Soluzioni degli esercizi per il 2 NOV

lunedì 2 novembre 2020 16:49

1) Verificare

a) $\lim_{n\to\infty} \frac{n+4}{n} = 1$

def. limite $a_n \to a$: $\forall \varepsilon > 0 \quad \exists \nu : |a_n - a| < \varepsilon \quad$ per $\forall n > \nu$

$|a_n - a| < \varepsilon \qquad \left|\frac{n+4}{n} - 1\right| < \varepsilon \qquad \left|\frac{n+4-n}{n}\right| < \varepsilon \qquad \left|\frac{4}{n}\right| < \varepsilon$

$\frac{4}{n} < \varepsilon \quad \to n > \frac{4}{\varepsilon} \qquad$ scelgo $\nu = \frac{4}{\varepsilon}$

b) $\lim_{n\to\infty} (n^2 - 1) = +\infty$

def. limite $a_n \to \infty$: $\forall M > 0 \quad \exists \nu : a_n > M \quad$ per $n > \nu$

$a_n > M \qquad n^2 - 1 > M \qquad n^2 > M + 1 \to n > \sqrt{M+1} \qquad$ scelgo $\nu = \sqrt{M+1}$

2) Calcolare :

a) $\lim_{n\to\infty} \frac{n^3+1}{2n-1} = \lim_{n\to\infty} \frac{n^3(1 + 1/n^3)}{n(2 - 1/n)} = \lim_{n\to\infty} n^2\left(\frac{1+1/n^3}{2-1/n}\right) = +\infty \left(\frac{1+0}{2-0}\right) = +\infty$

b) $\lim_{n\to\infty} \frac{1-n^2}{(n+2)^2} = \lim_{n\to\infty} \frac{n^2(1/n^2 - 1)}{n^2+4n+4} = \lim_{n\to\infty} \frac{\cancel{n^2}(1/n^2 - 1)}{\cancel{n^2}(1+4/n+4/n^2)} = \frac{(0-1)}{(1+0+0)} = -\frac{1}{1} = -1$

c) $\lim_{n\to\infty} \left(\sqrt{n^2+1} - \sqrt{n}\right)$

$\sqrt{n^2+1} - \sqrt{n} = \left(\sqrt{n^2+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n^2+1} + \sqrt{n}}{\sqrt{n^2+1} + \sqrt{n}} = \frac{n^2+1-n}{\sqrt{n^2+1} + \sqrt{n}} = \frac{n^{\cancel{2}}(1 + 1/n^2 - 1/n)}{\cancel{n}\left(\sqrt{1+1/n^2} + \sqrt{1/n}\right)}$

$\lim_{n\to\infty} \left(\sqrt{n^2+1} - \sqrt{n}\right) = \lim_{n\to\infty} \frac{n \quad (1 + 1/n^2 - 1/n)}{\sqrt{1+1/n^2} + \sqrt{1/n}} = \frac{\infty \cdot (1+0-0)}{\sqrt{1+0} + 0} = \infty \cdot \frac{1}{1} = +\infty$

d) $\lim_{n\to\infty} (e^n - 2^n) = \lim_{n\to\infty} \left[e^n\left(1 - \left(\frac{2}{e}\right)^n\right)\right] = +\infty\,[1-0] = +\infty$

$\hookrightarrow a^n \to 0$ se $-1 < a < 1$

e) $\lim_{n\to\infty} \sqrt[n]{\pi} = \lim_{n\to\infty} \pi^{1/n} = 1$

$\hookrightarrow a^{1/n} \to 1 \ \forall a > 0$ $\left[\frac{1}{n} \to 0 \quad \pi^0 = 1 \right]$

f) $\lim_{n\to\infty} \frac{2^{n+1} - 4^{n-1}}{3^n} = \lim_{n\to\infty} \left(\frac{2^{n+1}}{3^n} - \frac{4^{n-1}}{3^n} \right) =$

$\lim_{n\to\infty} \left(2 \cdot \left(\frac{2}{3}\right)^n - 4^{-1} \left(\frac{4}{3}\right)^n \right) = \left(2 \cdot 0 - \frac{1}{4}(-\infty) \right) = 0 - \infty = -\infty$

$\hookrightarrow a^n \to 0$ se $-1 < a < 1$ $\quad\quad \hookrightarrow a^n \to +\infty$ se $a > 1$

g) $\lim_{n\to\infty} \frac{2^n - 4^n}{3^n - n!} = \lim_{n\to\infty} \frac{\cancel{n!}(2^n/n! - 4^n/n!)}{\cancel{n!}(3^n/n! - 1)} =$

so $\frac{a^n}{n!} \to 0$ se $a > 1$ $\quad\quad = \frac{0}{0-1} = \frac{0}{-1} = 0$

h) $\lim_{n\to\infty} \frac{tg(1/n)}{1 - \cos(1/n)} \rightsquigarrow \frac{tg(0)}{1-\cos(0)} = \frac{0}{0}$

$\frac{tg(1/n)}{1-\cos(1/n)} = \frac{tg(1/n)}{1-\cos(1/n)} \cdot \frac{1+\cos(1/n)}{1+\cos(1/n)} = \frac{tg(1/n)(1+\cos(1/n))}{1-\cos^2(1/n)}$

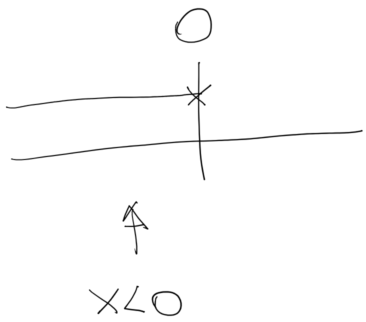
$= \frac{tg(1/n)(1+\cos(1/n))}{sen^2(1/n)} = \frac{1+\cos(1/n)}{\cos(1/n)\, sen(1/n)} \longrightarrow \frac{1+\cos(0)}{\cos(0)\cdot sen(0)} = \frac{2}{1\cdot 0} = \frac{2}{0} = \infty$

- RISCARICARE IL PDF DEL 29 OTT.

Sistema 2

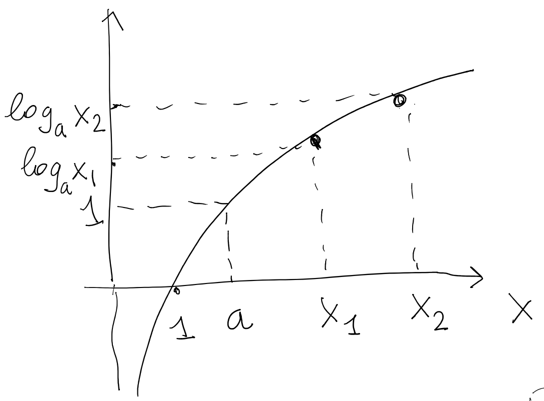
$$\begin{cases} x < 0 \\ (x-1)^2 \geq 0 \end{cases} \quad \begin{cases} x < 0 \\ \forall x \end{cases} \rightarrow x < 0$$

→ ES. 5



nell'es. 2, x > -5

DIS. LOG.



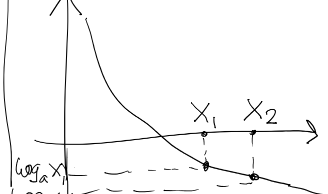
$$\log_a \underbrace{x_2} > \log_a \underbrace{x_1}$$

$$\boxed{x_2 > x_1}$$

$y = \log_a x$

$a > 1$

$0 < a < 1$



$\log_a x_1 > \log_a x_2$

$\boxed{x_1 < x_2}$

$\log_a x_2 < \log_a x_1$

$x_2 > x_1$

$\log x^2 > \log x \qquad x^2 > 0 \;,\; x > 0$

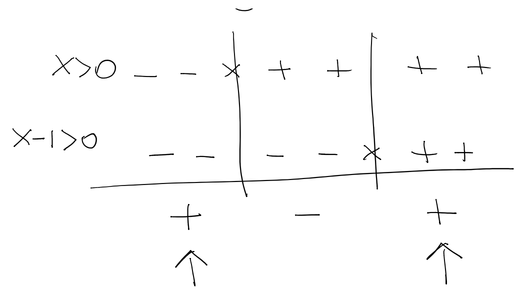
$a = e$

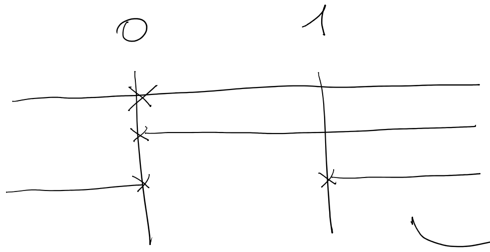
$x^2 > x \qquad x^2 - x > 0 \qquad \boxed{x(x-1) > 0}$

$$\begin{cases} x^2 > 0 \\ x > 0 \end{cases} \rightarrow \begin{cases} \forall x \neq 0 \\ x > 0 \end{cases}$$

		0		1	
x>0	- -	×	+ +		+ +

$$\begin{cases} x > 0 \\ x > 0 \\ x^2 > x \end{cases} \rightarrow \begin{cases} + x \neq 0 \\ x > 0 \\ x < 0 \ \cup \ x > 1 \end{cases}$$





$$\boxed{x > 1}$$

$$\log_{\frac{1}{7}} x < \sqrt{2}$$

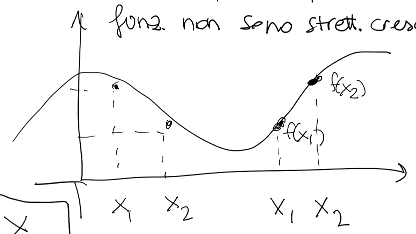
$$\sqrt{2} = \log_{\frac{1}{7}} y$$

$$\left(\frac{1}{7}\right)^{\sqrt{2}} = y$$

$$\underbrace{\log_{\frac{1}{7}} x}_{x_1} < \underbrace{\log_{\frac{1}{7}} \left[\frac{1}{7}\right]^{\sqrt{2}}}_{x_2}$$

$$x_1 > x_2 \qquad x > \left[\frac{1}{7}\right]^{\sqrt{2}}$$

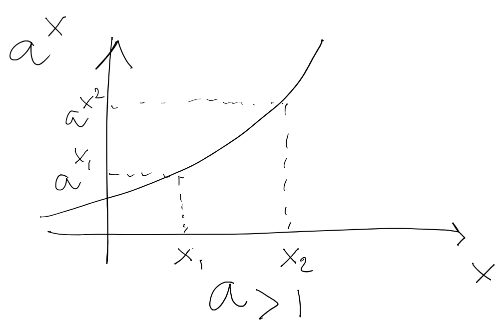
Perché non posso quando le funz. non sono strett. cresc. o decres.



$x_2 > x_1$ $f(x_2) < f(x_1)$

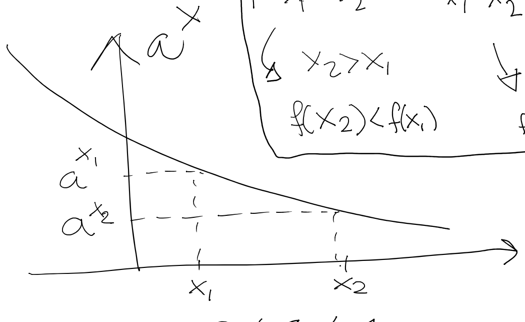
$x_2 > x_1$ $f(x_2) > f(x_1)$

$$\log_a x \rightarrow \underline{\underline{a^x}}$$



$a > 1$

$a^{x_2} > a^{x_1} \leftrightarrow x_2 > x_1$



$0 < a < 1$

$a^{x_1} > a^{x_2} \leftrightarrow x_1 < x_2$

SUCCESSIONI – limiti

1) $$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad x \in \mathbb{R}$$

② $a_n \to \pm\infty$ $\lim_{n\to\infty}\left(1+\frac{x}{a_n}\right)^{a_n} = e^x$ $x \in \mathbb{R}$

$\Downarrow$

$\lim_{n\to\infty} a_n = \pm\infty$

× $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^2 = \left[e^1\right]^2 = e^2$

× $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n \overset{x=-1}{=} \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x = e^{-1} = 1/e$

× $\lim_{n\to\infty}\left(1+\frac{2}{\sqrt{n}}\right)^{\sqrt{n}} =$ $a_n = \sqrt{n}$ $x = 2$

$= \lim_{n\to\infty}\left(1+\frac{x}{a_n}\right)^{a_n} = e^x = e^2$

$a_n \to \infty$ $\lim_{n\to\infty}\sqrt{n} = +\infty \leftarrow$

$\forall M > 0 \quad \exists \nu : \forall n > \nu \quad \underline{\underline{a_n > M}}$ $\rightarrow$

$\sqrt{n} > M \qquad n > M^2 \qquad \nu = M^2$

× $\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{n+1}$ $1+\frac{1}{x_n}$

$\left(\frac{n}{n-1}\right)^{n+1} = \left(\frac{n}{n-1}\right)^{n-1}\left(\frac{n}{n-1}\right)^2 =$

metodo 1

$\frac{n}{n-1} = \frac{n+1-1}{n-1} = \frac{(n-1)+1}{n-1} =$

$= \frac{n-1}{n-1} + \frac{1}{n-1} = 1 + \frac{1}{n-1}$

metodo 2

$\frac{n}{n-1} = 1 + \frac{1}{x_n} = 1 + \frac{1}{n-1}$

$n x_n = (x_n+1)(n-1)$

$\cancel{n x_n} = (\cancel{n}-1)x_n + (n-1)$

$x_n = n-1$

$\lim_{n\to\infty}\underbrace{\left(1+\frac{1}{n-1}\right)^{n-1}}_{(a_n = n-1)}\underbrace{\left(\frac{n}{n-1}\right)^2}_{\lim\left[\frac{n}{n(1-1/n)}\right]^2 =} =$

$$a_n = n-1 \qquad a_n \to \infty \qquad e^1 = e$$

$$\lim_{n\to\infty}\left[\frac{\cancel{n}}{\cancel{n}(1-1/n)}\right]^2 = \lim_{n\to\infty}\left[\frac{1}{1-1/n}\right]^2 = \left[\frac{1}{1-0}\right]^2 = 1^2 = 1$$

$$\cancel{n}x_n = (n-1)x_n + (n-1)$$

$$x_n = n-1$$

$$= e \cdot 1 = e$$

$$\lim_{n\to\infty}\left(\frac{n^2+n}{n^2-n+2}\right)^n$$

tento: $\displaystyle \lim_{n\to\infty}\left[\left[1+\frac{1}{a_n}\right]^{a_n}\right]^{\frac{n}{a_n}} = \lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{\frac{n\,a_n}{a_n}}$

$$\frac{n^2+n}{n^2-n+2} = 1+\frac{1}{a_n}$$ ← Risolvo per a_n

$$\boxed{\frac{n^2+n}{n^2-n+2} = \frac{a_n+1}{a_n}} \qquad a_n(n^2+n) = (a_n+1)(n^2-n+2)$$

$$a_n(n^2+n) = a_n(n^2-n+2) + n^2-n+2$$

$$a_n(n^2+n-n^2+n-2) = n^2-n+2$$

$$a_n(\cancel{n^2}+n-\cancel{n^2}+n-2) = n^2-n+2$$

$$a_n(2n-2) = n^2-n+2 \longrightarrow a_n = \frac{n^2-n+2}{2n-2}$$

$$\frac{n^2+n}{n^2-n+2} = 1+\frac{1}{\left(\frac{n^2-n+2}{2n-2}\right)}$$

$$\lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{a_n}\right)^{a_n}}_{\to e}\right]^{\frac{n}{a_n}} =$$

$a_n \to \pm\infty$

Studio ✓

$$\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2-n+2}{2n-2}} = \lim_{n\to\infty}\frac{\cancel{n}(n-1+2/n)}{\cancel{n}(2-2/n)} \sim \frac{\infty}{2} = +\infty$$

Studio $\boxed{\dfrac{n}{a_n}}$ → finito, → 0, → $\pm\infty$

$$\longrightarrow \lim_{n\to\infty}\left[e\right]^{\frac{n}{a_n}}$$

$$\lim_{n\to\infty}\frac{n}{a_n} = \frac{n(2n-2)}{n^2-n+2} = \frac{\cancel{n^2}(2-2/n)}{\cancel{n^2}(1-1/n+2/n^2)} = \frac{2-0}{} = \frac{2}{} = 2$$

$\longrightarrow \lim_{n\to\infty} [\quad] \cancel{\to} \quad n\to\infty \quad a_n \quad n^2-n+2 \quad \cancel{n^2}(1-1/n+2/\cancel{n^2})$

$\hookrightarrow [e]^2 = e^2$ $\qquad = \frac{2-0}{1-0+0} = \frac{2}{1} = 2$

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3) $\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} (a_{n+1} - a_n)$

4) $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ $\quad \forall n \; \underline{\underline{a_n>0}}$

$\Big\}$ SONO VERE SE $\exists$ i limiti a destra

Calcolare i limiti per $n \to \infty$ di:

$\times \; \sqrt[n]{n(n-1)}$

$\boxed{a_n = n(n-1)}$ $\qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \underline{\underline{\frac{a_{n+1}}{a_n}}}$

$a_{n+1} = (n+1)(n+1-1) = (n+1)n$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\cancel{n}}{\cancel{n}(n-1)} = \lim_{n\to\infty} \frac{n+1}{n-1} = \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(1-1/n)} \to \underline{\underline{1}}$

$\lim_{n\to\infty} \sqrt[n]{n(n-1)} = 1$

$\rhd \; a_{n+1}\,?$ $\qquad a_n(\underline{\underline{n}} = \underline{\underline{n+1}}) = a_{n+1} = (n+1)(n+1-1)$

CRITERIO DEL RAPPORTO

$a_n > 0 \qquad b_n = \frac{a_{n+1}}{a_n}$

se $b_n \to \begin{cases} \ell \in [0,1) & \Rightarrow a_n \to 0 \\ \ell \in (1,+\infty) & \Rightarrow a_n \to +\infty \\ \infty & \Rightarrow a_n \to +\infty \end{cases}$

$$\times \lim_{n\to\infty} \frac{n!}{n^{n+1}} = \lim_{n\to\infty} a_n \qquad \circ\ a_n = \frac{n!}{n^{n+1}}$$

$$\circ\ a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1+1}} = \frac{(n+1)!}{(n+1)^{n+2}}$$

$$b_n = \frac{a_{n+1}}{a_n} =$$

$$= \frac{(n+1)!}{(n+1)^{n+2}} \cdot \frac{n^{n+1}}{n!} = \frac{\cancel{(n+1)} \cdot \cancel{n!}}{\cancel{(n+1)}\,(n+1)^{n+1}} \cdot \frac{n^{n+1}}{\cancel{n!}} = \frac{n^{n+1}}{(n+1)^{n+1}} =$$

$$= \left(\frac{n}{n+1}\right)^{n+1} = \left(1 + \frac{1}{x_n}\right)^{n+1} \ast$$

$$\frac{n}{n+1} = 1 + \frac{1}{x_n} \qquad \frac{n}{n+1} = \frac{x_n+1}{x_n} \qquad x_n n = (x_n+1)(n+1)$$

$$\cancel{x_n n} = \cancel{x_n n} + x_n + n + 1 \qquad 0 = x_n + n + 1$$

$$x_n = -(n+1)$$

$$b_n = \ast = \left(1 + \frac{1}{-(n+1)}\right)^{n+1} = \left(1 + \frac{(-1)}{n+1}\right)^{n+1}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{x}{n+1}\right)^{n+1} \overset{x=-1}{=} e^x = e^{-1} = 1/e$$

$$\left.\lim_{n\to\infty} n+1 = \infty\right\} \to \text{vera} \qquad \text{ALLORA}$$

$$\Downarrow \qquad \ell \in [0,1]$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^{n+1}} = 0$$

Esercizi per giovedì 5 novembre

lunedì 2 novembre 2020 16:49

Calcolare i limiti delle seguenti successioni:

1) $\left(\dfrac{n^2+n}{n^2+n+1}\right)^{n^2}$

2) $\dfrac{n! - (n+1)!}{n^2 e^n}$

3) $\left(\dfrac{n^2-n}{n^2-n+3}\right)^{n}$

4) $\sqrt[n]{2^n+1}$

5) $\dfrac{(n^3+1)^n}{(n+1)^{3n}}$

6) $\dfrac{n!}{3^{n+1}}$ *

7) $\dfrac{n!}{n^{n/2}}$ *

* CRIT. RAPPORTO