac{2}{n} = 3 + \varepsilon \qquad \varepsilon = \frac{2}{n}$ piccolo positivo



MAI $x = 3$

$\sup B = \max B = 5$

$\inf B = 3$

ii) $C = \left\{x = \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

n pari $\quad x = \frac{1}{n} \qquad E_1 = \left\{\frac{1}{n} \quad n \in \mathbb{N} \text{ pari}\right\}$

n dispari $\quad x = -\frac{1}{n} \qquad E_2 = \left\{-\frac{1}{n} \quad n \in \mathbb{N} \text{ dispari}\right\}$

che possiamo scrivere come:

$E_1 = \left\{\frac{1}{2k} : k \in \mathbb{N}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

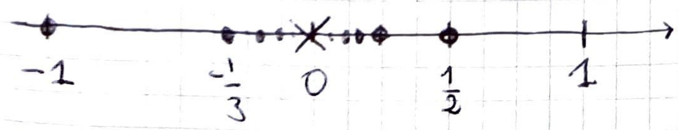
$E_2 = \left\{-\frac{1}{2k-1} : k \in \mathbb{N}\right\} = \left\{-1, -\frac{1}{3}, -\frac{1}{5}, \dots\right\}$

C lo vediamo come unione di E_1 ed E_2

$C = E_1 \cup E_2$

$\sup C = \max C = \frac{1}{2}$

$\inf C = \min C = -1$



Numeri complessi

i) $\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = \frac{-i}{1} = -i$

ii) $\frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{(1-i)^2}{(1)^2+(1)^2} =$

$= \frac{1-2i+i^2}{2} = \frac{1}{2}(1-2i-1) = -i$

iii) $13\,\frac{1+i}{2-3i} = 13\,\frac{1+i}{2-3i} \cdot \frac{2+3i}{2+3i} =$

$= 13\,\frac{(1+i)(2+3i)}{(2)^2+(3)^2} = \frac{13(2+3i+2i+3i^2)}{4+9} =$

$= \frac{\cancel{13}\,(2+5i+3(-1))}{\cancel{13}} =$

$= 2+5i-3 = -1+5i$

12/10/2020

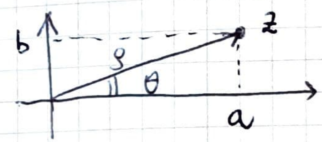
Numeri complessi

- corretto es. iii) per casa

Rappresentazione algebrica: $z = a+ib$

" trigonometrica: $z = \rho e^{i\theta} =$

$= \rho(\cos\theta + i\sin\theta)$



← rappresentazione geometrica

per passare tra le due rappresentazioni:

da (a,b) a (ρ,θ): $\begin{cases} \rho = \sqrt{a^2+b^2} \\ \cos\theta = a/\rho \\ \sin\theta = b/\rho \end{cases}$

da (ρ,θ) a (a,b): $\begin{cases} a = \rho\cos\theta \\ b = \rho\sin\theta \end{cases}$

//

potenza n-sima di z:

$z^n = \rho^n e^{in\theta} = \rho^n(\cos(n\theta) + i\sin(n\theta))$

//

→

radici n-sime di $z = \rho(\cos\theta + i\sin\theta)$:

↳ gli w tali che $w^n = z$:

$$w_k = \rho^{1/n}\left[\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right]$$

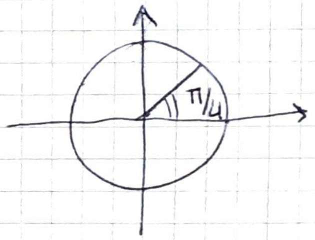
Sono n radici distinte (se $\rho \neq 0$) e corrispondono ai valori $k = 0, 1, 2, \dots, n-1$

$\boxed{4.4}$ a. scrivere in forma trigonometrica:

$z = 1 + i$

$\begin{cases} a = 1 \\ b = 1 \end{cases}$ $\quad \rho = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$$\begin{cases} \cos\theta = \dfrac{a}{\rho} = \dfrac{1}{\sqrt{2}} \\ \sin\theta = \dfrac{b}{\rho} = \dfrac{1}{\sqrt{2}} \end{cases} \Rightarrow \theta = \frac{\pi}{4} + 2k\pi$$



$$z = \sqrt{2}\, e^{i\left(\frac{\pi}{4}+2k\pi\right)} = \sqrt{2}\left[\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right]$$

$\boxed{4.5}$ b scrivere in forma algebrica:

$\rho = 3, \ \theta = -\frac{\pi}{2} \quad : \quad z = 3e^{-i\frac{\pi}{2}}$

$$\begin{cases} a = \rho\cos\theta = 3\cos(-\pi/2) = 0 \\ b = \rho\sin\theta = 3\sin(-\pi/2) = 3\cdot(-1) = -3 \end{cases}$$

$\rightarrow z = a + ib = -3i$

$\boxed{4.6}$ verificare che la potenza sesta di $z = \frac{1}{2}(\sqrt{3}+i)$ è -1.

passo nella rappresentazione (θ, ρ):

$$\rho = \sqrt{a^2+b^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

$$\begin{cases} \operatorname{sen}\theta = \dfrac{b}{\rho} = \dfrac{1/2}{1} = \dfrac{1}{2} \\ \cos\theta = \dfrac{a}{\rho} = \dfrac{\sqrt{3}/2}{1} = \dfrac{\sqrt{3}}{2} \end{cases} \Rightarrow \theta = \frac{\pi}{6} + 2k\pi$$

prendiamo $k = 0$: $\theta = \frac{\pi}{6}$: $z = \rho e^{i\theta} = e^{\pi/6\, i}$

$$z^6 = \left(e^{i\frac{\pi}{6}}\right)^6 = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i(0) = -1$$

4.21 risolvere: $4z^2 - 4z + 3 - 2\sqrt{3}\,i = 0$

Equazione di II grado: $az^2 + bz + c = 0$

$a \neq 0$. Ha due soluzioni (distinte se $\Delta = b^2 - 4ac \neq 0$):

$$z_\pm = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Nota: $\sqrt{\Delta}$ è nel campo dei complessi

$\sqrt{\Delta} = \Delta^{1/2}$ $\Rightarrow$ vedi radici n-sime di z

Nel nostro caso:

$$4z^2 - 4z + 3 - 2\sqrt{3}\,i = 0$$

$a = 4$, $b = -4$, $c = 3 - 2\sqrt{3}\,i$

$$z_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} =$$

$$= \frac{4 \pm \sqrt{16 - 4\cdot 4\cdot(3 - 2\sqrt{3}\,i)}}{8} =$$

$$= \frac{4 \pm \sqrt{16 - 16(3 - 2\sqrt{3}\,i)}}{8} =$$

16 a fattor comune

$$= \frac{4 \pm \sqrt{16\,[1 - (3 - 2\sqrt{3}\,i)]}}{8} =$$

$16 = 4^2$

$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b}$

$$= \frac{4 \pm 4\sqrt{1 - 3 + 2\sqrt{3}\,i}}{8} = \frac{4}{8}\left(1 \pm \sqrt{-2 + 2\sqrt{3}\,i}\right) =$$

$$= \frac{1}{2}\left(1 \pm \underbrace{\sqrt{-2 + 2\sqrt{3}\,i}}_{\text{questo lo dobbiamo mettere nella forma } a+ib}\right) \qquad *$$

$$\sqrt{-2 + 2\sqrt{3}\,i} = (-2 + 2\sqrt{3}\,i)^{1/2} = (v)^{1/2}$$

$\nwarrow$ complesso

$v = -2 + 2\sqrt{3}\,i = a + ib$ troviamo (ρ, θ):

$$\rho = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$$

$$\begin{cases} \cos\theta = \dfrac{a}{\rho} = -\dfrac{2}{4} = -\dfrac{1}{2} \\[2ex] \sin\theta = \dfrac{b}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \end{cases} \Rightarrow \theta = \frac{2}{3}\pi + 2k\pi$$

radici distinte per $k = 0, 1$

consideriamo $k = 0$:

$$v = \rho\, e^{i\theta_{k=0}} = 4\, e^{i\frac{2}{3}\pi}$$

$$(v)^{1/2} = \left(4\, e^{i\frac{2}{3}\pi}\right)^{\frac{1}{2}} = 2\, e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right) =$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + \sqrt{3}\,i$$

QUINDI ABBIAMO TROVATO CHE

$$\sqrt{-2 + 2\sqrt{3}\,i} = 1 + \sqrt{3}\,i$$

ora possiamo sostituirlo in $z_\pm$ (eq $*$)

$$z_\pm = \frac{1}{2}\left(1 \pm \sqrt{-2+2\sqrt{3}\,i}\right) =$$

$$= \frac{1}{2}\left[1 \pm (1+\sqrt{3}\,i)\right]$$

$$z_+ = \frac{1}{2}\left[1 + (1+\sqrt{3}\,i)\right] = \frac{1}{2}\left[2+\sqrt{3}\,i\right] =$$

$$= 1 + \frac{\sqrt{3}}{2}\,i$$

$$z_- = \frac{1}{2}\left[1 - (1+\sqrt{3}i)\right] = \frac{1}{2}\left[-\sqrt{3}\,i\right] =$$

$$= -\frac{\sqrt{3}}{2}\,i$$

z_+ e z_- sono le soluzioni dell'eq. iniziale

provate con $K=1$ ...

PER GIOVEDÌ 15/10

- Scrivere in rappresentazione trigonometrica:
  $(2-2i)$, $(\sqrt{3}+i)$, $(-1+i\sqrt{3})$
- Scrivere in rappresentazione algebrica:
  $(\rho=1,\ \theta=\frac{\pi}{2})$; $(\rho=4,\ \theta=\frac{\pi}{3})$; $(\rho=\sqrt{2},\ \theta=-\frac{\pi}{4})$;
  $(1+i)^8$; $\left(\frac{1}{i}\right)^4$; $(1-i)^{12}$
- trovare le radici terze di: 8 e -8
- Ripassare sin e cos degli ~~angoli~~ angoli noti
- Ripassare i logaritmi

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