Soluzioni degli esercizi per il 5 novembre 2020

giovedì 5 novembre 2020 16:12

Calcolare i limiti delle seguenti successioni:

1) $\left(\frac{n^2+n}{n^2+n+1}\right)^{n^2}$ cerco di scriverlo come $\Bigg[\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{\substack{\text{se } a_n\to\pm\infty \text{ questo} \\ \text{è limite notevole} \\ =e}}\Bigg]^{\frac{n^2}{a_n}}$

dovrò calcolare il limite

$$\frac{n^2+n}{n^2+n+1} = 1+\frac{1}{a_n}$$

$$\frac{n^2+n}{n^2+n+1} - 1 = \frac{1}{a_n} \qquad \frac{\cancel{n^2}+\cancel{n}-\cancel{n^2}-\cancel{n}-1}{n^2+n+1} = \frac{1}{a_n}$$

$$\frac{-1}{n^2+n+1} = \frac{1}{a_n} \Rightarrow a_n = -(n^2+n+1) \qquad \text{quindi:}$$

$$\left(\frac{n^2+n}{n^2+n+1}\right)^{n^2} = \left[\left(1+\frac{1}{a_n}\right)^{a_n}\right]^{\frac{n^2}{a_n}} \qquad \text{qui } a_n = -(n^2+n+1)$$

i) controllo che $a_n \to \pm\infty$ per $n\to\infty$ per usare il limite notevole

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -(n^2+n+1) = -\infty \qquad \text{ok}: \left(1+\frac{1}{a_n}\right)^{a_n} \to e$$

ii) calcolo il valore del limite dell'esponente n^2/a_n:

$$\lim_{n\to\infty} \frac{n^2}{a_n} = \frac{n^2}{-(n^2+n+1)} = \lim_{n\to\infty} \frac{-1}{(1+1/n+1/n^2)} = -1$$

quindi mettendo insieme:

$$\lim_{n\to\infty} \left(\frac{n^2+n}{n^2+n+1}\right) = \lim_{n\to\infty} \left[\left(1+\frac{1}{a_n}\right)^{a_n}\right]^{\frac{n^2}{a_n}} = [e]^{-1} = e^{-1} = 1/e$$

2) $\frac{n!-(n+1)!}{n^2 e^n}$

$$\frac{n!-(n+1)!}{n^2e^n} = \frac{n!-(n+1)n!}{n^2e^n} = \frac{n!(1-(n+1))}{n^2e^n} = \frac{-n!\,n}{n^2e^n} = -\frac{n!}{ne^n} =$$

$$= -\frac{n(n-1)!}{n\,e^n} = -\frac{(n-1)!}{e^n} = -\frac{(n-1)!}{e^{n+1-1}} = -\frac{(n-1)!}{e^1 e^{n-1}} = -\frac{(n-1)!}{e\,e^{n-1}}$$

lim. notevole $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ se $a > 1$ noi abbiamo ($y = n-1$) $\lim_{y\to\infty} -\frac{y!}{e\,e^y} = \lim_{y\to\infty} -\frac{1}{e}\,\frac{1}{e^y/y!}$

$$= \lim_{y\to\infty}\left(-\frac{1}{e}\right)\cdot\frac{1}{e^y/y!} = \left(-\frac{1}{e}\right)\cdot\frac{1}{0} = \left(-\frac{1}{e}\right)\cdot\infty = -\infty$$

lim. notevole $\to 0$ $(e > 1)$

$\rightsquigarrow$ con il criterio del rapporto per le successioni?

3) $$\left(\frac{n^2-n}{n^2-n+3}\right)^n = \left(1+\frac{1}{a_n}\right)^n = \left[\left(1+\frac{1}{a_n}\right)^{a_n}\right]^{\frac{n}{a_n}}$$

$$\frac{n^2-n}{n^2-n+3} = 1+\frac{1}{a_n} \qquad \frac{\cancel{n^2}-\cancel{n}-\cancel{n^2}+\cancel{n}-3}{n^2-n+3} = \frac{1}{a_n} \qquad \frac{-3}{n^2-n+3} = \frac{1}{a_n} \to a_n = -\frac{1}{3}(n^2-n+3)$$

i) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[-\frac{1}{3}(n^2-n+3)\right] = \lim_{n\to\infty}\left[-\frac{1}{3}n^2(1+1/n+3/n^2)\right] = -\infty$$

allora $\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$

ii) $$\lim_{n\to\infty}\frac{n}{a_n} = \lim_{n\to\infty}\frac{n}{(-\frac{1}{3})(n^2-n+3)} = \lim_{n\to\infty}\frac{-3}{n-1+3/n} = \frac{-3}{\infty} = 0$$

quindi: $$\lim_{n\to\infty}\left(\frac{n^2-n}{n^2-n+3}\right)^n = \lim_{n\to\infty}\left(\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{\to e\ [\text{punto i)}]}\right)^{\overset{\frac{n}{a_n}}{\to 0\ [\text{punto ii)}]}} = e^0 = 1$$

4) $$\sqrt[n]{2^n+1}$$

proprietà: $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

$a_n = 2^n+1$

$a_{n+1} = 2^{n+1}+1$

$$\lim_{n\to\infty}\sqrt[n]{2^n+1} = \lim_{n\to\infty}\frac{2^{n+1}+1}{2^n+1} = \lim_{n\to\infty}\frac{\cancel{2^n}(2+1/2^n)}{\cancel{2^n}(1+1/2^n)} = 2$$

5) $$\frac{(n^3+1)^n}{(n+1)^{3n}}$$

$A^3+B^3 = (A+B)(A^2-AB+B^2)$

impongo io

5) $\dfrac{(n^3+1)^n}{(n+1)^{3n}}$ $\qquad\qquad\qquad A^3+B^3=(A+B)(A^2-AB+B^2)$

impongo io

$$= \left[\frac{n^3+1}{(n+1)^3}\right]^n = \left[\frac{(n+1)(n^2-n+1)}{(n+1)^3}\right]^n = \left[\frac{n^2-n+1}{(n+1)^2}\right]^n = \left[1+\frac{1}{a_n}\right]^n \quad \star$$

$$\frac{n^2-n+1}{(n+1)^2} = 1+\frac{1}{a_n} \qquad \frac{n^2-n+1-(n+1)^2}{(n+1)^2} = \frac{1}{a_n} \qquad \frac{\cancel{n^2}-n+\cancel{1}-\cancel{n^2}-2n-\cancel{1}}{(n+1)^2} = \frac{1}{a_n}$$

$$\frac{-3n}{(n+1)^2} = \frac{1}{a_n} \longrightarrow a_n = -\frac{1}{3n}(n+1)^2$$

i) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(-\frac{1}{3}\frac{(n+1)^2}{n}\right) = -\infty \Rightarrow \lim\limits_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$

ii) $\lim\limits_{n\to\infty} \dfrac{n}{a_n} = \lim\limits_{n\to\infty} \dfrac{n\cdot n}{(-1/3)(n+1)^2} = \lim\limits_{n\to\infty} \dfrac{-3\,\cancel{n^2}}{\cancel{n^2}(1+2/n+1/n^2)} = \lim\limits_{n\to\infty}\dfrac{-3}{(1+2/n+1/n^2)} = -3$

quindi

$$\lim_{n\to\infty}\frac{(n^3+1)^n}{(n+1)^{3n}} = \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{\to e}\Bigg]^{\frac{n}{a_n}\to -3} = e^{-3} = 1/e^3$$

$\star$ Anche senza usare $A^3+B^3=(A+B)(A^2-AB+B^2)$:

$$\left(\frac{n^3+1}{n^3+3n^2+3n+1}\right)^n = \left(1+\frac{1}{a_n}\right)^n \qquad \frac{\cancel{n^3}+\cancel{1}-\cancel{n^3}-\cancel{1}-3n^2-3n}{n^3+3n^2+3n+1} = \frac{1}{a_n} \qquad \frac{-3n(n+1)}{(n+1)^3} = \frac{1}{a_n}$$

$$a_n = -\frac{1}{3}\frac{(n+1)^2}{n}$$

6) $\dfrac{n!}{3^{n+1}}$ $\qquad$ Con criterio del rapporto

$a_n = \dfrac{n!}{3^{n+1}} > 0 \qquad$ posso usare tale criterio

$$a_{n+1} = \frac{(n+1)!}{3^{(n+1)+1}} = \frac{(n+1)!}{3^{n+2}}$$

$$b_n = \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+2}}\,\frac{3^{n+1}}{n!} = \frac{(n+1)\,\cancel{n!}}{\cancel{n!}}\,\frac{\cancel{3^{n+1}}}{\cancel{3^{n+1}}\cdot 3} = \frac{1}{3}(n+1)$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3}(n+1) = +\infty \quad \Rightarrow \lim_{n\to\infty} a_n = +\infty$$

⑦ $\dfrac{n!}{n^{n/2}} = a_n$ $\qquad a_n > 0$ uso il criterio del rapporto

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)/2}}$$

$$b_n = \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{(n+1)/2}} \frac{n^{n/2}}{n!} = \frac{(n+1)\cancel{n!}}{\cancel{n!}} \frac{n^{n/2}}{(n+1)^{n/2}(n+1)^{1/2}} =$$

$$= (n+1)^{1-\frac{1}{2}} \left(\frac{n}{n+1}\right)^{n/2} = (n+1)^{1/2}\left(1-\frac{1}{n+1}\right)^{n/2} = (n+1)^{1/2}\left[\left(1-\frac{1}{n+1}\right)^{n+1}\right]^{\frac{n}{2(n+1)}}$$

$$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} \underbrace{(n+1)^{1/2}}_{+\infty} \underbrace{\left[\underbrace{\left(1-\frac{1}{n+1}\right)^{n+1}}_{e^{-1}}\right]^{\overbrace{\frac{n}{2(n+1)}}^{\frac{1}{2}}}}_{e^{-1/2}} = +\infty \cdot e^{-1/2} = +\infty$$

poiche $b_n \to +\infty$, anche $a_n \to +\infty$

Esercitazione 5 NOV

giovedì 5 novembre 2020 13:58

TEOR. CONFRONTO

(1) $\exists \nu$: $a_n \le b_n \quad \forall n > \nu$

$a_n \to \infty$ $\Longrightarrow$ $b_n \to +\infty$

(2) $a_n \le b_n$

$b_n \to -\infty$ $\Longrightarrow$ $a_n \to -\infty$

(3) $\begin{cases} a_n \le c_n \le b_n \\ a_n \to a \\ b_n \to a \end{cases}$ $\Longrightarrow$ $c_n \to a$

(4) $\begin{cases} a_n \to 0 \\ b_n \text{ limitata} \end{cases}$ $\Longrightarrow$ $a_n b_n \to 0$

- $\lim\limits_{n\to\infty} (\underbrace{n - \operatorname{sen} n}) = \lim\limits_{n\to\infty} b_n = +\infty$

$\underbrace{-1 \le \operatorname{sen} n \le 1}$

$\boxed{\operatorname{sen} n \le 1} \to \underbrace{-1 \le -\operatorname{sen} n}$ $\quad$ n - sen n

$\boxed{\underbrace{n-1}_{a_n} \le \underbrace{n - \operatorname{sen} n}_{b_n}}$

$\to a_n \le b_n$

$\lim\limits_{n\to\infty} n-1 = \lim\limits_{n\to\infty} a_n = +\infty \quad \Longrightarrow b_n \to +\infty$

- $\lim\limits_{n\to\infty} \underbrace{\cos(n)}_{b_n} \cdot \underbrace{\left(\log(\sqrt{n}-1) - \log\sqrt{n-1}\right)}_{a_n}$

$a_n = \log\left(\dfrac{\sqrt{n}-1}{\sqrt{n-1}}\right) = \log\left[\dfrac{\sqrt{n}\,(1-1/\sqrt{n})}{\sqrt{n}\,\sqrt{1-1/n}}\right]$ $\qquad \sqrt{n-1} = \sqrt{n(1-1/n)} = \sqrt{n}\,\sqrt{1-1/n}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \log\left[\dfrac{1-1/\sqrt{n}}{\sqrt{1-1/n}}\right] = \log(1) = 0$

$b_n = \cos(n)$ è limitata $\quad |\cos(n)| \le 1$

$-1 \le \cos n \le 1$

$\lim\limits_{n\to\infty} \underbrace{b_n \cdot a_n} = 0$ $\qquad$ teor. 4

limitata $\qquad a_n \to 0$

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limiti di funzioni

* verificare $\lim_{x\to\infty} (\underline{x - 2\sqrt{x}}) = +\infty$

$\forall M>0 \quad \exists k>0 : f(x) > M \quad \forall x \in X : x > k$

$\boxed{x - 2\sqrt{x} > M}$ diseq. in x

$t = \sqrt{x}$ $\quad t^2 - 2t - M > 0$

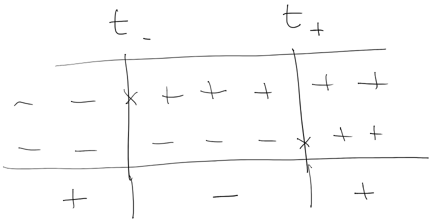
$t = \frac{2 \pm \sqrt{4+4M}}{2} = 1 \pm \sqrt{1+M}$

$t_- = 1-\sqrt{1+M} < 0$

$t_+ = 1+\sqrt{1+M} > 0$

segno $(t-t_-)(t-t_+)>0$

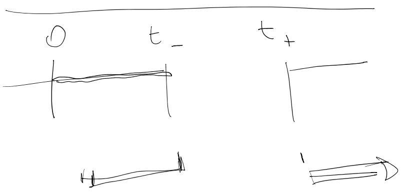
$\Rightarrow \frac{2\pm\sqrt{4(1+M)}}{2} = \frac{2\pm\sqrt{4}\cdot\sqrt{1+M}}{2} = \frac{\cancel{2}\pm \cancel{2}\sqrt{1+M}}{\cancel{2}} = 1\pm\sqrt{1+M}$



$\uparrow$ ~~$t<t_-$~~ $\qquad \uparrow t > t_+$

$\boxed{t=\sqrt{x}}$ $\quad t>t_+ : \sqrt{x} > t_+ \quad \boxed{x > t_+^2} \Rightarrow = k$

$x > (1+\sqrt{1+M})^2$ allora $f(x) > M$



$0 < t < t_-$ con $t = \sqrt{x}$

$t < t_- \to \sqrt{x} < t_-$

$\boxed{x < t_-^2}$

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* $\lim_{x\to1}\left(\frac{x+1}{x-1}\right)^2 = \lim_{x\to1} f(g(x))$

$\lim_{x\to x_0} g(x) = y_0$

$\lim_{y\to y_0} f(y) = \ell$

$\downarrow$

$\exists \delta>0 : \alpha |x-x_0|<\delta \to g(x) \neq y_0$

$\Rightarrow \lim_{x\to x_0} f(g(x)) = \ell$

$g(x) = \frac{x+1}{x-1}$

$f(y) = y^2$

$\lim_{x\to1}(g(x))^2 = \lim_{y\to\ldots} y^2$

$\boxed{\lim_{x\to1} g(x)} = \lim_{x\to1}\frac{x+1}{x-1} = \frac{2}{0} = +\infty$

$\lim_{y\to\infty} f(y) = \lim_{y\to\infty} y^2 = \underline{\infty}$

$\lim_{x\to x_0} f(g(x)) =$ $\quad g(x) = \frac{x+1}{x-1} \quad f(y) = y^2$

$\lim_{x\to x_0} f(g(x)) = \lim_{y\to y} f(y)$

$\lim_{x\to x_0} f(g(x)) =$ $\quad g(x) = \frac{x+1}{x-1}$, $\quad f(y) = y^2$ $\quad\Big|\quad \lim_{x\to x_0} f(g(x)) = \lim_{y\to y_0} f(y)$

$= \lim_{y\to y_0} y^2$ con $\boxed{y_0 = \lim_{x\to x_0} g(x)} = \lim_{x\to 1} \frac{x+1}{x-1} = \infty$

$= \lim_{y\to\infty} y^2 = +\infty$

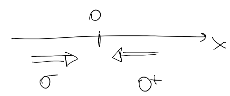
funz. comp.

$\lim_{x\to 0} \operatorname{arctg}^4 \frac{1}{x}$ $\qquad$ ① $\lim_{x\to 0} \frac{1}{x}$ NON ESISTE

$= \lim_{x\to 0} f(g(x))$

$g(x) = \frac{1}{x}$

$f(y) = \operatorname{arctg}^4(y)$

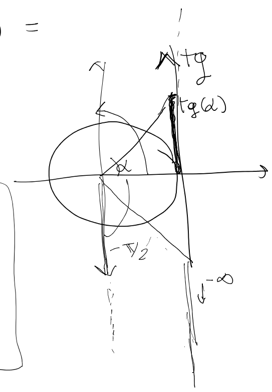


$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty$

i due y_0

$= \lim_{y\to y_0} f(y) = \lim_{y\to y_0} \operatorname{arctg}^4(y) =$



① $= \lim_{y\to -\infty} \operatorname{arctg}^4(y) = \left(-\frac{\pi}{2}\right)^4$

$\operatorname{arctg}(-\infty) = -\pi/2$

② $= \lim_{y\to\infty} \operatorname{arctg}^4(y) = \left(\frac{\pi}{2}\right)^4$

$\operatorname{arctg}(+\infty) = \pi/2$

$= \left(\frac{\pi}{2}\right)^4 = \pi^4/2^4 = \pi^4/16$

Calcolare:

* $\lim_{x\to +\infty} \operatorname{arctg} \underbrace{\log^3 x} = \lim_{y\to +\infty} \operatorname{arctg} y = \frac{\pi}{2}$

$\log^3 x \to +\infty$

* $\lim_{x\to +\infty} \frac{\log(x^2+1)}{2^x} = \lim_{x\to +\infty} \frac{\log(x^2+1)}{x^2+1} \cdot \frac{x^2+1}{2^x} = 0$

* $\lim\limits_{x\to+\infty} \dfrac{\log(x^2+1)}{2^x} = \lim\limits_{x\to\infty} \underbrace{\dfrac{\log(x^2+1)}{x^2+1}}_{\downarrow} \cdot \dfrac{x^2+1}{2^x} = 0$

LIM. NOTEV.

$$\frac{\log x}{x^b} \xrightarrow[x\to\infty]{} 0 \quad \forall b>0 \qquad \lim_{y\to\infty}\left(\frac{\log y}{y}\right) = 0$$

$y = x^2+1$

Se $x\to\infty$, $y\to\infty$

$$\frac{x^b}{a^x} \xrightarrow[x\to\infty]{} 0 \quad \forall b>0,\ a>1$$

$b=2$, $a=2$: $\dfrac{x^2}{2^x} \to 0$

* $\lim\limits_{x\to-\infty}\left(\dfrac{2x+3}{2x}\right)^{1-x} = \lim\limits_{x\to-\infty}\left(1+\dfrac{3}{2x}\right)^{1-x} =$

$= \lim\limits_{x\to-\infty}\left(1+\dfrac{3/2}{x}\right)^{1-x} =$

ANALOGO

$$\left(1+\frac{b}{a_n}\right)^{a_n} \xrightarrow[a_n\to\pm\infty]{} e^b$$

$= \lim\limits_{x\to-\infty}\left(1+\dfrac{(-3/2)}{-x}\right)^{\frac{1-x}{-x}\cdot(-x)} =$

$= \lim\limits_{x\to-\infty}\left[\underbrace{\left(1+\dfrac{(-3/2)}{-x}\right)^{-x}}_{e^{-3/2}}\right]^{\frac{1-x}{-x}}$ $\qquad \dfrac{1-x}{-x} \xrightarrow{x\to-\infty} 1$

* $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x} = \lim\limits_{x\to 0}\dfrac{1}{x}\log(1+x) =$

$= \lim\limits_{x\to 0}\log(1+x)^{1/x} =$ $\qquad y=\dfrac{1}{x}$

Se $x\to 0^+$, $y\to+\infty$

Se $x\to 0^-$, $y\to-\infty$

$= \lim\limits_{y\to\pm\infty}\log\left(1+\dfrac{1}{y}\right)^y = \log(e) = 1$

Esercizi per giovedì 12 novembre 2020

venerdì 6 novembre 2020 17:43

* Usando i teoremi di confronto, calcolare i limiti delle successioni:

(1) $\cos^2(n+1) - (n+1)^2$

(2) $n\left[2 - \operatorname{sen}(n^2+1)\right]$

(3) $\dfrac{3+\operatorname{sen} n}{n}$

LIMITI DI FUNZIONI (esercizi sul cap. 8)

* Utilizzando la definizione di limite, verificare:

(4) $\lim\limits_{x \to 0^+} \dfrac{x^2+x+|x|}{x} = 2$

(5) $\lim\limits_{x \to 0^-} \dfrac{x^2+x+|x|}{x} = 0$

* Studiare bene i limiti notevoli

* Calcolare i seguenti limiti:

(6) $\lim\limits_{x \to +\infty} \left(\dfrac{x+2}{x+1}\right)^x$ (metodo es. 3 per il 5 nov)

(7) $\lim\limits_{x \to 1} x^{\frac{2}{x-1}}$ $\left(y = \dfrac{2}{x-1} \ \ldots \right)$

(8) $\lim\limits_{x \to \infty} x^2\left[\lg(x^2+2) - 2\lg x\right]$

x^x $\left(e^{\lg a} = a \right)$

9 $\lim_{x \to 0^+} x^x$ $\left(e^{\lg a} = a \ .. \right)$

10 $\lim_{x \to 0} \frac{2^{3x} - 1}{x}$ ($y = 2^{3x} - 1$ + cambiamento di base ($2 \to e$) + ultimo esercizio visto in aula)

* Ripassare le funzioni trigonometriche inverse

Buon lavoro!