Soluzioni esercizi per il 18 gennaio 2021

40) a) $\sum_{1}^{\infty} \frac{(2n)!}{(n!)^2}$

$$\frac{(2n+2)!}{((n+1)!)^2} \frac{(n!)^2}{(2n)!} = (2n+2)(2n+1)\left(\frac{1}{n+1}\right)^2 = 2\,\frac{(2n+1)}{n+1}$$

$$\lim_{n\to\infty} 2\,\frac{(2n+1)}{n+1} = 2\cdot 2 = 4 \quad \text{diverge} \quad \text{(RAPP.)}$$

41) $\sum_{1}^{\infty} \frac{n!}{\sqrt{(2n)!}}$

$$\frac{(n+1)!}{\sqrt{(2n+2)!}} \frac{\sqrt{(2n)!}}{n!} = n+1\sqrt{\frac{1}{(2n+2)(2n+1)}} =$$

$$= \sqrt{\frac{(n+1)^2}{2(n+1)(2n+1)}} = \sqrt{\frac{n+1}{2(2n+1)}} \longrightarrow \sqrt{\frac{1}{4}} = \frac{1}{2} < 1$$

CONV.

Nota:

$$\frac{n!}{\sqrt{(2n)!}} = \frac{1\cdot 2\cdot \ldots \cdot n}{\sqrt{1\cdot 2\cdot \ldots \cdot n\cdot (n+1)\cdot \ldots \cdot (2n-1)\cdot 2n}} =$$

(RAPP.)

$$= \sqrt{\frac{1\cdot 2\cdot \ldots \cdot n}{(n+1)\cdot (n+2)\cdot \ldots \cdot (2n)}} < 1$$

ogni termine sotto è maggiore di ognuno di quelli sopra

④ $\sum_{n=1}^{\infty} \frac{\sqrt{n+e^n} - \sqrt{n}}{1+2^n}$

$$\frac{\sqrt{n+e^n} - \sqrt{n}}{1+2^n} \cdot \frac{\sqrt{n+e^n} + \sqrt{n}}{\sqrt{n+e^n} + \sqrt{n}} = \frac{e^n}{(1+2^n)(\sqrt{n+e^n} + \sqrt{n})}$$

$$\sqrt[n]{\frac{e^n}{(1+2^n)(\sqrt{n+e^n} + \sqrt{n})}} = \sqrt[n]{\frac{e^n}{2^n(2^{-n}+1)\, e^{n/2}(\sqrt{ne^{-n}+1} + \sqrt{ne^{-n}})}}$$

$$= \frac{\sqrt{e}}{2} \frac{1}{(2^{-n}+1)(\sqrt{ne^{-n}+1} + \sqrt{ne^{-n}})} \xrightarrow[n\to\infty]{} \frac{\sqrt{e}}{2} \frac{1}{(0+1)(\sqrt{0+1}+0)} =$$

$$= \frac{\sqrt{e}}{2}$$

Poiche' $\frac{\sqrt{e}}{2} < 1$ la serie converge per il criterio della radice

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$$\sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \operatorname{arctg} n\right)$$

$$\int_0^t \left(\frac{\pi}{2} - \operatorname{arctg} x\right) dx = t\left(\frac{\pi}{2} - \operatorname{arctg} t\right) - \int \frac{t}{1+t^2}\, dt =$$

$$= t\left(\frac{\pi}{2} - \operatorname{arctg} t\right) - \frac{1}{2} \lg(1+t^2) = f(t)$$

$\lim_{t \to \infty} f(t) = +\infty$ diverge CRIT. INTEGRALE

9) Studiare $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \lg\left(\frac{n+1}{n}\right)\right)$

a) CRIT. CONFRONTO

Serie a termini positivi, infatti dalla relazione:

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} \qquad \text{(A)}$$

si ha:

$\left(1+\frac{1}{n}\right)^n < e$ applico $\lg(\cdot)$:

$$n \lg\left(1+\frac{1}{n}\right) < \lg e \rightarrow n \lg\left(1+\frac{1}{n}\right) < 1$$

da cui $\lg\left(1+\frac{1}{n}\right) < \frac{1}{n} \rightarrow \frac{1}{n} - \lg\left(1+\frac{1}{n}\right) > 0$

$\rightarrow$ termini positivi

come diseq. di destra:

$e < \left(1+\frac{1}{n}\right)^{n+1}$ applico $\lg(\cdot)$:

$$1 < (n+1) \lg\left(1+\frac{1}{n}\right) \rightarrow \lg\left(1+\frac{1}{n}\right) > \frac{1}{n+1}$$

quindi (A) corrisponde a:

$$\frac{1}{n+1} < \lg\left(1+\frac{1}{n}\right) < \frac{1}{n} \qquad \text{(B)}$$

da: $lg\left(1+\frac{1}{n}\right) > \frac{1}{n+1}$ segue:

$$\frac{1}{n} - lg\left(1+\frac{1}{n}\right) < \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

quindi la nostra serie è maggiorata da $\Sigma \frac{1}{n^2}$ che è convergente, per il criterio del confronto deve convergere

b) CRIT. CONFRONTO ASINTOTICO

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - lg\left(1+\frac{1}{n}\right)\right)$$

per $n\to\infty$: $\frac{1}{n}\to 0$: $lg(1+t) = t - \frac{t^2}{2}$ $t\to 0$

$$lg\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n} + \frac{1}{2n^2}\right) = \sum_{n=1}^{\infty}\frac{1}{2n^2} \quad \text{CONV.}$$

la nostra serie si comporta asintoticamente come $\Sigma \frac{1}{2n^2} < \infty$, quindi deve convergere

Esercitazione 18/01/2021
+ qualche esercizio per casa

② $\sum_{n=1}^{\infty} \lg\left(1+\frac{x^{2n}}{2^n+1}\right)$ $x \in \mathbb{R}$

termini positivi

$\frac{x^{2n}}{2^n+1} = \left(\frac{x^2}{2}\right)^n \frac{1}{1+\frac{1}{2^n}}$ ↷ tende a 1 per $n \to \infty$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \lg\left[1+\left(\frac{x^2}{2}\right)^n \frac{1}{1+1/2^n}\right] = \begin{cases} +\infty & x^2/2 > 1 \quad Ⓐ \\ \lg 2 & x^2/2 = 1 \quad Ⓑ \\ 0 & x^2/2 < 1 \end{cases}$$

Ⓐ Ⓑ serie diverge per $x \leq -\sqrt{2}$ e $x \geq \sqrt{2}$

Ⓒ in $-\sqrt{2} < x < \sqrt{2}$ ho $\lim_{n\to\infty} a_n = 0$

devo vedere se converge o meno

la mia serie ha stesso carattere di $\left(\frac{x^2}{2}\right)^n$

infatti

$\rightarrow \lg(1+t) \sim t$ in $t \to 0$

$$\lim_{n\to\infty} \frac{\lg\left(1+\frac{x^{2n}}{2^{n+1}}\right)}{\left(\frac{x^2}{2}\right)^n} = \lim_{n\to\infty} \frac{\frac{x^{2n}}{2^{n+1}}}{\left(\frac{x^2}{2}\right)^n} = \lim_{n\to\infty} \frac{\cancel{\left(\frac{x^2}{2}\right)^n} \frac{1}{1+1/2^n}}{\cancel{\left(\frac{x^2}{2}\right)^n}}$$

$= 1$

poiché $\Sigma \left(\frac{x^2}{2}\right)^n$ convergono (geometrica con ragione $\frac{x^2}{2} < 1$)

allora per il crit. confronto asintotico converge anche la serie data

$\Rightarrow$ la serie converge per $x \in (-\sqrt{2}, \sqrt{2})$ e diverge altrimenti

⑨ $\displaystyle\sum_{n=0}^{\infty} \frac{(6-2x)^n}{2^n (x-1)^{2n}}$

$$\frac{(6-2x)^n}{2^n (x-1)^{2n}} = \left(\frac{2(3-x)}{2(x-1)^2}\right)^n = \left(\frac{3-x}{(x-1)^2}\right)^n$$

Conv. $\left|\frac{3-x}{(x-1)^2}\right| < 1$

$$A\begin{cases} \frac{3-x}{(x-1)^2} \geq 0 \\ \frac{3-x}{(x-1)^2} < 1 \end{cases} \qquad B\begin{cases} \frac{3-x}{(x-1)^2} < 0 \\ \frac{x-3}{(x-1)^2} < 1 \end{cases}$$

Ⓐ $\begin{cases} x \leq 3 \\ \dfrac{3-x-x^2+2x-1}{(x-1)^2} < 0 \end{cases}$ $\begin{cases} x \leq 3 \\ x<-1 \cup x>2 \end{cases}$ $\begin{cases} x<-1 \cup \\ 2<x\leq 3 \end{cases}$

$\rightarrow \dfrac{-x^2+x+2}{(x-1)^2} < 0$ $\quad \dfrac{x^2-x-2}{(x-1)^2} > 0$ $\quad \dfrac{(x+1)(x-2)}{(x-1)^2} > 0$

Ⓑ $\begin{cases} x > 3 \\ \dfrac{x-3-x^2+2x-1}{(x-1)^2} < 0 \end{cases}$ $\begin{cases} x > 3 \\ \forall x \end{cases}$ $\begin{cases} x > 3 \end{cases}$

$\rightarrow \dfrac{-x^2+3x-4}{(x-1)^2} < 0$ $\quad \dfrac{x^2-3x+4}{(x-1)^2} > 0$ $\rightarrow \Delta = 9-16$

$\Delta < 0$

Sempre +

TOT : $x<-1,\ x>2$

$\rightarrow$ in questo caso $S = \sum_{n=1}^{\infty} \left(\dfrac{3-x}{(x-1)^2} \right)^n = \dfrac{1}{1-\dfrac{3-x}{(x-1)^2}} =$

$= \dfrac{(x-1)^2}{x^2-2x+1-3+x} = \dfrac{(x-1)^2}{x^2-x-2}$

1) Det. i valori del parametro α tali che l'integrale improprio

$$\int_1^2 \frac{x^2-1}{(x-1)^\alpha}\,dx$$ sia convergente

$$\frac{x^2-1}{(x-1)^\alpha} = \frac{(x-1)(x+1)}{(x-1)^\alpha} = \frac{x+1}{(x-1)^{\alpha-1}}$$

$$\int_1^2 \frac{x^2-1}{(x-1)^\alpha}\,dx = \int_1^2 \frac{x+1}{(x-1)^{\alpha-1}}\,dx =$$

$t = x-1$
$dt = dx$
$x = 2 \rightarrow t = 1$
$x = 1 \rightarrow t = 0$
$x+1 = t+2$

$$= \int_0^1 \frac{t+2}{t^{\alpha-1}}\,dt = \int_0^1 \frac{t}{t^{\alpha-1}}\,dt + 2\int_0^1 \frac{1}{t^{\alpha-1}}\,dt =$$

$$= \int_0^1 \frac{1}{t^{\alpha-2}}\,dt + 2\int_0^1 \frac{1}{t^{\alpha-1}}\,dt$$

la divergenza è in $t=0$

$$\int_0^a \frac{1}{x^b} < \infty \quad \text{se} \quad b<1$$

$$\int_h^1 \frac{1}{y^\beta}\,dy = \frac{1}{1-\beta}\, y^{1-\beta}\Big|_h^1$$ se $\beta<1$

quindi il primo integrale converge se $\alpha-2<1$ e il secondo se $\alpha-1<1$:

$\begin{cases}\alpha<3\\ \alpha<2\end{cases}$ $\rightarrow$ l'integrale di partenza converge se convergono entrambi:

questo succede per $\alpha<2$

2) Det $a, b \in \mathbb{R}$ per i quali $\int_0^{\infty} \frac{1}{x^a(4+9x)^{b+1}} dx$ converge

$$\int_0^{\infty} \frac{dx}{x^a(4+9x)^{b+1}} = \underbrace{\int_0^{1} \frac{dx}{x^a(4+9x)^{b+1}}}_{A} + \underbrace{\int_1^{\infty} \frac{dx}{x^a(4+9x)^{b+1}}}_{B}$$

(A) per $x \to 0^+$:

$$\frac{1}{x^a(4+9x)^{b+1}} \sim \frac{1}{x^a \, 4^{b+1}} \sim \frac{1}{x^a}$$

CONVERGE se $a < 1$

(B) per $x \to +\infty$:

$$\frac{1}{x^a(4+9x)^{b+1}} \sim \frac{1}{9^{b+1} x^{a+b+1}} \sim \frac{1}{x^{a+b+1}}$$

CONVERGE se $a+b+1 > 1$

globalmente converge se $\begin{cases} a < 1 \\ a+b+1 > 1 \end{cases} \rightarrow \begin{cases} a < 1 \\ b > -a \end{cases}$

3) $\int_2^{3} \frac{x(\sin(x-2))^{\alpha}}{\sqrt{x^2-4}} dx$ Det. $\alpha \in \mathbb{R}$ per i quali converge

per $x \to 2^+$: $\sin(x-2)$ è infinitesimo: $\sin t \sim t$

$$\frac{x(\sin(x-2))^{\alpha}}{\sqrt{(x-2)(x+2)}} \sim \frac{2 \cdot (x-2)^{\alpha}}{2 \, (x-2)^{1/2}} = \frac{1}{(x-2)^{1/2-\alpha}}$$

che converge se $\frac{1}{2} - \alpha < 1 \Rightarrow \alpha > -\frac{1}{2}$

④ quali integrali convergono?

i) $\int_0^1 \frac{1}{tgx}\,dx$ ii) $\int_0^1 \frac{\sqrt{x}\,\sin x^2}{tgx - \sin x}\,dx$

iii) $\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx$

i) e iii) per casa

i) per $x \to 0^+$ $tgx = 0$ $1/tgx \to \infty$

taylor: $tgt \sim t + o(t)$ in $t \to 0$

$\frac{1}{tgx} \sim \frac{1}{x}$ ma $\int_0^1 \frac{1}{x}\,dx$ diverge, quindi per confronto asintotico diverge anche l'integrale $\int_0^1 1/tgx\,dx$

ii) per $x \to 0^+$: $\sin(x^2) \sim x^2$,

$tg\,x \sim x + \frac{x^3}{3}$

$\sin x \sim x - \frac{x^3}{6}$

$\sqrt{x}\,\sin x^2 \sim x^{1/2} x^2 = x^{5/2}$

$tgx - \sin x \sim x + \frac{x^3}{3} - x + \frac{x^3}{6} \sim x^3$

$$\int_0^1 \frac{\sqrt{x}\,\sin x^2}{tgx - \sin x}\,dx \sim \int_0^1 \frac{x^{5/2}}{x^3}\,dx = \underbrace{\int_0^1 \frac{1}{x^{1/2}}\,dx}$$

converge anche l'integrale dato $\leftarrow$ $\hookrightarrow$ converge $\int_0^1 \frac{1}{x^a}\,dx$ $a<1$

iii) $x \to +\infty$ va studiato $\quad \sin x \leq 1$

$\left(\frac{\sin x}{x}\right)^2 \leq \frac{1}{x^2}$ poiché $\int_1^\infty \frac{1}{x^2}\,dx$ (*) converge

allora converge anche l'integrale dato

(*) $\int_1^\infty \frac{1}{x^p}\,dx < \infty$ per $p > 1$

Esercizi per casa

5) $\int_0^\infty \frac{x^3 - \sin(x^3)}{x^\alpha (x^4+1)}\,dx$ convergenza al variare di α

qui vanno studiati entrambi gli estremi, quindi formalmente dividiamo l'integrale come

$\int_0^\infty = \int_0^1 + \int_1^\infty$ e vediamo quando convergono entrambi.

- per $x \to 0^+$: $\sin(x^3) \sim x^3 - \frac{(x^3)^3}{6} = x^3 - \frac{x^9}{6}$

$$\frac{x^3 - \sin(x^3)}{x^\alpha (x^4+1)} \sim \frac{x^9}{x^\alpha} = \frac{1}{x^{\alpha-9}}$$

CONVERGE se $\alpha - 9 < 1 \Rightarrow \alpha < 10$

- per $x \to +\infty$:

$$\frac{x^3 - \sin(x^3)}{x^\alpha (x^4+1)} \sim \frac{x^3}{x^\alpha x^4} = \frac{1}{x^{\alpha+1}}$$

CONVERGE se $\alpha + 1 > 1 \Rightarrow \alpha > 0$

l'integrale dato converge per $0 < \alpha < 10$

⑥ $\sum_{n=1}^{\infty} n^{\alpha}\left(e^{1/n}-1\right)$

per $n\to\infty$: $\frac{1}{n}\to 0$: $e^t \sim 1+t+o(t)$ $t\to 0$

$$e^{1/n} = 1+\frac{1}{n}$$

$$\sum_{n=1}^{\infty} n^{\alpha}\left(1+\frac{1}{n}-1\right) = \sum_{n=1}^{\infty}\frac{1}{n}$$

converge se $1-\alpha>1 \Rightarrow \alpha<0$
per confronto asintotico la serie converge per $\alpha<0$.

nota: termini sono positivi

3) $\sum_{n=1}^{\infty} \frac{n!\, x^n}{(3n)^n}$ convergenza assoluta al variare di $x \in \mathbb{R}$

A) $x = 0$ $\sum_{n=1}^{\infty} \frac{n!}{(3n)^n} \cdot 0 = 0$ conv.

B) $x \neq 0$: $\frac{n!\, x^n}{(3n)^n} = \frac{n!}{n^n} \left(\frac{x}{3}\right)^n$

Studiamo conv. assoluta con criterio del rapporto

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(n+1)!}{(n+1)^{n+1}}\left(\frac{x}{3}\right)^{n+1}\frac{n^n}{n!}\left(\frac{x}{3}\right)^{-n}\right| =$$

$$= \left|\frac{(n+1)!}{n!}\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}\left(\frac{x}{3}\right)\right| =$$

$$= \left(\frac{n}{n+1}\right)^n\left|\frac{x}{3}\right| = \left(\frac{n+1}{n}\right)^{-n}\left|\frac{x}{3}\right| = \left(1+\frac{1}{n}\right)^{-n}\left|\frac{x}{3}\right| =$$

$$= \left|\frac{x}{3}\right|\frac{1}{(1+1/n)^n}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left|\frac{x}{3}\right|\frac{1}{(1+1/n)^n} = \left|\frac{x}{3}\right|\cdot\frac{1}{e} = \ell$$

- $\ell < 1$ la serie converge assolutamente

$$\left|\frac{x}{3}\right|\frac{1}{e} < 1 \to |x| < 3e$$

- $\ell > 1$ la serie diverge : $|x| > 3e$

Se $\ell = 1$ il criterio non vale. Studio a parte:

$|x| = 3e$

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{x}{3}\right|\frac{1}{(1+1/n)^n} = \frac{e}{(1+1/n)^n} \geq 1$$

RICORDANDO $\boxed{\left(1+\frac{1}{n}\right)^n \leq e}$

per cui $|a_{n+1}| > |a_n|$: $|a_n|$ succ. cresc. → NON CONV (non può essere infinitesima)

(7) $\sum_{n=1}^{\infty} n \frac{3^n}{(x-1)^n}$

convergenza in modulo: $\sum |a_n|$

- infinitesima $|a_n|$:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n \left|\frac{3}{x-1}\right|^n \overset{*}{=} 0 \quad \text{se } \left|\frac{3}{x-1}\right| < 1$$

* limite notevole: $\lim_{n\to\infty} n^b \frac{1}{a^n} = 0 \quad \forall b>0, a>1$

$$= \lim_{n\to\infty} n^b \left(\frac{1}{a}\right)^n = 0 \quad \begin{matrix}\text{se } a>1 \\ 0<1/a<1\end{matrix}$$

- termini positivi, criterio rapporto:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{n+1}{n} \left|\frac{3}{x-1}\right| \to \left|\frac{3}{x-1}\right|$$

se $\left|\frac{3}{x-1}\right| < 1$ converge assolutamente

$$\left|\frac{3}{x-1}\right| < 1 \qquad \frac{3}{|x-1|} < 1 \qquad |x-1| > 3$$

$$\begin{cases} x-1 < -3 \\ x-1 > 3 \end{cases} \qquad \begin{cases} x < -2 \\ x > 4 \end{cases}$$