Risonanza parametrica

Pb: cambiando $\omega \to \omega(t)$: posso rendere instabile $\underline{x}_0 = \underline{0}$ per l'o. oscillatore armonico; $\underline{\dot{x}} = A\underline{x}$ con $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$

$$\omega(t) = \begin{cases} \omega + \varepsilon & 0 \leq t < \pi \\ \omega - \varepsilon & \pi \leq t < 2\pi \end{cases}$$

$$\omega(t + 2\pi) = \omega(t) \qquad 0 < \varepsilon \ll \omega$$

e opportuna scelta parametri

(1) $\qquad \underline{\dot{x}} = A(t)\,\underline{x}$

$$A(t) = \begin{cases} A_1 & \text{se} \quad t \in [0, \pi) \\ A_2 & \text{se} \quad t \in [0, 2\pi) \end{cases}$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -\omega_1^2 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -\omega_2^2 & 0 \end{pmatrix}$$

$$\omega_1 = \omega + \varepsilon \qquad \omega_2 = \omega - \varepsilon$$

→ def matrice trasporto

$$\phi_t \, \underline{x}_0 = e^{tA_1} \underline{x}_0 \quad \text{se} \quad t \in [0, \pi)$$

$$\phi_t \, \underline{x}_0 = e^{(t-\pi)A_2} e^{\pi A_1} \underline{x}_0 \qquad t \in [\pi, 2\pi)$$

$$\phi_{2\pi} \, \underline{x}_0 = \underbrace{e^{\pi A_2} e^{\pi A_1}}_{B \,=\, B_2 B_1} \underline{x}_0$$

$$\phi_{n 2\pi} \underline{x}_0 = B^n \underline{x}_0 \qquad \text{con} \quad \underset{i=1,2}{B_i} = \begin{pmatrix} \cos \omega_i \pi & \dfrac{\sin \omega_i \pi}{\omega_i} \\ -\omega_i \sin \omega_i \pi & \cos \omega_i \pi \end{pmatrix}$$

Proposizione

$\underline{x}_0$ è pto fisso di B $(B\underline{x}_0 = \underline{x}_0)$

$\Longleftrightarrow$ $\phi_t \, x_0$ è periodica

Soluz. periodica è stabile se

$\forall \varepsilon \quad \exists \delta$ t.c. se $|\underline{x} - \underline{x}_0| < \delta \Rightarrow |B^n \underline{x} - \underbrace{B^n \underline{x}_0}_{x_0}| < \varepsilon$

$\forall n$

$\underline{x}_0 = \underline{0}$ è pto fisso di B

Voglio dire che $\exists$ scelta ω, ε t.c.

$\underline{0}$ è instabile cioè

$\exists \, \varepsilon_0$ t.c. $\forall \delta \quad \exists \, \underline{x}_0$ con $|\underline{x}_0| < \delta$

t.c. $|\phi_t \, x_0| > \varepsilon_0$ per qualche t

con $t = n \, 2\pi$

$\Longleftrightarrow \quad |B^n \underline{x}_0| > \varepsilon_0$

Osservazione:

se B ha a.v. reali distinti $\lambda_1 < \lambda_2$

$\det B = \det B_1 \det B_2 = 1$

$\lambda_1 \cdot \lambda_2 = 1 \quad \Rightarrow \quad \lambda_2 > 1$

$\Rightarrow$ $\underline{x}_2$ a.vettore corrisp. a λ_2

$\underline{x}_0 = \delta \, \underline{x}_2 \qquad B^n \underline{x}_0 = \delta \, \lambda_2^n \, x_2 \Rightarrow |\phi_{2\pi n} x_0| = \delta \, \lambda_2^n$

eq. autovalori

$$\lambda^2 - \mathrm{tr} B\, \lambda + 1 = 0$$

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ (a-\lambda)(d-\lambda) - cb = 0 \\ \lambda^2 - \underbrace{(a+d)}_{\mathrm{tr}}\lambda + \underbrace{ad - cb}_{\det} = 0 \end{cases}$$

$$\lambda = \frac{\mathrm{tr} B \pm \sqrt{(\mathrm{tr} B)^2 - 4}}{2}$$

condizione $\lambda_1 < \lambda_2 \in \mathbb{R} \iff |\mathrm{tr} B| > 2$

abbiamo $B_i = \begin{pmatrix} c_i & s_i/\omega_i \\ -\omega_i s_i & c_i \end{pmatrix}$ $\quad i = 1, 2$

$$B = \begin{pmatrix} c_2 & s_2/\omega_2 \\ -\omega_2 s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1/\omega_1 \\ -\omega_1 s_1 & c_1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_1 c_2 - \frac{\omega_1}{\omega_2} s_1 s_2 & s_1 c_2/\omega_1 + c_1 s_2/\omega_2 \\ -c_1 s_2 \omega_2 - s_1 c_2 \omega_1 & -\frac{\omega_2}{\omega_1} s_1 s_2 + c_1 c_2 \end{pmatrix}$$

$$\mathrm{tr} B = 2 c_1 c_2 - \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) s_1 s_2 =$$

$$= \underbrace{2 c_1 c_2 - 2 s_1 s_2}_{2 \cos 2\pi\omega} + \left(2 - \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) \right) s_1 s_2$$

$$\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} = \frac{(\omega+\varepsilon)^2 + (\omega-\varepsilon)^2}{\omega^2 - \varepsilon^2} = \frac{2\omega^2 + 2\varepsilon^2}{\omega^2 - \varepsilon^2}$$

$$= \frac{2(\omega^2 - \varepsilon^2) + 4\varepsilon^2}{\omega^2 - \varepsilon^2} = 2 + \frac{4\varepsilon^2}{\omega^2 - \varepsilon^2}$$

$$=: 2(1 + \Delta)$$

$$\Delta := \frac{2\varepsilon^2}{\omega^2 - \varepsilon^2} = \frac{2\varepsilon^2}{\omega^2} + O(\varepsilon^4)$$

$$2 s_1 s_2 = \cos 2\pi\varepsilon - \cos 2\pi\omega$$

$$2\varepsilon = \omega_1 - \omega_2$$
$$2\omega = \omega_1 + \omega_2$$

$\Rightarrow$

$$\operatorname{tr} B = 2\cos 2\pi\omega - \Delta(\cos 2\pi\varepsilon - \cos 2\pi\omega)$$
$$= (2+\Delta)\cos 2\pi\omega - \Delta\cos 2\pi\varepsilon$$

$$\cos 2\pi\varepsilon = 1 - \frac{1}{2}(2\pi\varepsilon)^2 + \mathcal{O}(\varepsilon^4)$$

se $\omega = k$ $\qquad \operatorname{tr} B = 2+\Delta - \Delta(1 - 2\pi\varepsilon^2 + \mathcal{O}(\varepsilon^4))$
$$= 2 + \Delta 2\pi\varepsilon^2 + o(\varepsilon^4) > 2$$

analogo per $\omega = k + \frac{1}{2}$

per $\omega = k + \delta$:

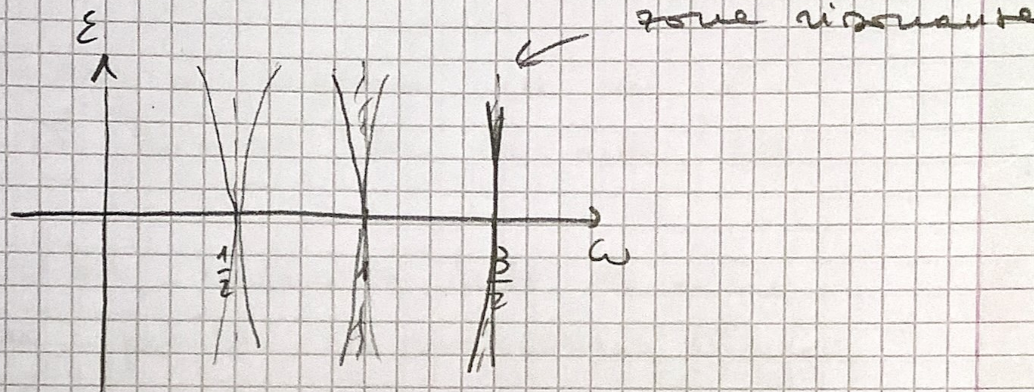
zona risonante

$$(2+\Delta)(1 - 2\pi^2\delta^2) + \mathcal{O}(\delta^4) + \Delta(1 - 2\pi^2\varepsilon^2) + o(\varepsilon^4) > 2$$
$$-4\pi^2\delta^2 - \Delta 2\pi^2\delta^2 + \Delta 2\pi^2\varepsilon^2 + \mathcal{O}(\delta^4) + o(\varepsilon^4) > 0$$
$$-(2+\Delta)\delta^2 + \Delta\varepsilon^2 > 0$$

$$\delta^2 < \frac{\Delta\varepsilon^2}{\Delta+2} \qquad \frac{\Delta}{\Delta+2} = \frac{\cancel{2}\varepsilon^2}{\omega^2}\frac{1}{\cancel{2}} + \mathcal{O}(\varepsilon^4)$$

per $\omega = k + \delta$

$$\Rightarrow \quad \delta^2 < \frac{\varepsilon^4}{k^2} + o(\varepsilon^4)$$



vd. V. I. Arnold
Metodi Matematici della Meccanica
Classica
Ed. Riuniti