ENUMERATIVE GEOMETRY OF PLANE CURVES

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1. Spaces of plane curves

Counting problems are among the most basic in mathematics. Enumerative geometry studies these problems when they concern geometric entities, but its interaction with other areas has been overwhelming over the past three decades. In this paper we focus on algebraic plane curves and highlight the interplay between enumerative issues and topics of a different type.

The classical ambient space for algebraic geometry is the complex projective space, $\mathbb{P}^r$, viewed as a topological space with the Zariski topology. The Zariski closed subsets are defined as the the zero loci of a given collection of homogeneous polynomials in $r+1$ variables, with coefficients in $\mathbb{C}$. These closed sets are called "algebraic varieties" when considered with the algebraic structure induced by the polynomials defining them.

Plane curves are a simple, yet quite interesting, type of algebraic variety. As sets, they are defined as the zeroes in the plane, $\mathbb{P}^2$, of a non-zero homogeneous polynomial in three variables. A homogeneous polynomial of degree d in three variables, x_0, x_1, x_2, has the form

$$G_d = \sum_{\substack{i+j+k=d \\ i,j,k\geq 0}} a_{i,j,k} x_0^i x_1^j x_2^k \tag{1}$$

where the coefficients $a_{i,j,k}$ are in $\mathbb{C}$. The set of all such polynomials is a complex vector space of dimension $\binom{d+2}{2}$ and, by definition, two non-zero polynomials determine the same curve if and only if they are multiple of one another. Therefore the set of all plane curves of degree d can be identified with the projective space of dimension $c_d := \binom{d+2}{2} - 1 = d(d+3)/2$,

$$P_d := \text{space of plane curves of degree } d = \mathbb{P}^{c_d}.$$

If d is small these spaces are well known. For $d = 1$ we have the space of all lines, which is a $\mathbb{P}^2$. For $d = 2$ we have the space of all "conics", a $\mathbb{P}^5$. This is more interesting as there are three different types of conics: (a) smooth conics, corresponding to irreducible polynomials; (b) unions of two distinct lines, corresponding to the product of two polynomials of degree 1 with different zeroes; (c) double lines, corresponding to the square of a polynomial of degree 1. Notice that conics of type (c) form a space of dimension 2 and conics of type (b) form a space of dimension 4, as each of the two lines varies in $\mathbb{P}^2$. Since the family of all conics has dimension 5 we see that most conics are smooth or, with a suggestive terminology, "the general conic is smooth", which is a shorthand for "the set of smooth conics is dense and open in the space of all conics".

The assortment of types of curves gets larger as the degree d gets larger, but for any d the general curve in P_d is smooth, i.e. given by a polynomial whose three partial derivatives have no common zeroes.

So, smooth curves form a Zariski open dense subset in P_d. This claim is an instance of a remarkable phenomenon in algebraic geometry. Indeed, let S_d be the subset in P_d parametrizing singular, i.e. non smooth, curves. By what we said, S_d is closed in P_d, hence the zero locus of some polynomials, therefore S_d is an algebraic variety. Moreover, as we shall see, the geometry of S_d is all the more interesting as it reflects some properties of the curves it parametrizes. The phenomenon we are witnessing is the fact that the sets parametrizing algebraic varieties of a certain type have themselves a natural structure of algebraic variety, they are usually called "moduli spaces" and are a central subject in current mathematics.

In this spirit, let us go back to plane curves and give an interpretation to the *dimension* of the spaces of curves we encountered so far. We introduced in (1) the general polynomial, G_d, of degree d; now we consider the projective space P_d with homogeneous coordinates $\{a_{i,j,k}, \ \ \forall i,j,k \geq 0: \ i+j+k=d\}$, and the product $P_d \times \mathbb{P}^2$. The polynomial G_d is bi-homogeneous of degree 1 in the $a_{i,j,k}$, and d in the x_i . Therefore the locus where G_d vanishes is a well defined subset of $P_d \times \mathbb{P}^2$, and it is an algebraic variety which we denote by $\mathcal{F}_d$. We view $\mathcal{F}_d$ as a "universal family" of plane curves of degree d, in fact we have the two projections, written π_1 and π_2,

$$\begin{tikzcd} & \mathcal{F}_d \subset P_d \times \mathbb{P}^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ P_d & & \mathbb{P}^2 \end{tikzcd} \tag{2}$$

and the restriction of π_1 to $\mathcal{F}_d$ expresses it as a family of plane curves: the preimage in $\mathcal{F}_d$ of a point, $[X] \in P_d$, parametrizing a curve, $X \subset \mathbb{P}^2$, is isomorphic to X, and it is mapped to X by the projection, π_2, to $\mathbb{P}^2$.

The fact that P_d has dimension $c_d = d(d+3)/2$ tells us that if we fix c_d points in $\mathbb{P}^2$ there will exist some curve of degree d passing through them, and the curve will be unique for a general choice of points. In fact, fix $p_1, \ldots, p_{c_d} \in \mathbb{P}^2$; a curve passes through p_i if the polynomial defining it vanishes at p_i. Therefore the curves passing through our points are determined by imposing $G_d(p_i) = 0$ for all $i = 1, \ldots, c_d$. This gives the following system of c_d homogeneous linear equations in $1 + c_d$ unknowns (the $a_{i,j,k}$):

$$G_d(p_1) = \ldots = G_d(p_{c_d}) = 0.$$

The solutions of this system form a vector space of dimension at least 1, with equality if and only if the equations are linearly independent, which will happen for general points $p_1, \ldots, p_{c_d}$. Since a one dimensional vector space of polynomials corresponds to a unique curve, we derive that there exists at least one curve through our fixed points, and the curve will be unique for a general choice of points. In short

$$c_d = \max\{n : \text{ any } n \text{ points in } \mathbb{P}^2 \text{ lie in a curve of degree } d\},$$

and we solved our first, however easy, enumerative problem by showing that the number of curves of degree d passing through c_d general points is equal to 1. The phrase "general points" means that the c_d points vary in a dense open subset of $(\mathbb{P}^2)^{c_d}$.

Let us now focus on S_d, the space of singular plane curves of degree d. It turns out that S_d is a hypersurface in P_d, i.e. the set of zeroes of one polynomial, hence $\dim S_d = c_d - 1$. Arguing as before, the dimension of S_d can be interpreted as the maximum number of points in the plane which are always contained in some singular curve of degree d.

For instance, four points always lie in some singular conic and it is easy to describe which. If the four points are general, i.e. no three collinear, there are exactly six lines lines passing through two of them, and our conics are given by all possible pairs of them. This gives a total of three conics pictured in Figure 1.

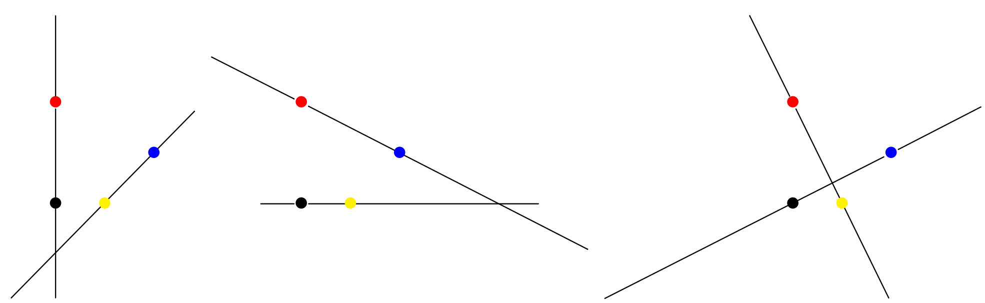

FIGURE 1. The 3 singular conics through 4 points

If three of the fixed points are collinear, we take all conics given by the union of the line through the three points with any line through the fourth point; since the set of lines through a point has dimension one, we get a one-dimensional space of conics. If the four points are collinear, we have the two-dimensional space of conics given by the union of the line through the points with an arbitrary line.

Summarizing, if (and only if) the four points are general, i.e. no three collinear, there exist finitely many singular conics through them, and the number of such conics is always 3, regardless of the choice of the four points.

As easy as this is for conics, things get more complicated already for $d = 3$. Here $\dim S_3 = 8$, and counting the "cubics" through eight points is much harder.

The key is to give this number a different interpretation[1] and identify it with another invariant of S_d, its *degree* as a subvariety of P_d.

2. The degree of the Severi variety

The degree of a subvariety in projective space is the number of points of intersection with as many generically chosen hyperplanes as its dimension. In P_d there are hyperplanes with a special geometric meaning, parametrizing curves passing through a fixed point. Indeed, let p be a point in $\mathbb{P}^2$ and let H^p be the locus in P_d of curves through p:

$$H^p = \{[X] \in P_d : \ p \in X\}. \tag{3}$$

Thus H^p is the zero locus in P_d of the homogeneous linear polynomial $G_d(p)$, hence H^p is a hyperplane. Therefore $\deg S_d$, the degree of S_d, is the number of singular curves passing through $\dim S_d$ general points, as claimed. Recalling that $\dim S_d = c_d - 1$, we want to solve the following:

Problem 1. *Compute the number of singular plane curves of degree d passing through $c_d - 1$ general points. Equivalently: compute the degree of S_d.*

Since S_d is a hypersurface, its degree is equal to the number of points of intersection with a general line. So, we fix a general line, L, in P_d and notice that L corresponds to a family of curves of degree d, a so-called "pencil of curves". More precisely, from diagram (2) we restrict the projection $\mathcal{F}_d \to P_d$ over L to get a map

$$\phi : \mathcal{X} \longrightarrow L \subset P_d$$

whose fiber over every point, $\ell \in L$, is the plane curve of degree d corresponding to the curve parametrized by ℓ. The word "pencil" indicates that the base of the family, L, is a line.

We identify L with $\mathbb{P}^1$ and denote by t_0, t_1 its homogeneous coordinates. Then our pencil is given by the zeroes in $\mathbb{P}^1_{t_0,t_1} \times \mathbb{P}^2_{x_0,x_1,x_2}$ of a polynomial

$$F(t_0, t_1; x_0, x_1, x_2),$$

bihomogeneous of degree 1 in t_0, t_1 and d in x_0, x_1, x_2, so that $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^2$ is set of zeroes of F. Since L is a general line in P_d, it intersects S_d transversally in finitely many points. These are the points of L such that the fiber of ϕ is singular and our goal is to count them. We first count the singular points of the fibers of ϕ, which are determined by the solutions of the following system, where F_{x_i} is the partial derivative with respect to x_i,

$$F_{x_0} = F_{x_1} = F_{x_2} = 0. \tag{4}$$

Since F is bihomogeneous of bidegree $(1, d)$, each F_{x_i} is bihomogenous of bidegree $(1, d-1)$ and corresponds to a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ of the same bidegree. The number of solutions of the system (4) is thus the number of points of intersection in $\mathbb{P}^1 \times \mathbb{P}^2$ of three hypersurfaces of bidegree $(1, d-1)$.

[1] *"Mathematics is the art of giving the same name to different things".* H.Poincaré

We do know how to compute this number because we can compute intersections in projective space.

We write $H^*(\mathbb{P}^r)$ for the cohomology ring with $\mathbb{Z}$-coefficients of the projective space $\mathbb{P}^r$, whose cup product can be interpreted as the intersection product. As a ring, $H^*(\mathbb{P}^r)$ is isomorphic to $\mathbb{Z}[x]/(x^{r+1})$ where x is identified with the cohomology class, $h_r \in H^2(\mathbb{P}^r)$, corresponding to a hyperplane. Hence $x^k \in H^{2k}(\mathbb{P}^r)$ corresponds to a linear subspace of complex dimension $r-k$ and real codimension $2k$. The intersection product in $\mathbb{P}^r$ depends only on the cohomology class, and the degree of the intersection of r hypersurfaces is the product of their degrees; in algebraic geometry, this is Bézout's theorem. This degree is the appropriate count for the number of points of intersection of the r hypersurfaces.

One usually identifies zero-dimensional classes, like the class of the intersection of two curves in $\mathbb{P}^2$, with their degree. This amounts to identifying the top cohomology group, $H^{2r}(\mathbb{P}^r)$, with $\mathbb{Z}$ so that the class of a point corresponds to 1.

The generator, h_1, for $\mathbb{P}^1$ is the dual of a point, and the generator, h_2, for $\mathbb{P}^2$ is the dual of a line, with $h_1^2 = 0$ and h_2^2 equal to the class of a point; with the above identification we write $h_2^2 = 1$.

What about $\mathbb{P}^1 \times \mathbb{P}^2$? It satisfies a Künneth type formula, so that its cohomology ring is generated by the pull backs of the generators of the two factors. Let us denote by $\mathfrak{h}_i$ the pull back of h_i for $i = 1, 2$.

By what we said, the number of solutions of the system (4) is the degree of the triple intersection of the class $\mathfrak{h}_1 + (d-1)\mathfrak{h}_2$. The following basic relations are easily seen to hold (again identifying the top cohomology group with $\mathbb{Z}$)

$$\mathfrak{h}_1^3 = \mathfrak{h}_1^2\mathfrak{h}_2 = \mathfrak{h}_2^3 = 0 \qquad \mathfrak{h}_1\mathfrak{h}_2^2 = 1.$$

Hence

$$(\mathfrak{h}_1 + (d-1)\mathfrak{h}_2)^3 = 3(d-1)^2.$$

Therefore the number of singularities in the fibers of ϕ is equal to $3(d-1)^2$.

By the generality of the line L, every singular fiber has exactly one singular point. Hence the number of singular fibers of ϕ is $3(d-1)^2$, and hence

$$\deg S_d = 3(d-1)^2 \tag{5}$$

is the answer to Problem 1. This confirms that there are three singular conics through four general points, and it tells us, for example, that there are twelve singular cubics passing through eight general points. Notice that, differently from what happens with conics, if the eight points are general (no three collinear, no six on a conic), each of these cubics will be irreducible, i.e. not the the union of a line and a conic.

In answering Problem 1 we mentioned that the general curve in S_d has exactly one singular point. Moreover, this point is a "node", the simplest type of singularity a curve can have, whose analytic local equation has the form $x^2 = y^2$.

We now consider curves with more singular points. We denote by $S_{d,\delta}$ the *Severi variety* of plane irreducible curves of degree d with at least δ nodes; see [S]. More precisely, $S_{d,\delta}$ is defined as the closure in P_d of the locus of irreducible curves of degree d with δ nodes:

$$S_{d,\delta} := \overline{\{[X] \in P_d : \ X \text{ irreducible with } \delta \text{ nodes}\}}.$$

If $\delta = 0$ then $S_{d,0} = P_d$, if $\delta = 1$ and $d \geq 3$ we have $S_{d,1} = S_d$.

It is clear that for $S_{d,\delta}$ to be non empty δ cannot be too big, for example it is easy to see that δ must be less than $(d-1)^2/2$. Indeed, suppose we have an irreducible curve, X, of degree $d \geq 3$ with at least $(d-1)^2/2$ nodes. Through these nodes there certainly passes a curve, Y, of degree $d-2$, because $c_{d-2} > (d-1)^2/2$. Hence the degree of the intersection of X and Y is at least $2(d-1)^2/2 = (d-1)^2$, but this contradicts Bézout's theorem, according to which the degree of the intersection of X and Y is $d(d-2)$. A more refined analysis gives $\binom{d-1}{2}$ as sharp upper bound on δ, and we have

Fact 2.1. *If $\delta > \binom{d-1}{2}$ then $S_{d,\delta}$ is empty. Assume $\delta \leq \binom{d-1}{2}$, then*

(a) $S_{d,\delta}$ is irreducible of dimension $c_d - \delta$;

(b) $S_{d,\delta}$ is smooth at points parametrizing irreducible curves with exactly δ nodes, and the locus of such points is open and dense in $S_{d,\delta}$.

The irreducibility of $S_{d,\delta}$ is proved in [H]. The number $\binom{d-1}{2}$ is the *arithmetic genus* of a plane curve of degree d. There are two types of genus for a curve: the *geometric genus* and the arithmetic genus, which coincide if the curve is smooth and irreducible.

The genus of a smooth curve is the topological genus of the real surface underlying the curve. For example, a smooth plane curve of degree 1 or 2 has genus zero and its underlying real surface is the sphere, S^2. In degree 3, the genus is 1 and the surface underlying a smooth cubic is a torus.

The geometric genus of an irreducible singular curve is defined as the genus of its desingularization.

The arithmetic genus can be thought of as the total energy of the curve, with the geometric genus being the potential energy. Just like the total energy of a system remains constant, so does the arithmetic genus in a family of curves of fixed degree. On the other hand the potential energy can be converted, all or part of it, to a "less useful" energy, and indeed the geometric genus of a curve can decrease in a specialization, but never increase. In particular, a family of curves of genus zero specializes to a curve of genus zero, all of whose irreducible components must have genus zero.

What about positive genus? Consider a family of smooth curves of degree $d \geq 3$ specializing to an irreducible curve with δ nodes. The underlying family of topological surfaces has, as general fiber, a surface of genus $g = \binom{d-1}{2}$, hence with g handles; see Figure 1 for a picture with $g = 1$ and $\delta = 1$. In this family every node of the specialization is generated by the contraction of a loop around a handle of the general fiber. The surface underlying the

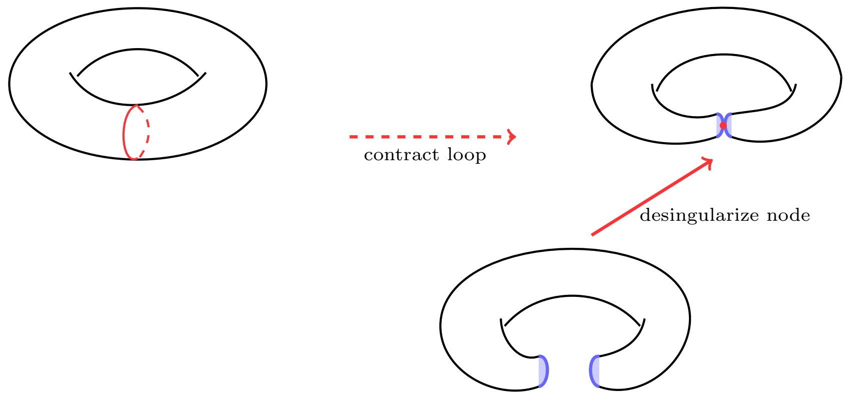

FIGURE 2. A singular specialization and its desingularization

special curve is no longer a topological manifold at the point where the loop got contracted, where the surface looks locally like two disks with centers identified. Separating the two disks desingularizes the curve so that the underlying surface has one fewer handle, hence its genus goes down by 1.

This was an informal explanation for the fact that the geometric genus of an irreducible curve of degree d with δ nodes is equal to $\binom{d-1}{2} - \delta$; hence if $\delta > \binom{d-1}{2}$ there exist no such curves. We refer to [ACGH] and [ACG] for the general theory of curves.

Recapitulating, $S_{d,\delta}$ can be defined as the the closure in P_d of the locus of irreducible curves with geometric genus $g = \binom{d-1}{2} - \delta$. A simple calculation gives a different expression for its dimension:

$$\dim S_{d,\delta} = 3d + g - 1.$$

The basic enumerative problem, generalizing Problem 1, is the following

Problem 2. *Compute the number of irreducible plane curves of degree d and genus g passing through $3d + g - 1$ general points.*

Or, compute the degree of $S_{d,\delta}$. If $\delta = 0$ then $g = \binom{d-1}{2}$ and the answer is 1. If $\delta = 1$ then $g = \binom{d-1}{2} - 1$ and we know the answer is $3(d-1)^2$. For $\delta \le 8$ the answer, given in [KP], is again a polynomial in d; see also [DI] and [G]. For bigger δ the first general solutions were discovered as recursive, rather than closed, formulas, as we shall illustrate in the rest of the paper.

3. Recursive enumeration for rational curves

A rational curve is an irreducible curve of geometric genus zero. By what we said earlier a family of rational curves can specialize only to a curve all of whose irreducible components are rational. This makes the enumeration problem in genus zero self-contained and solvable by a recursive formula, which expresses the degree of the Severi variety in degree d and genus zero in terms of the degrees of the Severi varieties in lower degrees and genus, again, zero.

We denote by R_d the Severi variety of rational curves of degree d:

$$R_d := S_{d,\binom{d-1}{2}}, \qquad r_d := \dim R_d = 3d-1, \qquad N_d := \deg R_d. \tag{6}$$

Of course, $N_1 = 1$. In case $g = 0$ the answer to Problem 2 is the following.

Theorem 3.1. *[Kontsevich's formula.] For $d \geq 2$*

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1 d_2 \left[\binom{3d-4}{3d_1-2} d_1 d_2 - \binom{3d-4}{3d_1-3} d_2^2 \right].$$

As explained in [KM], this formula was discovered in a rather different context, and it came as a beautiful surprise. While establishing the mathematical foundations for Gromov-Witten theory (a theory largely inspired by ideas from physics), Kontsevich and Manin gave an axiomatic construction of the Gromov-Witten invariants and of the quantum cohomology ring, a generalization of the classical cohomology ring of a projective algebraic variety. For $\mathbb{P}^2$ the quantum product on the quantum cohomology ring was defined using our numbers N_d, which appeared as Gromov-Witten invariants. The above formula was found as the condition characterizing the associativity of the quantum product.

The proof we shall illustrate was not among the first to be given (for which we refer to [KM] and [K], or to [RT]) but is close in spirit to our answer to Problem 1.

The shape of the formula indicates that we should use splittings of the curve into a union of two components of degrees d_1 and d_2 with $d = d_1 + d_2$. We will do that via a one-dimensional family of curves similar to the pencil we used to compute the degree of S_d, but with opposite point of view. In the previous case the unknown was the number of special curves, now the number of special curves will be easy to compute and will be used to determine the unknown, N_d.

To get our one-dimensional family we intersect R_d with general hyperplanes until we get a curve. Since intersecting with a general hyperplane decreases the dimension by one, we need to intersect with $r_d - 1$ hyperplanes. We use hyperplanes of type H^p, defined in (3). So, fix $q_1, \ldots, q_{r_d-1}$ general points in $\mathbb{P}^2$ and set

$$C = R_d \cap H^{q_1} \cap \ldots \cap H^{q_{r_d-1}}.$$

Now C is the curve in P_d parametrizing the family of rational curves of degree d through the base points $q_1, \ldots, q_{r_d-1}$, which we write as follows

$$C \times \mathbb{P}^2 \supset \mathcal{X} \longrightarrow C.$$

The basic idea is that our degree, N_d, is equal to the number of points of intersection between C and one more general hyperplane H^p, as this is

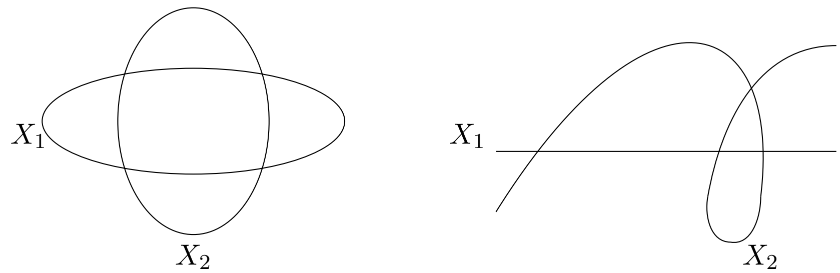

Figure 3. Reducible specializations of rational quartics

equal to the number of curves in R_d passing through $q_1, \ldots, q_{r_d-1}, p$. To put this idea to work we need to study the geometry of the family $\mathcal{X} \to C$.

By Fact 2.1 and Bertini's theorems, C is irreducible and the subset of points parametrizing irreducible curves with $\binom{d-1}{2}$ nodes and no other singularities is open, dense, and contained in the smooth locus of C. Let us look at the reducible curves parametrized by C. There are finitely many of them and, by what we said earlier, they must be unions of two rational curves of smaller degrees.

Example 3.2. Let $d = 4$, hence $\delta = 3$ and $r_4 = 11$. Our C parametrizes quartics with three nodes passing through ten points. The reducible curves parametrized by C, written $X_1 \cup X_2$, are of two types, drawn in Figure 3. First type: for any partition of the base points into two subsets, B_1 and B_2, of five points, let X_i be the conic through B_i for $i = 1, 2$ (drawn on the left in the picture). Second type: for any partition of the base points into a subset, B_1, of cardinality 2 and a subset, B_2, of cardinality 8, let X_1 be the line through B_1 and let X_2 be one of the twelve nodal cubics through B_2.

We use the following notation, $X_1 \cup X_2$ denotes a reducible curve of our family, with X_1 containing the base point q_1. The degree of X_1 will be d_1 and X_2 has degree $d_2 = d - d_1$. We say that such curves are of type (d_1, d_2). Let us count them; we have

$$r_{d_1} + r_{d_2} = r_d - 1,$$

which is the number of base points of our family. Therefore for every partition of the base points into two subsets, B_1 and B_2, of respective cardinalities r_{d_1} and r_{d_2}, there exist N_{d_i} rational curves of degree d_i passing through B_i. Since we are assuming that the base point q_1 lies on X_1, the number of curves of type (d_1, d_2) is equal to $N_{d_1} N_{d_2}$ times the number of partitions of the base points $q_2, \ldots, q_{r_d-1}$ into two subsets of cardinalities $r_{d_1} - 1$ and r_{d_2}, hence equal to

$$N_{d_1} N_{d_2} \binom{r_d - 2}{r_{d_1} - 1}. \tag{7}$$

Now, is C singular at such special points? As C is a general linear section of the Severi variety R_d, its singularity at any point reflects the local geometry of R_d, which depends on the singularities of the corresponding plane curve. Let us count the singular points of a curve of type (d_1, d_2). This amounts to counting the nodes of each component and the $d_1 d_2$ nodes in which the two components intersect, which gives a total of

$$\binom{d_1 - 1}{2} + \binom{d_2 - 1}{2} + d_1 d_2 = \binom{d-1}{2} + 1.$$

Now, when rational curves of degree d specialize to a curve of type (d_1, d_2) each of the $\binom{d-1}{2}$ nodes of the general fiber specializes to a node of the special fiber. By the above computation, there is exactly one node of the specialization which is not the limit of a "general" node. In other words, the special curve has exactly one node that gets "smoothed". But can all the nodes of the special curve be smoothed in this way? No, only the $d_1 d_2$ nodes lying in the intersection of the two components can.

To see why, suppose we have a curve, $X = X_1 \cup X_2$, of type (d_1, d_2) occurring as the specialization of a family as above parametrized by a smooth curve, U. Write $\mathcal{Z} \to U$ for this family and let $u_0 \in U$ be the point parametrizing X. Up to shrinking U near u_0 we can assume that all fibers away from u_0 are irreducible with $\binom{d-1}{2}$ nodes. The surface $\mathcal{Z}$ is necessarily singular along the nodes of the irreducible fibers, hence it has $\binom{d-1}{2}$ singular curves which we resolve by desingularizing $\mathcal{Z}$. In Figure 4 we have the example of rational quartics specializing to a reducible curve (see also Figure 3 and Example 3.2), the dotted red curves represent the singular curves of $\mathcal{Z}$. Denote by $\mathcal{Z}'$ the desingularization of $\mathcal{Z}$ so that we have a chain of maps

$$\phi : \mathcal{Z}' \longrightarrow \mathcal{Z} \longrightarrow U$$

whose composition is a new family of curves. This operation has the effect of desingularizing the general fibers, so that $\phi : \mathcal{Z}' \to U$ is a family whose fibers away from u_0 is a smooth rational curve.

The curve $X_1 \cup X_2$ will be desingularized at all nodes but the one which is smoothed in $\mathcal{Z}$ (i.e. the node which is not a limit of nodes, marked by a circle in Figure 4), hence the fiber of ϕ over u_0 is reducible with exactly one node. If, by contradiction, all points in $X_1 \cap X_2$ were limits of general nodes, then they will all be desingularized when passing to $\mathcal{Z}'$, and the fiber of ϕ over u_0 will be disconnected. But this contradicts the connectedness principle, as the fibers of ϕ away from u_0 are all connected.

In conclusion, the local geometry of R_d at a curve, X, of type (d_1, d_2) reflects that in a general deformation there is exactly one node of X which gets smoothed, so that R_d is the intersection of smooth branches, each of which corresponds to the node of X which gets smoothed along that branch.

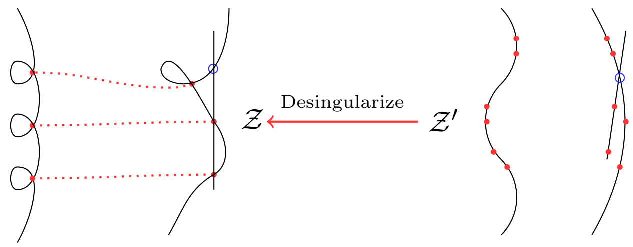

FIGURE 4. Reducible specialization and desingularization

By what we said, only the d_1d_2 "intersection" nodes can be smoothed, and the fact that all such nodes are actually smoothable follows by a symmetry argument. Concluding, R_d is, locally at X, the transverse intersection of d_1d_2 smooth branches. Hence the curve C has an ordinary d_1d_2-fold point and, under $B \to C$, the preimage of a point of type (d_1, d_2) is made of d_1d_2 distinct points.

We want to compute N_d using, as for Problem 1, intersection theory. We start from the family of curves parametrized by C and let $B \to C$ be its desingularization. We pull back to B the original family $\mathcal{X} \to C$ but, as we noted above, the so-obtained surface is singular along the nodes of its irreducible fibers, hence we replace it by its desingularization, $\mathcal{Y}$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d, "\varphi"'] \arrow[rrr, bend left, "\pi"] & \mathcal{X} \arrow[r, hook] \arrow[d] & \mathbb{P}^2 \times R_d \arrow[r] \arrow[d] & \mathbb{P}^2 \\
B \arrow[r] & C \arrow[r, hook] & R_d &
\end{tikzcd}$$

where φ is a family of rational curves with finitely many reducible fibers.

Our $\mathcal{Y}$ is a ruled surface, birational to $\mathbb{P}^1 \times B$, and its intersection product is something we can handle. First, we need a section of φ. Every base point of the family determines such a section, let Q be the section corresponding to q_1. Thus Q is the curve in $\mathcal{Y}$ intersecting each fiber of φ in one point and such that $\pi(Q) = q_1$.

We denote by $T \subset B$ the set of points over which the fiber of φ is reducible. For every $b \in T$ we denote its fiber by $Z_{b,1} \cup Z_{b,2}$ with the convention that the map π sends $Z_{b,i}$ to a plane curve of degree d_i, and the curve of degree d_1 contains q_1; so $Z_{b,2}$ does not intersect Q. We write $T(d_1, d_2)$ for the set of points parametrizing a curve of type (d_1, d_2), and $t(d_1, d_2)$ for its cardinality.

We computed in (7) the number of points in C parametrizing curves of type (d_1, d_2), over each such point there are d_1d_2 points of B, hence

$$t(d_1, d_2) = d_1 d_2 N_{d_1} N_{d_2} \binom{r_d - 2}{r_{d_1} - 1}. \tag{8}$$

As $\mathcal{Y}$ is a ruled surface its intersection ring is generated by the curves

$$\{Q, Y, Z_{b,2} \ \ \forall b \in T\},$$

and we will use the same symbols for curves and their cohomology classes. We have the following obvious relations, for every $\forall b \in T$

$$Q \cdot Y = 1, \quad Q \cdot Z_{b,2} = 0, \quad Y \cdot Z_{b,2} = 0, \quad Y^2 = 0 \quad Z_{b,2}^2 = -1,$$

identifying, as before, the top cohomology group of $\mathcal{Y}$ with $\mathbb{Z}$. To complete the intersection table we need Q^2. We compute it using a second section from the base points, so let Q' be the section corresponding to, say, q_{r_d-1}. We have $Q \cdot Q' = 0$ and $(Q')^2 = Q^2$, therefore $Q^2 = (Q - Q')^2/2$.

Now, the intersection numbers of Q and Q' differ only on the generators $Z_{b,2}$ for which the base point q_{r_d-1} lies on $\pi(Z_{b,2})$, in which case we have $Q' \cdot Z_{b,2} = 1$. Write $S \subset T$ for the set of such points, so that $Q - Q'$ and $\sum_{b \in S} Z_{b,2}$ have the same class in the intersection ring of $\mathcal{Y}$. Hence

$$Q^2 = \frac{1}{2}(Q - Q')^2 = \frac{1}{2}(\sum_{b \in S} Z_{b,2})^2 = -\frac{\#S}{2} = -\frac{1}{2}\sum_{d_1+d_2=d} s(d_1, d_2),$$

where $s(d_1, d_2)$ is the number of points in S of curves of type (d_1, d_2). Arguing similarly as for $t(d_1, d_2)$, we obtain

$$s(d_1, d_2) = d_1 d_2 N_{d_1} N_{d_2} \binom{r_d - 3}{r_{d_1} - 1}.$$

Now, the preimage of a general point, p, in $\mathbb{P}^2$ under the map π is the set of curves of our family passing through p, whose cardinality is, of course, $\deg \pi$. Hence $\deg \pi$ is the number of rational curves of degree d passing through $q_1, \ldots, q_{r_d-1}, p$, that is our unknown, N_d. On the other hand, let h_2 be the class of a line in $\mathbb{P}^2$, then $\deg \pi$ is equal to $(\pi^* h_2)^2$, an intersection number on $\mathcal{Y}$. Hence computing N_d is the same as computing $(\pi^* h_2)^2$. In the intersection ring of $\mathcal{Y}$ we can write

$$\pi^* h_2 = c_Q Q + c_Y Y + \sum_{b \in T} c_b Z_{b,2} \tag{9}$$

for some coefficients c_Q, c_Y, c_b. We can compute these coefficients by intersecting both sides of (9) with the three types of generators. We have

$$\pi^* h_2 \cdot Q = 0, \quad \pi^* h_2 \cdot Y = d, \quad \pi^* h_2 \cdot Z_{b,2} = d_2$$

by construction. These give linear relations which, with the intersection table, enable us to determine the coefficients and obtain

$$\pi^* h_2 = dQ - (dQ^2)Y - \sum_{d_1+d_2=d} \left(\sum_{b \in T(d_1,d_2)} d_2 Z_{b,2} \right).$$

Now, since $(\pi^* h_2)^2 = N_d$ we have

$$N_d = -d^2 Q^2 + \sum_{\substack{d_1+d_2=d \\ b\in T(d_1,d_2)}} (d_2 Z_{b,2})^2 = \sum_{d_1+d_2=d} \left[\frac{d^2}{2} s(d_1,d_2) - d_2^2 t(d_1,d_2)\right].$$

Hence

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1 d_2 \left[\frac{d^2}{2}\binom{r_d-3}{r_{d_1}-1} - d_2^2 \binom{r_d-2}{r_{d_1}-1}\right].$$

To see that this formula gives Theorem 3.1, set $n = r_d - 3 = 3d-4$ and $k = r_{d_1} - 1 = 3d_1 - 2$. For the term in square brackets we have

$$\frac{d^2}{2}\binom{n}{k} - d_2^2\binom{n+1}{k} = \frac{d_1^2+d_2^2}{2}\binom{n}{k} + d_1 d_2 \binom{n}{k} - d_2^2\binom{n}{k} - d_2^2\binom{n}{k-1} =$$

$$= d_1 d_2 \binom{n}{k} - d_2^2 \binom{n}{k-1} - \frac{d_2^2 - d_1^2}{2}\binom{n}{k}.$$

Summing up for $d_1 + d_2 = d$ the third summand vanishes and we are done.

For details we refer to [CH1], where this technique is used to obtain other recursions enumerating rational curves on rational surfaces.

4. Curves of positive genus

We now look at Problem 2 for $g > 0$. A complete answer is provided by means of a recursion, the precise description of which would require too many new technical details. We thus limit ourselves to illustrating the main idea and the comparison with the previous formulas.

If we consider, as we did for $g = 0$, the family of curves of degree d with δ nodes passing through a number of points equal to $\dim S_{d,\delta} - 1$, we will get a one-dimensional family which, in contrast with the case of rational curves, will parametrize no reducible curve, in general.

Example 4.1. Let $d = 5$ and $\delta = 2$ so that the geometric genus is 4 and $\dim S_{5,2} = 18$. Let C be the linear section of $S_{5,2}$ parametrizing all quintics with two nodes passing through seventeen points. There exists no reducible quintic passing through seventeen general points, as $c_1 + c_4 = 2 + 14 = 16$ and $c_2 + c_3 = 5 + 9 = 14$. Hence C parametrizes only irreducible curves.

To set up a recursive approach we need some further constraint to force reducible curves to appear. Our method is to impose that the base points lie all on a fixed line in the plane. Since an irreducible curve cannot meet a line in more points than its degree, imposing a high enough number of base points on a line will certainly cause the occurrence of reducible curves.

For this idea to work we must impose one base point at a time, in order to be able to tell at which step reducible curves appear, and to be able to describe them. This will occur at various steps, and we will have to handle sections our Severi variety having arbitrary dimension. Therefore we cannot limit our study to families over a one-dimensional base, as we did for the earlier formulas.

More precisely, the procedure starts by fixing fix a line, L, in $\mathbb{P}^2$ and a certain number of general points $q_1, q_2, \dots,$ on L. Now we intersect the Severi variety with the hyperplane H^{q_1}, then with H^{q_2}, and so on, by keeping track of how the intersection behaves at every step. After a certain number of steps the intersection splits into irreducible components, some of which parametrize curves of type $L \cup X$, so that X has lower degree and the recursion kicks in.

The price of this recursive method is that we must consider a new version of Severi variety, where the novelty is in the prescription of certain orders of contact with L at some of the base points q_i, and at arbitrary points. In other words, we need to introduce the Severi variety parametrizing plane curves of fixed (degree, genus, and) intersection profile with the line L.

We set $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_m)$, with a_i and b_i nonnegative integers such that $\sum ia_i + \sum ib_i = d$. On the line L we fix $\sum a_i$ general points, $\{q_{i,j}\}$ with $i = 1, \dots, n$ and $j = 1, \dots, a_i$. We define $S_{d,\delta}(\alpha, \beta)$ to be the closure in P_d of the locus of irreducible curves with δ nodes, having

(a) a point of contact of order i with L at $q_{i,j}$ for every $i = 1, \dots, n$ and $j = 1, \dots, a_i$;
(b) b_i points of contact of order i with L for every $i = 1, \dots, m$, different from the ones in (a).

For $\alpha = (0, \dots, 0)$ and $\beta = (d, 0, \dots, 0)$ we recover the classical Severi variety $S_{d,\delta}$.

One final point: in this set-up it is natural to drop the condition that the general curve be irreducible. So, next to $S_{d,\delta}(\alpha, \beta)$ we consider the *generalized Severi variety*, defined as before but omitting the irreducibility requirement on the general curve.

Example 4.2. Consider the generalized Severi variety with $d = 4$ and $\delta = 3$, defined as the closure in P_4 of the set of quartics with three nodes (so $\alpha = (0, \dots, 0)$ and $\beta = (4, 0, \dots, 0)$). This variety is the union of two irreducible components, both of dimension 11. One component is $S_{4,3}$, whose general point parametrizes irreducible rational quartics (the same considered in Example 3.2). The second component parametrizes reducible quartics given by the union of a line and a cubic. Since $c_1 = 2$ and $c_3 = 9$, this component has dimension 11.

The degree of the generalized Severi variety is computed in [CH2] by a recursive formula from which the (more intricate) recursion for the degree of $S_{d,\delta}(\alpha, \beta)$ follows. The proof is based on a thorough analisys of the geometry of the Severi variety, a subject of its own interest. More recently, a different proof of the same formula has been given in [GM] using tropical geometry.

In [V] an approach similar to [CH2] is used to enumerate curves of any genus in general rational surfaces. As for the enumerative geometry of curves on different surfaces, some results are known and some interesting conjectures are under investigation by means of diverse techniques. As we cannot, for lack of space, give here an exhaustive list of references we refer to [GS] and to its bibliography.

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