ESERCIZI PER GIOVEDÌ 10 DIC

① $\int \frac{b}{mx^2+n}\,dx$ (vedi int. k) ② $\int \frac{1}{x^2+2x+3}\,dx$

③ $\int \frac{x+3}{x^2+2x+2}\,dx$ ④ $\int \frac{1}{x^2+x+2}\,dx$

⑤ $\int \frac{3x+1}{x^2-4x+3}\,dx$ ⑥ $\int \frac{2x+1}{9x^2-6x+1}\,dx$

⑦ risolvere con la formula di Taylor

$$\lim_{x\to 0^+} \frac{\text{sen}^2 x + 3(x-\text{sen}^2\sqrt{x})}{x^2}$$

⑧ $$\lim_{x\to 0} \frac{e^{x^2}-1-\lg(1+x+\text{arctg}\, x)}{\sqrt{1+2x^4}-1}$$ * non esiste, fare con $\lg(1+x\,\text{arctg}\,x)$

⑨ calcolare:

$$\lim_{n\to\infty} \frac{n^{4/3}-\sqrt{n}\,\lg(1+2^n)}{n^{-1/3}+\sqrt{1+n^3}+[\lg n]^2}$$

— Soluzioni

① $\int \frac{b}{mx^2+n}\,dx = \frac{b}{n}\int \frac{1}{1+\frac{m}{n}x^2}\,dx = \frac{b}{n}\int \frac{1}{1+(\sqrt{\frac{m}{n}}x)^2}\,dx =$

$= \frac{b}{n}\sqrt{\frac{n}{m}}\int \frac{\sqrt{m/n}}{1+(\sqrt{\frac{m}{n}}x)^2}\,dx = \frac{b}{\sqrt{nm}}\,\text{arctg}\left(\sqrt{\frac{m}{n}}\,x\right)+c$

② $\int \frac{1}{x^2+2x+3}\,dx =$

D (denominatore): $\Delta = b^2-4ac = 4-12 = -8 < 0$

$D = x^2+2x+3 = (x+1)^2+2 = 2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]$

$= \int \frac{1}{2}\frac{1}{(\frac{x+1}{\sqrt{2}})^2+1}\,dx = \frac{1}{2}\sqrt{2}\int \frac{1/\sqrt{2}}{(\frac{x+1}{\sqrt{2}})^2+1}\,dx = \frac{1}{\sqrt{2}}\,\text{arctg}\left(\frac{x+1}{\sqrt{2}}\right)+c$

potete anche sostituire:

$t = \frac{x+1}{\sqrt{2}}$ $dt = \frac{1}{\sqrt{2}}dx \Rightarrow dx = \sqrt{2}\,dt$

$\int \frac{1}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{1}{(\frac{x+1}{\sqrt{2}})^2+1}\,dx = \frac{1}{2}\int \frac{1}{t^2+1}\sqrt{2}\,dt = \frac{1}{\sqrt{2}}\int \frac{1}{t^2+1}\,dt =$

$= \frac{1}{\sqrt{2}}\,\text{arctg}(t) + \text{cost} = \frac{1}{\sqrt{2}}\,\text{arctg}\left[\frac{x+1}{\sqrt{2}}\right]+c$

③ $\int \frac{x+3}{x^2+2x+2}\,dx$

D x^2+2x+2 $\Delta = b^2-4ac = 4-8 = -4 < 0$

$\frac{x+3}{x^2+2x+2} = \frac{A}{x^2+2x+2}\cdot\frac{d}{dx}(x^2+2x+2) + \frac{B}{x^2+2x+2} =$

$= \frac{A(2x+2)+B}{x^2+2x+2} = \frac{2Ax+2A+B}{x^2+2x+2}$

x: $\begin{cases} 1 = 2A \\ 3 = 2A+B \end{cases}$ $\begin{cases} A = 1/2 \\ B = 3-2A = 2 \end{cases}$
tn:

$\int \frac{x+3}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + 2\int \frac{1}{x^2+2x+2}\,dx$

$= \frac{1}{2}\lg(x^2+2x+2) + 2\int \frac{1}{x^2+2x+2}\,dx$

$\int \frac{1}{x^2+2x+2}\,dx$ con $\Delta<0$ (come es. di prima)

D: $x^2+2x+2 = (x+1)^2+1$

$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{1+(x+1)^2}\,dx = \text{arctg}(x+1)+c$

$= \frac{1}{2}\lg(x^2+2x+2) + 2\,\text{arctg}(x+1)+C$

④ $\int \frac{1}{x^2+x+2}\,dx$ D x^2+x+2 $\Delta = 1-8 = -7 < 0$

$x^2+x+2 = (x+\frac{1}{2})^2 - \frac{1}{4}+2 = (x+\frac{1}{2})^2+\frac{7}{4} =$

$= \frac{7}{4}\left[\left(\sqrt{\frac{4}{7}}(x+\frac{1}{2})\right)^2+1\right]$

$= \frac{7}{4}\left[\left(\frac{2}{\sqrt{7}}(x+\frac{1}{2})\right)^2+1\right] \rightarrow$

$$\int \frac{1}{x^2+x+2}\,dx = \frac{4}{7}\int \frac{1}{\left[\frac{2}{\sqrt{7}}(x-\frac{1}{2})\right]^2+1}\,dx \qquad t=\frac{2}{\sqrt{7}}(x-\tfrac{1}{2}) \qquad dt = \frac{2}{\sqrt{7}}\,dx \to dx = \frac{\sqrt{7}}{2}dt$$

$$= \frac{4}{7}\int \frac{1}{1+t^2}\frac{\sqrt{7}}{2}\,dt = \frac{2}{\sqrt{7}}\int \frac{1}{1+t^2}\,dt = \frac{2}{\sqrt{7}}\operatorname{arctg} t + c =$$

$$= \frac{2}{\sqrt{7}}\operatorname{arctg}\left[\frac{2x-1}{\sqrt{7}}\right] + c$$

5) $$\int \frac{3x+1}{x^2-4x+3}\,dx$$

D: $x^2-4x+3 \qquad \Delta = 16-12 = 4 > 0$

$x_\pm = \frac{4\pm\sqrt{\Delta}}{2} = \frac{4\pm 2}{2} = 1, 3$

$$\frac{3x+1}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{Ax-3A+Bx-B}{(x-1)(x-3)} = \frac{(A+B)x-3A-B}{(x-1)(x-3)}$$

$\to$ x: $\begin{cases} 3 = A+B \\ 1 = -3A-B \end{cases}$ tn: $\begin{cases} A = 3-B \\ 1 = -9+3B-B \end{cases}$ $\begin{cases} A = -2 \\ B = 5 \end{cases}$

$$= \int \frac{-2}{x-1}\,dx + \int \frac{5}{x-3}\,dx = -2\lg|x-1| + 5\lg|x-3| + c$$

6) $$\int \frac{2x+1}{9x^2-6x+1}\,dx$$

D: $9x^2-6x+1$

$\Delta = 36-4\cdot 9 = 0 \qquad x = \frac{6\pm\sqrt{\Delta}}{18} = 1/3$

$\to 9x^2-6x+1 = 9\cdot(x-1/3)^2$

$$= \frac{1}{9}\int \frac{2x+1}{(x-1/3)^2} =$$

$$\frac{2x+1}{(x-1/3)^2} = \frac{A}{x-1/3} + \frac{B}{(x-1/3)^2} = \frac{Ax-A/3+B}{(x-1/3)^2} \to \begin{cases} A=2 \\ B=5/3 \end{cases}$$

$$= \frac{1}{9}\left[\int \frac{2}{x-1/3}\,dx + \int \frac{5/3}{(x-1/3)^2}\,dx\right] =$$

$$= \frac{1}{9}\left[2\lg|x-1/3| + \frac{5}{3}(x-1/3)^{-1}\right] + c = \frac{2}{9}\lg|x-1/3| + \frac{5}{27}\frac{1}{x-1/3} + c$$

7) $$\lim_{x\to 0^+} \frac{\operatorname{sen}^2 x + 3(x-\operatorname{sen}^2\sqrt{x})}{x^2}$$

cerco ordine x^2 come il denominatore:

- $\operatorname{sen} x = x - \frac{x^3}{6} + o(x^3) \qquad (\operatorname{sen} x)^2 = x^2 + o(x^3)$
- $\operatorname{sen} t = t - \frac{t^3}{6} + o(t^3) \qquad$ con $t = \sqrt{x}$

$\operatorname{sen}\sqrt{x} = \sqrt{x} - \frac{x^{3/2}}{6} + o(x^{3/2}) \qquad (\operatorname{sen}\sqrt{x})^2 = x - \frac{1}{3}x^2 + o(x^2)$

$$= \lim_{x\to 0^+} \frac{x^2+3(x-x+\frac{1}{3}x^2)}{x^2} = \lim_{x\to 0^+}\frac{2x^2}{x^2} = 2$$

8) $$\lim_{x\to 0} \frac{e^{x^2}-1-\lg(1+x\operatorname{arctg} x)}{\sqrt{1+2x^4}-1}$$

- $\sqrt{1+t} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + o(t^2) \qquad$ con $t = 2x^4$

$\sqrt{1+2x^4} = 1 + \frac{1}{2}(2x^4) - \frac{1}{8}(2x^4)^2 + o(x^8) = 1 + x^4 + o(x^4)$

$\to$ il denominatore è $\sim x^4$ in zero

Sviluppiamo fino a x^4 il numeratore

- $e^t = 1 + t + \frac{1}{2}t^2 + o(t^2) \qquad t = x^2$

$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + o(x^4)$

- $\operatorname{arctg} x = \operatorname{arctg} 0 + \left[\frac{1}{1+x^2}\right]_{x=0} x + \left[-\frac{2x}{(1+x^2)^2}\right]_{x=0}\frac{x^2}{2} + \left[\frac{6x^2-2}{(x^2+1)^3}\right]_{x=0}\frac{x^3}{6} + o(x^3)$

$= 0 + x + 0\cdot\frac{x^2}{2} - 2\frac{x^3}{6} + o(x^3) = x - \frac{1}{3}x^3 + o(x^3)$

e mi sono fermata a x^3 perché l'arctg x è moltiplicata nel limite per x:

$x\operatorname{arctg} x = x\left(x - \frac{1}{3}x^3 + o(x^3)\right) = x^2 - \frac{1}{3}x^4 + o(x^4)$

- $\lg(1+t) = t - \frac{1}{2}t^2 + o(t^2) \qquad$ con $t = x\operatorname{arctg} x$

$\to$ mi fermo a t^2 perché: $t^3 = (x\operatorname{arctg} x)^3 = (x^2 - \frac{1}{3}x^4)^3 = x^6 + o(x^6)$

$\to$ è $o(x^4)$

$$\lg(1+x\,\mathrm{arctg}x) = x^2-\frac{1}{3}x^4-\frac{1}{2}\left(x^2+\frac{1}{3}x^4\right)^2 =$$
$$= x^2-\frac{1}{3}x^4-\frac{1}{2}\left(x^4+o(x^4)\right) =$$
$$= x^2-\frac{5}{6}x^4+o(x^4)$$

$$\lim_{x\to 0}\frac{e^{x^2}-1-\lg(1+x\,\mathrm{arctg}x)}{\sqrt{1+2x^4}-1} = \lim_{x\to 0}\frac{x^2+\frac{1}{2}x^4-x^2+\frac{5}{6}x^4}{x^4} =$$
$$= \lim_{x\to 0}\frac{8}{6}\frac{x^4}{x^4} = \frac{8}{6} = \frac{4}{3}$$

9) $$\lim_{n\to\infty}\frac{n^{4/3}-\sqrt{n}\,\lg(1+2^n)}{n^{-1/3}+\sqrt{1+n^3}+(\lg n)^2}$$

DENOMINATORE:

$$n^{-1/3}+\sqrt{1+n^3}+(\lg n)^2 = \frac{1}{\sqrt[3]{n}}+\sqrt{1+n^3}+(\lg n)^2$$

per $n\to\infty$ $$\simeq \sqrt{1+n^3}+(\lg n)^2 \sim n^{3/2}+(\lg n)^2 =$$
$$= n^{3/2}\left[1+\left(\frac{\lg n}{n^{3/4}}\right)^2\right] = n^{3/2}$$

lim. notevole $\to 0$

NUMERATORE:

$$n^{4/3}-n^{1/2}\lg(1+2^n) \overset{n\to\infty}{\sim} n^{4/3}-n^{1/2}\lg(2^n) =$$
$$= n^{4/3}-n^{1/2}\,n\lg(2) = n^{4/3}-n^{3/2}\lg(2) =$$
$$= n^{3/2}\left(n^{-1/6}-\lg(2)\right) = n^{3/2}\left(\frac{1}{\sqrt[6]{n}}-\lg(2)\right) \sim n^{3/2}(-\lg(2))$$

$$= \lim_{n\to\infty}\frac{n^{3/2}(-\lg(2))}{n^{3/2}} = -\lg(2)$$

ESERCITAZIONE GIOVEDÌ 10 DICEMBRE

1) $$\int\frac{1-x^4+x}{1+x^2}\,dx$$

$(1-x^4)+x = (1-x^2)(1+x^2)+x$ anche:

$-x^4$	0	0	x	1	x^2+1
x^4		$+x^2$			$-x^2+1$
/		x^2	x	1	
		$-x^2$		-1	
		/	x	/	

$\Rightarrow -x^4+x+1 = (x^2+1)(1-x^2)+x$

$$= \int\frac{(x^2+1)(1-x^2)+x}{1+x^2}\,dx = \int\frac{(x^2+1)(1-x^2)}{(1+x^2)}\,dx+\int\frac{x}{1+x^2}\,dx =$$
$$= x-\frac{1}{3}x^3+\frac{1}{2}\lg(1+x^2)+c$$

2) $$\int\frac{x^4-4x^3+3x^2+5x-4}{x^2-4x+4}\,dx$$

x^4	$-4x^3$	$3x^2$	$5x$	-4	x^2-4x+4
$-x^4$	$+4x^3$	$-4x^2$			x^2-1
0	0	$-x^2$	$5x$	-4	
		x^2	$-4x$	$+4$	
		0	x	0	

$\Rightarrow (x^4-4x^3+3x^2+5x-4) = (x^2-1)(x^2-4x+4)+x$

$$= \int\frac{(x^2-1)\cancel{(x^2-4x+4)}}{\cancel{x^2-4x+4}}\,dx+\int\frac{x}{x^2-4x+4}\,dx = \int(x^2-1)\,dx+\int\frac{x}{x^2-4x+4}\,dx =$$
$$= \frac{1}{3}x^3-x+\int\frac{x}{x^2-4x+4}\,dx$$

$$\frac{x}{x^2-4x+4} = \frac{A}{x-2}+\frac{B}{(x-2)^2} = \frac{Ax-2A+B}{(x-2)^2} \to \begin{cases}A=1\\B-2A=0\end{cases} \begin{cases}A=1\\B=2\end{cases}$$

perché $\Delta=0$

$$\int\frac{x}{x^2-4x+4}\,dx = \int\frac{1}{x-2}\,dx+2\int\frac{1}{(x-2)^2}\,dx = \lg|x-2|-\frac{2}{x-2}+c$$

$$= \frac{1}{3}x^3-x+\lg|x-2|-\frac{2}{x-2}+c$$

③ $\int \frac{1}{(x-1)^3 x^2} dx$

$$\frac{1}{(x-1)^3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

$$\frac{1}{(x-1)^3x^2} = \frac{Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2}{x^2(x-1)^3}$$

$$1 = (Ax+B)(x-1)^3 + Cx^2(x^2-2x+1) + Dx^3 - Dx^2 + Ex^2$$

$$1 = (Ax+B)(x^3-3x^2+3x-1) + Cx^4 - 2Cx^3 + Cx^2 + Dx^3 + (E-D)x^2$$

$$1 = Ax^4 - 3Ax^3 + 3Ax^2 - Ax + Bx^3 - 3Bx^2 + 3Bx - B + Cx^4 + (D-2C)x^3 + (C+E-D)x^2$$

$$1 = (A+C)x^4 + (-3A+B+D-2C)x^3 + (3A-3B+C+E-D)x^2 + (-A+3B)x - B$$

$$\begin{cases} A+C=0 \\ -3A+B+D-2C=0 \\ 3A-3B+C+E-D=0 \\ -A+3B=0 \\ -B=1 \end{cases} \qquad \begin{cases} A=3B=-3 \\ B=-1 \\ C=-A=3 \\ D=3A-B+2C=-9+1+6=-2 \\ E=D-3A+3B-C=-2+9-3-3=1 \end{cases}$$

$$\int \frac{1}{x^2(x-1)^3} dx = \int -\frac{3}{x} dx + \int \frac{-1}{x^2} dx + \int \frac{3}{x-1} dx + \int \frac{-2}{(x-1)^2} dx + \int \frac{1}{(x-1)^3} dx =$$

$$= -3\lg|x| + \frac{1}{x} + 3\lg|x-1| + \frac{2}{x-1} - \frac{1}{2}\frac{1}{(x-1)^2} + c$$

④ $\int sen(ax)\, sen(bx)\, dx \qquad a \neq \pm b$

prostaferesi: $\cos p - \cos q = -2\, sen\frac{p+q}{2}\, sen\frac{p-q}{2}$

$$\begin{cases} ax = \frac{p+q}{2} \\ bx = \frac{p-q}{2} \end{cases} \qquad \begin{cases} (a+b)x = p \\ (a-b)x = q \end{cases}$$ ricavo in ax e bx

$$\cos[(a+b)x] - \cos[(a-b)x] = -2\, sen(ax)\, sen(bx)$$

$$= -\frac{1}{2}\left[\int \cos[(a+b)x]\, dx - \int \cos[(a-b)x]\, dx\right] =$$

$$= -\frac{1}{2}\frac{sen[(a+b)x]}{(a+b)} + \frac{1}{2}\frac{sen[(a-b)x]}{(a-b)} + c$$

⑤ $\int ctg^2x\, dx = \int \frac{\cos^2 x}{sen^2 x} dx = \int \frac{1-sen^2x}{sen^2x} dx = \int \frac{1}{sen^2x} dx - \int dx =$

$$= -ctg\,x - x + c$$

⑥ $\int arctg\,x\, dx$ PARTI: $\int fg'dx = fg - \int f'g\, dx$

$f = arctg\,x \quad g' = 1 \ : \ f' = \frac{1}{1+x^2} \quad g = x$

$$= x\, arctg\,x - \int \frac{x}{1+x^2} dx = x\, arctg\,x - \frac{1}{2}\lg(1+x^2) + c$$

⑦ $\int \lg x\, dx =$

$f = \lg x \quad g' = 1 \quad f' = \frac{1}{x} \quad g = x$

$$= x\lg x - \int \frac{1}{x} x\, dx = x\lg x - \int dx = x\lg x - x + c$$

⑧ $\int x e^{2x} dx = \qquad f = x \quad g' = e^{2x} \quad f' = 1 \quad g = \frac{1}{2}e^{2x}$

$$= \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x} dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$$

⑨ $\int \lg x\,(sen x + x\cos x)\, dx$

$f = \lg x \qquad f' = 1/x$

$g' = sen x + x\cos x \qquad g = ?$

$$g = \int g'\, dx = \int (sen x + x\cos x)\, dx = -\cos x + \int x\cos x\, dx$$

per parti con $f = x,\ g' = \cos x$

$$= -\cos x + x\, sen x - \int sen x\, dx =$$

$$= -\cos x + x\, sen x + \cos x = x\, sen x$$

$$= \lg x\,(x\, sen x) - \int \frac{1}{x} x\, sen x\, dx =$$

$$= x \lg x\, sen x - \int sen x\, dx = x\lg x + \cos x + c$$

⑩ $\int x^3 e^{x^2} dx = \int x^2\, x e^{x^2} dx$:

$f = x^2 \quad f' = 2x \quad e \quad g'(x) = x e^{x^2} \quad g = \frac{1}{2}e^{x^2}$

$$= \frac{x^2}{2}e^{x^2} - \int 2x\frac{1}{2}e^{x^2} dx = \frac{x^2}{2}e^{x^2} - \int x e^{x^2} dx =$$

$$= \frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + c = \frac{1}{2}(x^2-1)e^{x^2}$$

(11) $\int \frac{lg(lg x)}{x}\, dx$ $\qquad f = lg(lg x) \quad f' = \frac{1}{x\, lg x}$

$\qquad g' = \frac{1}{x} \qquad g = lg|x| = lg x$

$= lg x \cdot lg(lg x)) - \int \frac{1}{x\, \cancel{lg x}} \cancel{lg x}\, dx = lg x \cdot lg(lg x) - lg x + c$

$= lg x\,(lg(lg x) - 1) + c$

(12) $\int \sqrt{a^2 - x^2}\, dx \qquad a > 0$

$x = a\, sen t \qquad dx = a \cos t\, dt$

$= \int \sqrt{a^2 - a^2 sen^2 t}\; a \cos t\, dt = a^2 \int \sqrt{1 - sen^2(t)}\, \cos(t)\, dt =$

$= a^2 \int \cos^2 t\, dt = \qquad \cos(2\alpha) = 2\cos^2(\alpha) - 1$

$= \frac{a^2}{2} \int (\cos 2t + 1)\, dt = \frac{a^2}{2}\left[\frac{sen\, 2t}{2} + t \right] + C = \qquad sen\, 2\alpha = 2\, sen\alpha \cos\alpha$

$= \frac{a^2}{2}\left[sen t \cos t + t \right] + c =$

$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2} + arcsin\frac{x}{a} \right] + c$

$= \frac{a^2}{2} arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c$

$\begin{cases} sen t = \frac{x}{a} \\ \cos t = \sqrt{1 - sen^2 t} = \sqrt{1 - x^2/a^2} \\ t = arcsin(x/a) \end{cases}$

PER LUNEDÌ 14 DICEMBRE : calcolare:

(1) $\int \cos(3x)\cos(2x)\, dx$

(2) $\int lg(1 + x^3)^{x^2}\, dx$

(3) $\int x^3\, sen(x^2)$

(4) $\int \frac{x^2 - 7x + 12}{(x-2)^3}\, dx$

(5) $\int \frac{\sqrt{x}}{2 + \sqrt{x}}\, dx$

(6) $\int \frac{e^x}{3e^{2x} + e^x + 2}\, dx$