TROPICAL METHODS IN THE MODULI THEORY OF ALGEBRAIC CURVES

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Contents

1. Introduction and Notation

In recent years a series of remarkable advances in tropical geometry and in non-Archimedean geometry have brought new insights to the moduli theory

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of algebraic curves and their Jacobians. The goal of this expository paper is to present some of the results in this area.

There are some important aspects about the interplay between the theories of tropical and algebraic curves which we do not discuss here, to keep the paper to a reasonable length. In particular, we leave out Brill-Noether theory, which has already seen a fruitful interchange between the two areas over the last years and is currently progressing at a fast pace.

We begin the paper by defining, in Section 2, tropical curves as weighted metric graphs, we then construct their moduli space together with its compactification, the moduli space of extended tropical curves.

In Section 3, after introducing Deligne-Mumford stable curves and their moduli space, we explain the connection between tropical curves and families of stable curves parametrized by a local scheme. We then show how to globalize this connection by introducing the Berkovich analytification of the moduli space of stable curves and showing that it has a canonical retraction onto the moduli space of extended tropical curves, which is in turn identified with the skeleton of the moduli stack of stable curves.

In Section 4, highlighting combinatorial aspects, we describe Jacobians of nodal curves and their models over discrete valuation rings, focusing on the Néron model and the Picard scheme. We then turn to compactified Jacobians, introduce a compactification of the moduli space of principally polarized abelian varieties and the compactified Torelli map.

We devote Section 5 to the Torelli theorem for graphs, tropical curves, and stable curves. Although the proofs of these theorems are logically independent, they use many of the same combinatorial ideas, which we tried to highlight in our exposition.

The order we chose to present the various topics does not reflect the actual chronology. As we said, we begin with tropical curves and their moduli space, then discuss the connection with algebraic stable curves also using analytic spaces; we treat the Torelli problem at the end. History goes almost in the opposite direction: moduli spaces for tropical curves and abelian varieties were rigorously constructed after the first Torelli theorem was proved, and in fact the need for such moduli spaces was explicitly brought up by the Torelli theorem; see the appendix in [CV10]. The connection with algebraic curves via non-Archimedean geometry was established a few years later. Although our presentation is not consistent with history, it is, in our opinion, a natural path through the theory, starting from the most basic objects (the curves) and ending with the most complex (the Jacobians).

In writing this paper we tried to address the non-expert reader, hence we included various well known statements in the attempt of making the material more accessible; of course, we claim no originality for that.

1.1. **Notation.**

Unless we specify otherwise, we shall use the following notations and conventions.

We work over an algebraically closed field k.

We denote by K a field containing k and assumed to be complete with respect to a non-Archimedean valuation, written

$$v_K : K \to \mathbb{R} \cup \{\infty\}.$$

Such a K is also called, as in [Ber90], a *non-Archimedean field*. We assume v_K induces on k the trivial valuation, $k^* \to 0$. We denote by R the valuation ring of K.

In this paper any field extension $K'|K$ is an extension of non-Archimedean fields so that K' will be endowed with a valuation inducing v_K.

Curves are always assumed to be reduced, projective and having at most nodes as singularities.

$g \geq 2$ is an integer.

$G = (V, E)$ a finite graph (loops and multiple edges allowed), V and E the sets of vertices and edges. We also write $V = V(G)$ and $E = E(G)$. Our graphs will be connected.

$\mathbf{G} = (G, w)$ denotes a weighted graph.

$\Gamma = (G, w, \ell)$ denotes a tropical curve, possibly extended.

We use the "Kronecker" symbol: for two objects x and y

$$\kappa_{x,y} := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

In drawing graphs we shall denote by a "$\circ$" the vertices of weight 0 and by a "$\bullet$" the vertices of positive weight.

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2. Tropical curves

2.1. **Abstract tropical curves.** We begin by defining abstract tropical curves, also known as "metric graphs", following, with a slightly different terminology, [Mik07b], [MZ08], and [BMV11].

Definition 2.1.1. A *pure tropical curve* is a pair $\Gamma = (G, \ell)$ such that $G = (V, E)$ is a graph and $\ell : E \to \mathbb{R}_{>0}$ is a *length* function on the edges.

More generally, a *(weighted) tropical curve* is a triple $\Gamma = (G, \ell, w)$ such that G and ℓ are as above, and $w : V \to \mathbb{Z}_{\geq 0}$ is a *weight* function on the vertices.

Pure tropical curves shall be viewed as tropical curves whose weight function is identically 0.

The *genus* of the tropical curve $\Gamma = (G, \ell, w)$ is defined as the genus of the underlying weighted graph (G, w), that is:

$$g(\Gamma) := g(G, w) := b_1(G) + \sum_{v \in V} w(v), \tag{1}$$

where $b_1(G) = rk_{\mathbb{Z}} H_1(G, \mathbb{Z})$ is, as usual, the first betti number of G.

Pure tropical curves arise naturally in tropical geometry, and they were the first ones to be studied for sometime; the need to generalize the definition by adding weights on the vertices appeared when studying families. These families are quite easy to generate by varying the lengths of the edges; they have a basic invariant in the genus defined above, which, for pure tropical curves, equals the first betti number.

Now, the problem one encounters when dealing with such families is that if some edge-length tends to zero the genus of the limiting graph may decrease, as in the following example.

Example 2.1.1. Let us pick a pure tropical curve of genus 2, drawn as the first graph from the left in the picture below. Now consider a degeneration in three steps by letting the edge lengths, l_1, l_2, l_3, go to zero one at the time while the other lengths remain fixed; the picture represents the three steps. It is clear that after the second and third step the genus decreases each time, and the last graph has genus 0.

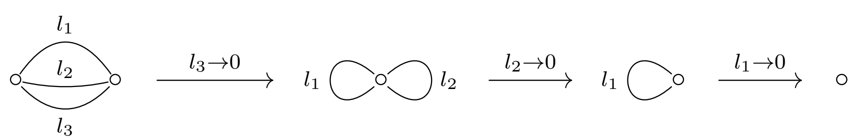


FIGURE 1. Specialization of pure tropical curves

A remedy to this "genus-dropping" problem, introduced in [BMV11], is to add a piece of structure, consisting of a weight function on the vertices, and define *weighted (edge) contractions* in such a way that as a loop-length goes to zero the weight of its base vertex increases by 1. Moreover, when the length of a non-loop edge, e, tends to zero, e gets contracted to a vertex whose weight is defined as the sum of the weights of the ends of e.

For example, let us modify the above picture by assuming the initial curve has both vertices of weight 1, so its genus is 4. As l_3 tends to zero the limit curve has a vertex of weight $1 + 1 = 2$. In the remaining two steps we have a loop getting contracted, so the last curve is an isolated vertex of weight 4.

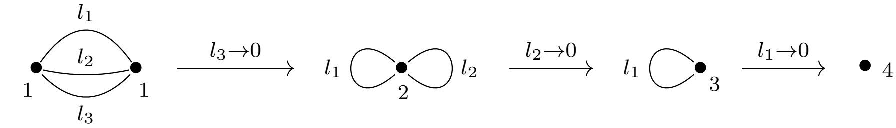


FIGURE 2. Specialization of (weighted) tropical curves

It is quite clear that weighted (edge) contractions can be defined combinatorially, disregarding the length function. Hence we shall denote by

$$(G, w) \to (G', w') \tag{2}$$

a weighted contraction as defined above. Notice that there is a natural inclusion $E(G') \subset E(G)$, and a surjection $V(G) \to V(G')$. As we said, weighted contractions leave the genus invariant.

One thinks of a vertex v of positive weight $w(v)$ as having $w(v)$ invisible loops of zero length based at it; this somewhat naive interpretation turns out to work well in developing a Riemann-Roch divisor theory for (weighted) tropical curves, as carried out in [AC13].

From the point of view of algebraic geometry, the notion of weighted tropical curve is quite natural for at least two reasons. First of all, it resembles the notion of non-reduced schemes (some subsets of which have a "multiple" structure); moreover, just as in the tropical set-up where weights are introduced to better understand families and their limits, non reduced-schemes in algebraic geometry are indispensable to study families of algebraic varieties. Secondly, weighted graphs (with no length function) have been used since the time of [DM69] to describe the moduli space of stable algebraic curves, as we shall explain in Section 3.

According to these analogies, a vertex v of weight $w(v)$ corresponds to a component of geometric genus $w(v)$ (the geometric genus of an irreducible curve is the genus of its desingularization). For example, in the last step of Figure 2, the last graph corresponds to a smooth curve of genus 4, and the previous graph corresponds to a curve of geometric genus 3 with one node. In this analogy, the arrow in the picture has to be reversed and interpreted as a family of smooth curves of genus 4 specializing to a curve (of geometric genus 3) with one node.

2.2. **Equivalence of tropical curves.** Two tropical curves, $\Gamma = (G, \ell, w)$ and $\Gamma' = (G', \ell', w')$ are *isomorphic* if there is an isomorphism between G and G' which preserves both the weights of the vertices and the lengths of the edges.

Isomorphism is thus a natural notion; on the other hand tropical curves are defined to be *equivalent* by a more general relation.

Definition 2.2.1. Two tropical curves, Γ and Γ', of genus at least 2 are *equivalent* if one obtains isomorphic tropical curves, $\overline{\Gamma}$ and $\overline{\Gamma'}$, after performing the following two operations until $\overline{\Gamma}$ and $\overline{\Gamma'}$ have no weight-zero vertex of valency less than 3.

- Remove all weight-zero vertices of valency 1 and their adjacent edge.
- Remove every weight-zero vertex v of valency 2 and replace it by a point, so that the two edges adjacent to v are transformed into one edge.

It is clear that the underlying graph, $(\overline{G}, \overline{w})$, of $\overline{\Gamma}$ has the property that every vertex of weight zero has valency at least 3 (the same holds for $\overline{\Gamma'}$, obviously). Weighted graphs of genus at least 2 with this property are called *stable*; they are old friends of algebraic geometers as we shall see in Section 3. One usually refers to $\overline{\Gamma}$ as the *stabilization* of Γ.

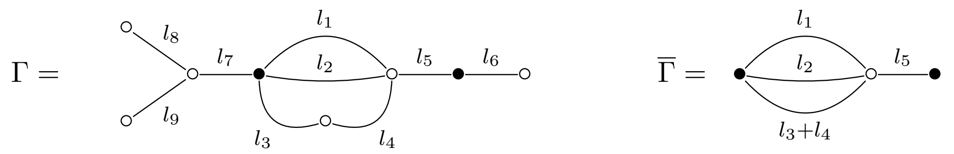


FIGURE 3. A tropical curve Γ and its stabilization, $\overline{\Gamma}$.

Observe that while for any fixed genus there exist infinitely many (non-isomorphic) graphs, only finitely many of them are stable.

The following instructive simple lemma will also be useful later.

Lemma 2.2.2. *Let (G,w) be a stable graph of genus $g \geq 2$. Then G has at most $3g-3$ edges, and the following are equivalent.*

(1) $|E(G)| = 3g-3$.
(2) Every vertex of G has weight 0 and valency 3.
(3) Every vertex of G has weight 0 and $|V(G)| = 2g-2$.

Proof. For any fixed genus there exist only finitely many stable graphs, hence there exists a maximum, M, on the number of edges that a stable graph of that genus can have. Let G be a graph having M edges, then no vertex of G has positive weight, for otherwise we could replace this vertex by a weight-zero vertex with as many loops as its weight attached to it; this would increase the number of edges without changing the genus or loosing stability.

So, every vertex of G has weight 0, hence, by stability, it has valency at least 3, and we have $3|V| \leq 2M$. We obtain

$$g = M - |V| + 1 + \sum_{v \in V} w(v) \geq M - 2M/3 + 1 = M/3 + 1$$

hence $M \leq 3g-3$, as claimed. By the above inequality, if $M = 3g-3$ every vertex has (weight 0 and) valency equal to 3, proving (1) $\Rightarrow$ (2).

For (2) $\Rightarrow$ (3), we have $3|V|/2 = |E|$ hence

$$g = 3|V|/2 - |V| + 1 = |V|/2 + 1$$

as wanted. For (3) $\Rightarrow$ (1), we have $g = |E| - 2g + 2 + 1 = |E| - 2g + 3$, and we are done. ♣

Now, the set of equivalence classes of tropical curves of genus $g \geq 2$ forms a nice moduli space, denoted here by M_g^{trop}, constructed in [BMV11] to which we refer for details (see also [Mik07a]). Let us, for the moment, content ourselves to recall that M_g^{trop} has a natural structure of topological space; it is a "stacky fan" in the terminology of [BMV11]. It is partitioned as follows

$$M_g^{\mathrm{trop}} = \bigsqcup_{(G,w)\in \mathrm{S}_g} M^{\mathrm{trop}}(G,w) \tag{3}$$

where S_g denotes the set of all stable weighted graphs of genus g and $M^{\mathrm{trop}}(G,w)$ is the set of all isomorphism classes of tropical curves whose underlying graph is (G,w).

Remark 2.2.3. The description of M_g^{trop} given in (3) implies that for every equivalence class of tropical curves there is a unique representative, up to isomorphism, whose underlying graph is stable. Therefore from now on we shall usually assume that our tropical curves have stable underlying graph. Such tropical curves are also called "stable". So, two stable tropical curves are isomorphic if and only if they are equivalent.

2.3. **Constructing the moduli space of tropical curves.** We shall here describe how to give a topologically meaningful structure to M_g^{trop}.

Let us start from the stratification given in (3). It is not hard to describe each stratum $M^{\mathrm{trop}}(G,w)$. Write $G=(V,E)$ and consider the open cone $\mathbb{R}_{>0}^{|E|}$ with its euclidean topology. To every point $(l_1,\ldots,l_{|E|})$ in this cone there corresponds a unique tropical curve whose i-th edge has length l_i. Now consider the automorphism group, $\mathrm{Aut}(G,w)$, of (G,w); by definition, $\mathrm{Aut}(G,w)$ is the set of automorphisms of G which preserve the weights on the vertices; in particular we have a homomorphism from $\mathrm{Aut}(G,w)$ to the symmetric group on $|E|$ elements. Hence $\mathrm{Aut}(G,w)$ acts on $\mathbb{R}_{>0}^{|E|}$ by permuting the coordinates, and the quotient by this action, endowed with the quotient topology, is the space of isomorphism classes of tropical curves having (G,w) as underlying graph:

$$M^{\mathrm{trop}}(G,w) = \mathbb{R}_{>0}^{|E|}/\mathrm{Aut}(G,w).$$

Now, the boundary of the closed cone $\mathbb{R}_{\geq 0}^{|E|}$ naturally parametrizes tropical curves with fewer edges that are specializations, i.e. weighted contractions, of tropical curves in the open cone. Indeed, the closure in M_g^{trop} of a stratum as above is a union of strata, and we have

$$M^{\mathrm{trop}}(G,w) \subset \overline{M^{\mathrm{trop}}(G',w')} \quad \Leftrightarrow \quad (G',w') \to (G,w) \tag{4}$$

(notation in (2)) which gives an interesting partial ordering on the set of strata appearing in (3).

It turns out that the action of $\mathrm{Aut}(G,w)$ extends on the closed cone in such a way that the quotient

$$\widetilde{M^{\mathrm{trop}}}(G,w) = \mathbb{R}_{\geq 0}^{|E|}/\mathrm{Aut}(G,w)$$

identifies isomorphic curves, and hence maps to M_g^{trop}.

Now, there are a few equivalent ways to construct M_g^{trop} as a topological space; we proceed starting from its "biggest" strata (i.e. strata of maximum dimension), as follows. By Lemma 2.2.2, the maximum number of edges of a stable graph of genus g is $3g-3$. Such graphs correspond to our biggest strata, and we proceed by considering the map

$$\bigsqcup_{\substack{(G,w)\in S_g:\\ |E|=3g-3}} \widetilde{M^{\rm trop}}(G,w) \longrightarrow M_g^{\rm trop}$$

mapping a curve to its isomorphism class, as we observed above. Now, by Proposition 2.3.1 below, this map is surjective, i.e. every stable tropical curve can be obtained as a specialization of a stable tropical curve with $3g-3$ edges. Therefore we can endow $M_g^{\rm trop}$ with the quotient topology induced by the space on the left.

Proposition 2.3.1. *Let (G,w) be a stable graph of genus g. Then there exists a stable graph (G',w') of genus g having $3g-3$ edges and such that $M^{\rm trop}(G,w)\subset \overline{M^{\rm trop}(G',w')}$.*

Proof. We can assume $|E(G)| < 3g-3$. By (4), it suffices to show that there exists a (G',w') with $3g-3$ edges such that $(G',w')\to(G,w)$. The following proof is illustrated on an explicit case in Example 2.3.1.

Let $V_+\subset V(G)$ be the set of vertices of positive weight; consider the graph G'' obtained from G by replacing every $v\in V_+$ by a weight-zero vertex with $w(v)$ new loops attached to it.

By construction, G'' has all vertices of weight zero, and specializes to (G,w) by contracting every one of the new loops ($\sum_{v\in V_+} w(v)$ of them).

If every vertex of G'' has valency 3 we are done by Lemma 2.2.2.

So, suppose G'' has some vertex, v, of valency $N\geq 4$. We shall construct a graph G''' which specializes to G'' by contracting to v one edge whose ends have both valency less than N. Iterating this construction until there are no vertices of valency more than 3 we are done.

We partition the set, H_v, consisting in the N half-edges adjacent to v, into two subsets, H_v^1 and H_v^2, of respective cardinalities $N_1=\lfloor N/2\rfloor$ and $N_2=\lceil N/2\rceil$. As $n\geq 4$ we have

$$2\leq N_i\leq N-2.$$

Consider the graph G''' obtained from G'' by replacing v by a non-loop edge e whose ends, u_1 and u_2, are attached to, respectively, H_v^1 and H_v^2. As u_i has valency N_i+1, the graph G''' is stable and its vertices u_i have both valency less than N. It is clear that contracting e in G''' gives back our G''. So we are done. ♣

Example 2.3.1. The following picture illustrates the proof of Proposition 2.3.1 on the genus-2 graph G consisting of one vertex with weight 2 and no edges.

On the right we see the two possible graphs G', corresponding (in the proof) to different distributions in G'' of the four half-edges adjacent to v.

Of course, G'' is obtained from G' contracting the edge e, and G is obtained by contracting both loops of G''.

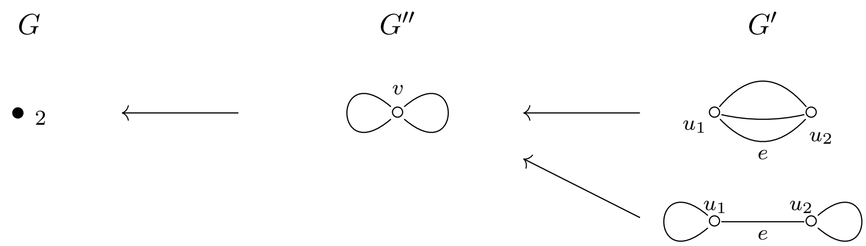


FIGURE 4. Proof of Proposition 2.3.1

As it turns out, the topological space M_g^{trop} is connected, Hausdorff, and of pure dimension $3g-3$ (i.e. it has a dense open subset which is a $(3g-3)$-dimensional orbifold over $\mathbb{R}$). More details can be found in [BMV11] and [Cap12]; see also [CGP16] for some recent results on the topology of M_g^{trop}.

2.4. **Extended tropical curves.** Let us now focus on the fact that, as it is easy to check, M_g^{trop} is not compact; this is caused by edge-lengths being allowed to grow arbitrarily. In fact, we explained before how adding weights on the vertices enabled us to control edge-lengths going to zero; on the other hand we still cannot control edge-lengths going to infinity.

This problem can be solved by further generalizing the definition of tropical curve, by allowing edges of infinite length (this is done in [Cap12] using an idea of G. Mikhalkin).

Definition 2.4.1. An *extended tropical curve* is a triple $\Gamma = (G, \ell, w)$ where (G, w) is a stable graph and $\ell : E \to \mathbb{R}_{>0} \cup \{\infty\}$ an "extended" length function.

Now, we compactify $\mathbb{R} \cup \{\infty\}$ by the Alexandroff one-point compactification, and consider its subspaces with the induced topology. The moduli space of extended tropical curves with fixed underlying weighted graph is

$$\overline{M^{\text{trop}}(G,w)} = \frac{(\mathbb{R}_{>0} \cup \{\infty\})^{|E|}}{\text{Aut}(G,w)}$$

with the quotient topology. Now, just as we did for M_g^{trop}, we extend the action of $\text{Aut}(G,w)$ on the closure of $(\mathbb{R}_{>0} \cup \{\infty\})^{|E|}$, so that the quotient

$$\widetilde{M_\infty^{\text{trop}}}(G,w) = \frac{(\mathbb{R}_{\geq 0} \cup \{\infty\})^{|E|}}{\text{Aut}(G,w)}$$

is a compact topological space which maps to the moduli space for extended tropical curves, written $\overline{M_g^{\text{trop}}}$. Arguing as in the previous subsection, we have a surjection

$$\bigsqcup_{\substack{(G,w)\in \mathrm{S}_g:\\ |E|=3g-3}} \widetilde{M_{\infty}^{\mathrm{trop}}}(G,w) \longrightarrow \overline{M_g^{\mathrm{trop}}} = \bigsqcup_{(G,w)\in \mathrm{S}_g} \overline{M^{\mathrm{trop}}(G,w)}. \tag{5}$$

It is not hard to prove that $\overline{M_g^{\mathrm{trop}}}$, with the quotient topology, is compact, normal, and contains M_g^{trop} as dense open subset.

Remark 2.4.2. As we shall see in the next section, the introduction of extended tropical curves has a precise meaning in relating tropical curves to algebraic curves. A tropical curve will be shown to correspond to families of smooth curves degenerating to nodal ones, while an extended tropical curve will correspond to families of nodal curves degenerating, again, to nodal ones. For instance, under this correspondence an extended tropical curve $\Gamma = (G, w, \underline{\infty})$, all of whose edges have length equal to ∞, will correspond to locally trivial families all of whose fibers have dual graph (G, w); see Proposition 3.3.3.

3. From algebraic curves to tropical curves

The primary goal of this section is to show that, as we just mentioned, tropical curves are associated to degenerations of smooth curves to singular ones. We first need to introduce some terminology and some conventions.

3.1. **Algebraic curves.** By *algebraic curve* we mean a projective variety of dimension one over an algebraically closed field, unless we specify otherwise. For reasons that will be explained in the next subsection, we shall be interested exclusively in *nodal* curves, i.e. reduced (possibly reducible) curves admitting at most nodes as singularities, and we shall usually omit to mention it.

To a curve X one associates its *(weighted) dual graph*, written (G_X, w_X) and defined as follows. The vertex-set of G_X is the set of irreducible components of X, and the weight of a vertex/component is its geometric genus (i.e. the genus of its desingularization); the edge-set is the set of nodes of X, with the ends of an edge/node equal to the irreducible components of X on which the node lies, so that loops correspond to nodes of irreducible components.

In symbols, for any vertex v of G_X we write $C_v \subset X$ for the corresponding irreducible component; $w_X(v)$ is the genus of the normalization, C_v^{ν}, of C_v. We shall abuse notation by using the same symbol for a node of X and the corresponding edge of G_X.

From now on, we shall assume that X is a connected curve.

We define X to be *stable* if so is its dual graph (G_X, w_X).

Remark 3.1.1. It is well known that a connected curve is stable if and only if it has finitely many automorphisms, if and only if its dualizing line bundle is ample; see [DM69], [HM98] or [ACG11].

We want to highlight the connection between a curve and its dual graph. Let us begin by proving the following basic

Claim. *A curve X has the same (arithmetic) genus as its dual graph.*

First of all, recall that the genus of X is defined as $g = h^1(X, \mathcal{O}_X)$.
Now, write $G_X = (V, E)$, and consider the normalization map

$$\nu : X^\nu = \bigsqcup_{v \in V} C_v^\nu \longrightarrow X.$$

The associated map of structure sheaves yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \nu_* \mathcal{O}_{X^\nu} \longrightarrow \mathcal{S} \longrightarrow 0 \tag{6}$$

where $\mathcal{S}$ is a skyscraper sheaf supported on the nodes of X; the associated exact sequence in cohomology is as follows (identifying the cohomology groups of $\nu_* \mathcal{O}_{X^\nu}$ with those of $\mathcal{O}_{X^\nu}$ as usual)

$$\begin{array}{rll} 0 \longrightarrow & H^0(X, \mathcal{O}_X) \longrightarrow & H^0(X^\nu, \mathcal{O}_{X^\nu}) \xrightarrow{\tilde{\delta}} k^{|E|} \longrightarrow \\ \longrightarrow & H^1(X, \mathcal{O}_X) \longrightarrow & H^1(X^\nu, \mathcal{O}_{X^\nu}) \longrightarrow 0. \end{array}$$

Hence

$$g = h^1(X^\nu, \mathcal{O}_{X^\nu}) + |E| - |V| + 1 = \sum_{v \in V} g_v + b_1(G_X) = g(G_X, w_X)$$

where $g_v = h^1(C_v^\nu, \mathcal{O}_{C_v^\nu})$ is the genus of C_v^ν. By (1) we conclude that X and (G_X, w_X) have the same genus. The claim is proved.

3.2. A graph-theoretic perspective on the normalization of a curve.

We shall continue the preceding analysis and prove that the map $\tilde{\delta}$ in the cohomology sequence above can be identified with the coboundary map of the graph G_X.

Let us first describe $\tilde{\delta}$. Let $\Phi = \{\phi_v\}_{v \in V}$ be an element in $H^0(X^\nu, \mathcal{O}_{X^\nu})$, so that $\phi_v \in H^0(C_v^\nu, \mathcal{O}_{C_v^\nu}) \cong k$ for every vertex v. Now $\tilde{\delta}$ is determined up to a choice of sign for each node of X, i.e. for each edge in E; this choice of sign amounts to the following. Let $e \in E$ be a node of X, then e corresponds to two (different) branch-points on the normalization, X^ν, of X, each of which lies in a component of X^ν; let us denote by v_e^+ and v_e^- the vertices corresponding to these two (possibly equal) components of X^ν (so that the node e lies in $C_{v_e^+} \cap C_{v_e^+}$). Now we have

$$\tilde{\delta}(\Phi) = \{\phi_{v_e^+} - \phi_{v_e^-}\}_{e \in E}.$$

To describe this map in graph-theoretic terms we need a brief excursion into graph theory, which will also be useful later in the paper.

For a pair of vertices, v, w, we set

$$[v,w] := \kappa_{v,w} = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{otherwise;} \end{cases} \tag{7}$$

this gives a bilinear pairing on $C_0(G_X, k)$, the k-vector space having V as basis.

An orientation on G_X is a pair of maps $s, t : E \to V$ so that every edge e is oriented from $s(e)$ to $t(e)$, where $s(e)$ and $t(e)$ are the ends of e. Now we can define the boundary map

$$C_1(G_X, k) \stackrel{\partial}{\longrightarrow} C_0(G_X, k); \qquad \sum_{e \in E} c_e e \mapsto \sum_{e \in E} c_e\big(t(e) - s(e)\big)$$

(where $C_1(G_X, k)$ is the k-vector space having E as basis). We need to define the coboundary map,

$$C_0(G_X, k) \stackrel{\delta}{\longrightarrow} C_1(G_X, k).$$

We define δ as the linear extension of the following: for every $v \in V$ and $e \in E$ we set, using (7),

$$\delta(v) := \sum_{e \in E} \big[v, \partial(e)\big] e$$

where we have

$$\big[v, \partial(e)\big] = \begin{cases} 1 & \text{if } v = t(e) \text{ and } t(e) \neq s(e_), \\ -1 & \text{if } v = s(e) \text{ and } t(e) \neq s(e_), \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Now, with the notation at the beginning of this subsection, we assume that the orientation maps s and t are such that $s(e) = v_e^-$ and $t(e) = v_e^+$. We have

$$H^0(X^\nu, \mathcal{O}_{X^\nu}) \stackrel{\alpha}{\longrightarrow} C_0(G_X, k) \stackrel{\delta}{\longrightarrow} C_1(G_X, k) \stackrel{\beta}{\longrightarrow} k^{|E|}$$

where α and β are the obvious isomorphisms. In particular, any vertex $u \in V$ viewed as element of $C_0(G_X, k)$ is the image by α of $\Phi^u = \{\phi_v^u\}_{v \in V}$ defined as follows

$$\phi_v^u := \kappa_{u,v}.$$

By definition, we have $\tilde{\delta}(\Phi^u) = \{\phi^u_{v_e^+} - \phi^u_{v_e^-}\}_{e \in E}$, where

$$\phi^u_{v_e^+} - \phi^u_{v_e^-} = \begin{cases} 1 & \text{if } u = v_e^+ \text{ and } v_e^+ \neq v_e^-, \\ -1 & \text{if } u = v_e^- \text{ and } v_e^+ \neq v_e^-, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with (8) we obtain

$$\tilde{\delta}(\Phi^u) = \beta \circ \delta(u) = \beta \circ \delta \circ \alpha(\Phi^u).$$

Since $\{\Phi^u,\ \forall u \in V\}$ is a basis for $H^0(X^\nu, \mathcal{O}_{X^\nu})$ we conclude that the map $\tilde{\delta}$ is naturally identified with the coboundary map δ of the graph G_X; more precisely we proved the following.

Proposition 3.2.1. *Notations as above. Then $\tilde{\delta} = \beta \circ \delta \circ \alpha$ and we have the following exact sequence*

$$0 \longrightarrow H_1(G_X, k) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X^\nu, \mathcal{O}_{X^\nu}) \longrightarrow 0.$$

3.3. **Families of algebraic curves over local schemes.** As we mentioned, from the point of view of tropical geometry, smooth algebraic curves are particularly interesting when considered in families specializing to singular curves. We shall need a local analysis, hence we shall concentrate on families parametrized by a valuation scheme or, even more "locally", by a complete valuation scheme.

Now, it is well known that in families of smooth curves singular degenerations are, in general, unavoidable. This phenomenon is reflected in the fact that $\overline{M}_g$, the moduli scheme of smooth curves of genus g, is not projective.

On the other hand for many purposes, including ours, it suffices to consider only degenerations to nodal curves, ruling out all other types of singularities; in moduli theoretic terms, the moduli space M_g is a dense open subset in an irreducible, normal, projective scheme, $\overline{M}_g$, which is the moduli space for stable curves.

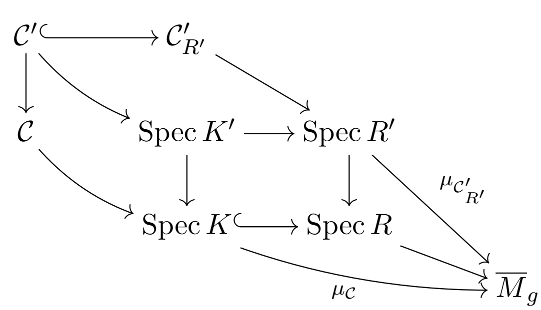
We shall get back to $\overline{M}_g$ later, let us now limit ourselves to the local picture, described by the following well known fact, a variation on the classical Deligne-Mumford Stable Reduction Theorem, in [DM69].

Recall that K denotes a complete valuation field and R its valuation ring.

Theorem 3.3.1. *Let $\mathcal{C}$ be a stable curve over K.*

Then there exists a finite field extension $K'|K$ such that the base change $\mathcal{C}' = \mathcal{C} \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ admits a unique model over the valuation ring of K' whose special fiber is a stable curve.

The theorem is represented in the following commutative diagram.



Proof. If K is a discrete valuation field (not necessarily complete) this is the original Stable Reduction Theorem [DM69, Cor. 2.7]. Our statement is a consequence of it, and of some properties of the moduli scheme $\overline{M}_g$.

Indeed, recall that $\overline{M}_g$ is projective, it is a coarse moduli space for stable curves, and admits a finite covering over which there exists a tautological family of stable curves (i.e. the fiber over a point is isomorphic to the curve parametrized by the image of that point in $\overline{M}_g$).

Now, since $\overline{M}_g$ is projective, by the valuative criterion of properness the moduli map $\mu_{\mathcal{C}} : \operatorname{Spec} K \to \overline{M}_g$ associated to $\mathcal{C}$ extends to a regular map $\operatorname{Spec} R \to \overline{M}_g$, which, however, needs not come from a family of stable curves over $\operatorname{Spec} R$. But, as we observed above, a finite extension $R \hookrightarrow R'$ exists such that the base-changed curve, $\mathcal{C}' = \mathcal{C} \times_{\operatorname{Spec} K} \operatorname{Spec} K'$, over the fraction field, K', of R', admits a stable model, $\mathcal{C}'_{R'}$, over $\operatorname{Spec} R'$. ♣

To avoid confusion, let us point out that the stable model, $\mathcal{C}'_{R'}$, does not, in general, coincide with the base change $\mathcal{C}_R \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.

Some explicit constructions of stable reduction can be found in [HM98, Sect. 3C].

Remark 3.3.2. The map $\operatorname{Spec} R \to \overline{M}_g$ in the diagram above is uniquely determined by $\mathcal{C}$; hence so is the image of the special point of $\operatorname{Spec} R$. This is a stable curve defined over the residue field, k, of R which will be denoted by $\mathcal{C}_k$. We can thus define a *(stable) reduction* map for our field K:

$$\operatorname{red}_K : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{M}_g; \qquad \mathcal{C} \mapsto \mathcal{C}_k \tag{9}$$

where $\overline{\mathcal{M}}_g(K)$ denotes the set of stable curves of genus g over K.

Let now $\overline{M}_g(K)$ denote, as usual, the set of K-points of $\overline{M}_g$. We have a natural map

$$\mu_K : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{M}_g(K); \qquad \mathcal{C} \mapsto \mu_{\mathcal{C}}$$

where $\mu_{\mathcal{C}} : \operatorname{Spec} K \to \overline{M}_g$ is the moduli map associated to $\mathcal{C}$, already appeared in Theorem 3.3.1. It is clear that the map red_K factors through μ, i.e. we have

$$\operatorname{red}_K : \overline{\mathcal{M}}_g(K) \xrightarrow{\mu} \overline{M}_g(K) \longrightarrow \overline{M}_g. \tag{10}$$

So far, in Theorem 3.3.1 and in Remark 3.3.2, the valuation of K, written $v_K : K \to \mathbb{R} \cup \{\infty\}$, did not play a specific role; it was only used to apply the valuative criterion of properness. It will play a more important role in what follows.

The next statement is a consequence of Theorem 3.3.1. It can be found in various forms in [BPR13], [BPR16], [Tyo12] and [Viv13].

Proposition 3.3.3. *Let $\mathcal{C}$ be a stable curve over K and let $\mathcal{C}_k$ be the stable curve over k defined in Remark 3.3.2. Then there exists an extended tropical curve $\Gamma_{\mathcal{C}} = (G_{\mathcal{C}}, \ell_{\mathcal{C}}, w_{\mathcal{C}})$ with the following properties.*

(1) $(G_{\mathcal{C}}, w_{\mathcal{C}})$ is the dual graph of $\mathcal{C}_k$.
(2) $\Gamma_{\mathcal{C}}$ is a tropical curve (i.e. all edges have finite length) if and only if $\mathcal{C}$ is smooth.

(3) If $K'|K$ is a finite extension and $\mathcal{C}'$ the base change of $\mathcal{C}$ over K', then $\Gamma_{\mathcal{C}} = \Gamma_{\mathcal{C}'}$.

Remark 3.3.4. As it will be clear from the proof, $\Gamma_{\mathcal{C}}$ depends on v_K even though our notation does not specify it.

Proof. The main point of the proof consists in completing the definition of the tropical curve $\Gamma_{\mathcal{C}}$ by explaining how the length function $\ell_{\mathcal{C}}$ is defined. Using the notation of the previous theorem, for some finite ring extension $R \subset R'$ we can assume that $\mathcal{C}_k$ is the special fiber of a family of stable curves over R'. Let e be a node of $\mathcal{C}_k$, then the equation of the family locally at e has the form $xy = f_e$, with f_e in the maximal ideal of R'. We set

$$\ell_{\mathcal{C}}(e) = v_{K'}(f_e)$$

where $v_{K'}$ is the valuation of the field K'. Indeed, as K is complete and the extension $K'|K$ is finite, the valuation $v_{K'}$ is uniquely determined by v_K, moreover K' is complete; see [Lan65, Prop. XII.2.5]. Now, if $\mathcal{C}$ is smooth then for every node e the function f_e above is not zero, hence $\ell_{\mathcal{C}}(e) \in \mathbb{R}_{>0}$ and $\Gamma_{\mathcal{C}}$ is a tropical curve. Conversely, if $\mathcal{C}$ has some node, then this node specializes to some node, e, of $\mathcal{C}_k$. For such an e the local equation, f_e, is equal to zero because the family is locally reducible. Therefore $\ell_{\mathcal{C}}(e) = \infty$, and part (2) is proved. It remains to prove that the definition of $\ell_{\mathcal{C}}$ does not depend on the choice of the local equation or of the field extension. This is standard, we refer to the above mentioned papers. ♣

Remark 3.3.5. Let K be our complete valuation field with valuation v_K, and $\overline{\mathcal{M}}_g(K)$ as defined in Remark 3.3.2. By what we said so far we can define a *local tropicalization map*, trop_K, as follows

$$\mathrm{trop}_K : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{M_g^{\mathrm{trop}}}; \qquad \mathcal{C} \mapsto \Gamma_{\mathcal{C}}. \tag{11}$$

As done in Remark 3.3.2 for the map red_K, one easily checks that trop_K factors as follows

$$\mathrm{trop}_K : \overline{\mathcal{M}}_g(K) \xrightarrow{\mu} \overline{M}_g(K) \longrightarrow \overline{M_g^{\mathrm{trop}}}. \tag{12}$$

We abuse notation and denote, again, $\mathrm{trop}_K : \overline{M}_g(K) \longrightarrow \overline{M_g^{\mathrm{trop}}}$.

Now, for every finite extension $K'|K$ we have a map of sets given by base-change

$$\beta_{K,K'} : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{\mathcal{M}}_g(K'); \qquad \mathcal{C} \mapsto \mathcal{C} \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, K'.$$

A consequence of Proposition 3.3.3 is that the local tropicalization map is compatible with finite base change, that is we have the following.

Corollary 3.3.6. *Let $K'|K$ be a finite extension of complete valuation fields. Then* $\mathrm{trop}_K = \mathrm{trop}_{K'} \circ \beta_{K,K'}$.

3.4. **Tropicalization and analytification of $\overline{\mathcal{M}_g}$.** In the previous subsection we have shown that to a local family of stable curves there corresponds an extended tropical curve, and this is compatible with base change. We can then ask for more, namely for a global version of this correspondence, where by "global" we mean involving the entire moduli spaces. That it ought to be possible to satisfy such a request is indicated by another type of evidence, consisting in some analogies between the moduli space of tropical curves and the moduli space of stable curves, as we shall now illustrate.

As we have seen in (3) and (5), the moduli space of tropical curves is partitioned via stable graphs. Now, this is true also for the moduli space of stable curves, for which we have

$$\overline{M}_g = \bigsqcup_{(G,w)\in \mathrm{S}_g} M(G,w)$$

where $M(G,w)$ denotes the locus of stable curves having (G,w) as dual graph (recall that S_g is the set of stable graphs of genus g). The stratum $M(G,w)$ turns out to be an irreducible quasi-projective variety of dimension $3g-3-|E(G)|$, equal to the codimension of the locus $M^{\mathrm{trop}}(G,w)$ in M_g^{trop}. Moreover, these strata can be shown to form a partially ordered set with respect to inclusion of closures, and we have:

$$M(G,w) \subset \overline{M_{(G',w')}} \quad \Leftrightarrow \quad (G,w) \to (G',w').$$

This is analogous, though reversing the arrow, to what happens in $\overline{M_g^{\mathrm{trop}}}$, as we have seen in (4).

Let us go back to the purpose of this subsection: to obtain a global picture from which the local correspondence described earlier can be derived. We proceed following [ACP15] to which we refer for details.

The above described analogies between M_g^{trop} and $\overline{M}_g$, together with the construction of M_g^{trop} by means of Euclidean cones, indicate that M_g^{trop} should be the "skeleton" of the moduli stack, $\overline{\mathcal{M}_g}$, of stable curves. We must now consider the stack, $\overline{\mathcal{M}_g}$, instead of the scheme, $\overline{M}_g$, because the former has a toroidal structure that the latter does not have. It is precisely the toroidal structure that enables us to construct the above mentioned skeleton as a *generalized cone complex* associated to $\overline{\mathcal{M}_g}$, denoted by $\Sigma(\overline{\mathcal{M}_g})$, and compactified by an *extended generalized cone complex*, written $\overline{\Sigma}(\overline{\mathcal{M}_g})$. Having constructed these spaces one proves that there are isomorphisms of generalized cone complexes represented by the vertical arrows of the following commutative diagram (see [ACP15, Thm. 1.2.1])

$$(13)\qquad \begin{array}{ccc} \Sigma(\overline{\mathcal{M}_g}) & \hookrightarrow & \overline{\Sigma}(\overline{\mathcal{M}_g}) \\ \cong \downarrow & & \cong \downarrow \varphi \\ M_g^{\mathrm{trop}} & \hookrightarrow & \overline{M_g^{\mathrm{trop}}} \end{array}$$

The above diagram offers an interpretation, from a broader context, of the global analogies between the moduli spaces of stable and tropical curves illustrated earlier.

We now turn to the local correspondences described in the previous subsection in order to place them into a more general framework. To do that, we introduce the *Berkovich analytification*, $\overline{M}_g^{\mathrm{an}}$, of $\overline{M}_g$.

In this paper, we cannot afford to introduce analytic geometry over non-Archimedean fields in a satisfactory way, despite its deep connections to tropical geometry of which a suggestive example is the application to the moduli theory of curves we are about to describe. On the one hand this would require a consistent amount of space, on the other hand it would take us too far from our main topic.

The starting point is that, following [Ber90, Chapt. 3], to any scheme X over k one associates its analytification, X^{an}, a more suitable space from the analytic point of view.

We apply this to the scheme $\overline{M}_g$, and limit ourselves to mention that $\overline{M}_g^{\mathrm{an}}$ is a Hausdorff, compact topological space, and to describe it set-theoretically.

To do that, for any non-Archimedean field $K|k$ consider the set, $\overline{M}_g(K)$, of K-points of $\overline{M}_g$. Then we have

$$(14)\qquad \overline{M}_g^{\mathrm{an}} = \frac{\bigsqcup_{K|k} \overline{M}_g(K)}{\sim_{\mathrm{an}}}$$

where the union is over all non-Archimedean extensions $K|k$, and the equivalence relation $\sim_{\mathrm{an}}$ is defined as follows. Let ξ_1 and ξ_2 be K_i-points of $\overline{M}_g$, i.e. $\xi_i : \operatorname{Spec} K_i \to \overline{M}_g$ for $i = 1, 2$. We set $\xi_1 \sim_{\mathrm{an}} \xi_2$ if there exists a third extension $K_3|k$ which extends also K_1 and K_2, and a point $\xi_3 \in \overline{M}_g(K_3)$ such that the following diagram is commutative

$$\begin{array}{ccccc} & & \operatorname{Spec} K_3 & & \\ & \swarrow & \downarrow \xi_3 & \searrow & \\ \operatorname{Spec} K_1 & & & & \operatorname{Spec} K_1 \\ & \xi_1 \searrow & & \swarrow \xi_2 & \\ & & \overline{M}_g & & \end{array}$$

We conclude by recalling that (14) induces a bijection between $\overline{M}_g^{\mathrm{an}}(K)$ and $\overline{M}_g(K)$ for every K; see [Ber90, Thm 3.4.1].

Remark 3.4.1. From the above description, a point of $\overline{M}_g^{\text{an}}$ is represented by a stable curve $\mathcal{C}$ over a non-Archimedean field K. By Theorem 3.3.1 we can furthermore assume (up to field extension) that $\mathcal{C}$ admits a stable model over the valuation ring of K.

Now, we can use the local tropicalization maps defined in (11) to define a map

$$\text{trop} : \overline{M}_g^{\text{an}} \longrightarrow \overline{M_g^{\text{trop}}}$$

such that its restriction $\overline{M}_g^{\text{an}}(K) = \overline{M}_g(K)$ coincides with the map trop_K defined in Remark 3.3.5.

The connection between this map and Diagram (13) is achieved using results of [Thu07], which enable us to construct a retraction (or projection) from the analytification of $\overline{M}_g$ to the extended skeleton of $\overline{\mathcal{M}_g}$. This remarkable retraction map is denoted by "p" in the commutative diagram below.

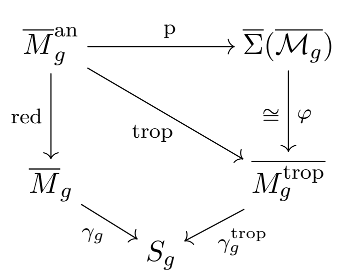
Finally, let us introduce the two forgetful maps from $\overline{M}_g$ and $\overline{M_g^{\text{trop}}}$ to the set S_g of stable graphs of genus g

$$\gamma_g : \overline{M}_g \longrightarrow S_g, \qquad \gamma_g^{\text{trop}} : \overline{M_g^{\text{trop}}} \longrightarrow S_g$$

mapping a stable, respectively tropical, curve to its dual, respectively underlying, weighted graph.

Now, using the above notation, we summarize the previous discussion with the following statement.

Theorem 3.4.2. *The following diagram is commutative.*



Notice that all the arrows in the diagram are surjective.

With respect to the previous subsections, the new and non-trivial part of the diagram is its upper-right corner, a special case of [ACP15, Thm.1.2.1], to which we refer for a proof.

We need to define the "reduction" map, red : $\overline{M}_g^{\text{an}} \longrightarrow \overline{M}_g$. Using the set-up of the previous subsection, this map sends a point in $\overline{M}_g^{\text{an}}$ represented by a stable curve $\mathcal{C}$ over the field K to the stable curve $\mathcal{C}_k$, defined over k, of Remark 3.3.2. The commutativity of the diagram follows immediately from the earlier discussions.

4. Curves and their Jacobians

Our main focus in this paper is the role of tropical and graph-theoretical methods in the theory of algebraic curves and their moduli. As is well known, the geometry of algebraic curves is closely connected to the geometry of their Jacobian varieties, to study which combinatorial methods have been used for a long time. This will be the topic of the present section.

4.1. Jacobians of smooth curves and their moduli.

To any smooth connected projective curve C of genus g defined over k one associates its Jacobian, $\mathrm{Jac}(C)$, an abelian variety of dimension g endowed with a canonical principal polarization, the theta divisor, $\Theta(C)$. The pair $(\mathrm{Jac}(C), \Theta(C))$ is a so-called *principally polarized abelian variety*, and will be denoted

$$\mathbf{Jac}(C) := (\mathrm{Jac}(C), \Theta(C)) \tag{15}$$

The structure of $\mathbf{Jac}(C)$ is extremely rich and a powerful instrument to study the curve C; it has been investigated in depth for a long time and from different points of view: algebro-geometric, arithmetic and analytic.

There exist various ways of describing the Jacobian, one of which is

$$\mathrm{Jac}(C) = \mathrm{Pic}^0(C), \tag{16}$$

that is, $\mathrm{Jac}(C)$ is the moduli space of line bundles (equivalently, divisor classes) of degree 0 on C. Now, we describe the Theta divisor in a way that will be useful later. First, we identify

$$\mathrm{Pic}^0(C) \cong \mathrm{Pic}^{g-1}(C)$$

(even though the isomorphism, mapping $L \in \mathrm{Pic}^0(C)$ to $L \otimes L_0$ for some fixed $L_0 \in \mathrm{Pic}^{g-1}(C)$, is not canonical), then we identify $\Theta(C)$ as follows

$$\Theta(C) = \{L \in \mathrm{Pic}^{g-1}(C) : \ h^0(C, L) \geq 1\}. \tag{17}$$

It is well known that $\Theta(C)$ is an irreducible, codimension-one closed subvariety of $\mathrm{Pic}^{g-1}(C)$, and an ample principal divisor.

If the base field k is $\mathbb{C}$, then $\mathrm{Jac}(C)$ is identified with a g-dimensional torus, as follows

$$\mathrm{Jac}(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) \cong \mathbb{C}^g/\mathbb{Z}^{2g}. \tag{18}$$

Using the notation (15) we recall the following famous Torelli Theorem.

Theorem 4.1.1. *Let C_1 and C_2 be two smooth curves; then $C_1 \cong C_2$ if and only if $\mathbf{Jac}(C_1) \cong \mathbf{Jac}(C_2)$.*

What about the moduli theory of Jacobians?

Since Jacobians are abelian varieties of a special type, it is natural to approach this issue within the moduli theory of principally polarized abelian varieties. In fact, this theory has been developed to a large extent in parallel to the moduli theory of smooth curves, and the broader area has seen an extraordinary development in the second half of the twentieth century. We have already discussed the case of curves, let us briefly discuss the case of abelian varieties.

The moduli space of principally polarized abelian varieties of dimension $g \geq 2$ is denoted by A_g. It is an irreducible algebraic variety of dimension $g(g+1)/2$. The set of all Jacobians of curves is a distinguished subspace of A_g, the so-called Schottky locus, written Sch_g. The Torelli theorem can be re-phrased in this setting by saying that the following map

$$\tau : M_g \longrightarrow A_g; \qquad C \mapsto \mathbf{Jac}(C)$$

is an injection. In fact τ, the *Torelli map*, is a morphism of algebraic varieties whose image is, of course, the Schottky locus.

Just like M_g, the space Sch_g is not projective and one is interested in constructing useful compactifications of it. Now, A_g is not projective either. So, one could compactify A_g first, and then study the closure of Sch_g; this can be done, as we shall see in Subsection 4.4. On the other hand, the problem of completing just the space of all Jacobians is somewhat special and can be dealt with independently, as we are going to explain.

A natural approach is to start from the compactification, $\overline{M}_g$, of M_g by stable curves, and try to describe, in terms of stable curves, the points to add to complete Sch_g.

With this in mind, the first object to study is the (generalized) Jacobian of a stable curve.

4.2. **Jacobians of nodal curves.** Let X be a nodal curve defined over k and $G_X = (V, E)$ its dual graph. Its (generalized) Jacobian, $\mathrm{Jac}(X)$, is defined extending (16), as the group of line bundles on X having degree 0 on every irreducible component of X. It is well known that $\mathrm{Jac}(X)$ is a connected algebraic variety and a commutative algebraic group, but it is not projective, i.e. it is not an abelian variety, with the exception of a few cases to be described in Remark 4.2.1.

More precisely, we are going to show that $\mathrm{Jac}(X)$ is a *semi-abelian* variety, i.e. it fits into an exact sequence of type

$$0 \longrightarrow T \longrightarrow \mathrm{Jac}(X) \longrightarrow A \longrightarrow 0 \tag{19}$$

where A is an abelian variety and $T \cong (k^*)^b$ with $b \in \mathbb{Z}_{\geq 0}$. The above sequence for our curve X can be recovered as follows. Recall that the normalization of X is written $X^\nu = \sqcup_{v \in V} C_v^\nu$, and the normalization map is $\nu : X^\nu \to X$. The sequence (6) can be restricted to the subsheaves of units,

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \nu_* \mathcal{O}_{X^\nu}^* \longrightarrow \mathcal{S}^* \longrightarrow 0$$

yielding the following cohomology sequence, where we identify $H^1(..., \mathcal{O}^*_{...})$ with Pic(...),

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X^\nu, \mathcal{O}_{X^\nu}^*) \to \ (k^*)^{|E|} \to \mathrm{Pic}(X) \xrightarrow{\nu^*} \mathrm{Pic}(X^\nu) \to 0.$$

Now, a line bundle $L \in \mathrm{Pic}(X)$ (or in $\mathrm{Pic}(X^\nu)$) has a *multidegree* $\underline{d} \in \mathbb{Z}^V$ such that $d_v = \deg_{C_v} L$ for all $v \in V$. Therefore in the above sequence, the two Picard schemes decompose into connected components, one for every multidegree, as follows (we do that only for X, the case of X^ν is similar)

$$\operatorname{Pic}(X) = \sqcup_{\underline{d}\in\mathbb{Z}^V} \operatorname{Pic}^{\underline{d}}(X)$$

where $\operatorname{Pic}^{\underline{d}}(X)$ denotes the subscheme of line bundles of multidegree $\underline{d}$.

The Jacobian varieties correspond to the multidegree $(0,\dots,0)$, therefore by restricting the sequence to the Jacobians we get

$$0 \to H^0(X,\mathcal{O}_X^*) \to H^0(X^\nu,\mathcal{O}_{X^\nu}^*) \xrightarrow{\hat\delta} (k^*)^{|E|} \to \operatorname{Jac}(X) \xrightarrow{\nu^*} \operatorname{Jac}(X^\nu) \to 0.$$

With the notation of the sequence (19), we set

$$A = \operatorname{Jac}(X^\nu) = \Pi_{v\in V}\operatorname{Jac}(C_v^\nu)$$

and we have identified the right side of Sequence (19). Now, using what we proved in Subsection 3.2, we see that the map $\hat\delta$ in the cohomology sequence above can be identified with the coboundary map on the graph G_X, and we have the following exact sequence

$$0 \longrightarrow H_1(G_X,k^*) \longrightarrow \operatorname{Jac}(X) \longrightarrow \operatorname{Jac}(X^\nu) \longrightarrow 0.$$

In conclusion, we have an identification

$$T = \operatorname{Ker}\,\nu^* = H_1(G_X,k^*) = (k^*)^{b_1(G_X)}$$

completing the explicit description of (19).

Remark 4.2.1. From the above discussion we easily derive that $\operatorname{Jac}(X)$ is projective (i.e. an abelian variety) if and only if $b_1(G_X)=0$, i.e. if and only if G_X is a tree (the weights on the vertices play no role). If this is the case X is said to be a curve of *compact type*.

4.3. **Models for Jacobians over a DVR.** We have described Jacobians for a fixed curve, now we want to consider families of curves and their corresponding families of Jacobians, and concentrate on the fact that such families are seldom complete.

Let R be a discrete valuation ring whose fraction field is written K. In this subsection we can drop the assumption that K is complete. Now let

$$\phi : \mathcal{C}_R \to \operatorname{Spec} R$$

be a family of curves with smooth generic fiber; denote by $\mathcal{C}_k$ the special fiber of ϕ and assume $\mathcal{C}_k$ is a nodal curve over k. We write $\mathcal{C}_k \cong X$ to tie in with the notation in the previous subsection, which we continue to use. We then consider the associated relative Jacobian, which is a morphism

$$\mathcal{J}_R \longrightarrow \operatorname{Spec} R$$

such that the generic fiber, written $\mathcal{J}_K$, is the Jacobian of the generic fiber of ϕ, and the special fiber, $\mathcal{J}_k$, is the Jacobian of X. The above morphism is smooth and separated but, as explained in Remark 4.2.1, is not proper unless X is a curve of compact type. Furthermore, it is "too small" from the point of view of the moduli theory of line bundles. More precisely, recall that the Jacobian of a smooth curve is the moduli space for line bundles of degreee 0 on the curve, i.e. subvarieties of the Jacobian are in bijective correspondence with families of line bundles (of degree 0) on the curve. Now, this fails for $\mathcal{J}_R$, and most families of degree 0 line bundles on the fibers of $\mathcal{C}_R$ do not correspond to subvarieties of $\mathcal{J}_R$.

Example 4.3.1. Suppose the special fiber, $\mathcal{C}_k$, of our family has two irreducible components and consider a line bundle $\mathcal{L}$ on $\mathcal{C}_R$ such that its restriction to $\mathcal{C}_k$ has multidegree $(1,-1)$. Our $\mathcal{L}$ gives a family of degree 0 line bundles on the fibers of ϕ which does not correspond to any subvariety of $\mathcal{J}_R$, simply because its bidegree on the special fiber is not $(0,0)$.

To remedy these problems one concentrates on the generic fiber, $\mathcal{J}_K$, and asks whether it admits a model over $\operatorname{Spec} R$ with better properties than $\mathcal{J}_R$.

The following commutative diagram summarizes the current state of our knowledge with respect to models that are "canonical", i.e. independent of any specific choice. All arrows represent morphisms over R, the three arrows on the right are smooth morphisms.

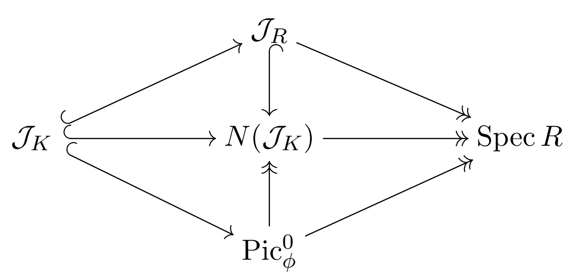
In the diagram, referring to [Nér64], [Ray70], and [BLR90] for details, we have:

• $N(\mathcal{J}_K) \to \operatorname{Spec} R$ is the Néron model of $\mathcal{J}_K$; it is a group scheme, separated over R but not complete, in general.

• $\operatorname{Pic}^0_\phi \to \operatorname{Spec} R$ is the relative degree-0 Picard scheme of the original family of curves, $\phi : \mathcal{C}_R \to \operatorname{Spec} R$. Its special fiber is the group of all line bundles of degree 0 on X. It is, in general, neither quasi-projective nor separated, but it is a moduli space for line bundles of degree 0 on the fibers of ϕ.

• The surjection $\operatorname{Pic}^0_\phi \to N(\mathcal{J}_K)$ can be described as the largest separated quotient of $\operatorname{Pic}^0_\phi$ over $\operatorname{Spec} R$. We notice in passing that it does admit sections.

(20)



What about the special fibers of the schemes appearing in the middle of the diagram? We already know the special fiber of $\mathcal{J}_R$, of course; what about the other two?

The special fiber of $\operatorname{Pic}^0_\phi$ is a disjoint union of copies of $\mathcal{J}_k$, the Jacobian of X. To describe it explicitly, for any $\underline{d} \in \mathbb{Z}^V$ consider $\operatorname{Pic}^{\underline{d}}(X)$. We have non canonical isomorphisms

$$\mathrm{Pic}^{\underline{d}}(X) \cong \mathcal{J}_k.$$

Now, the special fiber of the Picard scheme $\mathrm{Pic}^0_\phi \to \operatorname{Spec} R$ is

$$\mathrm{Pic}^0(X) = \bigsqcup_{\underline{d} \in \mathbb{Z}^V : \sum_v d_v = 0} \mathrm{Pic}^{\underline{d}}(X).$$

Hence $\mathrm{Pic}^0(X)$ is an infinite union of copies of $\mathcal{J}_k$, unless X is irreducible in which case $\mathrm{Pic}^0(X)$ is also irreducible.

To describe the special fiber, N_k, of the Néron model $N(\mathcal{J}_K)$, we shall make an additional assumption, namely that the total space of the family of curves, $\mathcal{C}_R$, is nonsingular (see Remark 4.3.1 below). We have

$$N_k \cong \bigsqcup_{i \in \Phi_{G_X}} \mathcal{J}_k^{(i)},$$

in other words, N_k is, again, a disjoint union of copies of the Jacobian of X. These copies are indexed by an interesting finite group, Φ_{G_X}, which has been object of study since Néron models were constructed in [Nér64] and, for Jacobians, in [Ray70]. As the notation suggests, Φ_{G_X} depends only on the dual graph of X. Indeed we have

$$\Phi_X = \Phi_{G_X} \cong \frac{\partial C_1(G_X, \mathbb{Z})}{\partial\delta C_0(G_X, \mathbb{Z})}, \tag{21}$$

where, for some orientation on the graph G_X (the choice of which is irrelevant) "δ" and "∂" denote the coboundary and boundary maps defined in subsection 3.2; see [OS79]. See [Ray70], [BLR90], [Lor89], [BMS06] and [BMS11] for more details on the group Φ_X.

Remark 4.3.1. Suppose the total space $\mathcal{C}_R$ is singular. Then its desingularization, $\mathcal{C}'_R \to \mathcal{C}_R$, is an isomorphism away from a set, F, of nodes of the special fiber, X, and the preimage of every node in F is a chain of rational curves. The new special fiber is thus a nodal curve, X', whose dual graph is obtained by inserting some vertices in the edges of G_X corresponding to F. One can show that if F is not entirely made of bridges of G_X, then the group $\Phi_{X'}$ is not isomorphic to Φ_X.

4.4. **Compactified Jacobians and Torelli maps.** In the previous section we described Néron models for Jacobians. Although they have many good properties (they are canonical and they have a universal mapping property, see [BLR90]), their use is limited to certain situations. First of all, they are defined over one-dimensional bases (in fact in the previous subsection we assumed R was a discrete valuation ring). Secondly, they are not projective.

The problem of constructing projective models for Jacobians, or compactifications of Jacobians for singular curves, has been investigated for a long time, also before stable curves were introduced and M_g compactified.

Building upon the seminal work of Igusa, Mumford, Altman-Kleiman and Oda-Seshadri, substantial progress was made in the field and, since the beginning of the twenty-first century, we have a good understanding of compactified Jacobians of stable curves and compactified moduli spaces for Abelian varieties.

Compactified Jacobians for families of stable curves were constructed in [Cap94], [Sim94], [Ish78]. Some of these constructions have been compared to one another in [Ale04] and related to the compactification of the moduli space of abelian varieties constructed in [Ale02].

Let us turn to the Torelli map τ defined earlier for smooth curves, and study how to extend it over the whole of $\overline{M_g}$ in a geometrically meaningful way. This problem is a classical one, and has been studied for a long time; see [Nam76] for example. Let us present a solution in the following diagram.

$$\begin{array}{ccc} \overline{M_g} & \xrightarrow{\overline{\tau}} & \overline{A_g}; \qquad [X] \longmapsto [\overline{\mathbf{Jac}}(X)] = \big[\mathrm{Jac}(X) \curvearrowright (\overline{P_{g-1}(X)}, \overline{\Theta(X)})\big] \\ \uparrow & & \uparrow \\ M_g & \xrightarrow{\tau} & A_g \end{array}$$

In order to define the extension $\overline{\tau}$ we followed [Ale04] and used the main irreducible component, $\overline{A_g}$, of the compactification of the moduli space of principally polarized abelian variety constructed in [Ale02]; this is a moduli space for so-called "semi-abelic stable pairs" (which we shall not define here). Clearly, the image of $\overline{\tau}$ contains and compactifies the Schottky locus, Sch_g, defined earlier.

Remark 4.4.1. For future use, in analogy with Remark 3.3.2, we denote by $\mathcal{A}_g(K)$ the set of principally polarized abelian varieties of dimension g defined over the field K, and define the *reduction map*

$$\mathrm{red}_K^{A_g} : \mathcal{A}_g(K) \longrightarrow \overline{A_g}.$$

The map $\mathrm{red}_K^{A_g}$ sends a principally polarized abelian variety $\mathcal{P}$ defined over K to the image of the special point of the map $\operatorname{Spec} R \to \overline{A_g}$, extending the moduli map $\operatorname{Spec} K \to A_g$ associated to $\mathcal{P}$.

For a stable curve X, let us describe its image via $\overline{\tau}$, written $[\overline{\mathbf{Jac}}(X)]$ in the above diagram. This is the isomorphism class of the pair $(\overline{P_{g-1}(X)}, \overline{\Theta(X)})$ acted upon by the group $\mathrm{Jac}(X)$. Here $\overline{P_{g-1}(X)}$ denotes the compactified Jacobian constructed in [Cap94], namely the moduli space for so-called "balanced" line bundles of degree $g-1$ on semistable curves stably equivalent to X. The variety $\overline{P_{g-1}(X)}$ is connected, reduced, of pure dimension g. Next, $\overline{\Theta(X)}$, called again the *Theta divisor*, is defined in analogy with (17) as the closure in $\overline{P_{g-1}(X)}$ of the locus of line bundles on X having some non-zero section. $\overline{\Theta(X)}$ is an ample Cartier divisor; see [Est01] and [Cap09]. We shall describe $\overline{\mathbf{Jac}}(X)$ in more details in subsections 4.4 and 5.5.

In this set-up we propose the Torelli problem for stable curves, as follows: describe the fibers of the morphism $\overline{\tau}$, and in particular the loci where it is injective. That $\overline{\tau}$ could not possibly be injective is well known; see [Nam80] for example. Let us focus on this aspect. First of all, we have

$$\overline{\tau}^{-1}(A_g) = \overline{M}_g^{\text{cpt}}$$

where $\overline{M}_g^{\text{cpt}}$ is the locus in $\overline{M}_g$ of curves of compact type. Indeed, we have seen in Remark 4.2.1 that the generalized Jacobian of a curve of compact type is an abelian variety. Now we have the following well known fact.

Proposition 4.4.2. *Let X_1 and X_2 be stable curves of compact type. If $X_1^\nu \cong X_2^\nu$ then $\overline{\tau}(X_1) = \overline{\tau}(X_2)$. The converse holds if X_1 and X_2 have the same number of irreducible components.*

Proof. Let us begin by proving the first part in the simplest nontrivial case, of a curve $X = C_1 \cup C_2$ with only two (necessarily smooth) components. Let g_i be the genus of C_i, so that $g_i \geq 1$ by the stability assumption.

As we said above, we can identify $\overline{\tau}(X)$ with the image of X via the classical Torelli map τ. As already shown, we have

$$\text{Jac}(X) \cong \text{Jac}(X^\nu) = \text{Jac}(C_1) \times \text{Jac}(C_2).$$

It remains to look at the theta divisor. It is well known that, $\Theta(X)$ is the union of two irreducible components as follows

$$\Theta(X) \cong \Big(\Theta(C_1) \times \text{Jac}(C_2)\Big) \cup \Big(\text{Jac}(C_1) \times \Theta(C_2)\Big),$$

hence $\Theta(X)$ depends only on X^ν, as claimed.

Now, if X has more than two components the proof is similar. It suffices to observe that, writing as usual $X^\nu = \cup_{v \in V} C_v$, the components C_v are smooth, hence

$$\text{Jac}(X) \cong \prod_{v \in V} \text{Jac}(C_v) = \prod_{v : g_v > 0} \text{Jac}(C_v),$$

where g_v is the genus of C_v. For the Theta divisor we have

$$\Theta(X) \cong \bigcup_{v : g_v > 0} \Big(\Theta(C_v) \times \prod_{w \in V \smallsetminus \{v\}} \text{Jac}(C_w)\Big)$$

so $\overline{\tau}(X)$ only depends on the normalization of X.

For the converse, by what we observed above and by the Torelli theorem for smooth curves, the Jacobian of X detects precisely the components of positive genus of X, which are therefore the same for X_1 and X_2. Hence, as X_1 and X_2 have the same number of components, they also have the same number of components of genus zero, so we are done. ♣

4.5. **Combinatorial stratification of the compactified Jacobian.** Let X be a stable curve and (G_X, w_X) its dual graph. We are going to describe an interesting stratification of $\overline{P_{g-1}(X)}$ preserved by the action of $\mathrm{Jac}(X)$. The strata depend on certain sets of nodes of X, to which we refer as the strata *supports*, so that each stratum is isomorphic to the Jacobian of the desingularization of X at the corresponding support. As we are going to see, this stratification is purely combinatorial, governed by the dual graph.

First of all, for any graph $G = (E, V)$, we define the partially ordered set of *supports*, $\mathcal{SP}_G$, as follows.

$$\mathcal{SP}_G := \{S \subset E : \ G - S \text{ is free from bridges}\}, \tag{22}$$

where a *bridge*, or *separating edge*, is an edge not contained in any cycle (equivalently, an edge whose removal disconnects the connected component containing it); notice that $G - S$ needs not be connected. $\mathcal{SP}_G$ is partially ordered by inclusion: $S \geq S'$ if $S \subset S'$.

Now, for any $S \in \mathcal{SP}_{G_X}$ we denote by X_S^ν the curve obtained from X by desingularizing the nodes in S. The dual graph of X_S^ν is thus $(G_X - S, w_X)$. On the curve X_S^ν we have a distinguished finite set, written Σ_S, of so-called "stable" multidegrees for line bundles of total degree $g(X_S^\nu) - 1$; we omit the precise definition as it is irrelevant here.

We are ready to introduce the stratification of $\overline{P_{g-1}(X)}$:

$$\overline{P_{g-1}(X)} = \bigsqcup_{\substack{S \in \mathcal{SP}_{G_X} \\ \underline{d} \in \Sigma_S}} P_S^{\underline{d}} \tag{23}$$

and for every stratum $P_S^{\underline{d}}$ above we have a canonical isomorphism

$$P_S^{\underline{d}} \cong \mathrm{Pic}^{\underline{d}} X_S^\nu.$$

Now, the Jacobian $\mathrm{Jac}(X)$ parametrizes line bundles of degree zero on every component of X. Its action on $\overline{P_{g-1}(X)}$ can be described, in view of the above isomorphisms, by tensor product of line bundles pulled-back to the various normalizations, X_S^ν. Therefore we have the following important fact.

Remark 4.5.1. The strata $P_S^{\underline{d}}$ appearing in (23) coincide with the orbits of the action of $\mathrm{Jac}(X)$.

We denote by $\mathcal{ST}_X$ the set of all strata, i.e. $\mathrm{Jac}(X)$-orbits, appearing in (23). Remark 4.5.1 implies that the closure of any $P_S^{\underline{d}} \in \mathcal{ST}_X$ is a union of strata in $\mathcal{ST}_X$. In particular, $\mathcal{ST}_X$ is partially ordered as follows:

$$P_S^{\underline{d}} \geq P_{S'}^{\underline{d}'} \quad \text{if} \quad P_{S'}^{\underline{d}'} \subset \overline{P_S^{\underline{d}}}.$$

Now, it is a fact that the following map, associating to a stratum its support,

$$\mathrm{Supp} : \mathcal{ST}_X \longrightarrow \mathcal{SP}_{G_X}; \qquad P_S^{\underline{d}} \mapsto S \tag{24}$$

is surjective and order-preserving (it is a morphism of posets). So, in order to understand the geometry of $\overline{P_{g-1}(X)}$, we turn to the combinatorial properties of $\mathcal{SP}_{G_X}$, for which we need a detour into graph theory. A pleasant surprise is awaiting us: our detour will naturally take us to study Jacobians of graphs and tropical curves, and their Torelli problem.

5. Torelli theorems

In this section we discuss the Torelli problem for graphs, tropical curves, and stable curves, by respecting the time sequence in which these problems were solved, using many of the same combinatorial techniques. We devote Subsection 5.4 to the Torelli problem for weighted graphs, which we prove as an easy consequence of the Torelli theorem for tropical curves and which, to our knowledge, does not appear in the literature.

5.1. **Some algebro-geometric graph theory.** In the early 90's, a line of research in graph theory began establishing an analogue of the classical Riemann-Roch theory, viewing a finite graph as the analogue of a Riemann surface; see [Big97], [BdlHN97], [BN07], for example. So, a divisor theory for graphs was set-up and developed in that direction; we will now give just a short overview.

The group of all divisors on a graph $G = (V, E)$, denoted by $\mathrm{Div}(G)$, is defined as the free abelian group on V. Hence $\mathrm{Div}(G)$ can be naturally identified with $\mathbb{Z}^V$; the degree of a divisor is the sum of its V-coordinates.

In analogy with algebraic geometry one defines "principal" divisors as divisors associated to "functions". A function on G is defined as a map $f : V \to \mathbb{Z}$, hence every function has the form $f = \sum_{v\in V} c_v f_v$, where $c_v \in \mathbb{Z}$ and f_v is the function such that $f_v(w) = \kappa_{v,w}$ for any $w \in V$. The set of all functions on G is the free abelian group generated by $\{f_v,\ \forall v \in V\}$.

We can now define, for any function f as above, the *principal* divisor, $\mathrm{div}(f)$, associated to f. By what we said it suffices to define $\mathrm{div}(f_v)$ for every $v \in V$ and extend the definition by linearity. We set

$$\mathrm{div}(f_v) = -\mathrm{val}(v) + 2\,\mathrm{loop}(v) + \sum_{w\in V\smallsetminus v} (v, w)w$$

where $\mathrm{val}(v)$ is the valency of v, $\mathrm{loop}(v)$ is the number of loops based at v, and (v, w) the number of edges between v and w.

Example 5.1.1. Let f be a constant function, i.e. $f = c\sum_{v\in V} f_v$ for some $c \in \mathbb{Z}$. Then $\mathrm{div}(f) = 0$.

See Example 5.1.2 for more examples.

Principal divisors have degree zero and form a subgroup, written $\mathrm{Prin}(G)$, of $\mathrm{Div}^0(G)$, the group of divisors of degree 0 on G.

Two divisors are linearly equivalent if their difference is in $\mathrm{Prin}(G)$. The quotient $\mathrm{Div}^0(G)/\mathrm{Prin}(G)$ is a finite group called, not surprisingly with (16) in mind, the *Jacobian group* of G.

The Jacobian group turns out to be isomorphic to a finite group we encountered earlier, namely we have

$$\frac{\mathrm{Div}^0(G)}{\mathrm{Prin}(G)} \cong \Phi_G.$$

The above isomorphism follows from (21) together with the following simple observation

$$\frac{\mathrm{Div}^0(G)}{\mathrm{Prin}(G)} \cong \frac{\partial C_1(G,\mathbb{Z})}{\partial\delta C_0(G,\mathbb{Z})}.$$

Indeed, we can naturally identify $\mathrm{Div}(G) = C_0(G,\mathbb{Z})$, and then we have $\partial C_1(G,\mathbb{Z}) \subset \mathrm{Div}^0(G)$; now, as G is connected, this containment is easily seen to be an equality. Next, we have

$$\partial\delta(v) = \partial\Big(\sum_{t(e)=v} e - \sum_{s(e)=v} e\Big) = \mathrm{val}(v) - 2\,\mathrm{loop}(v) - \sum_{w\in V\smallsetminus v}(v,w)w = -\mathrm{div}(f_v)$$

proving that $\mathrm{Prin}(G)$ is identified with $\partial\delta C_0(G,\mathbb{Z})$.

Several results have been obtained in this area, the Riemann-Roch theorem among others; we refer to [BN07] for weightless graphs, and to [AC13] for the case of weighted graphs.

By contrast, an analog of the Torelli theorem for graphs does not find its place in this setting. Indeed, the problem is that the group Φ_G is not a good tool to characterize graphs, as it is easy to find examples of different graphs having the same Jacobian group.

Example 5.1.2. Let us show that all graphs in Figure 5 have $\mathbb{Z}/3\mathbb{Z}$ as Jacobian group.

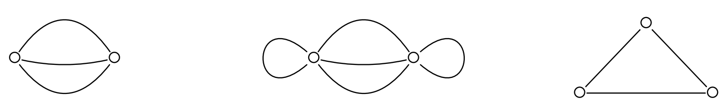


FIGURE 5. Graphs with isomorphic Jacobian group.

It is clear that the definition of the Jacobian group does not take loops into account, hence the first two graphs have isomorphic Jacobian groups. Now, that this group is $\mathbb{Z}/3\mathbb{Z}$ is easily seen as follows. Denote by G the graph on the left, then

$$\mathrm{Div}^0(G) \cong \{(n,-n),\ \forall n\in\mathbb{Z}\} \qquad \mathrm{Prin}(G) \cong \{(3n,-3n),\ \forall n\in\mathbb{Z}\};$$

indeed, if v, w are the two vertices of G, then $\mathrm{div}(f_v) = -3v + 3w$ and $\mathrm{div}(f_w) = -\mathrm{div}(f_v)$.

Next, denote by H the "triangle" graph on the right. We have

$$\mathrm{Div}^0(H) \cong \{(n,m,-n-m),\ \forall n,m\in\mathbb{Z}\} \cong \mathbb{Z}\oplus\mathbb{Z}$$

and (recalling that $\mathrm{div}(\sum_{v\in V} f_v) = 0$)

$$\mathrm{Prin}(H) \cong \langle\{(1,1,-2),(1,-2,1)\}\rangle$$

(if u, v, w are the three vertices of H, then $\mathrm{div}(f_v) = -2v + w + u$). Hence

$$\Phi_H \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,1)(1,-2)\rangle} \cong \mathbb{Z}/3\mathbb{Z}.$$

5.2. **Torelli problem for graphs.** We have seen another description of the Jacobian of an algebraic curve, namely the "classical" one given in (18). This description also has its graph theoretic analogue in the following definition due to Kotani-Sunada, [KS00]. For a weightless graph G, its *Jacobian torus* is defined as follows

$$\mathbf{Jac}(G) := \Big(H_1(G,\mathbb{R})/H_1(G,\mathbb{Z}); (\ ,\ )\Big) = \Big(\mathbb{R}^g/\Lambda; (\ ,\ )\Big) \tag{25}$$

where $(\ ,\ )$ denotes the bilinear form defined on $C_1(G,\mathbb{R})$ (and hence on $H_1(G,\mathbb{R})$) by linearly extending

$$(e,e') := \kappa_{e,e'}, \quad \forall e,e' \in E.$$

Remark 5.2.1. In [KS00] the notation "Alb(G)" is used instead of $\mathbf{Jac}(G)$, and the above space is called the *Albanese torus* of G, whereas the name "Jacobian torus", and the notation $\mathrm{Jac}(G)$, is used for something else. In this paper however, for consistency with the terminology for tropical Jacobians we shall introduce later, we use the notation $\mathbf{Jac}(G)$ as in (25).

Now we can ask the "Torelli" question: is it true that two graphs having isomorphic Jacobian tori are isomorphic?

One sees easily that the answer is no. It is in fact clear from the definition that the Jacobian torus, defined in terms of the cycle spaces of the graph, should not change if some bridge of the graph gets contracted to a vertex.

Therefore one should refine the question. First, for any graph G we denote by $G^{(2)}$ the graph obtained from G by contracting every bridge to a vertex (in graph-theoretic terms, $G^{(2)}$ is "2-edge-connected", which explains the notation). Notice that $G^{(2)}$ is well defined and has the same genus as G.

Now we ask: is it true that if two graphs G_1 and G_2 have isomorphic Jacobian tori, then $G_1^{(2)} \cong G_2^{(2)}$?

The answer is, again, no. The reason now is more subtle: pick two non-isomorphic bridgeless graphs, G_1 and G_2. Suppose, as in Figure 6 below, that there exists a bijection between $E(G_1)$ and $E(G_2)$ which induces an isomorphism between the cycle spaces $H_1(G_1,\mathbb{Z})$ and $H_1(G_2,\mathbb{Z})$. In this case one says that G_1 and G_2 are *cyclically equivalent* and writes $G_1 \equiv_{cyc} G_2$. Then it is not hard to see that $\mathbf{Jac}(G_1) \cong \mathbf{Jac}(G_2)$.

We mention that the notion of cyclic equivalence we just defined, sometime called "2-isomorphism", has been given by Withney, in [Whi33], a constructive characterization which is a helpful tool in proving the results we are presently describing.

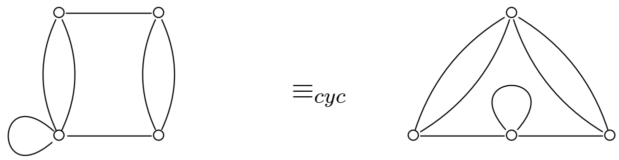


FIGURE 6. Cyclically equivalent graphs.

We are ready to state an analogue of the Torelli theorem for graphs, [CV10, Thm 3.1.1].

Theorem 5.2.2. *Let G_1 and G_2 be two graphs. Then* $\mathbf{Jac}(G_1) \cong \mathbf{Jac}(G_2)$ *if and only if* $G_1^{(2)} \equiv_{cyc} G_2^{(2)}$.

As we mentioned, the "if" part of the above theorem is easy.

What about weighted graphs? We shall see in Theorem 5.4.1 that the above theorem can be extended. For the moment, we limit ourselves to explain how to generalize the basic definitions.

Following [BMV11], the definition of cyclic equivalence extends trivially as follows: two weighted graphs, (G, w) and (G', w'), having the same genus are *cyclically equivalent* if so are G and G'.

Furthermore, for a weighted graph $\mathbf{G} = (G, w)$ we can define the graph $\mathbf{G}^{(2)} = (G^{(2)}, w^{(2)})$ obtained by contracting every bridge of G and by defining the weight function $w^{(2)}$ accordingly (recall that by definition of weighted contraction, whenever a bridge e gets contracted to a vertex v, the weight of v is the sum of the weights of the ends of e). Again, $\mathbf{G}^{(2)}$ has the same genus as $\mathbf{G}$.

Example 5.2.1. In the picture we have a stable graph $\mathbf{G} = (G, w)$ of genus 9 on the left, the corresponding graph $\mathbf{G}^{(2)} = (G^{(2)}, w^{(2)})$ in the middle, and, on the right, a graph cyclically equivalent, but not isomorpic, to $\mathbf{G}^{(2)}$.

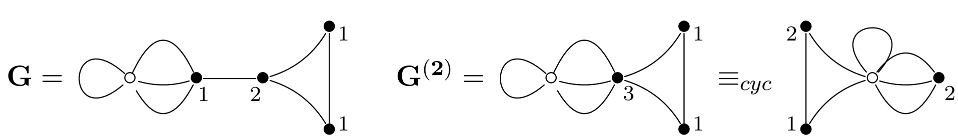


FIGURE 7

5.3. **Torelli problem for tropical curves.** The results we described in the previous subsection were obtained around the first decade of the twenty-first century, during a time when tropical geometry was flourishing (see [Mik06] and [MS15] for more on the subject). In particular the Jacobian of a pure tropical curve was introduced and studied in [MZ08] in close analogy to the Jacobian torus of a graph we introduced earlier.

Having just settled the Torelli problem for graphs, the question on how to extend it to metric graphs, i.e. to tropical curves, arose as a very natural one. This is the topic of the present subsection.

Let us start with the main definition. Following [MZ08], the Jacobian of a pure tropical curve, $\Gamma = (G, \ell)$, is defined as the following polarized torus

$$\mathbf{Jac}(\Gamma) := \left(\frac{H_1(G, \mathbb{R})}{H_1(G, \mathbb{Z})}; (\ ,\ )_\ell \right) \tag{26}$$

where $(\ ,\ )_\ell$ is the bilinear form defined on $C_1(G, \mathbb{R})$ by linearly extending

$$(e, e')_\ell := \kappa_{e,e'} \ell(e), \quad \forall e, e' \in E.$$

For simplicity, we postpone the definition of the Jacobian for an arbitrary tropical curve, and argue as in the previous subsection to arrive at the statement of the Torelli theorem for pure tropical curves.

Let $\Gamma_1 = (G_1, \ell_1)$ and $\Gamma_2 = (G_2, \ell_2)$ be two pure tropical curves. We say that Γ_1 and Γ_2 are *cyclically equivalent*, and write $\Gamma_1 \equiv_{\text{cyc}} \Gamma_2$, if there is a length preserving bijection between $E(G_1)$ and $E(G_2)$ which induces an isomorphism between $H_1(G_1, \mathbb{Z})$ and $H_1(G_2, \mathbb{Z})$. Of course if two curves are cyclically equivalent, so are the underlying graphs.

Next, for a tropical curve $\Gamma = (G, \ell)$ we denote by $\Gamma^{(2)} = (G^{(2)}, \ell^{(2)})$ the curve whose underlining graph, $G^{(2)}$, is defined as in the previous subsection and whose length function $\ell^{(2)}$ coincides with ℓ (as we have a natural inclusion $E(G^2) \subset E(G)$).

Reasoning as in the previous subsection, we see that if $\Gamma_1^{(2)} \equiv_{\text{cyc}} \Gamma_2^{(2)}$ then $\mathbf{Jac}(\Gamma_1) \cong \mathbf{Jac}(\Gamma_2)$. Does the converse hold?

The answer is no. Consider the two tropical curves Γ and Γ' in the picture below (ignoring the weights on the vertices for the moment) and assume that the ten unmarked edges (five on each curve) have the same length, say equal to 1. Now, Γ and Γ' are clearly not cyclically equivalent, but they turn out to have isomorphic Jacobians.

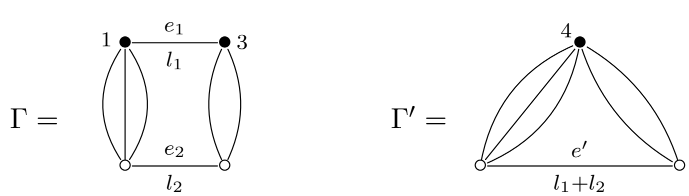


FIGURE 8. $\Gamma' = \Gamma^{(3)}$

In fact, observe that in Γ the pair of edges (e_1, e_2) has the property that every cycle containing e_1 also contains e_2, and conversely. This property is equivalent to the fact that (e_1, e_2) is a *separating pair* of edges, i.e. it disconnects Γ. Now, there is a bijection between the cycles in Γ and those of Γ' mapping bijectively cycles containing e_1 and e_2 to cycles containing e'. Finally, and this is a key point, although this bijection does not preserve the number of edges in the cycles, it does preserve the total length of the cycles.

Let us turn this example into a special case of something we have yet to define. For a tropical curve $\Gamma = (G, \ell)$ we introduce a curve $\Gamma^{(3)} = (G^{(3)}, \ell^{(3)})$ defined as follows. First, we eliminate all the bridges by replacing Γ with $\Gamma^{(2)}$. Then for any separating pair of edges, (e_1, e_2), we contract one of them to a vertex and set the length of the remaining edge equal to $\ell(e_1)+\ell(e_2)$. We repeat this process until there are no separating pairs left; the resulting curve is written $\Gamma^{(3)} = (G^{(3)}, \ell^{(3)})$ (because the graph $G^{(3)}$ is "3-edge connected"). Now, the curve $\Gamma^{(3)}$ is not uniquely determined by Γ but, as we will show in Lemma 5.3.4, its cyclic equivalence class is.

For future purposes, we need a definition.

Definition 5.3.1. Let $G = (E, V)$ be a graph free from bridges. We define an equivalence relation on E as follows: $e_1 \sim e_2$ if (e_1, e_2) is a separating pair, i.e. every cycle of G containing e_1 contains e_2 and conversely. The equivalence classes of this relation are called the *C1-sets* of G.

It is clear that "$\sim$" is an equivalence relation.

As we will explain in Subsection 5.5, the terminology "C1-set" is motivated by the corresponding algebro-geometric setting.

The C1-sets turn out to be elements of the poset $\mathcal{SP}_G$ defined earlier, in subsection 4.5.

It is not hard to see that a bridgeless graph is 3-edge connected if and only if its C1-sets have all cardinality 1.

Remark 5.3.2. Given a tropical curve $\Gamma = (G, w, \ell)$, the curve $\Gamma^{(3)} = (G^{(3)}, w^{(3)}, \ell^{(3)})$ can be constructed as follows. For every C1-set of G, contract all but one of its edges. Then set the length, $\ell^{(3)}(e)$, of the remaining edge, e, equal to the sum of the lengths of all the edges in the C1-set.

The weight function $w^{(3)}$ is defined as for any weighted edge-contraction.

Example 5.3.1. In Figure 7, the curve Γ has only one C1-set with more than one element, namely $\{e_1, e_2\}$. By contracting e_1 and defining the length as in the picture we get $\Gamma^{(3)} = \Gamma'$.

In [CV10, Thm 4.1.10] it is proved that two pure tropical curves Γ_1 and Γ_2 have isomorphic Jacobians if and only if $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$ are cyclically equivalent. This has been generalized to all tropical curves in [BMV11]. Before stating the theorem we need to define, following [BMV11], the Jacobian of a tropical curve $\Gamma = (G, w, \ell)$ of genus g, which is as follows

$$\mathbf{Jac}(\Gamma) := \left(\frac{H_1(G,\mathbb{R}) \oplus \mathbb{R}^{g-b_1(G)}}{H_1(G,\mathbb{Z}) \oplus \mathbb{Z}^{g-b_1(G)}}; (\ ,\ )_\ell \right) \tag{27}$$

where $(\ ,\ )_\ell$ is defined as in (26) on $H_1(G,\mathbb{R})$, and extended to 0 on $\mathbb{R}^{g-b_1(G)}$.

Next, two tropical curves $\Gamma_1 = (G_1, w_1, \ell_1)$ and $\Gamma_2 = (G_2, w_2, \ell_2)$ are defined to be *cyclically equivalent* if so are the pure tropical curves (G_1, ℓ_1) and (G_2, ℓ_2).

We can finally state the tropical Torelli theorem, [BMV11, Thm 5.3.3] (in different words).

Theorem 5.3.3. *Let* Γ_1 *and* Γ_2 *be tropical curves. Then* $\mathbf{Jac}(\Gamma_1) \cong \mathbf{Jac}(\Gamma_2)$ *if and only if* $\Gamma_1^{(3)} \equiv_{cyc} \Gamma_2^{(3)}$.

As a last comment on this theorem we shall clarify its statement by proving that the cyclic equivalence class of $\Gamma^{(3)}$ is uniquely determined by Γ.

Lemma 5.3.4. *Let* $\Gamma = (G, w, \ell)$ *be a tropical curve. Then the cyclic equivalence class of* $\Gamma^{(3)} = (G^{(3)}, w^{(3)}, \ell^{(3)})$*, defined in Remark 5.3.2, is uniquely determined.*

Proof. Fix $\Gamma^{(3)}$ as above. Since cyclic equivalence does not take weights into account, we are free to ignore w and $w^{(3)}$.

We shall prove that there is a canonical length-determining bijection between the edges of $G^{(3)}$ and the C1-sets of G which, moreover, induces a canonical bijection between the cycles of $G^{(3)}$ and those of G. By definition of cyclic equivalence this will conclude the proof.

By Remark 5.3.2 every edge, e, of $G^{(3)}$ corresponds to a unique C1-set of G and every C1-set of G is obtained in this way. Next, $\ell^{(3)}(e)$ equals the cardinality of this C1-set. This gives a canonical length-determining bijection between edges of $G^{(3)}$ and C1-sets of G.

By Definition 5.3.1, if a cycle of G contains a certain edge, then it contains the entire C1-set containing it. In other words, every cycle is partitioned into C1-sets. Hence in the contraction $G \to G^{(3)}$ no cycle gets collapsed to a vertex, and different cycles of G go to different cycles of $G^{(3)}$ (C1-sets are disjoint). We thus get a canonical bijection between the cycles of G and those of $G^{(3)}$. Now we are done. ♣

5.4. **Torelli problem for weighted graphs.** We are now ready to deal with the Torelli problem for weighted graphs.

Let $\mathbf{G} = (G, w)$ be a weighted graph of genus g. Let its Jacobian torus be defined as follows

$$\mathbf{Jac}(\mathbf{G}) := \left(\frac{H_1(G,\mathbb{R}) \oplus \mathbb{R}^{g-b_1(G)}}{H_1(G,\mathbb{Z}) \oplus \mathbb{Z}^{g-b_1(G)}}; (\ ,\ ) \right)$$

where $(\ ,\ )$ is defined as for (27).

We shall now derive the following theorem as a consequence of Theorem 5.3.3. It could probably be proved directly, but, as far as we know, this has not been done.

Theorem 5.4.1. *Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be two stable weighted graphs of genus g. Then $\mathbf{Jac}(\mathbf{G_1}) \cong \mathbf{Jac}(\mathbf{G_2})$ if and only if ${\mathbf{G_1}}^{(2)} \equiv_{cyc} {\mathbf{G_2}}^{(2)}$.*

Proof. We prove the "only if" part (the other part is clear); so assume $\mathbf{Jac}(\mathbf{G_1}) \cong \mathbf{Jac}(\mathbf{G_2})$. By the other direction, we can assume $\mathbf{G_1}$ and $\mathbf{G_2}$ are free from bridges.

We introduce the locus in $M_g^{\rm trop}$ of tropical curves all of whose edges have length 1:

$$M_g^{\rm trop}[1] := \{[\Gamma] = [(G, w, \ell)] \in M_G^{\rm trop} : \ \ell(e) = 1, \ \forall e \in E\}.$$

We identify it with the set of stable graphs of genus g, as follows

$$\mathrm{S}_g \xrightarrow{\cong} M_g^{\rm trop}[1]; \qquad \mathbf{G} = (G, w) \mapsto [(G, w, \underline{1})]$$

where $\underline{1}$ denotes the constant length function equal to 1. Denote by Γ_1 and Γ_2 the tropical curves in $M_g^{\rm trop}[1]$ corresponding to $\mathbf{G_1}$ and $\mathbf{G_2}$; as $\mathbf{Jac}(\mathbf{G_1}) \cong \mathbf{Jac}(\mathbf{G_2})$ we have, of course, $\mathbf{Jac}(\Gamma_1) \cong \mathbf{Jac}(\Gamma_2)$, and hence $\Gamma_1^{(3)} \equiv_{\rm cyc} \Gamma_2^{(3)}$, by Theorem 5.3.3.

Let $i = 1, 2$; as Γ_i has all edges of length 1, all the edges of $\Gamma_i^{(3)}$ have integer length. Recall from Remark 5.3.2 that the edges of $\Gamma_i^{(3)}$ are in bijection with the C1-sets of G_i, and, under this bijection, the C1-set corresponding to an edge, e, of $\Gamma_i^{(3)}$ has cardinality $\ell^3(e)$.

Since $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$ are cyclically equivalent, there is a length preserving bijection between the edges of $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$, hence we obtain a cardinality preserving bijection between the C1-sets of G_1 and those of G_2. By [CV10, Prop. 2.3.9] we conclude that G_1 and G_2 are cyclically equivalent, and hence so are $\mathbf{G_1}$ and $\mathbf{G_2}$. Our statement is proved. ♣

5.5. **Torelli problem for stable curves.** We now go back to the set up of subsection 4.4, and study the fibers of the Torelli map $\overline{\tau} : \overline{M}_g \longrightarrow \overline{A_g}$. To simplify the exposition we shall assume our curves are free from separating nodes (i.e. their dual graph is bridgeless). As we have seen in Proposition 4.4.2, separating nodes don't play a significant role but some technicalities are needed to treat them, and the statement of Theorem 5.5.1 needs to be modified; see [CV11, Thm. 2.1.4].

Recall that for a stable curve X we set

$$\overline{\tau}([X]) = \overline{\mathbf{Jac}}(X) = \big[\mathrm{Jac}(X) \curvearrowright (\overline{P_{g-1}(X)}, \overline{\Theta(X)})\big].$$

We shall look more closely at the stratification of $\overline{P_{g-1}(X)}$. By combining algebro-geometric arguments with the combinatorial tools we developed for the tropical Torelli problem one can prove the following facts.

(A) With respect to the stratification (23) and the support map (24), the codimension-one strata of $\overline{P_{g-1}(X)}$ are in bijection with the C1-sets of G_X; cf. Definition 5.3.1. (The terminology "C1-set" stands for "Codimension 1 set", and is motivated precisely by the present situation).

Hence for every C1-set, S, there is a unique stratum corresponding to it (in other words $|\Sigma_S| = 1$); we shall denote by P_S this stratum.

(B) Let P_S be the the stratum corresponding to a C1-set, S. The intersection $\overline{\Theta(X)} \cap P_S$ determines the set of branch points over S, i.e. the set $\nu^{-1}(S) \subset X^\nu$ (abusing notation as usual viewing $S \subset \mathrm{Sing}(X)$).

(C) For every C1-set S, the number of irreducible components of $\overline{\Theta(X)} \cap P_S$ equals the cardinality of S.

(D) On the opposite side, in (23) the maximum codimension of a stratum is equal to $b_1(G_X)$, and there is a unique such "smallest" stratum, $P_E \subset \overline{P_{g-1}(X)}$, where $E = E(G_X)$, identified as follows

$$P_E \cong \prod_{v \in V} \mathrm{Pic}^{g_v - 1} C_v^\nu.$$

For any $v \in V$ denote by $\pi_v : P_E \to \mathrm{Pic}^{g_v-1} C_v^\nu$ the projection; then the intersection of $\overline{\Theta(X)}$ with the smallest stratum satisfies

$$\pi_v(\overline{\Theta(X)} \cap P_E) = \Theta(C_v^\nu).$$

Now, consider two stable curves X_1 and X_2, such that $\overline{\tau}([X_1]) = \overline{\tau}([X_2])$; how are they related? Our goal is to answer this question and show they need not be isomorphic.

By hypothesis we have an isomorphism between $\mathrm{Jac}(X_1) \curvearrowright \overline{P_{g-1}(X_1)}$ and $\mathrm{Jac}(X_2) \curvearrowright \overline{P_{g-1}(X_2)}$, hence, by Remark 4.5.1, we have a bijection between the stratifications which preserves the dimension of the strata. Hence, with the notation of Subsection 4.4, we have an isomorphism of posets $\mathcal{ST}_{X_1} \cong \mathcal{ST}_{X_2}$. Now, this turns out to imply that there is an isomorphism of posets $\mathcal{SP}_{X_1} \cong \mathcal{SP}_{X_2}$. Using the combinatorial set-up described in the previous sections and [CV10, Thm 5.3.2], this implies

$$G_1^{(3)} \equiv_{cyc} G_2^{(3)}. \tag{28}$$

In particular, we have a bijection between the C1-sets of G_1 and G_2.

On the other hand, we have the isomorphism between the Theta divisors. Hence for every C1-set, S_1, of G_1 the intersection $\overline{\Theta(X_1)} \cap P_{S_1}$ is mapped isomorphically to $\overline{\Theta(X_2)} \cap P_{S_2}$, where S_2 is the C1-set of G_2 corresponding to S_1. Part (C) implies that S_1 and S_2 have the same cardinality. Combining this with (28) and arguing as in the final step of the proof of Theorem 5.4.1, we conclude

$$G_1 \equiv_{cyc} G_2.$$

The above relation implies that X_1 and X_2 have the same number of irreducible components, and the same number of nodes.

Now, the isomorphism $\overline{P_{g-1}(X_1)} \cong \overline{P_{g-1}(X_2)}$ maps the smallest stratum isomorphically to the smallest stratum. By what we said in (D), we can apply the Torelli theorem for smooth curves to obtain an isomorphism

$$X_1^\nu \smallsetminus \{\text{rational components}\} \cong X_2^\nu \smallsetminus \{\text{rational components}\}.$$

On the other hand, we already proved that X_1 and X_2 have the same number of components, hence $X_1^\nu \cong X_2^\nu$. Now, by (B), this isomorphism maps the branch points of X_1 to those of X_2.

Summarizing, we showed that X_1 and X_2 have

(1) the same normalization, let us denote it by Y and let $Y \xrightarrow{\nu_i} X_i$ be the normalization maps for $i = 1, 2$;
(2) cyclically equivalent graphs, and hence the "same" C1-sets;
(3) the same set of branch points $B = \nu_1^{-1}(\mathrm{Sing}(X_1)) = \nu_2^{-1}(\mathrm{Sing}(X_2))$.

Let us focus on B, a set of points in Y which must be pairwise glued to form the nodes of X_1 and X_2. How are the points of B to be glued? As we said, B is determined together with a partition induced by the partition in C1-sets of $E(G_1)$ and $E(G_2)$. Indeed, let us identify the C1-sets of G_1 with those of G_2 and let S be one of them. Since S corresponds to a set of singular points, B contains a subset, B_S, of $2|S|$ branch points mapping to S. As S varies the sets B_S form a partition of B.

The problem now is: we do not know how to pairwise glue the branch points in B_S, unless S has cardinality one, of course. Hence the curves X_1 and X_2 may fail to be isomorphic if they have C1-sets of cardinality greater than one. This is the case in the following example.

Example 5.5.1. Let C_1 and C_2 be two non-isomorphic smooth curves of genus 2, pick two distinct points on each, $p_i, q_i \in C_i$ which are not mapped to one another by an automorphism of C_i. Set $Y = C_1 \sqcup C_2$ and $B = \{p_1, p_2, q_1, q_2\}$ and let the dual graph be as in Figure 9 (G is unique in its cyclic equivalence class). G has only one C1-set, namely $\{e_1, e_2\}$, hence the

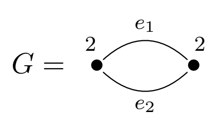


FIGURE 9

partition induced on B is the trivial one (only one set); therefore we can form exactly two non isomorphic curves corresponding to our data, namely

$$X_1 = \frac{C_1 \sqcup C_2}{p_1 = p_2, q_1 = q_2} \quad \text{and} \quad X_2 = \frac{C_1 \sqcup C_2}{p_1 = q_2, q_1 = p_2}$$

For such curves we have, indeed, $\overline{\mathbf{Jac}}(X_1) \cong \overline{\mathbf{Jac}}(X_2)$.

Let us contrast the above example by the following variation

Example 5.5.2. Let C_1, C_2 and Y be as in the previous example, pick three distinct points, $p_i, q_i, r_i \in C_i$. Set $B = \{p_1, p_2, q_1, q_2, r_1, r_2\}$ and fix the following dual graph (unique in its cyclic equivalence class)

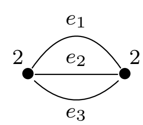
$G =$ 

FIGURE 10

Now G has three C1-sets namely $\{e_1\}, \{e_2\}, \{e_3\}$ to each of which there corresponds a pair of points in B which must, therefore, be glued together. For example, if the partition induced on B is $\{p_1, p_2\}, \{q_1, q_2\}, \{r_1, r_2\}$ then the only stable curve corresponding to these data is

$$X = \frac{C_1 \sqcup C_2}{p_1 = q_1, p_2 = q_2, r_2 = r_2}.$$

We are ready for the Torelli theorem.

Theorem 5.5.1. *Let X_1 and X_2 be stable curves of genus at least 2 without separating nodes. Then $\overline{\mathbf{Jac}}(X_1) \cong \overline{\mathbf{Jac}}(X_2)$ if and only if the following holds.*

1. *There is an isomorphism $\phi : X_1^\nu \to X_2^\nu$.*
2. *There is a bijection between the C1-sets of X_1 and those of X_2 such that for every C1-set, S_1, of X_1 and the corresponding C1-set, S_2, of X_2 we have $\phi(\nu^{-1}(S_1)) = \nu^{-1}(S_2)$.*

In particular, if the dual graphs of X_1 and X_2 are 3-edge connected, then $\overline{\mathbf{Jac}}(X_1) \cong \overline{\mathbf{Jac}}(X_2)$ if and only if $X_1 \cong X_2$.

The last part of the theorem follows from what we observed right before Remark 5.3.2. We refer to [CV11] for more general and detailed results on the fiber of the Torelli map $\overline{\tau}$.

6. CONCLUSIONS

As we discussed in the previous section, in the paper [BMV11] the authors prove the tropical Torelli theorem and construct the moduli space for tropical curves. Furthermore, they construct a moduli space for principally polarized tropical abelian varieties which we have yet to introduce. In fact, as its title indicates, the principal goal of that paper was the definition of the tropical Torelli map, in analogy with the classical Torelli map $\tau : M_g \to A_g$. In order to do that, they had to construct both the moduli space for tropical curves and the moduli space for tropical abelian varieties.

A principally polarized tropical abelian variety of dimension g is defined as a polarized torus

$$\left(\frac{\mathbb{R}^g}{\Lambda};(\ ,\ )\right),$$

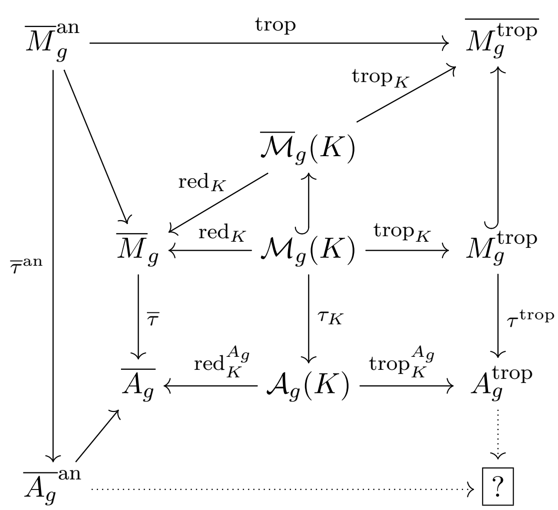
where Λ is a lattice of maximal rank in $\mathbb{R}^g$ and $(,)$ a bilinear form on $\mathbb{R}^g$ defining a semi-definite quadratic form whose null space admits a basis in $\Lambda \otimes \mathbb{Q}$. The Jacobian of a tropical curve defined in (27) is a principally polarized abelian variety.

We here denote by $A_g^{\rm trop}$ the moduli space of tropical abelian varieties constructed in [BMV11], where a different notation is used. $A_g^{\rm trop}$ is a topological space (proved to be Hausdorff in [Cha12]) and a stacky fan, and has remarkable combinatorial properties. The authors also construct and study the tropical Torelli map,

$$\tau^{\rm trop}: M_g^{\rm trop} \longrightarrow A_g^{\rm trop}; \qquad \Gamma \mapsto \mathbf{Jac}(\Gamma);$$

see also in [Cha12], [CMV13] and [Viv13].

Our goal now is to summarize, through the commutative diagram below, the connections between algebraic and tropical moduli spaces surveyed in this paper.



We have already encountered most of the maps appearing in the diagram. Only the left vertical arrow, $\overline{\tau}^{\rm an}: \overline{M_g}^{\rm an} \longrightarrow \overline{A_g}^{\rm an}$, is really new. The space $\overline{A_g}^{\rm an}$ is the Berkovich analytification of the main irreducible component of compactified moduli space of principally polarized abelian varieties, introduced in Subsection 4.4, and $\overline{\tau}^{\rm an}$ is the "analytic" Torelli map, defined as the map of Berkovich analytic spaces associated to the Torelli map $\overline{\tau}$; see [Ber90, Sect. 3.3.].

Next, recall that $\mathcal{M}_g(K)$ and $\mathcal{A}_g(K)$ are the sets of, respectively, smooth curves of genus g over K and principally polarized abelian varieties of dimension g over K; then τ_K is the corresponding Torelli map.

The map red_K is defined in Remark 3.3.2, the map $\mathrm{red}_K^{A_g}$ is defined in Remark 4.4.1, and the map trop_K is defined in (11).

The square diagram in the bottom right is a straightforward generalization of [Viv13, Thm. 4.1.7].

As the diagram illustrates, we don't known how to fill in its right-bottom corner. This would amount to constructing a compactification of the moduli space of tropical abelian varieties which could be identified with the tropicalization of $\overline{A_g}$. In this way we could try and complete the picture by gluing together the local reduction and tropicalization maps, as done for stable curves via the moduli space of extended tropical curves.

The same question could be asked by focusing on Jacobians and by replacing, in the diagram, the moduli spaces of abelian varieties by the Schottky loci. In this special case also the problem is awaiting to be solved.

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