PER LUNEDI 21 DIC

1) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}-x}$

2) $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x}}$

3) $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2-2}}$

4) $\displaystyle\int \frac{dx}{\sqrt{2x^2+3x-2}}$

5) $\displaystyle\int \frac{x}{\sqrt{x^2-3x+2}}\,dx$

6) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}}$

hints: $\sqrt{\ } = t \pm x$

$\cdot\ \alpha t \pm \beta x$

SOLUZIONI ESERCIZI PER IL 21 DIC

1) $\int \frac{dx}{\sqrt{x^2-1}-x}$ $\qquad x+t=\sqrt{x^2-1}$

$x=-\frac{(t^2+1)}{2t} \qquad dx=-\frac{t^2-1}{2t^2}\,dt$

$$=-\int \frac{1}{t}\,\frac{t^2-1}{2t^2}\,dt=-\frac{1}{2}\int\left(\frac{t^2}{t^3}-\frac{1}{t^3}\right)dt=$$

$$=-\frac{1}{2}\int\left(\frac{1}{t}-\frac{1}{t^3}\right)dt=-\frac{1}{2}\lg|t|-\frac{1}{4t^2}+c=$$

$$=-\frac{1}{2}\lg\left|\sqrt{x^2-1}-x\right|-\frac{1}{4}\,\frac{1}{(\sqrt{x^2-1}-x)^2}+c$$

2) $\int \frac{dx}{x\sqrt{x^2+2x}}$ $\qquad x+t=\sqrt{x^2+2x}$

$x=\frac{t^2}{2(1-t)} \qquad dx=\frac{2t-t^2}{2(1-t)^2}\,dt$

$$\sqrt{x^2+2x}=x+t=\frac{-t^2+2t}{2(1-t)}$$

$$=\int \frac{2(1-t)}{t^2}\,\frac{2(1-t)}{-t^2+2t}\,\frac{2t-t^2}{2(1-t)^2}\,dt=\int\frac{2}{t^2}\,dt=$$

$$=-\frac{2}{t}+c=-\frac{2}{\sqrt{x^2+2x}-x}+c$$

3) $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$ $\qquad \sqrt{x^2-2}=x+t$

$x=-\frac{t^2+2}{2t} \qquad dx=\frac{2-t^2}{2t^2}\,dt$

$\sqrt{x^2-2}=\frac{t^2-2}{2t} \qquad x-1=\frac{-(t^2+2t+2)}{2t}$

$$= \int \frac{-2t}{t^2+2t+2}\,\frac{2t}{t^2-2}\,\frac{-t^2+2}{2t^2}\,dt = \int \frac{2\,dt}{t^2+2t+2} =$$

$$= \int \frac{2}{(t+1)^2+1}\,dt = 2\operatorname{arctg}(t+1) + c =$$

$$= 2\operatorname{arctg}\left(\sqrt{x^2-2} - x + 1\right) + c$$

4) $$\int \frac{dx}{\sqrt{2x^2+3x-2}}$$

$$\sqrt{2x^2+3x-2} = \sqrt{2}\,(x+t)$$

$$x = \frac{2(t^2+1)}{3-4t}$$

$$dx = \frac{4(-2t^2+3t+2)}{(3-4t)^2}\,dt$$

$$\sqrt{2x^2+3x-2} = \sqrt{2}\,(x+t) = \sqrt{2}\,\frac{-2t^2+3t+2}{3-4t}$$

$$= \int \frac{1}{\sqrt{2}}\,\frac{\cancel{3-4t}}{\cancel{-2t^2+3t+2}}\,\frac{4(\cancel{-2t^2+3t+2})}{(3-4t)^{\cancel{2}}}\,dt =$$

$$= \frac{4}{\sqrt{2}} \int \frac{dt}{3-4t} = -\frac{1}{\sqrt{2}} \lg|3-4t| + c =$$

$$= -\frac{1}{\sqrt{2}} \lg\left|3 - 4\left(\frac{\sqrt{2x^2+3x-2}}{\sqrt{2}} - x\right)\right| + c =$$

$$= -\frac{1}{\sqrt{2}} \lg\left|3 - 2\sqrt{2}\sqrt{2x^2+3x-2} - 4x\right| + c$$

5) $$\int \frac{x}{\sqrt{x^2-3x+2}}\,dx$$

$$t + x = \sqrt{x^2-3x+2}$$

$$x = \frac{2-t^2}{2t+3}$$

$$dx = \frac{-2(t^2+3t+2)}{(2t+3)^2}\,dt$$

$$\sqrt{\ } = \frac{t^2+3t+2}{2t+3}$$

$$= \int \frac{2-t^2}{\cancel{2t+3}}\,\frac{\cancel{2t+3}}{\cancel{t^2+3t+2}}\,\frac{-2(\cancel{t^2+3t+2})}{(2t+3)^2}\,dt =$$

$$= -2\int \frac{2-t^2}{(2t+3)^2}\,dt = -2\int \frac{2dt}{(2t+3)^2} + 2\int \frac{t^2}{(2t+3)^2}\,dt =$$

divisione polinomi

$$= 2(2t+3)^{-1} + 2\int \left[\frac{1}{4} - 3\,\frac{t+3/4}{(2t+3)^2}\right] dt =$$

$$= 2(2t+3)^{-1} + \frac{t}{2} - 3\int \frac{\cancel{2}t+3/2}{(\cancel{2}t+3)^2}\,dt =$$

$$= 2(2t+3)^{-1} + \frac{t}{2} - 3\int \left[\frac{1}{2t+3} - \frac{3}{2}\,\frac{1}{(2t+3)^2}\right] dt$$

$$= 2(2t+3)^{-1} + \frac{t}{2} - \frac{3}{2}\lg|2t+3| - \frac{9}{4}(2t+3)^{-1} + c$$

$$= -\frac{1}{4}(2t+3)^{-1} + \frac{t}{2} - \frac{3}{2}\lg|2t+3| + c =$$

$$= \frac{1}{4}\left[\frac{-1}{2\sqrt{x^2-3x+2}-2x+3} + 2\sqrt{x^2-3x+2} - 2x\right] - \frac{3}{2}\lg\left|2\sqrt{x^2-3x+2}-2x+3\right| + c$$

6) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}}$ $\sqrt{x^2+x+1} = x+t$

$x = \dfrac{1-t^2}{2t-1}$

$dx = \dfrac{2(t-t^2-1)}{(2t-1)^2}dt$ $\sqrt{x^2+x+1} = \dfrac{t^2-t+1}{2t-1}$

$$= \int \frac{2t-1}{1-t^2} \frac{2t-1}{t^2-t+1} \frac{-2(t^2-t+1)}{(2t-1)^2} dt =$$

$$= -2\int \frac{1}{1-t^2} dt = 2\int \frac{1}{t^2-1} dt = 2\int \frac{1}{(t-1)(t+1)} dt =$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \lg|t-1| - \lg|t+1| + c =$$

$$= \lg\left|\sqrt{x^2+x+1} - x - 1\right| - \lg\left|\sqrt{x^2+x+1} - x + 1\right| + c$$

ESERCITAZIONE 21 DIC. 2020

① INTEGRALE 3 per oggi

② $\displaystyle\int \left(\frac{1}{\sqrt{3-x^2-2x}} \right)^3 dx$

$3 - x^2 - 2x = 4 - (x+1)^2 = 4\left(1 - \left(\frac{x+1}{2}\right)^2\right)$

$\dfrac{x+1}{2} = \cos t$ $dx = -2\,\text{sen}\,t\,dt$

$$= \int \frac{1}{8} \frac{-2\,\text{sen}\,t\,dt}{(\sqrt{1-\cos^2 t})^3} = -\frac{1}{4}\int \frac{\text{sen}\,t}{(\text{sen}\,t)^3} dt =$$

$$= -\frac{1}{4}\int \frac{dt}{\text{sen}^2 t}$$

$\text{tg}\,t = y$ $\dfrac{dy}{1+y^2} = dt$

$\dfrac{1}{\text{sen}^2 t} = 1 + \dfrac{1}{y^2} = \dfrac{y^2+1}{y^2}$

$$= -\frac{1}{4}\int \frac{y^2+1}{y^2} \frac{dy}{1+y^2} = -\frac{1}{4}\int \frac{dy}{y^2} = \frac{1}{4y} + c =$$

$$= \frac{1}{4\,\text{tg}\,t} + c = \frac{1}{4} \frac{\cos t}{\text{sen}\,t} + c =$$

$\cos t = \cos\left[\arccos \dfrac{x+1}{2}\right] = \dfrac{x+1}{2}$

$\text{sen}\,t = \sqrt{1-\cos^2 t} = \dfrac{1}{2}\sqrt{3-x^2-2x}$

$$= \frac{1}{4} \frac{x+1}{\sqrt{3-x^2-2x}} + c$$

③ $\displaystyle\int \frac{dx}{4-5\sin x}$ $\qquad t = tg\frac{x}{2}$

$\operatorname{sen}x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt$

$$= \int \frac{1}{4-\frac{10t}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{1}{2t^2-5t+2}\,dt =$$

$$= \int \frac{dt}{(t-2)(2t-1)} \qquad \frac{1}{(t-2)(2t-1)} = \frac{A}{t-2} + \frac{B}{2t-1}$$

$A = 1/3 \quad B = -2/3$

$$= \int \frac{1}{3(t-2)}\,dt - \int \frac{2}{3(2t-1)}\,dt = \frac{1}{3}\left[\lg|t-2| - \lg|2t-1|\right] + c$$

$$= \frac{1}{3}\lg\left|\frac{t-2}{2t-1}\right| + c = \frac{1}{3}\lg\left|\frac{tg\,x/2 - 2}{2\,tg\,x/2 - 1}\right| + c$$

④ Stabilire il carattere di:

$\displaystyle\int_{-3}^{1} \frac{1}{x^2}\,dx =$ $\qquad 1/x^2$ discontinua e non limitata nell'intervallo $(-3,1)$

$$= \underbrace{\int_{-3}^{0} \frac{dx}{x^2}}_{A} + \underbrace{\int_{0}^{1} \frac{dx}{x^2}}_{B}$$

Ⓐ $\displaystyle\lim_{t\to 0^+} \int_{-3}^{t} \frac{dx}{x^2} = \lim_{t\to 0^-} \left[-\frac{1}{x}\right]_{-3}^{t} = \lim_{t\to 0^-} \left(-\frac{1}{t} + \frac{1}{3}\right) = \frac{-1}{0^-} = \infty$

Ⓑ $\displaystyle\lim_{t\to 0^+} \int_{t}^{1} \frac{dx}{x^2} = \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_{t}^{1} = \lim_{t\to 0^+} \left[-1 + \frac{1}{t}\right] = \frac{+1}{0^+} = +\infty$

A e B sono divergenti, quindi anche l'integrale dato (è la somma)

⑤ $\displaystyle\int_{0}^{2} \frac{1}{(x-2)^2}\,dx$

$$= \lim_{t\to 2^-} \int_{0}^{t} \frac{1}{(x-2)^2}\,dx = \lim_{t\to 2^-} \left[-\frac{1}{t-2} - \frac{1}{2}\right] = -\frac{1}{0^-} = +\infty$$

⑥ $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \pi$ $\qquad$ verificare

in: $-1 \leq t \leq 0$

Ⓐ $\displaystyle\lim_{t\to -1^+} \int_{t}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to -1^+} (\operatorname{arcsen} 0 - \operatorname{arcsen} t) = -\left(-\frac{\pi}{2}\right) = \pi/2$

in: $0 \leq t \leq 1$

Ⓑ $\displaystyle\lim_{t\to 1^+} \int_{0}^{t} \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} (\operatorname{arcsen} t - \operatorname{arcsen} 0) = \pi/2$

$A + B = \pi/2 + \pi/2 = \pi$

(7) Stabilire il carattere:

$$\int_1^{\infty} \frac{1}{x^4(1+x^4)}\,dx$$

$$f(x) = \frac{1}{x^4(1+x^4)} \qquad f(x) < \frac{1}{x^4}$$

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3t^3} + \frac{1}{3}\right] = \frac{1}{3}$$

quindi $\int_1^{\infty} f(x)\,dx$ è convergente

(8) $$\int_1^{\infty} \frac{\sqrt{x^4+1}}{x^3}\,dx$$

$$f(x) = \frac{\sqrt{x^4+1}}{x^3} \qquad f(x) > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x}$$

$$\int_1^{\infty} \frac{dx}{x} = +\infty$$

quindi diverge anche l'integrale dato

(9) $$\int_0^{\infty} \frac{\sin^2 x}{1+x^2}\,dx$$

$$f(x) = \frac{\sin^2 x}{1+x^2} \qquad f(x) \leq \frac{1}{1+x^2}$$

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \left[\arctan t - \arctan 0\right] = \frac{\pi}{2}$$

$\Rightarrow$ l'integrale dato converge