**Università degli studi Roma Tre - Corso di Laurea in Matematica**
**Tutorato di ST1 - A.A. 2005/2006**
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Soluzioni del tutorato n.1 del 9/3/2006

**Esercizio 0.** Funzione generatrice dei momenti di una Normale bidimensionale:

$$\mathbb{E}(e^{tZ}) = \mathbb{E}(e^{t_1X+t_2Y}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{t_1x+t_2y} f(x,y)dxdy =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{t_1x+t_2y} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} exp\Bigg\{ -\frac{1}{2(1-\rho^2)}\Bigg[\Big(\frac{x-\mu_x}{\sigma_x}\Big)^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y} +$$

$$+\Big(\frac{y-\mu_y}{\sigma_y}\Big)^2\Bigg]\Bigg\}dxdy =$$

Applichiamo ora la sostituzione: $u = \frac{x-\mu_x}{\sigma_x}$ e $v = \frac{y-\mu_y}{\sigma_y}$, quindi $x = \sigma_x u + \mu_x$ e $y = \sigma_y v + \mu_y$, $du = \frac{dx}{\sigma_x}$ e $dv = \frac{dy}{\sigma_y}$.

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{t_1\mu_x+t_2\mu_y} \quad e^{t_1\sigma_x u+t_2\sigma_y v}}{2\pi\sqrt{1-\rho^2}} exp\Bigg\{ -\frac{1}{2(1-\rho^2)}\Bigg[u^2 - 2\rho uv + v^2\Bigg]\Bigg\}dudv =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{t_1\mu_x+t_2\mu_y}}{2\pi\sqrt{1-\rho^2}} exp\Bigg\{ -\frac{1}{2(1-\rho^2)}\Bigg[u^2 - 2\rho uv + v^2 - 2(1-\rho^2)t_1\sigma_x u +$$

$$-2(1-\rho^2)t_2\sigma_y v\Bigg]\Bigg\}dudv =$$

Consideriamo ora solo l'argomento dell'esponenziale dipendente dalle due variabili, completiamo il quadrato rispetto ad u e v aggiungendo e togliendo, quindi, opportune quantità, esso diventa:

$$= \Bigg\{ -\frac{1}{2(1-\rho^2)}\Bigg[\Big(u - \rho v - (1-\rho^2)t_1\sigma_x\Big)^2 +$$

$$-(1-\rho^2)^2 t_1^2\sigma_x^2 - \rho^2 v^2 - 2(1-\rho^2)\rho t_1\sigma_x v + v^2 - 2(1-\rho^2)t_2\sigma_y v\Bigg]\Bigg\} =$$

$$= \Bigg\{ -\frac{1}{2(1-\rho^2)}\Bigg[\Big(u - \rho v - (1-\rho^2)t_1\sigma_x\Big)^2 + (1-\rho^2)\Big(v^2 - (1-\rho^2)t_1^2\sigma_x^2 +$$

$$-2\rho t_1\sigma_x v - 2t_2\sigma_y v\Big)\Bigg]\Bigg\} =$$

$$= \Bigg\{ -\frac{1}{2(1-\rho^2)}\Bigg[\Big(u - \rho v - (1-\rho^2)t_1\sigma_x\Big)^2 + (1-\rho^2)(v - \rho t_1\sigma_x - t_2\sigma_y)^2 +$$

$$-(1-\rho^2)(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)\Bigg]\Bigg\} =$$

Definiamo ora $w = \frac{u-\rho v-(1-\rho^2)t_1\sigma_x}{\sqrt{1-\rho^2}}$ e $z = v - \rho t_1\sigma_x - t_2\sigma_y$ e quindi $dw = \frac{du}{\sqrt{i-\rho^2}}$ e $dz = dv$, l'esponenziale diventa:

$$= exp\Big\{ -\frac{1}{2}\Big[(w)^2 + (z)^2 - (t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)\Big]\Big\} =$$
$$= exp\Big\{ -\frac{1}{2}\Big[(w)^2 + (z)^2\Big]\Big\} exp\Big\{\frac{1}{2}\Big[ -(t_1^2\sigma_x^2 + t_2^2\sigma_y^2 + 2\rho t_1 t_2\sigma_x\sigma_y)\Big]\Big\}$$

Quindi tornando all'intero integrale:

$$\mathbb{E}(e^{tZ}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{t_1\mu_x+t_2\mu_y} \quad e^{\frac{1}{2}(t_1^2\sigma_x^2+t_2^2\sigma_y^2+2\rho t_1t_2\sigma_x\sigma_y)}}{2\pi} e^{-\frac{1}{2}(w^2+z^2)}dwdz$$
$$= e^{t_1\mu_x+t_2\mu_y+\frac{1}{2}(t_1^2\sigma_x^2+t_2^2\sigma_y^2+2\rho t_1t_2\sigma_x\sigma_y)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(w^2+z^2)}dwdz =$$
$$= e^{t_1\mu_x+t_2\mu_y+\frac{1}{2}(t_1^2\sigma_x^2+t_2^2\sigma_y^2+2\rho t_1t_2\sigma_x\sigma_y)}$$

poiché

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(w^2+z^2)}dwdz = 1$$

essendo la funzione integranda la densità di una normale bidimensionale che integrata dà quindi 1.

**Esercizio 1.**

(a) $\mathbb{P}(\min\{X,Y\} \leqslant x) = 1 - \mathbb{P}(\min\{X,Y\} > x) = 1 - \mathbb{P}(X > x, Y > x) =$

$= \text{(indipendenza)} 1 - \mathbb{P}(X > x)\mathbb{P}(Y > x) = 1 - \mathbb{P}(X > x)^2 = 1 - \left[\sum_{k=x+1}^{+\infty} 2^{-k}\right]^2,$

(b) $\mathbb{P}(Y > X) = \sum_{k=1}^{+\infty} \mathbb{P}(Y > X, X = k) = \sum_{k=1}^{+\infty} \mathbb{P}(Y > X | X = k)\mathbb{P}(X = k) =$

$= \sum_{k=1}^{+\infty} \mathbb{P}(Y > k)\mathbb{P}(X = k) = \sum_{k=1}^{+\infty} 2^{-k}2^{-k} = \sum_{k=1}^{+\infty} 4^{-k} = \sum_{j=1}^{+\infty}\Big(\frac{1}{4}\Big)^{j+1} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3},$

(c) $\mathbb{P}(X = Y) = \sum_{k=1}^{+\infty} \mathbb{P}(X = Y = k) = \sum_{k=1}^{+\infty} \mathbb{P}(X = k)\mathbb{P}(Y = k) =$

$= \sum_{k=1}^{+\infty} 2^{-k}2^{-k} = \sum_{k=1}^{+\infty} 4^{-k} = \frac{1}{3},$

(d) $\mathbb{P}(X \geqslant kY) = \sum_{x=1}^{+\infty} \mathbb{P}(X \geqslant kY, Y = x) = \sum_{x=1}^{+\infty} \mathbb{P}(X \geqslant kx)\mathbb{P}(Y = x) = \sum_{x=1}^{+\infty} 2^{-kx+1}2^{-x} =$

$= 2\sum_{x=1}^{+\infty} 2^{-x(k+1)} = 2\sum_{x=1}^{+\infty}\left[\Big(\frac{1}{2}\Big)^{k+1}\right]^x = 2\sum_{j=0}^{+\infty}\left[\Big(\frac{1}{2}\Big)^{k+1}\right]^{j+1} =$

$= 2\Big(\frac{1}{2}\Big)^{k+1}\frac{1}{1-(\frac{1}{2})^{k+1}} = \frac{2}{2^{k+1}-1}.$

**Esercizio 2.**

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases} \qquad i = 1, \dots, N, \quad S_N = \sum_{i=1}^{N} X_i$$

$$\mathbb{P}(S_N = k) = \sum_{n=k}^{+\infty} \mathbb{P}(S_N = k | N = n)\mathbb{P}(N = n) = \sum_{n=k}^{+\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n e^{-\lambda}}{n!} =$$

$$= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \sum_{n=k}^{+\infty} \frac{\lambda^{n-k}(1-p)^{n-k} e^{-\lambda(1-p)}}{(n-k)!} = \frac{(\lambda p)^k e^{-\lambda p}}{k!},$$

poiché $\frac{\lambda^{n-k}(1-p)^{n-k} e^{-\lambda(1-p)}}{(n-k)!}$ è la distribuzione di una $\mathrm{Po}(\lambda(1-p))$.

Quella ottenuta è la distribuzione di una $\mathrm{Po}(\lambda p)$.

**Esercizio 3.**

(a) $$\mathbb{E}(aY + bZ|X) = \sum_{y,z} (ay + bz)\ \mathbb{P}(Y = y, Z = z | X = x) =$$

$$= a \sum_{y,z} y\ \mathbb{P}(Y = y, Z = z | X = x) + b \sum_{y,z} z\ \mathbb{P}(Y = y, Z = z | X = x) =$$

$$= a \sum_{y} y\ \mathbb{P}(Y = y | X = x) + b \sum_{z} z\ \mathbb{P}(Z = z | X = x) = a\mathbb{E}(Y|X) + b\mathbb{E}(Z|X),$$

(b) $$\mathbb{E}(X|Y) = \sum_x x\ \mathbb{P}(X = x | Y = y) = \sum_x x\ \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \mathbb{E}(X).$$

**Esercizio 4.**

(a) $$1 = \int_0^{+\infty} dx \int_x^{+\infty} dy\ cx(y-x)e^{-y} = \int_0^{+\infty} dx\ cx \left[ \int_x^{+\infty} ye^{-y} dy - x \int_x^{+\infty} e^{-y} dy \right] =$$

$$= \int_0^{+\infty} dx\ cx \left[ -ye^{-y} \Big|_x^{+\infty} + \int_x^{+\infty} e^{-y} dy - x(-e^{-y}) \Big|_x^{+\infty} \right] =$$

$$= \int_0^{+\infty} dx\ cx \left[ xe^{-x} + (-e^{-y}) \Big|_x^{+\infty} - xe^{-x} \right] = \int_0^{+\infty} dx\ cxe^{-x} =$$

$$= c \left[ -xe^{-x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-x} dx \right] = c(-e^{-x}) \Big|_0^{+\infty} = c \quad \Rightarrow \quad c = 1.$$

(b) $$f_Y(y) = \int_0^y f(x,y)dx = \int_0^y x(y-x)e^{-y}dx = e^{-y}\left[\int_0^y xy\ dx - \int_0^y x^2\ dx\right] =$$
$$= e^{-y}\left[y\ \frac{y^2}{2} - \frac{y^3}{3}\right] = \frac{1}{6}e^{-y}y^3.$$
$$f_X(x) = \int_x^{+\infty} x(y-x)e^{-y}dy = x\left[\int_x^{+\infty} ye^{-y}dy - x\int_x^{+\infty} e^{-y}dy\right] =$$
$$= x\left[-ye^{-y}\Big|_x^{+\infty} + \int_x^{+\infty} e^{-y}dy - x(-e^{-y})\Big|_x^{+\infty}\right] =$$
$$= x\left[xe^{-x} + (-e^{-y})\Big|_x^{+\infty} - xe^{-x}\right] = xe^{-x}.$$
$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x(y-x)e^{-y}}{\frac{1}{6}e^{-y}y^3} = 6x\frac{y-x}{y^3}$$
$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x(y-x)e^{-y}}{xe^{-x}} = (y-x)e^{x-y}$$

(c) $$\mathbb{E}(X|Y) = \int_0^y xf_{X|Y}(x|y)dx = \int_0^y 6x^2\frac{y-x}{y^3}dx = \frac{6}{y^3}\left[\int_0^y x^2y\ dx - \int_0^y x^3\ dx\right] =$$
$$= \frac{6}{y^3}\left[\frac{y^4}{3} - \frac{y^4}{4}\right] = \frac{6}{y^3}\frac{y^4}{12} = \frac{y}{2} \quad \text{e} \quad \mathbb{E}(X|Y) = \frac{1}{2}Y.$$
$$\mathbb{E}(Y|X) = \int_x^{+\infty} f_{Y|X}(y|x)dy = \int_x^{+\infty} y(y-x)e^{x-y}dy =$$
$$= \int_x^{+\infty} y^2e^{x-y}dy - x\int_x^{+\infty} ye^{x-y}dy =$$
$$= e^x\left[-y^2e^{-y}\Big|_x^{+\infty} + \int_x^{+\infty} 2ye^{-y}dy\right] - xe^x\left[-ye^{-y}\Big|_x^{+\infty} + \int_x^{+\infty} e^{-y}dy\right] =$$
$$= e^x\left[x^2e^{-x} + 2\big(-ye^{-y}\Big|_x^{+\infty} + \int_x^{+\infty} e^{-y}dy\big)\right] - xe^x\left[xe^{-x} - e^{-y}\Big|_x^{+\infty}\right] =$$
$$= e^x\left[x^2e^{-x} + 2xe^{-x} - 2e^{-y}\Big|_x^{+\infty}\right] - xe^x\left[xe^{-x} + e^{-x}\right] = x^2 + 2x + 2 - x^2 - x =$$
$$= x + 2 \cdot \frac{1}{2} \quad \text{e} \quad \mathbb{E}(Y|X) = X + 2.$$

**Esercizio 5.**

$$\mathbb{E}(e^{tX}) = \sum_{k=0}^{+\infty} e^{tk}\mathbb{P}(X=k) = \sum_{k=0}^{+\infty} e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} = \frac{e^{-\lambda}}{e^{-\lambda e^t}}\sum_{k=0}^{+\infty}\frac{(\lambda e^t)^k}{k!}e^{-\lambda e^t} = e^{-\lambda(1-e^t)}.$$
$$\mathbb{E}(X) = \frac{d}{dt}\mathbb{E}(e^{tX})\Big|_{t=0} = e^{-\lambda(1-e^t)}(\lambda e^t)\Big|_{t=0} = \lambda.$$
$$\mathbb{E}(X^2) = \frac{d^2}{dt^2}\mathbb{E}(e^{tX})\Big|_{t=0} = e^{-\lambda(1-e^t)}(\lambda e^t)^2 + e^{-\lambda(1-e^t)}(\lambda e^t)\Big|_{t=0} = \lambda^2 + \lambda.$$
$$\Rightarrow Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

$$\mathbb{E}(e^{tS}) = \mathbb{E}\left(e^{t\sum_{i=1}^n X_i}\right) = \mathbb{E}\left(\prod_{i=1}^n e^{tX_i}\right) = \left(\mathbb{E}(e^{tX_1})\right)^n = \left(e^{-\lambda(1-e^t)}\right)^n = e^{-n\lambda(1-e^t)}$$

e questa è la f.g.m. di una $Po(n\lambda)$.

**Esercizio 6.**

$$\mathbb{E}(e^{tX}) = \int_0^{+\infty} e^{tx}e^{-\lambda}dx = \int_0^{+\infty} \lambda e^{-(\lambda-t)x}dx = -\lambda \frac{e^{-(\lambda-t)x}}{\lambda - t}\Bigg|_0^{+\infty} = \frac{\lambda}{\lambda - t}.$$

$$\mathbb{E}(X) = \frac{d}{dt}\mathbb{E}(e^{tX})\Big|_{t=0} = \frac{\lambda}{(\lambda-t)^2}\Bigg|_{t=0} = \frac{1}{\lambda}.$$

$$\mathbb{E}(X^2) = \frac{d^2}{dt^2}\mathbb{E}(e^{tX})\Big|_{t=0} = \frac{2\lambda}{(\lambda-t)^3}\Bigg|_{t=0} = \frac{2}{\lambda^2}.$$

$$\Rightarrow Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$