PERIODIC SOLUTIONS OF BIRKHOFF–LEWIS TYPE FOR THE NONLINEAR WAVE EQUATION

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ABSTRACT. We prove the existence of infinitely many periodic solutions accumulating to zero for the one-dimensional nonlinear wave equation (vibrating string equation). The periods accumulate to zero and are both rational and irrational multiples of the string length.

1. A Poincaré conjecture. The wave equation is an infinite dimensional Hamiltonian system. The importance of periodic solutions for finite dimensional Hamiltonian system was pointed out by Poincaré in [18]: “D’ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c’est qu’elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu’ici réputée inabordable.” Although periodic solutions are “few” “en effet, il ya une probabilité nulle pour que les conditions initiales du mouvement soient précisément celles qui correspondent à une solutions périodique,” however Poincaré stressed their importance formulating the following conjecture: “...voici un fait que je n’ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 1[3] et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu’on le veut, pendant un temps aussi long qu’on le veut.”

This conjecture stimulates the systematic study of periodic solutions by Poincaré himself, Lyapunov, Birkhoff, Moser, Weinstein etc.

In [19] a positive answer to the conjecture was given, but only in a generic sense (namely in the C^2-category of Hamiltonian functions): the periodic orbits are dense on every compact and regular energy surface.

On the other hand, for specific systems, the conjecture is still open (and far from being proved).

As an intermediate step, one can look for periodic orbits accumulating onto invariant manifolds. Indeed it was proved that periodic orbits accumulate onto:

1. elliptic periodic orbits, by Birkhoff and Lewis in the thirties, see [10];
2. maximal KAM tori, by Conley and Zehnder in the eighties, see [12];
3. elliptic tori of every dimension, recently in [6].
In this note we will discuss the existence of periodic orbits of the nonlinear wave equation accumulating to the origin which is an elliptic equilibrium of the associated infinite dimensional hamiltonian system.

The hamiltonian structure. We look for periodic in time solutions of the one–dimensional nonlinear wave equation with Dirichlet boundary conditions (vibrating string):

\[
\begin{aligned}
&\begin{cases}
  u_{tt} - u_{xx} + \mu u + f(u) = 0 \\
  u(t,0) = u(t,\pi) = 0,
\end{cases}
\end{aligned}
\]

where \(\mu > 0\) is the “mass” and \(f\), with \(f(0) = f'(0) = 0\), is the nonlinearity.

Introducing \(v = u_t\) the Hamiltonian is

\[
H(v,u) = \int_0^\pi \left( \frac{v^2}{2} + \frac{u_x^2}{2} + \mu \frac{u^2}{2} + g(u) \right) \, dx,
\]

where \(g = \int_0^u f'(s) \, ds\). The Hamilton’s equations are

\[
\begin{aligned}
u_t &= \partial H / \partial v = v, \\
v_t &= -\partial H / \partial u = u_{xx} - \mu u - f(u).
\end{aligned}
\]

Introducing coordinates \(q = (q_1, q_2, \ldots) \in \ell^{a,s}\), \(p = (p_1, p_2, \ldots) \in \ell^{a,s}\) with

\[
\ell^{a,s} := \left\{ \left| q \right|_{a,s}^2 := \sum |q_i|^2 2^{2s} e^{2ai} < \infty \right\}
\]

by means of the relations

\[
v(x) = \sum_i \sqrt{\omega_i} p_i \chi_i(x), \quad u(x) = \sum_i \frac{q_i}{\sqrt{\omega_i}} \chi_i(x),
\]

where \(\chi_i(x) := \sqrt{2/\pi} \sin ix\) and

\[
\omega_i := \sqrt{i^2 + \mu},
\]

the Hamiltonian takes the form

\[
H = \frac{1}{2} \sum_i \omega_i (q_i^2 + p_i^2) + P(p,q).
\]

The origin is an elliptic equilibrium.

The term \(P\) is regularizing, in the sense that its gradient gains a derivative

\[
\nabla P : \ell^{a,s} \times \ell^{a,s} \rightarrow \ell^{a,s+1} \times \ell^{a,s+1}.
\]

2. The linear equation. The orbits of the quadratic Hamiltonian \(\frac{1}{2} \sum \omega_i (q_i^2 + p_i^2)\) are superpositions of the harmonic oscillations \(q_i(t) = A_i \cos (\omega_i t + \varphi_i)\) where \(A_i \geq 0\), \(\varphi_i \in \mathbb{R}\) and \(\omega_i\) are, respectively, the amplitude, the phase and the frequency of the \(i\)-th harmonic oscillator. Analogously, the solutions of the linear equations

\[
\begin{aligned}
u_{tt} - \nu_{xx} + \mu u = 0,
\end{aligned}
\]

are

\[
u(t,x) = \sum_{i \geq 1} a_i \cos (\omega_i t + \varphi_i) \sin ix,
\]

with \(a_i = A_i \sqrt{2/\pi \omega_i} \geq 0\). These solutions are

(a) periodic if one mode is excited, namely \(\exists i_1 \geq 1\) such that \(a_{i_1} = 0\), \(\forall i \neq i_1\),
(b) quasi–periodic if $N \geq 2$ modes are excited, namely $\exists 1 \leq i_1 < \ldots < i_N$ such that $a_{i_i} \neq 0$ for $i = i_1$ and $a_i = 0$ for $i \neq i_j, 1 \leq j \leq N$.

(c) almost–periodic if infinitely many modes are excited, namely when infinitely many $a_i$ are different from zero.

We note that in case (b) (except a countable set of $\mu > 0$) the solutions are not periodic since $\forall I := \{i_1, \ldots, i_N\} \subset \mathbb{N}^+, N \geq 2$, the vector $\omega := (\omega_{i_1}, \ldots, \omega_{i_N})$ is rationally independent.

3. Extensions of the Lyapunov center theorem. A natural way to find periodic solutions of the wave equation (1) is to extend the Lyapunov Center Theorem for finite dimensional hamiltonian systems to the infinite dimensional Hamiltonian $H$ in (2): for any $i_0 \geq 1$, one constructs a family of small amplitude periodic orbits of $H$ bifurcating from the $i_0$–th mode. If $\mu > 0$, the linear frequency $\omega_{i_0}$ is non–resonant with $\omega_i, i \neq i_0$. However, differently from the finite dimensional case, the non–resonance condition on the frequencies deteriorates when $i$ goes to infinity and this gives rise to a small divisors problem. The frequencies $\tilde{\omega}$ of the solutions will be close to the linear frequency $\omega_{i_0}$ and the corresponding periods $2\pi/\tilde{\omega}$ will be close to the linear one $2\pi/\omega_{i_0}$. Note that these solutions are nonlinear continuations of the solutions of the linear problem (see case (a) above); in particular they have no large period. Various extensions of the Lyapunov Center Theorem can be found in [15],[17],[20],[16],[13],[11],[1].

4. A different kind of periodic solutions. The orbits we will find are not of “Lyapunov type”, they are rather of “Birkhoff–Lewis type” (recall [10]). They are not continuations of periodic orbits of the linear case. Indeed we will excite $N \geq 2$ modes, that corresponds to have quasi–periodic (not periodic!) solutions of the linear equation (see case (b) above). Therefore the solutions we find are a merely nonlinear phenomenon. Moreover they have very large minimal periods (really going to infinity), representing an interesting example of the complexity of the dynamics.

In the following we will discuss the existence of these solutions extending the finite dimensional techniques of [3] to our infinite dimensional situation. Two difficulties arise: the extension of the Birkhoff Normal Form and, in particular, a small divisors problem. The small divisors we will deal with (see (8) below) are different from the ones arising in extending the Lyapunov Center Theorem. We stress that, due to the small divisors, to obtain a positive measure set of periods we will have to use the Nash–Moser Implicit Function Theorem.

“Birkhoff–Lewis type” solutions for the beam equation and the NLS equation, were found in [2]. In that case the standard Implicit Function Theorem can be successfully applied (see Remark 1 for comparison).

5. Partial Birkhoff normal form. Following [18], we suppose that $f$ is real analytic and odd $f = \sum_{m \geq 3} f_m u^m$ with $f_3 \neq 0$ and we fix a finite subset $I := \{i_1, \ldots, i_N\} \subset \mathbb{N}$. Since, for $\mu > 0$, the linear frequencies $\omega_i = \sqrt{i^2 + \mu}, i \in I$, satisfy suitable non–resonance conditions, we can perform a canonical transformation putting $H$ in partial Birkhoff Normal Form:

$$H = \Lambda + \bar{G} + \bar{G} + K$$

where

- $\Lambda = \sum_{i \geq 1} \omega_i (p_i^2 + q_i^2)/2$ quadratic part
• $\bar{G} + \hat{G}$ is of order 4 in $p, q$
• $K$ is of order 6 in $p, q$
• $\bar{G}$ depends only by the “actions” $(p_i^2 + q_i^2)/2$
• $\hat{G}$ depends only by $p_i, q_i, i \notin I$,

Since we look for small amplitude solutions, we introduce the perturbative parameter $\eta$ (distance from the origin). Therefore we use action–angle variables $(I, \phi)$ on the $I$–modes:

$$(p_i, q_i) = \eta\left(\sqrt{I_i} \sin \phi_i, \sqrt{I_i} \cos \phi_i\right), \quad i \in I,$$

$$(p_i, q_i) = \eta(p_i, q_i), \quad i \notin I.$$

In the new variables $(I, \phi, p, q)$ the Hamiltonian writes:

$$\omega \cdot I + \Omega \cdot Z + \eta^2 \left[\frac{1}{2} AI \cdot I + BI \cdot Z + \hat{G}(p, q)\right] + \eta^4 K$$

where

$$\omega = (\omega_i)_{i \in I}, \quad \Omega = (\omega_i)_{i \notin I},$$

$$Z = (Z_i)_{i \notin I}, \quad Z_i := (p_i^2 + q_i^2)/2,$$

$$A \in \text{Mat}(N \times N) \quad B \in \text{Mat}(\infty \times N).$$

Notice that the “twist” condition $\det A \neq 0$ holds (since $f_3 \neq 0$).

6. The geometry of the problem: the integrable Hamiltonian. Let us consider the solutions of the integrable Hamiltonian

$$H_{\text{int}}(I, \phi, p, q) := \omega \cdot I + \Omega \cdot Z + \eta^2 \left[\frac{1}{2} AI \cdot I + BI \cdot Z\right]$$

$$\begin{cases}
  I(t) = I_0 \\
  \phi(t) = \phi_0 + \tilde{\omega}t + \eta^2 B^t(p_0^2 + q_0^2)t/2 \\
  (p_i(t), q_i(t)) = (p_{i0}, q_{i0}) \cos \tilde{\Omega}_it, \quad i \notin I
\end{cases}$$

where

$$\tilde{\omega} := \omega + \eta^2 Al_0$$

is the vector of the shifted linear frequencies and

$$\tilde{\Omega}_i := \Omega_i + \eta^2 (Bl_0)_i, \quad i \notin I$$

is the vector of the shifted elliptic frequencies. The manifold $\{p = q = 0\}$ is invariant for the $H_{\text{int}}$–flow and it is filled up by the $N$–dimensional tori

$$T(I_0) := \{I = I_0, \phi \in T^N, p = q = 0\}.$$

On $T(I_0)$ the $H_{\text{int}}$–flow

$$t \mapsto (I_0, \phi_0 + \tilde{\omega}t, 0, 0)$$

is $T$–periodic, $T > 0$, if and only if

$$\tilde{\omega}T/2\pi = k \in \mathbb{Z}^N. \quad (4)$$

In this case $T(I_0)$ is completely resonant and supports the infinite $T$–periodic orbits of the family

$$\mathcal{F} := \{I(t) = I_0, \phi(t) = \phi_0 + \tilde{\omega}t, \ p(t) = q(t) = 0\}.$$

$\mathcal{F}$ will not persist in its entirety for the flow of the complete Hamiltonian $H$. But if $\mathcal{F}$ is isolated, namely there are no other $T$–periodic solutions close to it, i.e.

$$\ell - \tilde{\Omega}_i T/2\pi \neq 0, \quad (5)$$
then we can think that at least $N$ solutions of $H$ bifurcate from $F$. To satisfy the periodicity condition \[ I_0 = I_0(T) := -A^{-1}\{\omega T/2\pi\} \]
\[ k = k(T) := [\omega T/2\pi] \]
\[ \eta := \sqrt{2\pi/T} \]
(connecting the small perturbative parameter $\eta$ with the period $T$). Then (6) becomes
\[ \ell - \Omega_i T/2\pi + A^{-1}B \left\{ \omega T/2\pi \right\} \neq 0, \quad \forall \ell \in \mathbb{Z}, \ i \notin I \]
clearly appearing as a non–resonance condition between the linear frequencies and the elliptic ones.

7. The Lyapunov-Schmidt decomposition. We look for periodic solutions close to the family $F$ namely
\[ (I(t), \phi(t), p(t), q(t)) = (I_0, \tilde{\omega}t + \phi_0, 0) + \zeta(t) \]
with $I_0$ defined above, $\phi_0 \in \mathbb{T}^N$ parameter to determinate and $\zeta(t) = (J(t), \psi(t), p(t), q(t))$ an analytic and $T$-periodic curve with $\int_0^T \psi = 0$. The Hamilton equations reduce to the following functional equation for $\zeta$
\[ L\zeta = N(\zeta; \phi_0), \quad (7) \]
where $L$ is the linear operator
\[ L\zeta := (\dot{\psi} - \eta^2 AJ, \dot{J}, \dot{q} - \tilde{\Omega}p, \dot{p} + \tilde{\Omega}q) \]
and $N$ is the nonlinearity. The kernel $\mathcal{K}$ and the range $\mathcal{R}$ of $L$ are
\[ \mathcal{K} = \{ \psi \equiv \text{const} \} \]
and
\[ \mathcal{R} = \{ \int_0^T \tilde{\psi} = 0 \}. \]
The equation $L\zeta = N(\zeta; \phi_0)$ splits into the kernel equation
\[ 0 = \Pi_\mathcal{K}N(\zeta; \phi_0) \]
and into the range equation
\[ L\zeta = \Pi_\mathcal{R}N(\zeta; \phi_0). \]

8. Estimating the small divisors: two different strategies. The expressions in (6) are exactly the small divisors. We will quantitatively impose that the $T$–admissible periods verify, $\forall \ell \in \mathbb{Z}, \ i \notin I$
\[ \left| \ell - \omega_i T/2\pi + A^{-1}B \left\{ \omega T/2\pi \right\} \right| \geq \text{cost.} \frac{\text{cost.}}{|i|^\tau} \]
for a suitable
\[ \tau \geq 1. \]
The small divisors estimate is the crucial step:
(a) if we impose the strong non–resonance condition,
\[ \tau = 1, \]
we can apply the standard IFT but $T \in \pi \mathbb{Q}$;
(b) if we impose the weaker diophantine condition

\[ \tau > 1 , \]

we must use the Nash-Moser IFT but \( T \in \pi(\mathbb{R} \setminus \mathbb{Q}) \).

**Case \( \tau = 1 \); periods which are rational multiples of \( \pi \).** For technical reasons we can deal only with rational periods for which

\[ \mu^2 T \ll 1 , \]

namely we consider the mass \( \mu \) as a perturbative parameter (\( \mu \approx \eta \)). For \( \mu \to 0 \) the cardinality of the set of admissible periods will be finite but increasing at infinity.

Since \( \tau = 1 \), the small divisors cause the loss of only one “derivative”, which is compensated by the fact that \( N \) “gains one derivative” (recall (3)); therefore the range equation can be solved by the standard Implicit Function Theorem. After solving the range equation, we solve the kernel equation finding \( \phi_0 \) by variational methods. The following result was proved in [8].

**Theorem 1.** Let \( f \) be an odd real analytic function \( f(u) = \sum_{m \geq 3} f_m u^m \), with \( f_3 \neq 0 \). Let \( N \geq 2 \) and \( \mathcal{I} := \{ i_1 , \ldots , i_N \} \subset \mathbb{N}^+ \). There exist at least \( O(1/\mu) \) geometrically distinct analytic periodic solutions \( u(t,x) \) of (1), verifying

\[ u(t,x) - \mu \sum_{i \in \mathcal{I}} a_i \cos(\tilde{\omega}_i t + \varphi_i) \sin ix = O(\mu^2) , \]

for suitable \( a_i \geq \text{const} > 0 \), \( \varphi_i \in \mathbb{R} \) and \( \tilde{\omega}_i \in \mathbb{R} \)

\[ \tilde{\omega}_i - \omega_i = O(\mu^2) . \]

The minimal period \( T_{\text{min}} \) of every solution belongs to \( \pi \mathbb{Q} \) and satisfies

\[ T_{\text{min}} \geq \frac{\text{cost}}{\mu} . \]

**Remark 1.** In [2], a positive measure set of admissible periods is obtainable just using the Implicit Function Theorem. Indeed, taking \( 1 < \tau < 2 \), there is a positive measure set of \( T \) satisfying (5); the small divisors cause the loss of “\( \tau \) derivative”, which is compensated by the fact that the nonlinearity of the beam equation “gains two derivatives”. On the other hand if one wants a positive measure set of admissible periods for the wave equation, then \( \tau > 1 \) and the loss of “\( \tau \) derivatives” is no more compensated by the gain on one derivative due to the nonlinearity \( N \). A different strategy is needed: we will use the Nash–Moser Implicit Function Theorem.

**Case \( \tau > 1 \); periods which are irrational multiples of \( \pi \).** As it is well known the crucial point in the Nash–Moser scheme is the inversion of the linearized operator. Moreover, arguing as in the case \( \tau = 1 \) above, due to the excision procedure used in the iterative algorithm to control small divisors, one should solve the bifurcation equation on a Cantor set (see [4],[3],[5]).

To overcome this last difficulty we invert the procedure solving first the kernel equation using suitable symmetries of the Hamiltonian and thereafter the range equation by a Nash–Moser scheme.

Unfortunately the resulting linearized operator is a first order not symmetric one. This fact requires a not standard spectral analysis (a sort of “weighted asymmetric diagonalization”) and the estimate of the norms of not Toeplitz (namely not product, as usual) operator. The following result is proved in [9] (see also [14]).
Theorem 2. Fix \( \mu > 0 \) and let \( f \) be a real analytic, odd function of the form 
\[ f(u) = \sum_{m \geq 3} f_m u^m, \quad f_3 \neq 0. \]
Let \( N \geq 2 \) and \( I := \{i_1, \ldots, i_N\} \subset \mathbb{N}^+ \). Then there exists a set \( C \) satisfying 
\[ \lim_{r \to 0^+} \frac{\text{meas}(C \cap (0, r])}{r} = 1, \]
such that for all \( \varpi = 2\pi/T \in C \) there exists a \( T \)-periodic analytic solution \( u(t, x) \) of (1), satisfying 
\[ u(t, x) - \sqrt{\varpi} \sum_{i \in I} a_i \cos(\tilde{\omega}_i t) \sin ix = O(\varpi), \]
for suitable \( a_i \geq \text{const} > 0, \quad \tilde{\omega}_i \in \mathbb{R} \)
\[ \tilde{\omega}_i - \omega_i = O(\varpi). \]
Moreover, fix \( 0 < \rho < 1/2 \), then, except a zero measure set of \( \mu \)'s, the minimal period \( T_{\text{min}} \) of the \( T \)-periodic orbit satisfies 
\[ T_{\text{min}} \geq \text{const} T^\rho. \]

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