Algebraic and combinatorial Brill-Noether theory.

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Abstract. The interplay between algebro-geometric and combinatorial Brill-Noether theory is studied. The Brill-Noether locus $W^r_d(\Gamma)$ of a genus-g (non-metric) graph Γ is shown to be non-empty if the Brill-Noether number $\rho^r_d(g)$ is non-negative, as a consequence of the analogous fact for smooth projective curves. Similarly, the existence of a graph Γ for which $W^r_d(\Gamma)$ is empty implies the emptiness of $W^r_d(C)$ for a general curve C of genus g. The main tool is a refinement of Baker's Specialization Lemma.

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1. Introduction

In this paper we investigate the interplay between the divisor theory on algebraic curves and the divisor theory on finite graphs. Recent progress in combinatorics shows that the analogies between the two fields are quite strong. For divisors on graphs there are notions of principal divisors, linear equivalence, degree and rank, canonical class, Brill-Noether loci; the corresponding basic theory has the same shape as for algebraic curves, and some fundamental facts, such as the the Riemann-Roch formula and the Clifford inequality, hold.

We focus on some classical theorems in Brill-Noether theory, and their analogs for graphs. Precise statements for what follows can be found at the beginning of Section 6. The two basic algebro-geometric theorems are the Existence Theorem, [**K**], [**KL1**], stating that if the Brill-Noether number $\rho^r_d(g)$ is non-negative, then the variety $W^r_d(C)$ is non-empty for every smooth projective curve C of genus g; and the Brill-Noether Theorem, [**GH**],

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according to which if $\rho^r_d(g)$ is negative, then $W^r_d(C)$ is empty for a general smooth projective curve C of genus g.

It is thus quite natural to ask whether the same results hold for graphs. As far as we know, the first place where these issues have been explicitly raised is M. Baker's paper [**B**]. One of the main goals of that paper was to prove a remarkable result, called the "Specialization Lemma", which is somewhat technical to be explained in this introduction (see Section 4), but which can be applied to connect the Brill-Noether theory of curves to the Brill-Noether theory of graphs, as explained in [**B**]. As it turns out, the applications of the Specialization Lemma work better for metric graphs, or tropical curves, rather than for ordinary graphs. For instance, one of the most striking is the fact that the Brill-Noether Theorem for curves follows from the existence of one tropical curve of genus g for which W^r_d is empty (such a curve is constructed in [**CDPR**]).

We are in this paper interested in the Brill-Noether theory of graphs (with no metric). Our first step is thus to strengthen the Specialization Lemma so as to make it applicable for us. This consists in extending it from strongly semistable, regular curves over discrete valuation rings (as assumed in [**B**]), to all one-parameter families of curves with regular total space and nodal singular fibers; and from divisors defined on the total space (as in [**B**]), to families of divisors (i.e. sections of the Picard scheme) not necessarily gluing to a globally defined divisor. We do that in Section 4 and treat a more refined version for graphs with loops in Section 5.

Then we use our refined version of the Specialization Lemma, together with the Existence Theorem for curves, to prove the Existence Theorem for graphs: see Theorem 6.3.

We conclude the paper with a discussion on the Brill-Noether Theorem for graphs, noticing, in Proposition 6.9, that the classical Brill-Noether Theorem for curves follows from the existence of a (non-metric) graph for which W^r_d is empty (again, this is known thanks to [**CDPR**]). We also include some speculations about which graphs are Brill-Noether general, i.e. have an empty W^r_d whenever $\rho^r_d(g) < 0$. There are several examples of graphs that are not Brill-Noether general, and it would be interesting to have a classification of them, even only for 3-regular ones. This last problem also relates to the recently very active area of research relating moduli spaces of algebraic curves, moduli spaces of tropical curves, and Berkovich spaces; see [**BPR**] for example. There is a direct correspondence between the moduli spaces of tropical curves and of algebraic curves, based only on the underlying (non-metric) graphs; see [**C2**, Th, 4.2.1]. It would be interesting to understand how the Brill-Noether theory fits in with this correspondence.

The paper is organized as follows. Sections 2 and 3 recall some useful definitions and results from Algebraic Geometry (Section 2) and Graph Theory (Section 3) and contain no original results. In Section 4 we prove a first refinement of the Specialization Lemma, Proposition 4.4. In Section 5 we prove a second refinement using a more precise definition of rank, taking loops into account; see Proposition 5.5. In Section 6 we prove the Existence Theorem for graphs, and further discuss the interplay between algebraic and combinatorial Brill-Noether theory.

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2. Algebro-geometric preliminaries.

2.1. *Algebraic curves.* Unless we specify otherwise, we work over an algebraically closed field k; the word "point" stands for closed point; the word "curve" stands for connected, reduced, projective one-dimensional scheme defined over k.

A nodal curve is a curve having at most nodes as singularities.

Let X_0 be a nodal curve. We denote by Γ the dual graph of X_0, so that its vertex set, $V(\Gamma)$, is identified with the set of irreducible components of X_0, and its edge set, $E(\Gamma)$, is identified with the set of nodes of X_0, with an edge joining two (possibly equal) vertices if the corresponding (possibly equal) components intersect at the corresponding node. Note that Γ is an ordinary graph (no orientation, metric, or weight function). We denote by

$$X_0 = \bigcup_{v \in V(\Gamma)} C_v \tag{2.1}$$

the decomposition of X_0 into irreducible components.

For a Cartier divisor D (or a line bundle L) on X_0, the multidegree $\underline{\deg}\, D$ is

$$\underline{\deg}\, D := \{\deg_{C_v} D\}_{v \in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)}$$

where $\deg_{C_v} D$ is the degree of the restriction of D to C. We denote $r(X_0, D) := h^0(X_0, D) - 1$.

2.2. *Picard scheme.* Details about what follows may be found in [**BLR**]. Let $\phi : \mathcal{X} \to B$ be a family of generically smooth curves, i.e. a projective morphism whose fibers are curves, such that B contains a dense open subset, $B^* \subset B$, over which the fibers of ϕ are smooth. For $b \in B$ we denote $X_b := \phi^{-1}(b)$. We assume B smooth and irreducible for simplicity. We denote by $\phi^* : \mathcal{X}^* \to B^*$ the restriction of ϕ over B^*. We have the associated (relative) Picard scheme

$$\pi : \operatorname{Pic}_\phi \longrightarrow B.$$

The notation $\operatorname{Pic}_{\mathcal{X}/B}$ is often used for what we here denote by Pic_ϕ; our notation, almost the same as the one used in [**ACG**], is more convenient for our purposes. Denote by $\pi^0 : \operatorname{Pic}^0_\phi \to B$ the (relative) Jacobian. So, the fiber of π over $b \in B$ is $\operatorname{Pic} X_b$ and the fiber of π^0 is $\operatorname{Pic}^0 X_b$, the generalized Jacobian of X_b, denoted often by $\operatorname{Jac} X_b$. Recall that for $b \in B^*$ we have

$$\operatorname{Pic}^0 X_b = \{L \in \operatorname{Pic} X_b : \deg L = 0\}.$$

Now, a "pathology" of the Picard scheme is that the morphism $\operatorname{Pic}^d_\phi \to B$ is not separated if ϕ admits reducible fibers (see below). It is thus desirable to have a separated model for $\operatorname{Pic}^d_{\phi^*} \to B^*$ over B.

2.3. *Néron model.* By fundamental results of A. Néron (we refer to [**BLR**] for details) there exists a universal solution to the above problem, the Néron model, provided one restricts to the case $\dim B = 1$, which we shall henceforth assume. For our purposes, it suffices to treat the case $d = 0$. The Néron model of $\mathrm{Pic}^0_{\phi^*} \to B^*$ is a smooth, separated group scheme of finite type over B, here denoted by $N^0_\phi \to B$, whose restriction over B^* is $\mathrm{Pic}^0_{\phi^*} \to B^*$. More exactly, N^0_ϕ is the largest separated quotient of $\mathrm{Pic}^0_\phi \to B$. We are going to describe it explicitly in the special case of interest for us.

We shall assume that $B \smallsetminus B^*$ is a unique point b_0, so that ϕ has only one singular fiber, $X_0 := \phi^{-1}(b_0)$. We shall refer to b_0 and X_0 as the special point and the special fiber. We shall assume that X_0 is a nodal curve.

Now we introduce the B-scheme $E \to B$ defined as the schematic closure in Pic^0_ϕ of the unit section $B^* \to \mathrm{Pic}^0_\phi$ mapping $b \in B^*$ to $\mathcal{O}_{X_b} \in \mathrm{Pic}(X_b)$. The fiber of E over the special point b_0 is a remarkable subgroup of $\mathrm{Pic}^0(X_0)$, called the subgroup of *ϕ-twisters* and denoted by $\mathrm{Tw}_\phi(X_0)$, described as follows

$$\mathrm{Tw}_\phi(X_0) := \{\mathcal{O}_\mathcal{X}(\sum_{v \in V(\Gamma)} n_v C_v)_{|X_0}, \quad n_v \in \mathbb{Z}\}_{/\cong} \subset \mathrm{Pic}^0(X_0). \tag{2.2}$$

Equivalently, the ϕ-twisters are those line bundles on X_0 which occur as specializations of the trivial line bundle $\mathcal{O}_{\mathcal{X}^*}$.

Now, N^0_ϕ is the quotient Pic^0_ϕ/E . Let us point out that the Néron model is compatible with finite étale base changes, but not with non-étale ones (details in [**BLR**, Chapter 9] or [**C1**, Section 3]).

2.4. *Component group of the Néron model.* Consider $N^0_\phi \to B$. Its fiber over b_0 depends only on X_0 and on the singularities of $\mathcal{X}$. We need the following standard terminology.

Definition 2.5. Let $\phi : \mathcal{X} \to B$ be a flat projective morphism satisfying the following properties. B is a smooth irreducible one-dimensional quasiprojective scheme; $b_0 \in B$ is a (closed) point. $\mathcal{X}$ is a nonsingular surface. $X_0 := \phi^{-1}(b_0)$ is a projective curve.

Then we say that ϕ is a *regular one-parameter smoothing* of X_0.

Assume that $\phi : \mathcal{X} \to B$ is a regular one-parameter smoothing of X_0. Since it turns out that the special fiber of the Néron model does not depend on ϕ, we shall denote it by N_{X_0}. We have that N_{X_0} is non canonically isomorphic to the disjoint union of finitely many copies of the generalized jacobian of X_0:

$$N_{X_0} \cong \coprod_{i \in \Delta_{X_0}} (\mathrm{Pic}^0 X_0)_i \tag{2.3}$$

where Δ_{X_0} is a finite group, often called the *group of components of the Néron model.* The group Δ_{X_0} has been extensively studied; in the present situation it depends only on the intersection product of divisors of the surface $\mathcal{X}$, whose definition we now recall. Using the notation (2.1), we have

$$(C_v \cdot C_w) := \begin{cases} |C_v \cap C_w| & \text{if } v \neq w, \\ -|C_v \cap \overline{X_0 \smallsetminus C_v}|, & \text{if } v = w. \end{cases}$$

Observe that this product depends only on X_0, not on ϕ. Let us connect with definition (2.2); for every $v \in V(\Gamma)$ we have

$$\underline{\deg}\ \mathcal{O}_{\mathcal{X}}(C_v)_{|X_0} = \{(C_v \cdot C_w)\}_{w\in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)}.$$

Remark 2.6. For every $v \in V(\Gamma)$ and every $b \in B$ we have

$$\deg\ \mathcal{O}_{\mathcal{X}}(C_v)_{|X_0} = \deg \mathcal{O}_{\mathcal{X}}(C_v)_{|X_b} = \deg(\mathcal{O}_{\mathcal{X}})_{|X_b} = 0.$$

Then Δ_{X_0} is the quotient of degree-0 multidegrees by the multidegrees of all twisters, i.e.

$$\Delta_{X_0} = \frac{\underline{d} \in \mathbb{Z}^{V(\Gamma)} : |\underline{d}| = 0}{< \underline{\deg}\ \mathcal{O}_{\mathcal{X}}(C_v),\ \forall v \in V(\Gamma) >} = \frac{\underline{d} \in \mathbb{Z}^{V(\Gamma)} : |\underline{d}| = 0}{\underline{\deg}\,(\mathrm{Tw}_\phi(X_0))} \tag{2.4}$$

where $|\underline{d}| = \sum_{v\in V(\Gamma)} d_v$ for $\underline{d} = \{d_v\}_{v\in V(\Gamma)}$ and $\underline{\deg} : \mathrm{Pic}(X_0) \to \mathbb{Z}^{V(\Gamma)}$ is the multidegree homomorphism.

Observe that $\mathrm{Tw}_\phi(X_0)$ depends on ϕ, whereas $\underline{\deg}\ \mathrm{Tw}_\phi(X_0)$, Δ_{X_0}, and hence N_{X_0}, do not.

3. Divisor theory on graphs

3.1. *Divisors and intersection product.* Let Γ be a finite connected graph, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The genus of Γ is its first Betti number. The following definitions originate from [**BN**] and [**B**], but we do allow loops; see Remark 3.9.

The group of divisors of Γ, denoted by $\mathrm{Div}(\Gamma)$, is the free abelian group generated by its vertices:

$$\mathrm{Div}(\Gamma) := \{ \sum_{v\in V(\Gamma)} n_v v,\ n_v \in \mathbb{Z}\} \cong \mathbb{Z}^{V(\Gamma)}.$$

The degree of a divisor $D = \sum n_v v$ is defined as $\deg D := \sum n_v$; we denote by $\mathrm{Div}^d(\Gamma)$ the set of divisors of degree d. If $n_v \geq 0$ for all v we say that D is *effective*, and write $D \geq 0$.

There is an intersection product on $\mathrm{Div}(\Gamma)$ given by linearly extending the following definition

$$(v \cdot w) = \begin{cases} \text{number of edges joining } v \text{ and } w & \text{if } v \neq w \\ -\deg(v) + 2\,\mathrm{loop}(v) & \text{if } v = w \end{cases}$$

where $\deg(v)$ is the degree, or valency, of v, and $\mathrm{loop}(v)$ is the number of loops based at v.

Remark 3.2. If Γ is the dual graph of the nodal curve X_0, we have

$$(v \cdot w) = (C_v \cdot C_w),$$

with the right-hand side as defined earlier.

3.3. *Principal divisors.* To define principal divisor in analogy with the case of algebraic curves, one considers the set of functions on Γ, that is, the set $k(\Gamma) := \{f : V(\Gamma) \longrightarrow \mathbb{Z}\}$. Then the "order" of $f \in k(\Gamma)$ at $v \in V(\Gamma)$ is the following integer

$$\mathrm{ord}_v(f) := \sum_{w\in V(\Gamma)} (v \cdot w) f(w) \in \mathbb{Z}.$$

Now, to any $f \in k(\Gamma)$ we associate the divisor $\operatorname{div}(f) \in \operatorname{Div}(\Gamma)$ defined as follows:

$$\operatorname{div}(f) := \sum_{v \in V(\Gamma)} \operatorname{ord}_v(f) v.$$

We denote by $\operatorname{Prin}(\Gamma)$ the set of all divisors of the form $\operatorname{div}(f)$.

DEFINITION 3.4. For $v \in V(\Gamma)$ let $f_v : V(\Gamma) \to \mathbb{Z}$ be the function such that $f_v(v) = 1$ and $f_v(w) = 0$ for all $w \in V(\Gamma) \smallsetminus v$. We set

$$T_v := \operatorname{div}(f_v) = \sum_{w \in V(\Gamma)} (w \cdot v) w. \tag{3.1}$$

Remark 3.5. Let us relate this to the situation presented in Section 2.4: if ϕ is a regular one-parameter smoothing of X_0 then by Remark 3.2 we have

$$T_v = \underline{\deg}_{X_0} \mathcal{O}_{\mathcal{X}}(C_v). \tag{3.2}$$

The set $\{T_v, \ \forall v \in V(\Gamma)\}$ clearly generates $\operatorname{Prin}(\Gamma)$. Therefore we can identify $\operatorname{Prin}(\Gamma)$ as the group of all multidegrees of ϕ-twisters, i.e.

$$\operatorname{Prin}(\Gamma) = \{\underline{\deg}\, T, \ \forall T \in \operatorname{Tw}_\phi(X_0)\}.$$

By Remark 2.6, or by an easy combinatorial argument, it follows that principal divisors on Γ have degree 0.

The *Jacobian group* of Γ is $\operatorname{Jac}(\Gamma) := \operatorname{Div}^0(\Gamma)/\operatorname{Prin}(\Gamma)$.

From the previous discussion we obtain the following fact, well known in algebraic geometry (with a different terminology on the combinatorial side).

FACT 3.6. *Let $\phi : \mathcal{X} \to B$ be a regular one-parameter smoothing of a nodal curve X_0, and let Γ be the dual graph of X_0. Then there is a canonical isomorphism*

$$\Delta_{X_0} \cong \operatorname{Jac}(\Gamma)$$

between the component group of the Néron model of $\operatorname{Pic}^0_{\phi^*} \to B^*$ and the Jacobian group of Γ.

We wish to emphasize the simple but important fact that the regularity assumption on $\mathcal{X}$ cannot be removed.

3.7. *Combinatorial rank.* We now go back to the purely graph-theoretic setting. We say that $D, D' \in \operatorname{Div}(\Gamma)$ are equivalent, and write $D \sim D'$, if $D - D' \in \operatorname{Prin}(\Gamma)$. It is clear that if $D \sim D'$ then $\deg D = \deg D'$. Set

$$|D| := \{E \in \operatorname{Div}(\Gamma) : E \geq 0, \ E \sim D\}$$

and

$$r_\Gamma(D) = \begin{cases} -1 & \text{if } |D| = \emptyset \\ \\ max\{k \geq 0 : \ \forall E \in \operatorname{Div}^k_+(\Gamma) \quad |D - E| \neq \emptyset\} & \text{otherwise,} \end{cases}$$

where $\operatorname{Div}^k_+(\Gamma)$ denotes the set of effective divisors of degree k.

Remark 3.8. If $D \sim D'$ then $|D| = |D'|$ and $r_\Gamma(D) = r_\Gamma(D')$.
Also, $r_\Gamma(D) \leq \max\{-1, \deg D\}$.

Remark 3.9. It is clear that the previous definitions do not depend on the loops of Γ. In fact, throughout [**BN**] and [**B**] the authors assume that the graphs are free from loops. A good reason for doing that is that the definition of rank given above is somewhat "rough", for example, it does not satisfy the Riemann-Roch formula if Γ contains loops. Anyways, even if Γ has some loops, with the above rough definition of rank the results of the present paper continue to hold, but are less tight. Therefore, in Section 5 we will give a more precise definition for the rank of a divisor on a graph admitting loops, and show that our results generalize with that definition.

4. Baker Specialization Lemma refined

Let $\phi : \mathcal{X} \to B$ be a family of curves. The associated Picard scheme Pic_ϕ may be viewed as a functor from the category of schemes over B to the category of sets; see [**BLR**, Chapter 8]. In particular, $\text{Pic}_\phi(B)$ denotes the set of regular sections of $\pi : \text{Pic}_\phi \to B$:

$$\text{Pic}_\phi(B) = \{\sigma : B \to \text{Pic}_\phi : \quad \pi \circ \sigma = id_B\}.$$

Remark 4.1. There is a natural map

$$\text{Pic}(\mathcal{X}) \longrightarrow \text{Pic}_\phi(B); \qquad \mathcal{L} \mapsto \sigma_{\mathcal{L}} \tag{4.1}$$

such that for every $b \in B$ we have $\sigma_{\mathcal{L}}(b) = \mathcal{L}_{|X_b}$. Observe that the map (4.1) may very well fail to be surjective; see [**BLR**, Ch. 8, Prop. 4].

Let $\phi : \mathcal{X} \to B$ be a family of curves as above, and let $b_0 \in B$ be a fixed (closed) point. As usual, we set $X_0 = \phi^{-1}(b_0)$; we assume that X_0 is a nodal curve and denote by Γ the dual graph of X_0. We identify $\text{Div}(\Gamma) = \mathbb{Z}^{V(\Gamma)}$, so that we have a map

$$\text{Pic}(X_0) \longrightarrow \text{Div}(\Gamma) = \mathbb{Z}^{V(\Gamma)}; \quad L \mapsto \underline{\deg}\, L.$$

Now, we have a *specialization* map $\tau = \tau_{\phi,b_0}$ mapping a section of π to the multidegree of its value on b_0:

$$\begin{array}{ccc} \text{Pic}_\phi(B) & \stackrel{\tau}{\longrightarrow} & \text{Div}(\Gamma) \\ \sigma & \mapsto & \underline{\deg}\ \sigma(b_0). \end{array} \tag{4.2}$$

Remark 4.2. To connect with Baker's work we need to temporarily drop the general conventions stated at the begining of Section 2. The definition of the map τ above is inspired by the specialization map, denoted by ρ, defined in [**B**, Subsection 2.1]. Our definition is a slight generalization: the map ρ coincides with the composition of our τ with the canonical map (4.1) (defined at the level of divisors, rather than linear equivalence classes). More precisely: let $B = \text{Spec}\, R$ with R a complete DVR with algebraically closed residue field (which is the set-up of [**B**], to which our previous definitions are easily seen to extend), then ρ can be defined as follows

$$\rho : \text{Div}(\mathcal{X}) \longrightarrow \text{Pic}(\mathcal{X}) \longrightarrow \text{Pic}_\phi(B) \stackrel{\tau}{\longrightarrow} \text{Div}(\Gamma).$$

So, we use the terminology "specialization map" for consistency with [**B**], since no confusion should arise.

Remark 4.3. Let ϕ be a regular one-parameter smoothing of a reducible nodal curve X_0, let Γ be the dual graph of X_0; pick $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$. By the classical upper-semicontinuity theorem we have, for all $b \in B$,

$$r(X_b, \mathcal{L}_{X_b}) \leq r(X_0, \mathcal{L}_{X_0}). \tag{4.3}$$

Let d be the ϕ-relative degree of $\mathcal{L}$, i.e. $d = \deg \mathcal{L}_{X_b}$ for all $b \in B$. Now, denote by $\mathcal{L}^*$ the restriction of $\mathcal{L}$ away from X_0. Then there are infinitely many different completions of $\mathcal{L}^*$ to a line bundle on the whole of $\mathcal{X}$, and hence infinitely many line bundles on X_0 appearing as specializations of $\mathcal{L}^*$. The point is that, as $\mathcal{L}^*$ is fixed, the term on the right of the inequality (4.3) is unbounded. Indeed, let $C \subset X_0$ be an irreducible component, and set $d_C := \deg_C \mathcal{L}$. Then for every $n \in \mathbb{Z}$ the line bundle

$$\mathcal{L}^{(n)} := \mathcal{L}(-nC) \in \mathrm{Pic}(\mathcal{X})$$

is a completion of $\mathcal{L}^*$, having ϕ-relative degree d, just like $\mathcal{L}$, of course. If $n \geq 1$ then

$$\deg_C \mathcal{L}^{(n)} = d_C - nC \cdot C \geq d_C + n,$$

and if $n \gg 0$ the degree of $\mathcal{L}^{(n)}$ on $\overline{X_0 \smallsetminus C}$ is negative. Also, if $n \gg 0$ we have

$$H^0(X_0, \mathcal{L}^{(n)}_{X_0}) \supset H^0(C, \mathcal{L}^{(n)}_C(-C \cap \overline{X_0 \smallsetminus C}))$$

and the dimension of the space on the right goes to $+\infty$ as n grows. Therefore $r(X_0, \mathcal{L}^{(n)}_{X_0})$ goes to $+\infty$ with n.

We now look at the corresponding situation for the combinatorial ranks. Consider the specialization of $\mathcal{L}^{(n)}$ via the map τ defined in (4.2):

$$D^{(n)} := \tau(\sigma_{\mathcal{L}^{(n)}}) \in \mathrm{Div}(\Gamma).$$

Then for every $n, m \in \mathbb{Z}$ we have $D^{(n)} \sim D^{(m)}$ hence, by Remark 3.8

$$r_\Gamma(D^{(n)}) = r_\Gamma(D^{(m)}) \leq d.$$

Concluding, the combinatorial rank behaves better under specialization, than the algebro-geometric rank, as it depends only on $\mathcal{L}^*$ (and not on the choice of completion) and it is bounded by the relative degree of $\mathcal{L}^*$.

The following is a refinement of Baker's Specialization Lemma, which we like to view as a mixed upper semicontinuity result.

PROPOSITION 4.4 (Mixed semicontinuity). *Let $\phi : \mathcal{X} \to B$ be a regular one-parameter smoothing of a nodal curve X_0 having dual graph Γ; consider the map τ defined in (4.2). Then for every $\sigma \in \mathrm{Pic}_\phi(B)$ there exists an open neighborhood $U \subset B$ of b_0 such that for every $b \in U$ with $b \neq b_0$*

$$r(X_b, \sigma(b)) \leq r_\Gamma(\tau(\sigma)). \tag{4.4}$$

Remark 4.5. A remarkable special case is that of an element in $\mathrm{Pic}_\phi(B)$ of type $\sigma_{\mathcal{L}}$, for some $\mathcal{L} \in \mathrm{Pic}\,\mathcal{X}$, as defined in (4.1). Then 4.4 states that there is a neighborhood $U \subset B$ of b_0 such that

$$h^0(X_b, \mathcal{L}_{|X_b}) - 1 \leq r_\Gamma(\underline{\deg}\, \mathcal{L}_{|X_0})$$

for every $b \in U$ with $b \neq b_0$.

Remark 4.6. By Remark 4.3 it is clear that the assumption $b \neq b_0$ is necessary, as $r(X_0, \sigma(b_0))$ is unbounded (if X_0 is reducible), whereas, if d denotes the ϕ-relative degree of σ, then $r_\Gamma(\tau(\sigma)) \leq d$.

Proof of Proposition 4.4. We begin with the following

Claim 4.7. *We can work up to finite étale base change.*

To prove the claim, let $\epsilon : B' \to B$ be a finite étale morphism and let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\hat{\epsilon}} & \mathcal{X} \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ B' & \xrightarrow{\epsilon} & B \end{array} \tag{4.5}$$

be the corresponding base change, so that $\hat{\epsilon} : \mathcal{X}' = \mathcal{X} \times_B B' \to \mathcal{X}$ is the projection. As ϵ is étale, the total space $\mathcal{X}'$ is nonsingular. Let $b_0' \in B'$ be such that $\epsilon(b_0') = b_0$; of course, the preimage of ϕ' over b_0' is isomorphic to X_0, and hence it has the same dual graph Γ. Now, let $\tau' = \tau_{\phi', b_0'}$ be the specialization map of ϕ' with respect to b_0' (see (4.2)). We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}_\phi(B) & \xrightarrow{\epsilon^*} & \mathrm{Pic}_{\phi'}(B') \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau'} \\ \mathrm{Div}(\Gamma) & = & \mathrm{Div}(\Gamma) \end{array} \tag{4.6}$$

where ϵ^* is the pull-back of sections (for $\sigma \in \mathrm{Pic}_\phi(B)$ and $b' \in B'$ we have

$$\epsilon^*(\sigma)(b') := \hat{\epsilon}^* \sigma(\epsilon(b')) \in \mathrm{Pic}\, X'_{b'}$$

where $\hat{\epsilon}^* : \mathrm{Pic}\, X_{\epsilon(b')} \to \mathrm{Pic}\, X'_{b'}$ is the ordinary pull-back.)

Up to replacing B' with an open neighborhood of b_0' we can assume that ϕ' has smooth fibers away from b_0', so that ϕ' satisfies the same hypotheses as ϕ. Now assume the result holds for ϕ'. Let $\sigma \in \mathrm{Pic}_\phi(B)$ and let

$$\sigma' = \epsilon^*(\sigma).$$

Let $U' \subset B'$ be a neighborhood of b_0' such that

$$r(X'_{b'}, \sigma'(b')) \leq r_\Gamma(\tau'(\sigma')) \tag{4.7}$$

for every $b' \in U' \smallsetminus \{b_0'\}$. Pick an open neighborhood $U \subset \epsilon(U')$ of b_0; for every $b \in U$ and every $b' \in \epsilon^{-1}(b)$ we have $X_b \cong X'_{b'}$ and $r(X_b, \sigma(b)) = r(X'_{b'}, \sigma'(b'))$. On the other hand the commutativity of (4.6) gives $\tau(\sigma) = \tau'(\epsilon^*(\sigma)) = \tau'(\sigma')$. Combining with (4.7) we get that (4.4) holds. The claim is proved.

By the claim, we can assume that ϕ has sections; in particular we shall assume that for every irreducible component $C_v \subset X_0$, the map ϕ has a section s_v intersecting C_v.

From now on we shall work up to replacing B by an open subset containing b_0, which we can obviously do.

Fix $\sigma \in \mathrm{Pic}_\phi(B)$. The existence of a section of ϕ ensures that the canonical map $\mathrm{Pic}\, \mathcal{X} \longrightarrow \mathrm{Pic}_\phi(B)$ introduced in (4.1) is surjective; see [**BLR**, Ch.

8, Prop. 4]. Hence there exists $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ such that for every $b \in B$ we have $\mathcal{L}_{|X_b} = \sigma(b)$. We write $L_b = \mathcal{L}_{|X_b}$ and $L_0 = \mathcal{L}_{|X_0}$.

We shall prove that if $r(X_b, L_b) \geq r$ then $r_\Gamma(\underline{\deg} L_0) \geq r$ (which is of course equivalent to (4.4)).

If $r = -1$ there is nothing to prove; so we may assume $r \geq 0$.

Since $r \geq 0$ we have that $\phi_* \mathcal{L}$ has positive rank (ϕ is proper). There exists an $M \in \operatorname{Pic} B$ such that $h^0(B, \phi_* \mathcal{L} \otimes M) \geq 1$. Therefore, as $\phi_* \phi^* M = M$,

$$h^0(\mathcal{X}, \mathcal{L} \otimes \phi^* M) = h^0(B, \phi_* \mathcal{L} \otimes M) \geq 1.$$

Hence there exists an effective divisor E on $\mathcal{X}$ such that $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(E)$. We can decompose

$$E = E_{\text{hor}} + E_{\text{ver}},$$

with E_{hor} an effective "horizontal" divisor (i.e. the support of E_{hor} contains no component of the fibers of ϕ) and E_{ver} an effective "vertical" divisor (i.e. E_{ver} is entirely supported on fibers of ϕ). Up to shrinking B we can further assume that E_{ver} is supported only on X_0.

Now we are ready to finish the proof of our result, using induction on r. What now follows is similar to Baker's proof of his Specialization Lemma. If $r = 0$ we need to show that $\underline{\deg} L_0$ is equivalent to an effective divisor on Γ. We have

$$\underline{\deg} L_0 = \underline{\deg}_{X_0} E_{\text{hor}} + \underline{\deg}_{X_0} E_{\text{ver}}.$$

Now,

$$\underline{\deg}_{X_0} E_{\text{hor}} \geq 0$$

because E_{hor} is a horizontal effective divisor. On the other hand, by hypothesis, E_{ver} is of type

$$E_{\text{ver}} = \sum_{v \in V(\Gamma)} n_v C_v$$

where $X_0 = \cup_{v \in V(\Gamma)} C_v$.

Therefore, using Remark 3.5 and (3.2) , we have

$$\underline{\deg}_{X_0} E_{\text{ver}} = \sum_{v \in V(\Gamma)} n_v T_v \in \operatorname{Prin}(\Gamma).$$

In conclusion, $\underline{\deg} L_0 \sim \underline{\deg}_{X_0} E_{\text{hor}}$, so we are done. The case $r = 0$ is finished.

Assume $r \geq 1$. For every vertex $v \in \Gamma$, let s_v be the previously introduced section of ϕ passing through C_v, and let $A_v := s_v(B)$ be the corresponding effective divisor on $\mathcal{X}$. We have, of course,

$$r(X_b, \mathcal{L}(-A_v)_{|X_b}) \geq r(X_b, L_b) - 1 \geq r - 1$$

for all $b \in B$. Denote by

$$D_0 = \underline{\deg} L_0 \in \operatorname{Div}(\Gamma).$$

Let

$$\hat{\tau} : \operatorname{Pic} \mathcal{X} \longrightarrow \operatorname{Pic}_\phi(B) \xrightarrow{\tau} \operatorname{Div}(\Gamma)$$

be the composition of τ with the canonical map (4.1); we have

$$\hat{\tau}(\mathcal{L}(-A_v)) = D_0 - v \in \operatorname{Div}(\Gamma).$$

By the induction hypothesis, we have

$$r_\Gamma(D_0 - v) \geq r - 1.$$

This holds for every $v \in V(\Gamma)$. Therefore, using [**B**, Lemma 2.7] we get $r_\Gamma(D_0) \geq r$ and we are done. ■

5. Specialization for graphs with loops

Let Γ be a graph admitting some loops. The definition of rank of a divisor given before is independent of the loops, and in fact it is not a satisfactory one; for example, it trivially violates the Riemann-Roch formula which does hold on graphs free from loops, by [**BN**]. We shall now give a better definition, and extend the Specialization Lemma to this definition. First, we shall introduce some useful terminology.

Definition 5.1. Let Γ be a graph. A *refinement* of Γ is a graph $\widehat{\Gamma}$ obtained by inserting a (finite) set of vertices in the interior of the edges of Γ. We have a natural inclusion $V(\Gamma) \subset V(\widehat{\Gamma})$ which induces an injective group homomorphism

$$\iota_{\Gamma,\widehat{\Gamma}} : \operatorname{Div}\Gamma \hookrightarrow \operatorname{Div}\widehat{\Gamma}. \tag{5.1}$$

We shall often write simply $\iota = \iota_{\Gamma,\widehat{\Gamma}}$.

In general, the map (5.1) is not compatible with linear equivalence, nor does it preserve the rank (see Example 5.4). There is, however, a useful situation in which the rank is preserved.

Remark 5.2. Let Γ be a graph and let $\Gamma^{(n)}$ be the refinement of Γ given by inserting n vertices in the interior of every edge of Γ. Then, if Γ has no loops, for every $D \in \operatorname{Div}(\Gamma)$ we have

$$r_\Gamma(D) = r_{\Gamma^{(n)}}(\iota_{\Gamma,\Gamma^{(n)}}(D)).$$

This follows from [**HKN**, Corollary 22]; see also [**L**, Thm 1.3].

There is another type of refinement which preserves the rank, and which enables us to define the rank for a graph with loops in a sharper way. So, let Γ be a graph admitting l loops; denote by $\{\ell_1, \ldots, \ell_l\} \subset E(\Gamma)$ the set of loop-edges.

Let $n_1, \ldots, n_l$ be positive integers and let $\underline{n} := (n_1, \ldots, n_l)$ be their ordered sequence. Let $\Gamma^{\underline{n}}$ be the graph obtained by inserting n_i vertices in the interior of every loop-edge ℓ_i, and leaving the other edges untouched. As $\Gamma^{\underline{n}}$ is a refinement of Γ there is a natural map $\iota : \operatorname{Div}\Gamma \to \operatorname{Div}\Gamma^{\underline{n}}$. For every $D \in \operatorname{Div}(\Gamma)$ we define

$$r_\Gamma^{\#}(D) := r_{\Gamma^{\underline{n}}}(\iota(D)). \tag{5.2}$$

Remark 5.3. The above definition is independent on the choice of the numbers $n_1, \ldots, n_l$, provided $n_i \geq 1$ for all $i = 1, \ldots, l$. More precisely, for every $\underline{m} := (m_1, \ldots, m_l)$ with $m_i \geq 1$ for every $i = 1, \ldots, m$, from [**L**, Theorems 1.3 and 1.5] we obtain

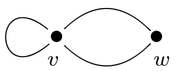
$$r_{\Gamma^{\underline{n}}}(\iota_{\Gamma,\Gamma^{\underline{n}}}(D)) = r_{\Gamma^{\underline{m}}}(\iota_{\Gamma,\Gamma^{\underline{m}}}(D)).$$

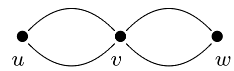
With definition (5.2) it is not hard to see that the Riemann-Roch formula holds.

Comparing with the rank r_Γ defined earlier, it is not hard to see that

$$r_\Gamma^\#(D) \leq r_\Gamma(D).$$

Example 5.4. Consider the graph Γ of genus 2 drawn in the picture below, so Γ has one loop-edge attached to the vertex v. To compute $r_\Gamma^\#$ we can use the graph $\Gamma^{(1)}$ obtained by inserting one vertex u in the loop-edge of Γ.

$\Gamma =$ 

$\Gamma^{(1)} =$ 

Consider the divisor $v + w \in \operatorname{Div}\Gamma$. Then one easily checks that

$$r_\Gamma(v+w) = 1, \qquad r_\Gamma^\#(v+w) = 0.$$

On the other hand $r_\Gamma(2v) = 1 = r_\Gamma^\#(2v)$.

We now prove that Proposition 4.4 holds with this definition of rank.

Proposition 5.5. *Assume that $\phi : \mathcal{X} \to B$ satisfies the same assumptions as Proposition 4.4. Then for every $\sigma \in \operatorname{Pic}_\phi(B)$ there exists an open neighborhood $U \subset B$ of b_0 such that for every $b \in U \smallsetminus b_0$ we have*

$$r(X_b, \sigma(b)) \leq r_\Gamma^\#(\tau(\sigma)). \tag{5.3}$$

Proof. Let l be the number of loops of Γ. As usual, we shall identify the edges of Γ with the nodes of X_0.

Let $\widehat{\mathcal{X}} \to \mathcal{X}$ be the blow up at all the l loops of Γ. By the regularity assumption on $\mathcal{X}$, every exceptional divisor of this blow-up is a (-1)-curve of $\widehat{\mathcal{X}}$, appearing with multiplicity 2 in the fiber which contains it. Hence the family of curves $\widehat{\phi} : \widehat{\mathcal{X}} \to B$ has non-reduced fiber over b_0.

We now apply the same construction as [**BPV**, Sect. III.9] to obtain a family of nodal curves with regular total space. Let $B^1 \to B$ be a degree-2 covering ramified over b_0 and let $\widehat{\mathcal{X}}^1 \to B^1$ be the base change of $\widehat{\phi}$. Let $\mathcal{Y} \to \widehat{\mathcal{X}}^1$ be the normalization, so that we have a family $\mathcal{Y} \to B^1$ of curves with nodal special fiber. The dual graph of the special fiber of $\mathcal{Y} \to B^1$ is the refinement $\widehat{\Gamma}$ of Γ obtained by adding one vertex in the interior of every loop edge of Γ. For further use, we denote by $\{e_i^+, e_i^-, i = 1, \ldots, l\} \subset E(\widehat{\Gamma})$ these new edges replacing the loops of Γ.

Now, $\mathcal{Y}$ has an A_1-singularity at every edge of $\widehat{\Gamma}$ other than the $2l$ edges $\{e_i^+, e_i^-\}$, and no other singularity ([**BPV**, proof of Prop.(III.9.2)]). Let $\mathcal{Z} \to \mathcal{Y}$ be the resolution of all such A_1 singularities; now $\mathcal{Z}$ is a regular surface and $\mathcal{Z} \to B^1$ is a regular smoothing of its special fiber.

Denote by Γ_Z the dual graph of the special fiber of $\mathcal{Z}$; then Γ_Z is a refinement of $\widehat{\Gamma}$ and hence of Γ. By construction, the edges $e_i^+, e_i^- \in E(\widehat{\Gamma})$

are not refined in Γ_Z, so they correspond to edges of Γ_Z. We abuse notation setting

$$\{e_i^+, e_i^-, \ i = 1, \dots, l\} \subset E(\Gamma_Z).$$

Finally, we introduce the refinement, $\widehat{\Gamma}^{(1)}$ of $\widehat{\Gamma}$ obtained by adding one vertex in the interior of every edge; of course, $\widehat{\Gamma}^{(1)}$ is also a refinement of Γ_Z. We have a sequence of maps

$$\operatorname{Div}\Gamma \xrightarrow{\iota} \operatorname{Div}\widehat{\Gamma} \longrightarrow \operatorname{Div}\Gamma_Z \longrightarrow \operatorname{Div}\widehat{\Gamma}^{(1)}.$$

The following picture helps to keep track of the above set-up.

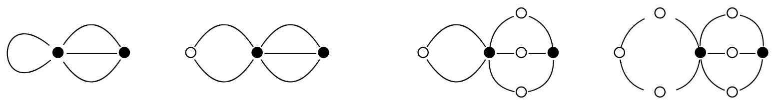


FIGURE 1. Refinements from left to right: $\Gamma, \widehat{\Gamma}, \Gamma_Z, \widehat{\Gamma}^{(1)}$. The original vertices are drawn as $\bullet$, the others as $\circ$.

Let $D \in \operatorname{Div}\Gamma$. We have

$$r^{\#}_{\Gamma}(D) = r_{\widehat{\Gamma}}(\iota(D)) = r_{\widehat{\Gamma}^{(1)}}(\iota_{\widehat{\Gamma},\widehat{\Gamma}^{(1)}}(\iota(D))) = r_{\Gamma_Z}(\iota_{\widehat{\Gamma},\Gamma_Z}(\iota(D))) = r_{\Gamma_Z}(\iota_{\Gamma,\Gamma_Z}(D)).$$

Indeed, the first "=" is the definition of $r^{\#}$; the second follows from Remark 5.2 (the refinement $\widehat{\Gamma}^{(1)}$ of $\widehat{\Gamma}$ adds one vertex in the interior of every edge); the third is the invariance mentioned in 5.3 ($\widehat{\Gamma}^{(1)}$ adds one vertex in the interior of every edge e_i^+ and e_i^- of Γ_Z); the last is $\iota_{\widehat{\Gamma},\Gamma_Z} \circ \iota = \iota_{\Gamma,\Gamma_Z}$.

Therefore we have reduced our statement to the loop-free situation for the family $\mathcal{Z} \to B^1$, which has been proved in Proposition 4.4. ∎

6. On the emptyness of Brill-Noether loci

From now on we shall assume $g \geq 2$, as, by the Riemann-Roch Theorem, the content of this section is interesting only in this case. For an algebraic curve C one defines the "Brill-Noether variety" of C as follows: $W^r_d(C) = \{L \in \operatorname{Pic}^d C : r(C, L) \geq r\}$.

Consider the Brill-Noether number $\rho^r_d(g) := g - (r+1)(g - d + r)$. We recall two fundamental theorems about $W^r_d(C)$. The "Existence Theorem" due to Kempf [**K**] and Kleiman-Laksov, [**KL1**] [**KL2**]; see also [**ACGH**, Thm. (1.1) Chapt. V]:

6.1. Existence Theorem. *If $\rho^r_d(g) \geq 0$ then for every smooth projective curve C of genus g we have $W^r_d(C) \neq \emptyset$. Moreover, if $r \geq d - g$, then every irreducible component of $W^r_d(C)$ has dimension at least $\rho^r_d(g)$.*

The assumption $r \geq d - g$ above and in Theorem 6.2 below is needed simply because if $r < d - g$ then, by Riemann-Roch, $W^r_d(C) = \operatorname{Pic}^d C$ and hence $\dim W^r_d(C) = g < \rho^r_d(g)$.

Next is the "Brill-Noether Theorem" proved by Griffiths-Harris [**GH**]; see [**ACGH**, Thm. (1.5) Chapt. V]:

6.2. Brill-Noether Theorem. *If $\rho^r_d(g) < 0$ then for a general smooth projective curve C of genus g we have $W^r_d(C) = \emptyset$. Moreover if $r \geq d - g$ then every irreducible component of $W^r_d(C)$ has dimension equal to $\rho^r_d(g)$.*

The word "general" above means: for every C in a nonempty Zariski open subset of the moduli space of smooth curves of genus g.

We now investigate whether analogous results hold for graphs. Let Γ be a graph of genus g; set

$$W^r_d(\Gamma) := \{[D] \in \operatorname{Pic}^d \Gamma : \ r^{\#}_{\Gamma}(D) \geq r\}.$$

The following Theorem 6.3 proves conjecture [**B**, Conj. 3.9 (1)] (and also [**B**, Conj. 3.10 (1)]).

Theorem 6.3 (Existence Theorem for graphs). *Let g, d, r be integers such that $\rho^r_d(g) \geq 0$. Then for every graph Γ of genus g we have $W^r_d(\Gamma) \neq \emptyset$.*

Proof. For later use, observe that we will prove that if r, d, g are such that $W^r_d(C)$ is non-empty for a general smooth curve of genus g, then $W^r_d(\Gamma) \neq \emptyset$ for every graph Γ of genus g.

Notice also that if $r < d - g$ the result is trivial, by Riemann-Roch.

Let X_0 be a general nodal curve whose dual graph is Γ. Fix a regular one-parameter smoothing $\phi : \mathcal{X} \to B$ of X_0; the existence of such a one-parameter smoothing is well known: it follows, for example, from [**DM**, Prop 1.5, pp. 81,82]. Moreover, a general one-parameter smoothing of X_0 will be a regular one. Denote by b_0 the special point of B; we can work up to replacing B with an open neighborhood of b_0. Furthermore, we choose ϕ so that it has a section (which we can do up to étale base change).

By the Existence Theorem 6.1 for curves, the assumption $\rho^r_d(g) \geq 0$ implies that for every curve C of genus g we have $W^r_d(C) \neq \emptyset$.

Now, by [**AC**, Sect. 2] (or [**ACG**, Ch. 21, Sect. 3]), for any family of smooth projective curves $\psi: \mathcal{C} \to B$ admitting a section, there exists a B-scheme $W^r_{d,\psi} \to B$ whose fiber over every $b \in B$ is $W^r_d(C_b)$. Moreover there is a natural injective morphism of B-schemes,

$$W^r_{d,\psi} \hookrightarrow \operatorname{Pic}_\psi$$

which we view as an inclusion. We want to use this construction for our one-parameter smoothing ϕ of X_0, but ϕ admits a singular fiber. Since the restriction $\phi^* : \mathcal{X}^* \to B^* = B \smallsetminus \{b_0\}$ is a family of smooth curves, the above set-up gives the relative Brill-Noether variety $W^r_{d,\phi^*} \to B^*$. We let $\overline{W^r_{d,\phi}} \to B$ be the closure of W^r_{d,ϕ^*} in the compactified Picard scheme $\overline{P^d_\phi} \to B$; recall that the restriction of $\overline{P^d_\phi}$ over B^* coincides with $\operatorname{Pic}^d_{\phi^*}$, and every point of its fiber over b_0 corresponds to a line bundle on a partial normalization of X_0; see [**C1**] for details. We define $W^r_{d,\phi}$ as the intersection of $\overline{W^r_{d,\phi}}$ with $\operatorname{Pic}^d_\phi$. By the above discussion, one obtains that this B-scheme $W^r_{d,\phi} \to B$ has non empty fiber over b_0, and, by the upper-semicontinuity of h^0, this fiber is contained in $W^r_d(X_0)$. We simplify the notation and write

$$W_\phi := W^r_{d,\phi} \longrightarrow B.$$

There exists a finite covering $\delta : B^1 \to B$, totally ramified over b_0, such that the base change $W_\phi \times_B B^1 \longrightarrow B^1$ admits a section. Now, denote by

$\phi^1 : \mathcal{X}^1 \to B^1$ the base change of ϕ; notice that the dual graph of the special fiber of ϕ^1 is again Γ. We have

$$W_{\phi^1} = W_\phi \times_B B^1 \longrightarrow B^1,$$

as compatibility with base change holds; see [**ACG**, Ch. 21, Sect. 3]. We denote by $\sigma : B^1 \to W_{\phi^1}$ the section mentioned above.

The family of curves ϕ^1 is no longer a regular smoothing of its special fiber; the situation we are about to describe is detailed in [**BPV**, Sect. III.9]. We denote by $\widetilde{\mathcal{X}^1} \to \mathcal{X}^1$ the normalization of $\mathcal{X}^1$ and by $\mathcal{Y} \to \widetilde{X^1}$ its resolution of singularities. More precisely, if, locally at b_0, the covering δ has the form $t \mapsto t^{n+1}$, then the surface $\widetilde{\mathcal{X}^1}$ has a singular point of type A_n at every node of its special fiber. Therefore the map $\mathcal{Y} \to \widetilde{X^1}$ replaces every node of the special fiber by a chain of n exceptional components. Denote by

$$\chi : \mathcal{Y} \longrightarrow B^1$$

the family over B^1 obtained by composing $\mathcal{Y} \to \widetilde{X^1} \to \mathcal{X}^1 \to B^1$. Now χ is a regular smoothing of its special fiber, Y_0. The dual graph of Y_0 is obtained from the dual graph, Γ, of the special fiber of ϕ^1, by inserting n vertices in the interior of every edge. Hence we denote by $\Gamma^{(n)}$ the dual graph of Y_0. We have a natural map

$$\iota = \iota_{\Gamma,\Gamma^{(n)}} : \mathrm{Div}(\Gamma) \longrightarrow \mathrm{Div}(\Gamma^{(n)})$$

and this map preserves the rank, i.e. $r_\Gamma^\#(D) = r_{\Gamma^{(n)}}^\#(\iota(D))$ for every $D \in \mathrm{Div}(\Gamma)$, by Remark 5.2. Now, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\beta} & \mathcal{X}^1 \\ \chi\downarrow & & \downarrow\phi^1 \\ B^1 & = & B^1 \end{array} \tag{6.1}$$

and an associated B^1-map $\mathrm{Pic}_{\phi^1} \to \mathrm{Pic}_\chi$, hence also a map

$$\beta^* : \mathrm{Pic}_{\phi^1}(B^1) \longrightarrow \mathrm{Pic}_\chi(B^1).$$

The previously defined section $\sigma : B^1 \to W_{\phi^1}$ is an element of $\mathrm{Pic}_{\phi^1}(B^1)$, so that $\beta^*(\sigma) \in \mathrm{Pic}_\chi(B^1)$. By Proposition 5.5 applied to χ and $\beta^*(\sigma)$, we have for every $b \in B^1$, $b \neq \delta^{-1}(b_0)$

$$r \leq r\Big(Y_b, \beta^*(\sigma)(b)\Big) \leq r_{\Gamma^{(n)}}^\#\Big(\tau\big(\beta^*(\sigma)\big)\Big).$$

On the other hand, by construction, the divisor $\tau(\beta^*(\sigma)) \in \mathrm{Div}(\Gamma^{(n)})$ corresponds to $\tau(\sigma) \in \mathrm{Div}(\Gamma)$ under the refinement map $\iota : \mathrm{Div}\,\Gamma \to \mathrm{Div}\,\Gamma^{(n)}$; in symbols

$$\tau(\beta^*(\sigma)) = \iota(\tau(\sigma)).$$

Since ι preserves the ranks, we obtain

$$r \leq r_{\Gamma^{(n)}}^\#(\tau(\beta^*(\sigma))) = r_{\Gamma^{(n)}}^\#(\iota(\tau(\sigma))) = r_\Gamma^\#(\tau(\sigma)).$$

Hence we have $\tau(\sigma) \in W_d^r(\Gamma)$. We have thus proved that $W_d^r(\Gamma)$ is not empty, so we are done. ■

Although the previous theorem is purely graph theoretic, our proof uses algebraic geometry. So we wish to propose the following problem.

Problem 1. *Find a purely combinatorial proof for Theorem 6.3.*

We now turn to the Brill-Noether Theorem 6.2. There is an important difference with the Existence Theorem, namely the Brill-Noether Theorem is valid for a general curve, and is well known to fail for some particular curves (for example, hyperelliptic curves of genus at least 3). This difference reflects itself in the subsequent discussion.

Fact 6.4. *(*[**B**, Conjecture 3.9 (2)] - [**CDPR**, Theorem 1.1]*) Assume $\rho_d^r(g) < 0$. There exists a graph of Γ of genus g such that $W_d^r(\Gamma) = \emptyset$.*

Remark 6.5. Theorem 1.1 of [**CDPR**] is actually a stronger result, from which the above fact follows. In particular, the authors obtain the following. Let Γ be a chain of g cycles $\Delta_1, \ldots \Delta_g$ each of which has $2g-1$ cyclically ordered vertices $V(\Delta_i) = \{v_1^i, \ldots, v_{2g-1}^i\}$ and such that

$$v_{2g-1}^1 = v_1^2, \quad v_{2g-1}^2 = v_1^3, \; \ldots, \; v_{2g-1}^i = v_1^{i+1}, \; \ldots, \; v_{2g-1}^{g-1} = v_1^g,$$

with no other identifications. Then $W_d^r(\Gamma) = \emptyset$ if $\rho_d^r(g) < 0$.

Recall that a graph is called 3-*regular* if all of its vertices have valency 3, and that a 3-regular graph is 3-connected if and only if it is 3-edge connected.

Conjecture 6.6. *Assume $g \geq 2$ and $\rho_d^r(g) < 0$.*

(1) There exists a 3-regular graph Γ of genus g for which $W_d^r(\Gamma) = \emptyset$.
(2) Let Γ be a graph of genus g with the highest number of automorphisms (among graphs of genus g). Then $W_d^r(\Gamma) = \emptyset$.

Remark 6.7. These two conjectures are quite different, but they are both inspired by the analogies between the moduli space $\overline{M_g}$ of Deligne-Mumford stable curves and the moduli space M_g^{trop} of tropical curves (see [**BMV**] and [**C2**]).

Both $\overline{M_g}$ and M_g^{trop} admit a partition into strata which are indexed by graphs (the dual graphs of stable curves for $\overline{M_g}$, the underlying graphs of tropical curves for M_g^{trop}); in both cases, the set of strata is partially ordered under inclusion of closures. Recall that the generic points of M_g^{trop}, i.e. the points in the top dimensional strata, parametrize tropical curves whose underlying graph is 3-regular. As we said, for a general point of $\overline{M_g}$, i.e. for a general smooth curve of genus g, the Brill-Noether variety is empty whenever ρ is negative. By analogy, we may ask whether some of the generic (from the tropical point of view) graphs, i.e. the 3-regular graphs, have an empty Brill-Noether locus when ρ is negative. This explains part (1). Finally, notice that there do exist 3-regular graphs that are not Brill-Noether general (see the next remark); hence if (1) holds, it would be interesting to characterize the graphs that satisfy it.

Next, to motivate the second part, we recall that by [**C2**, Thm. 4.7], the natural bijection between the partitions of $\overline{M_g}$ and M_g^{trop} is order reversing. This suggests that a general point of $\overline{M_g}$ corresponds to a special point of M_g^{trop}. Let us focus on the top dimensional strata: smooth curves on one side, 3-regular graphs on the other side. It is well known that a general smooth curve in $\overline{M_g}$ has no nontrivial automorphisms, hence, with the above "reversion" phenomenon in mind, we may think of a 3-regular graph with the greatest number of symmetries as its analog. Since, as we said, a general curve has empty Brill-Noether locus when ρ is negative we can ask whether the same holds for the analogous graphs.

Remark 6.8. Suppose $g \geq 3$. A 3-regular and 3-connected graphs is known to be non-hyperelliptic. Morover, for every g there exists a 3-regular hyperelliptic graph of connectivity 2 (such graphs are, of course, not Brill-Noether general).

In the next proposition, the new fact with respect to the tropical proof of the Brill-Noether Theorem of [**CDPR**] is that it suffices to have an ordinary graph, rather than a tropical curve, for which the Brill-Noether locus is empty (and checking the emptyness of $W^r_d(\Gamma)$ for a graph Γ is a finite amount of work).

PROPOSITION 6.9. *Suppose that for some integers d, g, r there exists a graph Γ of genus g such that $W^r_d(\Gamma) = \emptyset$. Then $W^r_d(C) = \emptyset$ for a general smooth projective curve C defined over an algebraically closed field.*

In particular, if d, g, r are such that $\rho^r_d(g) < 0$, then there exists such a graph, and hence the Brill-Noether Theorem 6.2 holds.

PROOF. As we already observed, the proof of Theorem 6.3 consists in showing that if $W^r_d(C)$ is non-empty for a general curve C, then $W^r_d(\Gamma)$ is non empty for every genus g graph. Hence the first part of the statement follows.

Now, if $\rho^r_d(g) < 0$, the existence of a genus g graph Γ for which $W^r_d(\Gamma) = \emptyset$ is proved in [**CDPR**], as stated in Fact 6.4. Hence we are done. ∎

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