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Uniformity of rational points an up-date and corrections

Lucia Caporaso, Joe Harris and Barry Mazur

The purpose of this note is to correct, and enlarge on, an argument in a paper we published a quarter century ago (*J. Amer. Math. Soc.* **10**:1 (1997), 1–35). The question raised is a simple one to state: given that a curve C of genus $g \geq 2$ defined over a number field K has only finitely many rational points, we ask if the number of points is bounded as C varies.

1. Introduction

In [Caporaso et al. 1997] it is asserted that, assuming the truth of the strong Lang conjecture (Conjecture 8), a very strong form of boundedness holds: for every $g \geq 2$ there is a finite bound $N(g)$ — not depending on K! — such that for any number field K there are only finitely many isomorphism classes of curves of genus g defined over K with more than $N(g)$ K-rational points. The issue is, in that statement do we mean finitely many isomorphism classes over K, or over the algebraic closure $\overline{K}$? The paper asserts the statement in the stronger form — up to isomorphism over K — but the proof establishes only the weaker statement that there are finitely many curves with more than $N(g)$ points up to isomorphism over $\overline{K}$.

The main purpose of this note is to give a complete argument of the stronger form, which we will do in Sections 3 and 4. Of course, if indeed there is a "universal" bound $N = N(g)$ on the number of points on a curve of genus g defined over an arbitrary number field — with finitely many exceptions for any given K — the question of how large $N(g)$ has to be is an intriguing one, and we devote the final chapter to a preliminary discussion of this and related questions.

2. Moduli spaces

Fix a genus $g > 1$.

The coarse moduli space. Let $M = M_g$, the coarse moduli space of smooth projective curves of genus g; so M is a variety defined over $\mathbb{Q}$.

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The rigidified moduli space.

Definition 1. A point p in a variety V over a field K is *rigid in* V if there are no nontrivial automorphisms of V (over the algebraic closure $\overline{K}$) that fix p; i.e., for any automorphism $\alpha : V \to V$ if $\alpha(p) = p$ then α is the identity.

Let $\mathcal{M}_{g,1}$ be the Deligne–Mumford stack of smooth projective curves C of genus g with one marked point $p \in C$. We will denote by $\mathcal{M}^*$ the open substack of $\mathcal{M}_{g,1}$ corresponding to pairs (C, p) where C is a smooth projective curve of genus g and p is a rigid point in C. (Call such a pair (C, p) a *rigidified curve*.) The stack $\mathcal{M}^*$ has trivial inertia and so is a fine moduli space representable by an algebraic space M^* (see 92.13 in [Stacks 2005–]). The algebraic space M^* is a quasiprojective scheme (see the classical results of Knudsen [1983] and Kollár [1990]). We note that M^* is a scheme of finite type over $\mathbb{Q}$ and: there is a universal family $\phi : \mathcal{C}_{M^*} \to M^*$, such that for any rigidified curve (C, p) defined over K there is a K-point $[(C, p)] \in M^*$ such that the fiber of $\mathcal{C}_{M^*}$ over the point $[(C, p)]$ is isomorphic to C.

The forgetful projection $(C, p) \mapsto C$ gives us a mapping

$$M^* \longrightarrow M$$

defined over $\mathbb{Q}$ (with one-dimensional fibers).

Proposition 2. *For $g > 1$ there is a finite bound B_g with the property that if K is a* (*number*) *field and C a smooth projective curve of genus g, defined over K, such that $|C(K)| > B_g$ there is a K-rational rigid point p in C. The curve C is* (*therefore*) *represented by a K-rational point of M^*.*

We thank Jakob Stix for communicating a proof of the fact that one can take B_g to be equal to $82(g-1)$. See Appendix B.

The moduli space with level structure. Here it will suffice for us to work over $\mathbb{C}$. Let $\ell \gg 0$ be a prime and $\widetilde{M}_{g,1} := M_{g,1}[\ell]$ the moduli space of smooth pointed curves of genus g with full level ℓ structure. That is, $M_{g,1}[\ell]$ classifies pairs (C, λ) where C is a smooth pointed curve of genus g (over $\mathbb{C}$) and (the "level structure") λ is an isomorphism of $\mathbb{F}_\ell$-vector spaces

$$\lambda : \mathbb{F}_\ell^{2g} \xrightarrow{\simeq} H_1(C_{\mathbb{C}}; \mathbb{F}_\ell).$$

Note that $\widetilde{M}_{g,1}$ is not connected, but this won't bother us. The finite group

$$G := \mathrm{GL}_{2g}(\mathbb{F})$$

acts on $\widetilde{M}_{g,1}$ with quotient $M_{g,1}$.

Define $\widetilde{M}^*$ by the following diagram, the upper square being exact:[1]

$$\begin{array}{ccc} \widetilde{M}^* & \longrightarrow & \widetilde{M}_{g,1} \\ \Big\downarrow G & & \Big\downarrow G \\ M^* & \longrightarrow & M_{g,1} \\ \Big\downarrow & & \Big\downarrow \\ M & \xrightarrow{=} & M \end{array} \tag{1}$$

So the group G acts on $\widetilde{M}^*$ with quotient M^* rendering $\widetilde{M}^*$ a G-torsor over M^* as well. The fine moduli space $\widetilde{M}^*$ classifies triples (C, p, λ) and we have an exact square of universal families:

$$\begin{array}{ccc} \mathcal{C}_{\widetilde{M}^*} & \xrightarrow{G} & \mathcal{C}_{M^*} \\ \Big\downarrow \tilde{\phi} & & \Big\downarrow \phi \\ \widetilde{M}^* & \xrightarrow{G} & M^* \end{array} \tag{2}$$

These (i.e., the vertical morphisms) are flat families of smooth projective curves of genus g, and the group G acts equivariantly, rendering the domains of the horizontal morphisms *G-torsors* over the corresponding ranges.[2]

General families of rigid curves. Let B be a scheme of finite type over $\mathbb{C}$, and $\phi_B : \mathcal{C}_B \to B$ a flat family of smooth projective *rigidified* curves of genus g (over B) — that is, such that there is a section $p : B \to \mathcal{C}_B$ having the property that for every point b of B the image point $p(b)$ in the fiber $\mathcal{C}_b$ over b is a rigid point of that curve $\mathcal{C}_b$. Since M^* is the fine moduli space for such objects, this family comes by pullback from a unique morphism

$$j : B \to M^*$$

---

[1] This is sometimes called a "cartesian square:" An *exact* (synonymously: *cartesian*) square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \Big\downarrow & & \Big\downarrow \\ D & \longrightarrow & C \end{array}$$

is a commutative square, where the mapping $A \to B \times_C D$ determined by the diagram is an isomorphism.

[2] E.g., the mapping

$$G \times \widetilde{M}^* \;\longrightarrow\; \widetilde{M}^* \times_{M^*} \widetilde{M}^*$$

given by $(g, m) \mapsto (m, g(m))$ is an isomorphism.

and ϕ_B fits into a diagram, the upper square being exact:

$$\begin{tikzcd}
& \mathcal{C}_B \arrow[r] \arrow[d, "\phi_B"] & \mathcal{C}_{M^*} \arrow[d, "\phi"] \\
& B \arrow[r, "j"] \arrow[d] \arrow[dr, "i"] & M^* \arrow[d, "k"] \\
& B_0 = i(B) \arrow[r, hook] & M
\end{tikzcd} \tag{3}$$

Here, by Chevalley's classical theorem, the image of B in M^* (via the mapping j) and in M (via the mapping i) are constructible sets, so the first is a finite union of locally closed (irreducible) subvarieties of M^*, and the second is a finite union of locally closed (irreducible) subvarieties of M. We will deal, inductively with all of these subvarieties; but

- let B_0' be any one of the locally closed (irreducible) subvarieties in M that is among components of the constructible set which is the image of B in M, and
- let B' be a locally closed (irreducible) subvariety of M^* that is
  - among components of the constructible set which is the image of B in M^*, and
  - that contains a Zariski-dense open in the inverse image of B_0' under k.

We have an analogous diagram as (3) but

- with B replaced with B'; and B_0 replaced with B_0'; but such that
- all morphisms are morphisms of varieties, and
- where B_0' and B' are locally closed subvarieties of M and M^*, respectively.

Removing the primes ($'$) from the terminology we have:

$$\begin{tikzcd}
& \mathcal{C}_B \arrow[r, hook] \arrow[d, "\phi_B"] & \mathcal{C}_{M^*} \arrow[d, "\phi"] \\
& B \arrow[r, hook] \arrow[d] \arrow[dr, "i"] & M^* \arrow[d, "k"] \\
& B_0 = i(B) \arrow[r, hook] & M
\end{tikzcd} \tag{4}$$

In diagram (4) it is *only* the upper square that is exact. These are the diagrams we will be studying. Call such a family of rigid curves, $\mathcal{C}_B \to B$, *clean*. From now on we will assume that our families $\mathcal{C}_B \to B$ are "clean."

Augmenting such a *clean* family with level structure by tensoring with $\widetilde{M}$ (over M) with we might form

$$
\begin{tikzcd}
\mathcal{C}_B \arrow[r] & B \arrow[r, "j"] & M^* & \mathcal{C}_{M^*} \arrow[l] \\
\mathcal{C}_{\widetilde{B}} \arrow[r] \arrow[u, "G"] \arrow[d] & \widetilde{B} \arrow[r, "\tilde{j}"] \arrow[u, "G"] \arrow[d] & \widetilde{M}^* \arrow[u, "G"] \arrow[d] & \mathcal{C}_{\widetilde{M}^*} \arrow[l] \arrow[u, "G"] \arrow[d] \\
\mathcal{C}_{\widetilde{B}_0} \arrow[r] & \widetilde{B}_0 \arrow[r, hook] \arrow[d, "G"] & \widetilde{M} \arrow[d, "G"] & \mathcal{C}_{\widetilde{M}} \arrow[l] \\
 & B_0 \arrow[r, hook] & M &
\end{tikzcd}
\tag{5}
$$

Here the vertical mappings in the two exact diagrams

$$
\begin{tikzcd}
\mathcal{C}_B \arrow[r] \arrow[d] & \mathcal{C}_{M^*} \arrow[d] \\
B \arrow[r, hook] & M^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_{\widetilde{B}_0} \arrow[r] \arrow[d] & \mathcal{C}_{\widetilde{M}} \arrow[d] \\
\widetilde{B}_0 \arrow[r, hook] & \widetilde{M}
\end{tikzcd}
$$

are flat families of (smooth projective rigidified curves of genus g) and — respectively — flat families of (smooth projective curves of genus g with level structure). The arrows labeled "G" are morphisms obtained by passing to the quotient by the natural action of G. All squares where the vertical arrows are labeled "G" are cartesian and G-equivariant. And note that the schemes on the bottom line of diagram (5) — i.e., $B_0 \hookrightarrow M$ — do not possess "universal families."

3. A strengthened correlation theorem

Note: the results of this section are purely geometric, rather than arithmetic; objects will be varieties defined over $\mathbb{C}$. Moreover, we will be dealing entirely with birational properties, so we will feel free to restrict to open subsets where convenient. Thus, for example, when we say that the fibers of a morphism are curves of genus g, we will mean that they are open subsets of a curve whose normalization is a smooth projective curve of genus g.

For our purposes, we will need the following slightly strengthened version of the *correlation theorem*, the key geometric lemma (i.e., Proposition 3.1) of [Caporaso et al. 1997]:

Proposition 3. *With the notation of the previous section, if the map* $B \xrightarrow{j} M^*$ *is generically finite, then for* $n \gg 0$ *the fiber power* $\mathcal{C}_B^n$ *(over* B*) is of general type.*

Remarks. (1) This is stronger than the correlation theorem in just one respect: we are only assuming that the map $j : B \to M^*$ is generically finite, not that the projection $B \to B_0 \hookrightarrow M$ is generically finite:

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "h"] \arrow[dr, "j_0"] & M^* \arrow[d] \\ B_0 \arrow[r, hook] & M \end{tikzcd}$$

(2) There is an obvious bifurcation: either the map $j_0 : B \to M$ is generically finite, or it has generically one-dimensional fibers. In the former case, Proposition (3.1) of [loc. cit.] applies, and we're done; thus we can, and will, assume that the general fiber of j_0 has dimension 1, and more specifically that

$$B \subset M^* \text{ is the inverse image of } B_0 \text{ in } M. \tag{6}$$

Lemma 4. *Under hypothesis* (6) *above*, *the morphism*

$$\widetilde{B} \to \widetilde{B}_0 \tag{7}$$

is a smooth morphism with fibers that are curves of genus g.

Proof. First, the morphism $\widetilde{M}^* \to \widetilde{M}$ has the property that its fibers are curves (whose smooth projective completions are) of genus g. This is because $\widetilde{M}$ is a fine moduli space, and the operation of "tilde" ($\tilde{}$) and "star" ($*$) commute, so that the fiber of a point $[(C, \lambda)]$ in $\widetilde{M}$ is given by $([(C, \lambda)], p)$, where p ranges through the locus of all rigid points of C.

Also, by (6), we also have that:

$$\widetilde{B} \subset \widetilde{M}^* \text{ is the inverse image of } \widetilde{B}_0 \text{ in } \widetilde{M} \tag{8}$$

so that

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\tilde{j}"] \arrow[d, "\tilde{h}"] & \widetilde{M}^* \arrow[d] \\ \widetilde{B}_0 \arrow[r, hook] & \widetilde{M} \end{tikzcd}$$

is an exact square, and therefore the fibers of $\widetilde{B} \to \widetilde{B}_0$ are pullbacks of the fibers of $\widetilde{M}^* \to \widetilde{M}$. $\square$

(3) However if it were true (but it is not true, generally) that $h : B \to B_0$ has fibers that are curves of genus g we would then be done: a high fiber power $\mathcal{C}^n_{B_0}$ (over B_0) would be of general type by the correlation theorem, and the projection

$$\mathcal{C}^n_B := \mathcal{C}^n_{B_0} \times_{B_0} B \xrightarrow{1 \times h} \mathcal{C}^n_{B_0} \times_{B_0} B_0 = \mathcal{C}^n_{B_0}$$

would have fibers that generically are curves of genus g. so — by [Kollár 1987] — it would be of general type as well. Another way of thinking about the obstruction to proving Proposition 3 is that there may not exist a tautological family over B_0.

To prove Proposition 3 we use a proposition supplied by Kenneth Ascher and Amos Turchet. Consider the diagonal action of G on fiber powers $\mathcal{C}^n_{\widetilde{B}}$ and $\mathcal{C}^n_{\widetilde{B}_0}$ (these powers being taken over $\widetilde{B}$ and $\widetilde{B}_0$ respectively).[3]

Proposition 5. *Keeping to the notation and hypotheses of Section 2, for n sufficiently large the quotient $\mathcal{C}^n_{\widetilde{B}_0}/G$ of $\mathcal{C}^n_{\widetilde{B}_0}$ (under the diagonal action of G) is of general type.*

Proof. This is just Theorem 1.7 in [Ascher and Turchet 2016], in the special case $D = 0$. The hypotheses in [Ascher and Turchet 2016] require that the base B be smooth and projective, but we can always achieve this by completing the family, applying stable reduction and resolving the singularities of the new base. □

It should be noted that a major part of the work in [Ascher and Turchet 2016] is to extend the original theorem to the setting of log varieties, which does not concern us; what is new and useful for us is the incorporation of the group G.

3.1. ***Fiber powers.*** The group G acts equivariantly on the objects in the exact diagram

$$\begin{tikzcd} \mathcal{C}_{\widetilde{B}} \arrow[r] \arrow[d] & \mathcal{C}_{\widetilde{B}_0} \arrow[d] \\ \widetilde{B} \arrow[r] & \widetilde{B}_0 \end{tikzcd} \tag{9}$$

The square (9) is exact since the $\mathcal{C}$ involved are the universal families of curves over $\widetilde{B} \to \widetilde{B}_0$ (that is, pullbacks of the universal family over the fine moduli space $\widetilde{M}_g$). For any $n \geq 1$ let

$$\mathcal{C}^n_{\widetilde{B}} := \mathcal{C}_{\widetilde{B}} \times_{\widetilde{B}} \mathcal{C}_{\widetilde{B}} \times_{\widetilde{B}} \cdots \times_{\widetilde{B}} \mathcal{C}_{\widetilde{B}},$$

i.e., the n-fold power of $\mathcal{C}_{\widetilde{B}}$ over $\widetilde{B}$, with the group G acting on $\mathcal{C}^n_{\widetilde{B}}$ by the diagonal action. This action is equivariant for the natural projection $\mathcal{C}^n_{\widetilde{B}} \to \widetilde{B}$. The map

$$\mathcal{C}^n_{\widetilde{B}} \to \mathcal{C}^n_{\widetilde{B}_0} \tag{10}$$

is a morphism of G-torsors.

Lemma 6. *For $n \geq 1$ the natural map $\mathcal{C}^n_{\widetilde{B}} \to \mathcal{C}^n_B$ identifies $\mathcal{C}^n_B$ (the corresponding fiber power $\mathcal{C}^n_B$ of our original family $\mathcal{C} \to B$) with $\mathcal{C}^n_{\widetilde{B}}/G$, the quotient of $\mathcal{C}^n_{\widetilde{B}}$ by the action of G.*

[3]See Section 3.1. The action of $g \in G$ is induced, in the evident way, from the action on isomorphism classes $(C, \lambda) \mapsto (C, \lambda \cdot g)$.

Proof. The natural map referred to arises from the following natural map, valid for any three schemes over a scheme S, call them

$$\begin{tikzcd} X \arrow[dr] & \tilde{S} \arrow[d] & Y \arrow[dl] \\ & S & \end{tikzcd}$$

Put: $\widetilde{X} := X \times_S \tilde{S}$ and $\widetilde{Y} := Y \times_S \tilde{S}$. We have canonical isomorphisms of $\tilde{S}$-schemes:

$$X \times_S Y \times_S \tilde{S} \simeq (X \times_S \tilde{S}) \times_{\tilde{S}} (Y \times_S \tilde{S}) \simeq \widetilde{X} \times_{\tilde{S}} \widetilde{Y},$$

E.g., on points $x, \tilde{s}, y$ of $X, \tilde{S}, Y$ all of which map to the same point s of S, it's given by

$$x \times y \times \tilde{s} \mapsto (x \times \tilde{s}) \times (y \times \tilde{s}).$$

Proceeding inductively on n this gives us a canonical isomorphism

$$\mathcal{C}^n_B \times_B \widetilde{B} := \mathcal{C}_B \times_B \mathcal{C}_B \times_B \cdots \mathcal{C}_B \times_B \widetilde{B} \xrightarrow{\simeq} \mathcal{C}^n_{\widetilde{B}} := \mathcal{C}_{\widetilde{B}} \times_{\widetilde{B}} \mathcal{C}_B \times_{\widetilde{B}} \cdots \mathcal{C}_B \times_B \widetilde{B}, \quad (11)$$

by taking $S := B$, $\tilde{S} := \widetilde{B}$, $X := \mathcal{C}_B$, $Y := \mathcal{C}^{n-1}_B$. Equation (11) is an equivariant isomorphism for the action of the group G, which acts diagonally on the right hand side and as for the left-hand side, an element $g \in G$ acts on the fiber product $\mathcal{C}^n_B \times_B \widetilde{B}$ by the identity on the first factor; and as it has been defined to act, on the second. The map $\widetilde{B} \to B = \widetilde{B}/G$ (i.e., the map that exhibits B as the quotient of $\widetilde{B}$ under the action of G) induces a mapping $\mathcal{C}^n_B \times_B \widetilde{B} \to \mathcal{C}^n_B \times_B B = \mathcal{C}^n_B$.

Since the quotient of $\widetilde{B}$ under the action of G is B, the quotient of $\mathcal{C}^n_B \times_B \widetilde{B}$ under the action of G is B is canonically isomorphic to $\mathcal{C}^n_B$, and we have the commutative diagram:

$$\begin{tikzcd} \mathcal{C}^n_B \times_B \widetilde{B} \arrow[r, "\simeq"] \arrow[d] & \mathcal{C}^n_{\widetilde{B}} \arrow[d] \\ \mathcal{C}^n_B \arrow[r, "\simeq"] & \mathcal{C}^n_{\widetilde{B}}/G \end{tikzcd}$$

□

We also have the following lemma:

Lemma 7. *For $n \geq 1$ the fibers of the map of quotients by the action of G*

$$\mathcal{C}^n_{\widetilde{B}}/G \to \mathcal{C}^n_{\widetilde{B}_0}/G \quad (12)$$

are generically curves of genus g.

The proof of Lemma 7 is given in Appendix A.

Proof of Proposition 3. By Proposition 5 we have that for $n \gg 0$ $\mathcal{C}^n_{\widetilde{B}_0}/G$ is of general type. By Lemmas 6 and 7, the mapping $\mathcal{C}^n_B \to \mathcal{C}^n_{\widetilde{B}_0}/G$ has fibers that are curves of genus ≥ 2, i.e., that are of general type. By [Kollár 1987], it follows that $\mathcal{C}^n_B$ is of general type. □

4. The boundedness argument

Let us first state the version of the Lang conjecture we will be invoking.

Conjecture 8 (strong Lang). *Let X be a variety of general type, defined over a number field K. There is then a proper subvariety $Z \subset X$ such that for any finite extension L of K, $\#(X \setminus Z)(L) < \infty$; that is, all but finitely many L-rational points of X lie in Z.*

Given this and Proposition 3 of Section 3, we can deduce:

Theorem 9. *Assume the SLC (Conjecture 8). If $\pi : \mathcal{C} \to B$ is a family of pointed curves without automorphisms, defined over $\mathbb{Q}$, such that the induced map $\phi : B \to M^*$ is finite, then there is then an integer N such that for any number field K,*

$$\#\{b \in B(K) \mid \#C_b(K) > N\} < \infty$$

Proof. We will prove an a priori weaker form of this: we will show that there exists a nonempty open subset $U \subset B$ and an integer N such that for any number field K,

$$\#\{b \in U(K) \mid \#C_b(K) > N\} < \infty;$$

Theorem 9 will then follow by Noetherian induction.

To prove this, let $\pi_n : \mathcal{C}^n_B \to B$ be the n-th fiber power of the family $\mathcal{C} \to B$. By Proposition 3, for large n the fiber power $\mathcal{C}^n_B$ will be of general type. By the Strong Lang Conjecture, then, there will be a proper subvariety $Z \subset \mathcal{C}^n_B$ such that for any number field K, all but finitely many K-rational points of $\mathcal{C}^n_B$ lie in Z; that is,

$$\#(\mathcal{C}^n_B \setminus Z)(K) < \infty.$$

We now define a sequence of subvarieties $Z_k \subset \mathcal{C}^k_B$ inductively as follows. We start with $Z_n = Z$, and let

$$Z_{n-1} = \{b \in \mathcal{C}^{n-1}_B \mid \pi^{-1}_{n,n-1}(b) \subset Z_n\},$$

where $\pi_{n,n-1} : \mathcal{C}^n_B \to \mathcal{C}^{n-1}_B$ is the projection; similarly, given Z_k we set

$$Z_{k-1} = \{b \in \mathcal{C}^{k-1}_B \mid \pi^{-1}_{k,k-1}(b) \subset Z_k\},$$

where $\pi_{k,k-1} : \mathcal{C}_B^k \to \mathcal{C}_B^{k-1}$ is the projection. We arrive at a tower of spaces and closed subvarieties:

$$\begin{tikzcd}
Z = Z_n \arrow[r, hook] & \mathcal{C}_B^n \arrow[d, "\pi_{n,n-1}"] \\
Z_{n-1} \arrow[r, hook] & \mathcal{C}_B^{n-1} \arrow[d, "\pi_{n-1,n-2}"] \\
& \vdots \arrow[d, "\pi_{2,1}"] \\
Z_1 \arrow[r, hook] & \mathcal{C} \arrow[d, "\pi=\pi_{1,0}"] \\
Z_0 \arrow[r, hook] & B
\end{tikzcd}$$

where the k-th story in this tower has the structure:

$$\begin{tikzcd}
& & \vdots \arrow[d, "\pi_{k+1,k}"] \\
\pi_{k,k-1}^{-1}(Z_{k-1}) \arrow[r, hook] \arrow[dr] & Z_k \arrow[r, hook] & \mathcal{C}_B^k \arrow[d, "\pi_{k,k-1}"] \\
& Z_{k-1} \arrow[r, hook] & \mathcal{C}_B^{k-1} \arrow[d, "\pi_{k-1,k-2}"] \\
& & \vdots
\end{tikzcd}$$

Note that since $Z \subsetneq \mathcal{C}_B^n$ and $\pi_n^{-1}(Z_0) \subset Z$, we necessarily have $Z_0 \subsetneq B$.

Now fix for the moment a value of k with $1 \le k \le n$. Every irreducible component $W_\alpha \subset Z_k$ will either be the preimage of a subvariety in $\mathcal{C}_B^{k-1}$, or will map onto its image in $\mathcal{C}_B^{k-1}$ with degree d_α. Let d_k be the sum of the degrees d_α, so that for any $p \in \mathcal{C}_B^{k-1}$, either $\#(\pi_{k,k-1}^{-1}(p) \cap Z_k) \le d_k$, or $\pi_{k,k-1}^{-1}(p) \subset Z_k$.

Finally, let N be the maximum of the d_k, and set $U = B \setminus Z_0$. We claim that for any number field K,

$$\#\{b \in U(K) \mid \#C_b(K) > N\} < \infty;$$

as noted above, Theorem 9 will follow by Noetherian induction. To see this, restrict our family and all fiber powers to the open subset $U \subset B$; similarly, replace Z by its intersection with $\pi_n^{-1}(U)$. Fix a number field K, and let

$$\Sigma = \{(\mathcal{C}_U^n \setminus Z)(K)\},$$

and let $\Sigma_0 = \pi_n(\Sigma)$ be its image; by hypothesis, this is a finite subset of U.

We claim finally that *for any $b \in B(K) \setminus \Sigma_0$, we have $\#(X_b(K)) \leq N$.* To see this, let $b \in B(K)$ be any K-rational point, and suppose that $\#(X_b(K)) > N$. Since $b \notin \Sigma_0$, all K-rational points of $\mathcal{C}_B^n$ lying over b must lie in Z. Pick any $n-1$ points $p_1, \ldots, p_{n-1} \in X_b(K)$, and consider the points

$$\big\{(X_b, p_1, \ldots, p_{n-1}, p) \mid p \in X_b(K)\big\} \subset \pi_{n,n-1}^{-1}\big((X_b, p_1, \ldots, p_{n-1})\big)$$

in the fiber of $\mathcal{C}_B^n$ over $(X_b, p_1, \ldots, p_{n-1}) \in \mathcal{C}_B^{n-1}$. Since there are by hypothesis more than $N \geq d_n$ such points, we conclude that $Z = Z_n$ *must contain the fiber of $\mathcal{C}_B^n$ over $(X_b, p_1, \ldots, p_{n-1}) \in \mathcal{C}_B^{n-1}$*; in other words, $(X_b, p_1, \ldots, p_{n-1}) \in Z_{n-1}$.

The same argument applies sequentially to show that $(X_b, p_1, \ldots, p_{n-2}) \in Z_{n-2}$, and so on; ultimately, we deduce that $b \in Z_0$, establishing our claim. □

5. Behavior of $N(g)$ as g tends to ∞

For C a smooth projective, irreducible curve of genus $g > 1$ defined over a number field K let $\mathrm{Aut}_K(C)$ be the group of automorphisms of C defined over K. The group $\mathrm{Aut}_K(C)$ acts naturally on the set $C(K)$ of K-rational points of C. Let $\nu(C; K)$ denote the number of $\mathrm{Aut}_K(C)$-orbits in $C(K)$ under that natural action. So, of course, $\nu(C; K) \leq |C(K)|$ and therefore any uniform upper bound established for $|C(K)|$ is valid for $\nu(C; K)$ as well.

Define $\nu(g)$ to be the smallest integer that has the property that for each number field K there are only finitely many curves C of genus g defined over K with the property that $\nu(C; K)$ is strictly greater than $\nu(g)$. By what we have shown, assuming the SLC, $\nu(g)$ is finite for every $g > 1$.

If one feels that there is a fair chance for Conjecture 8 to be true, and hence for $\nu(g)$ to be finite, one might wonder about the asymptotic behavior of $\nu(g)$ as g tends to infinity. Needless to say, we have no real evidence to make any conjectures, or precise predictions, but we set

$$\nu_* := \liminf_{g \to \infty} \nu(g)/g \quad \text{and} \quad \nu^* := \limsup_{g \to \infty} \nu(g)/g.$$

Note that curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, g+1)$ are of arithmetic genus g, and form a linear system of dimension $3(g+2)-1$. Given $3(g+2)-1$ general points $p_1, \ldots, p_{3g+5} \in \mathbb{P}^1 \times \mathbb{P}^1(\mathbb{Q})$, accordingly, there will be a smooth curve C defined over $\mathbb{Q}$ and passing through them. Moreover, since C is a general hyperelliptic curve, its automorphism group is equal to $\mathbb{Z}/2$, consisting of the identity and the hyperelliptic involution; and since no two of the points p_i lie in the same fiber of

$\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{P}^1$, no two are conjugate under the automorphism group of C. Thus we have $\nu(C, \mathbb{Q}) \geq 3g+5$ and hence $\nu(g) \geq 3g+5$.

We have accordingly:

$$3 \leq \nu_* \leq \nu^*. \tag{13}$$

Some natural questions:

(1) Is ν^*, or perhaps only ν_*, or neither of them, finite?

(2) Are both inequalities in Equation (13) equalities? (or is one of them, or neither)?

(3) Let $M^*_{g,n}$ denote the moduli space of projective smooth curves of genus g with n *distinct* marked rigid points. For K a number field let $d_{g,n}(K)$ denote the dimension of the Zariski-closure in $M^*_{g,n}$ of the set of K-rational points $M^*_{g,n}(K)$. Now define $d_{g,n} := \max_K d_{g,n}(K)$ where the maximum is taken over all number fields K. The discussion in this note shows that the SLC implies that — for fixed $g \geq 2$ — if $n \gg_g 0$, then $d_{g,n} = 0$. What else can one say — or even just conjecture — about these dimensions? For example, might $d_{g,n}$ be decreasing (albeit not necessarily strictly) for fixed g and increasing n?

Appendix A: Proof of Lemma 7

Recall:

Lemma 7. *For $n \geq 1$ the fibers of the map of quotients by the action of G*

$$\mathcal{C}^n_{\widetilde{B}}/G \to \mathcal{C}^n_{\widetilde{B}_0}/G \tag{12}$$

are generically curves of genus g.

The statement of Lemma 7 being geometric, we work over $\mathbb{C}$; and since we are only interested in fibers, we may assume that B_0 is a point. This point B_0 (in $\mathcal{M}_g$) classifies a single isomorphism class of curves (of genus $g > 1$); call one curve in that isomorphism class C. If we want to refer to that isomorphism class as a whole, we'll denote it $[C]$.

A.1. ***What is $\widetilde{B}_0$?*** Consider now $\widetilde{B}_0$ which classifies isomorphism classes of pairs (C, λ) where C is a curve in the isomorphism class $[C]$ equipped with a level structure λ on it. We have chosen our level structure so that such pairs are rigid: C has no nontrivial automorphisms that preserve that level structure λ. Let G be, as we had before, the group of automorphisms of the level structure.

More specifically, for any curve X (of our fixed genus $g > 1$) we have specified an ℓ such that no automorphism of a curve of genus g leaves fixed a basis of $H_1(X, Z/\ell Z) \simeq (Z/\ell Z)^{2g}$. By definition a *level structure* on X is a specific isomorphism $H_1(X, Z/\ell Z) \xrightarrow{\lambda} (Z/\ell Z)^{2g}$; and $G = \mathrm{GL}_{2g}(Z/\ell Z)$ acts naturally on level structures (by right-composition: $\lambda \mapsto \lambda \cdot g^{-1}$); hence — since B_0 is just one point — G acts transitively on the set $\widetilde{B}_0$.

Consider $\Gamma :=$ the full automorphism group of the curve C (the curve classified by the point B_0). Any automorphism of a curve X induces an automorphism of $H_1(X, \mathbb{Z}/\ell\mathbb{Z})$ and so induces a permutation of level structures on X. Fixing such a curve $X = C$ we get a homomorphism $\Gamma \to G$; it is injective since the curve C with a level structure is rigid. In other words — given our fixed curve C — the image of Γ in G is the isotropy subgroup of G relative to its (transitive) action on the finite set $\widetilde{B}_0$. Consequently,

Lemma 10. *Making a choice of curve and level structure* (C, λ) *there is a natural identification,*

$$\widetilde{B}_0 \xrightarrow{\simeq} G/\Gamma. \tag{14}$$

A.2. ***What is*** $\mathcal{C}_{\widetilde{B}_0}$***?*** Now let's pass to considering $\mathcal{C}_{\widetilde{B}_0}$; i.e., the union of the actual curves in the isomorphism class "$[C]$" with their level structures λ (that are classified by the corresponding points (C, λ) in the finite set $\widetilde{B}_0$). A point in $\mathcal{C}_{\widetilde{B}_0}$ is a triple $(C, p; \lambda)$ where C is — as will always be, in this discussion — "classified by" the point B_0, $p \in C$ and

$$(Z/\ell Z)^{2g} \xrightarrow{\lambda} H_1(C, Z/\ell Z)$$

is a level structure. There is a natural action of G on $\mathcal{C}_{\widetilde{B}_0}$. That is

$$g(C, p; \lambda) := (C, g(p); \lambda \cdot g^{-1}). \tag{15}$$

giving us G-equivariant mappings

$$\mathcal{C}_{\widetilde{B}_0} \xrightarrow{\pi} \widetilde{B}_0 \simeq G/\Gamma \tag{16}$$

every fiber of which is a curve of genus g — these being just our curves "C" with different level structures.

A.3. *What is the quotient of* $\mathcal{C}_{\widetilde{B}_0}$ *by the action of G?*

Lemma 11. *Fix a curve and level structure* (C, λ) *classified by a point in* $\widetilde{B}_0$*. After passing to the quotient by* G *the* (G*-equivariant*) *mapping* (16) *induces*

$$\mathcal{C}_{\widetilde{B}_0}/G \xrightarrow{\pi} \widetilde{B}_0/G = B_0, \tag{17}$$

the fibers being curves isomorphic to the quotient curve C/Γ.

Proof. This follows from the fact that the image of Γ in G is the isotropy subgroup of G relative to its (transitive) action on $\widetilde{B}_0$. □

A.4. ***What is B?*** B consists of isomorphism classes of pairs (C, q) where C is a curve classified by the point B_0 and q is a *rigid* point on C.

Lemma 12. *Fixing a curve C with moduli point $B_0 \in M_g$, let C^* denote the Zariski open subset of rigid points in C. We have an isomorphism*

$$B \xrightarrow{\simeq} C^*/\,\Gamma.$$

Proof. This is evident, but one might also notice that C^* is a Γ-torsor over B, as follows from the definition of rigidity. □

A.5. ***What is $\widetilde{B}$?*** The cover $\widetilde{B}$ of B consists of isomorphism classes of triples $(C, q; \lambda)$ with C having moduli point $[C] = B_0$, q a rigid point on C and λ a level structure on C. Now just consider the pair (C, λ). This pair has no nontrivial automorphisms, so as q ranges through the (rigid) points of C, we get that

Lemma 13. *Fixing a curve C with moduli point B_0 ,*

(1) *The (G-equivariant) mapping*

$$\widetilde{B} \xrightarrow{\psi} \widetilde{B}_0 = G/\,\Gamma \tag{18}$$

is surjective with fibers isomorphic to C^*.

(2) *The quotient of* (18) *by the action of G induces a mapping*

$$\widetilde{B}/G \xrightarrow{\bar{\psi}} \widetilde{B}_0/G = B_0 \tag{19}$$

with fibers isomorphic to $C^*/\,\Gamma$.

A.6. ***What is $\mathcal{C}_{\widetilde{B}}$?*** Consider the mapping

$$\mathcal{C}_{\widetilde{B}} \to \widetilde{B}. \tag{20}$$

A point $\tilde{c}$ of $\mathcal{C}_{\widetilde{B}}$ is given by an isomorphism class of 4-tuples $(C, q; \lambda; p)$ where $(C, q; \lambda)$ comprises the coordinates of the point of $\widetilde{B}$ over which $\tilde{c}$ lies, and $p \in C$ is a point of C. So (20) is a family of curves whose fibers are all isomorphic to C (over the base which is isomorphic to C^*).

Lemma 14. *We have an exact commutative 'G-equivariant' diagram*

$$\begin{tikzcd} \mathcal{C}_{\widetilde{B}} \arrow[r] \arrow[d] & \mathcal{C}_{\widetilde{B}_0} \arrow[d] \\ \widetilde{B} \arrow[r] & \widetilde{B}_0 = G/\,\Gamma \end{tikzcd}$$

where the fibers of the vertical maps are isomorphic to C and the fibers of the horizontal maps are isomorphic to C^*.

Proof. The vertical map sends the point $\tilde{c} \in \mathcal{C}_{\widetilde{B}}$ represented by the 4-tuple $(C, q; \lambda; p)$ to the point in $\widetilde{B}$ represented by the triple $(C, q; \lambda)$ while the horizontal map sends it to $(C, \lambda; p)$. In either case the "retention" of a level structure λ (under either of these "forgetful mappings") — guaranteeing the fact that (C, λ) admits no nontrivial automorphisms — tells us that the fibers of these projections are as claimed in the lemma. ☐

A.7. ***Specializing Lemma 14 to a point $\tilde{b}_0 \in \widetilde{B}_0$.*** Consider, now, the pullback of the above commutative square to a point $\tilde{b}_0 \in \widetilde{B}_0 = G/\Gamma$. Let $\mathcal{F} \subset \mathcal{C}_{\widetilde{B}}$ denote the fiber over $\tilde{b}_0 \in \widetilde{B}_0$ of the mapping

$$\mathcal{C}_{\widetilde{B}} \to \widetilde{B}_0 = G/\Gamma,$$

so that the pullback of the diagram in Lemma 14 to the point $\tilde{b}_0 \in \widetilde{B}_0$ yields an exact commutative "Γ-equivariant" diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & C \cong \mathcal{C}_{\tilde{b}_0} \arrow[d] \\ C^* \cong \widetilde{B}_{\tilde{b}_0} \arrow[r] & \tilde{b}_0 \end{tikzcd} \tag{21}$$

This diagram may be written simply as a "Γ-equivariant" isomorphism

$$\mathcal{F} \cong C \times C^* \tag{22}$$

where we note that the restriction of the action of G (on $\mathcal{C}_{\widetilde{B}}$) to $\Gamma \subset G$ stabilizes $\mathcal{F}$, and the action of Γ on the range $C \times C^*$ is the natural diagonal action; i.e.,

$$\gamma(x, y) = (\gamma(x), \gamma(y)).$$

We propose to show that the fibers of the mapping

$$\mathcal{F}/\Gamma \longrightarrow C/\Gamma \tag{23}$$

(in the quotient by the action of Γ on the top horizontal morphism of the above diagram (21) are (generically) curves in the isomorphism class $[C]$. More specifically, this is true for the fibers of (23) over points in the Zariski dense open $C^*/\Gamma \subset C/\Gamma$. We focus, then, on

$$(C^* \times C^*)/\Gamma \subset (C^* \times C)/\Gamma \cong \mathcal{F}.$$

Lemma 15. *Consider the projection*

$$(C^* \times C^*)/\Gamma \to C^*/\Gamma. \tag{24}$$

Fixing any point $x \in C^*$, the mapping

$$C^* \xrightarrow{\alpha} (C^* \times C^*)/\,\Gamma$$

given by

$$y \mapsto \text{the image of } (x, y) \text{ in } (C^* \times C^*)/\,\Gamma$$

identifies C^* with the fiber of (24) *over the image of x in $C/\,\Gamma$.*

Proof. That α maps C^* surjectively onto that fiber is clear: if $(x', y') \in C^* \times C^*$ maps to a point z in that fiber, we can find a $\gamma \in \Gamma$ such that $\gamma(x') = x$. Taking $y := \gamma(y')$ we have that the image of y is z. But α is also injective, since if for $y, y' \in C^*$ there were an element $\gamma \in \Gamma$ such that $\gamma(x, y) = \gamma(x, y')$ we would have $\gamma(x) = x$, which would contradict the rigidity of the point $x \in C^*$. □

A.8. ***Returning to Lemma 14.*** We are now ready to consider the quotient of the diagram in Lemma 14 by the (equivariant) action of the group G.

We get the commutative (*but not necessarily exact*) diagram:

$$\begin{tikzcd}
\mathcal{C}_B \arrow[d] & \mathcal{C}_{\widetilde{B}}/G \arrow[l, "\simeq"'] \arrow[r, "f"] \arrow[d] & \mathcal{C}_{\widetilde{B}_0}/G \arrow[d, "\bar{\pi}"] \\
B & \widetilde{B}/G \arrow[l, "\simeq"'] \arrow[r, "\bar{\psi}"] & \widetilde{B}_0/G = B_0
\end{tikzcd} \tag{25}$$

where $\bar{\psi}$ has fibers isomorphic to $C^*/\,\Gamma$ and $\bar{\pi}$ has fibers isomorphic to $C/\,\Gamma$. The two unlabeled vertical morphisms have fibers isomorphic to the curve C.

Returning to the notation of diagram (25) we have:

Proposition 16. *The fibers of the mapping*

$$\mathcal{C}_{\widetilde{B}}/G \xrightarrow{f} \mathcal{C}_{\widetilde{B}_0}/G$$

are (*generically*) *curves of genus g.*

Let $n \geq 1$. Let

$$\mathcal{C}^n_{\widetilde{B}} = \mathcal{C}_{\widetilde{B}} \times_{\widetilde{B}} \mathcal{C}_{\widetilde{B}} \times_{\widetilde{B}} \cdots \times_{\widetilde{B}} \mathcal{C}_{\widetilde{B}}, \quad \text{i.e., } n \text{ times},$$

as in Section 3.1 above; and ditto for $\mathcal{C}^n_{\widetilde{B}_0}$.

We let the group G act diagonally.[4] It was only for notational convenience that we worked, above, with the case $n = 1$. The same arguments, word for word, allow us (for general $n \geq 1$) to get, after passing to quotients by G:

Proposition 17. *The fibers of the mapping*

$$\mathcal{C}^n_{\widetilde{B}}/G \to \mathcal{C}^n_{\widetilde{B}_0}/G$$

are generically curves of genus g.

[4]As in Section 3.1 and as in Equation (15).

Appendix B: Automorphisms of curves: a lemma of Jakob Stix

Proposition 18. *Let C be a smooth projective curve of genus > 1, and let $\Sigma \subset C$ be the set of points of C fixed by some automorphism of C other than the identity. Then $|\Sigma|$ admits some finite upper bound $B_g < \infty$, dependent only on the genus $g > 1$.*

Remark. Although we only need to know that there is some finite upper bound $B_g < \infty$ for the purposes of application to Proposition 2 in Section 2 we are grateful to Jakob Stix for providing the following sharp bound.

A *Hurwitz curve* is a smooth projective curve X which admits a branched Galois cover $X \to \mathbb{P}^1$ with only three branch points and ramification index 2, 3 and 7. These are precisely the curves for which the Hurwitz-bound $|\mathrm{Aut}(X)| \leq 84(g-1)$ is an equality.

Lemma 19 (Stix). *Let X be a smooth projective geometrically connected curve of genus $g \geq 2$ over an algebraically closed field $k = \bar{k}$ of characteristic* 0. *The number of points in X which are fixed by a nontrivial automorphism of X is bounded above by* $82(g-1)$

$$\left|\left\{P \in X\,;\ \exists\,\mathrm{id} \neq \sigma \in \mathrm{Aut}(X) \colon \sigma(P) = P\right\}\right| \leq 82(g-1).$$

The bound is sharp and attained if and only if X is a Hurwitz curve.

Proof. Let $G = \mathrm{Aut}(X)$ be the automorphism group and let e_P denote the ramification index for points above $P \in X/G$ in the cover

$$X \to Y = X/G.$$

The number of points that we want to estimate is

$$T = |G| \cdot \sum_{P \in Y} \frac{1}{e_P}.$$

Let $B = |\{P \in Y\,;\ e_P > 1\}|$ be the number of branch points. The Riemann–Hurwitz formula tells us

$$\begin{aligned} 2g-2 &= |G|(2g_Y - 2) + \sum_{P \in Y} |G|\Big(1 - \frac{1}{e_P}\Big) = |G|(2g_Y - 2 + B) - T \\ &= |G|\Big(2g_Y - 2 + B - \sum_{P \in Y} \frac{1}{e_P}\Big). \end{aligned}$$

If $g_Y \geq 1$, then since $1 - 1/e_p \geq \frac{1}{2} \geq 1/e_P$ we are done because of

$$T \leq \sum_{P \in Y} |G|\Big(1 - \frac{1}{e_P}\Big) = 2g - 2 - |G|(2g_Y - 2) \leq 2g - 2.$$

So from now on we assume $g_Y = 0$. Since $2g-2>0$, we must have that

$$B-2 > \sum_{P\in Y} \frac{1}{e_P}.$$

If $B \geq 5$, then

$$B-2-\sum_{P\in Y}\frac{1}{e_P} \geq B\cdot\tfrac{1}{2}-2 \geq \tfrac{1}{2}$$

and so

$$|G| = \frac{2g-2}{B-2-\sum_{P\in Y} 1/e_P} \leq 4(g-1).$$

It follows that

$$T \leq \sum_{P\in Y} |G|\Big(1-\frac{1}{e_P}\Big) = 2g-2+2|G| \leq 10(g-1).$$

If $B=4$, then

$$B-2-\sum_{P\in Y}\frac{1}{e_P} \geq 2-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{3} = \tfrac{1}{6},$$

hence

$$|G| \leq 12(g-1) \quad \text{and} \quad T \leq 26(g-1).$$

It remains to discuss the case of $B=3$. Here, as in the proof of the Hurwitz bound, the minimal positive value of

$$B-2-\sum_{P\in Y}\frac{1}{e_P}$$

is attained for ramification indices 2, 3 and 7 leading to the Hurwitz bound $|G| \leq 84(g-1)$. But now

$$T = |G|\cdot(2g_Y-2+B)-2(g-1) = |G|-2(g-1) \leq 82(g-1). \qquad \square$$

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