TROPICAL CURVES OF UNIBRANCH POINTS AND HYPERTANGENCY

LUCIA CAPORASO AND AMOS TURCHET

ABSTRACT. We study integral plane curves meeting at a single unibranch point and show that such curves must satisfy two equivalent conditions. A numeric condition: the local invariants of the curves at the contact point must be arithmetically related. A geometric condition: the tropical curves that we associate to the contact point must be isomorphic. Moreover, we prove closed formulas for the delta-invariant of a unibranch singularity, and for the dimension of the loci of curves with an assigned unibranch point. Our work is motivated by interest in the Lang exceptional set.

CONTENTS

1. INTRODUCTION

1.1. **Results.** In this paper we study unibranch points of curves, i.e. points which have a single preimage in the normalization of the curve, and our main goal is to answer the following question.

Under what conditions can two plane projective curves of degree at least 2 intersect in exactly one point, q, such that q is unibranch for both of them?

This question is also relevant for the study of the Lang exceptional set appearing in some conjectures related to arithmetic and hyperbolic geometry, as we shall explain later in this introduction.

To a unibranch point q of a plane curve $C \subset \mathbb{P}^2$ of degree at least 2 we associate a type made of a pair of integers (m, n), where $m = \text{mult}_q(C) \geq 1$ is the multiplicity of C at q, and n is the multiplicity of the intersection of C

1991 *Mathematics Subject Classification.* 14H20, 14H50, 14T90.

Key words and phrases. Unibranch singularity, plane curve, hypertangency, euclidean algorithm, tropical curve.

with its tangent line L at q, i.e. $(L \cdot C)_q = n \geq 2$; we say that q is a (m,n)-point for C. The pair (m,n) encodes information on the local geometry of C at q. The first main result, answering the above question, numerically describes the connection between the local geometry of two curves meeting at a single unibranch point.

Theorem 1.1.1. *Let $B, C \subset \mathbb{P}^2$ be integral curves of degree at least 2 such that $B \cap C = \{q\}$, where q is an (m',n')-point for B and an (m,n)-point for C. Then $m'/n' = m/n$.*

The stronger conclusion $(m',n') = (m,n)$ holds under additional assumptions, see Corollary 4.1.3.

This theorem can be equivalently stated in a more geometric fashion, by associating a tropical curve to a unibranch point of type (m,n). We do that in Section 3 and denote this tropical curve by $\Gamma^{(m,n)}$. Then we prove the following statement, which is an alternative to the previous one

Theorem 1.1.2. *Hypotheses of Theorem 1.1.1. Then there is an isomorphism of tropical curves between $\Gamma^{(m,n)}$ and $\Gamma^{(m',n')}$.*

The core of the proof of these theorems is in Section 4.

Tropical curves are metric graphs, i.e. graphs enriched by the assignment of a length (a positive real number) to each edge; see [MZ08]. In the classical literature non-metric graphs, such as Enriques diagrams, have been widely used to study singularities of plane curves; see [EC85] or [Wal04]. Even though such graphs do not fit our purpose, they inspired our construction of the tropical curve $\Gamma^{(m,n)}$; see Section 3 for more details.

The techniques we develop here enable us to obtain, as a byproduct, closed formulas for the delta-invariant of a unibranch singularity (Proposition 5.1.1) and for the dimension of the space of curves with an assigned unibranch point (Proposition 5.2.1). These formulas occupy the last section of this article.

1.2. **Context.** Our work in this paper is motivated by the algebro-geometric counterparts of some outstanding conjectures from diophantine and hyperbolic geometry. To introduce them, let $B \subset \mathbb{P}^2$ be a complex projective curve having at most nodal singularities. The so-called algebraic exceptional set of B, introduced by Lang, is denoted as follows

$$\mathcal{E}(B) = \{C \subset \mathbb{P}^2 : \ C \text{ rational}, \ \ |\nu_C^{-1}(C \cap B)| \leq 2\} \tag{1}$$

where $\nu_C : C^\nu \to C$ is the normalization, so that $C^\nu \cong \mathbb{P}^1$ if C is rational. One of the main ideas behind the circle of conjectures of Lang, Vojta, Demailly and Campana, is that $\mathcal{E}(B)$ should contain all the infinite families of integral points in the quasi-projective surface $\mathbb{P}^2 \setminus B$, and all the images of nonconstant holomorphic maps $\mathbb{C} \to \mathbb{P}^2 \setminus B$.

Conjecturally, $\mathcal{E}(B)$ is finite as soon as $\deg B \geq 4$, which corresponds to the pairs $(\mathbb{P}^2, B)$ being of log general type. More generally, Lang conjectured that the fact that $\mathcal{E}(B)$ is a proper closed subset of $\mathbb{P}^2$ should be equivalent to the fact that $(\mathbb{P}^2, B)$ is of log general type (and this should hold more in general for projective varieties X with a normal crossing divisor B). For $\mathbb{P}^2$ this conjecture is known only for B reducible with at least three components; the finiteness follows from [CZ13, GNSW23], and we refer to [CT24] for further references and explicit bounds. It remains open when B is irreducible or has only two irreducible components, although [Che04, PR07, CRY23] show that it holds for a very general B.

If $C \in \mathcal{E}(B)$ and $C \cap B = \{q, q'\}$, then it is clear that q and q' must be unibranch points of C. Therefore one is naturally led to investigate unibranch points, and curves having high order of contact at such points. In particular when B has (at least) three components, any $C \in \mathcal{E}(B)$ will intersect at least one of the components of B in a single unibranch point. This is precisely the setting of Theorem 1.1.1 and 1.1.2.

As an example of application of this type of results we mention that, in our previous paper [CT24], we used as a crucial ingredient a special case of Theorem 1.1.1 (Theorem 2.2.1 in loc.cit. which assumes B is a smooth at q) to show that $\mathcal{E}(B)$ is empty for a general B with three irreducible components of total degree at least 5.

1.3. **Notation.** We work over $\mathbb{C}$. Throughout the paper, S will be a smooth projective surface endowed with a birational morphism onto $\mathbb{P}^2$, and $C \subset S$ an integral projective curve.

Given a point $p \in C$, we write $\mathrm{mult}_p(C)$ for the multiplicity of C at p; if $m = \mathrm{mult}_p(C)$ we say that p is an m-fold point . We denote by $\nu_C : C \to C$ the normalization. If $|\nu_C^{-1}(p)| = 1$ we say that p is *unibranch*.

If $C, B \subset S$ are reduced curves with no components in common, and $p \in C \cap B$, we write $(C \cdot B)_p$ for their multiplicity of intersection at p.

If $(C \cdot B)_p = \mathrm{mult}_p(C)\mathrm{mult}_p(B)$ we say that B and C are *transverse* at p.

If $|\nu_C^{-1}(B \cap C)| = 1$, we say that C is *hypertangent* to B.

Acknowledgements. We are grateful to Pietro Corvaja, Kristin DeVleming, Luca Ferrigno and Nicola Ottolini for useful comments. Special thanks to Laura Capuano for showing us Lemma 5.1.2. LC is partially supported by PRIN 2017SSNZAW and PRIN 2022L34E7W, Moduli spaces and birational geometry. AT is partially supported by PRIN 2022HPSNCR: Semiabelian varieties, Galois representations and related Diophantine problems and PRIN 2020KKWT53: Curves, Ricci flat Varieties and their Interactions, and is a member of the INDAM group GNSAGA.

2. Unibranch points

2.1. Resolving unibranch points.

Definition 2.1.1. Let $q \in C \subset S$ be a unibranch point; set $m = \mathrm{mult}_q(C)$. Let $F \subset S$ be a smooth integral curve, $F \neq C$; set $n = (C \cdot F)_q$. If $n > m$ we say that q is an (m, n)*-point of* C *with respect to* F.

For brevity we write: "$q \in C$ is an (m, n)-point w.r.t. F".

If $S = \mathbb{P}^2$ and F is the tangent line to C at q, we simply say that q is an (m, n)-point of C.

By definition, if $q \in C$ is an (m, n)-point with respect to F, then F and C are not transverse at q.

For example, a $(1, 2)$-point of a curve $C \subset \mathbb{P}^2$ is a smooth point which is not a flex.

Our surface S is an iterated blow-up of $\mathbb{P}^2$, hence every point $q \in S$ has an open neighborhood isomorphic to $\mathbb{A}^2$; we will always choose coordinates in such a neighborhood so that q is the origin. We say that a curve $F \subset S$ such that $q \in F$ is a *line locally at* q if there is an open neighborhood $U \cong \mathbb{A}^2$ of q such that the equation of $F \cap U$ has degree 1. In particular, if $E \subset S$ is an exceptional divisor, then E is a line locally at any of its points.

For a unibranch point $q \in C \subset S$ we now introduce an iterated sequence of blow-ups. We set $S_1 = S$, let $S_2 \to S_1$ be the blow-up at q, and $E_2 \subset S_2$ the exceptional divisor. We denote by $C^2 \subset S_2$ the proper transform of C and by $q^2 \in C^2$ the point lying over q, i.e. $q^2 = C^2 \cap E_2$. We use superscripts for strict transforms, and subscripts otherwise, but we denote $C = C^1$ and $q = q^1$ for convenience. We iterate to get a sequence of arbitrary length

$$\dots C^{h+1} \longrightarrow C^h \longrightarrow C^{h-1} \longrightarrow \dots \longrightarrow C^1 = C$$

where $C^h \subset S_h$ is the proper transform of C^{h-1} under the blow-up of S_{h-1} at $q^{h-1} \in C^{h-1}$, and $q^h = C^h \cap E_h$, where E_h is the exceptional divisor of $S_h \to S_{h-1}$. If $F \subset S$ is a curve, we write $F^h \subset S_h$ for its proper transform.

We need the following

Lemma 2.1.2. *Let $q \in C$ be an (m, n)-point w.r.t. L, where L is a line locally at q; set $k = \lfloor n/m \rfloor$. Then $q^h \in C^h$ is an $(m, n - (h-1)m)$-point w.r.t. L^h for every $h \leq k - 1$, and q^k is an m-fold point. Moreover*

(a) if $n/m \neq k$ then $q^k \in C^k$ is an $(m, n - (k-1)m)$-point w.r.t. L^k, and $q^{k+1} \in C^{k+1}$ is an $(n - km, m)$-point w.r.t. E_{k+1};
(b) if $n/m = k$ then C^k is transverse to both E_k and L^k.

Proof. We proceed as in the proof of [CT24, Lemma 2.1.1], of which the present lemma is a generalization. We write, as above, $q^1 = q \in C^1 = C \subset S^1 = S$. We use local coordinates at $q^1 \in U \cong \mathbb{A}^2_{x_1,y_1}$ so that $q^1 = (0, 0)$ and $L = L^1$ has equation $y_1 = 0$. As $q^1 \in C^1$ is an (m, n)-point w.r.t. L^1, the defining polynomial, $f_1(x_1, y_1)$, of C^1 is

$$f_1(x_1, y_1) = a_{0,m} y_1^m + \sum_{m+1 \leq i+j \leq d} a_{i,j} x_1^i y_1^j$$

with

$$a_{0,m} \neq 0, \quad a_{n,0} \neq 0, \quad a_{i,0} = 0 \quad \forall i < n. \tag{2}$$

We blow-up at q^1 and use local coordinates (x_2, y_2) at $q^2 = (0, 0)$, with $x_1 = x_2$ and $y_1 = y_2 x_2$. The local equation of the exceptional divisor E_2 is

$x_2 = 0$, and the local equation of L^2 is $y_2 = 0$. Let $f_2(x_2, y_2) = 0$ be the local equation of C^2, so that

$$f_2(x_2, y_2) = a_{0,m} y_2^m + \sum_{m+1 \le i+j \le d} a_{i,j} x_2^{i+j-m} y_2^j.$$

We iterate: for every $h \le k$ we use use local coordinates (x_h, y_h) at $q^h = (0, 0)$, with $x_{h-1} = x_h$ and $y_{h-1} = y_h x_h$, so that the local equation of E_h is $x_h = 0$ and the local equation of L^h is $y_h = 0$. The defining polynomial of C^h is

$$f_h(x_h, y_h) = a_{0,m} y_h^m + \sum_{m+1 \le i+j \le d} a_{i,j} x_h^{i+j-(h-1)m} y_h^j.$$

By (2), the smallest power of y_h appearing as a summand above is y_h^m, and the smallest power of x_h is $x_h^{n-(h-1)m}$. We have

$$n - (h-1)m = n - hm + m \ge n - km + m \ge n - n + m = m \tag{3}$$

using $h \le k$ in the first inequality and $k \le n/m$ in the second.

If $h \le k - 1$, the first inequality is strict, hence $n - (h-1)m > m$. Therefore $f_h(x_h, y_h)$ contains the summand y_h^m but not the summand x_h^m. Hence, as $q^h \in C^h$ is unibranch, all terms of degree at most m divisible by $x_h y_h$ must vanish. Hence $\text{mult}_{q^h}(C^h) = m$ and, as $y_h = 0$ is the local equation of L^h, we get $(L^h \cdot C^h)_{q^h} = n - (h-1)m$, hence $q^h \in C^h$ is an $(m, n-(h-1)m)$-point w.r.t. L^h, as claimed.

Let $h = k$. If $n/m \ne k$ then $k < n/m$ and the second inequality of (3) is strict. Hence, by the above argument, $q^k \in C^k$ is an $(m, n-(k-1)m)$-point w.r.t. L^k. Now look at $q^{k+1} \in C^{k+1}$, we have

$$f_{k+1}(x_{k+1}, y_{k+1}) = a_{0,m} y_{k+1}^m + \sum_{m+1 \le i+j \le d} a_{i,j} x_{k+1}^{i+j-km} y_{k+1}^j.$$

As before, y_{k+1}^m is the smallest power of y_{k+1} and x_{k+1}^{n-km} the smallest power of x_{k+1}, but now $n - km < m$. Since q^{k+1} is unibranch, by the same argument, $\text{mult}_{q^{k+1}}(C^{k+1}) = n - km$ and, as $x_{k+1} = 0$ is the local equation of E_{k+1}, we conclude that $q^{k+1} \in C^{k+1}$ is an $(n-km, m)$-point w.r.t. E_{k+1}.

Assume $n/m = k$. Then in (3) we have equality. Therefore we have both $x_k^{n-km} = x_k^m$ and y_k^m appearing in f_k. Since q^k is unibranch for C^k, this is possible only if the homogeneous part, $(f_k)_m$, of degree m of f_k, has form $(f_k)_m = (\alpha y_k + \beta x_k)^m$ with $\alpha, \beta \ne 0$. Hence $\text{mult}_{q^k}(C^k) = m$, and C^k is transverse to both E_k and L^k at q^k. ■

Remark 2.1.3. If $k = 1$ the first part of the statement is vacuous and the only interesting part is (a), which states that $q^2 \in C^2$ is an $(n-m, m)$-point with respect to the exceptional divisor E_2

If $q \in C \subset \mathbb{P}^2$ is a singular unibranch point, hence an (m, n)-point, one can resolve the singularity by a finite sequence of blow-ups of $\mathbb{P}^2$, each centered at a point lying over q. This process leads to consider the euclidean algorithm for the pair (m,n). This connection is known; see [Wal04], for example. Our setting here is different, as we need to extend our analysis to smooth points of curves and to tangent lines.

To a pair of integers, (m,n), with $n > m > 0$, the euclidean algorithm associates a chain of integers determining their greatest common divisor, $c := \gcd(m,n)$, as follows

$$l_0 = n > l_1 = m > l_2 > \ldots > l_{r-1} > l_r = c > l_{r+1} = 0 \tag{4}$$

where l_j is defined inductively for $2 \le j \le r+1$

$$l_j := l_{j-2} - l_{j-1}\lfloor l_{j-2}/l_{j-1}\rfloor.$$

We refer to (4) as the *euclidean sequence* of the pair (m,n). We set

$$k_j = \lfloor l_{j-1}/l_j\rfloor, \quad j = 1,\ldots,r.$$

As it turns out, in the resolution sequence resolving the (m,n)-point q, the first multiplicities that appear over q are $l_1, l_2, \ldots, l_r$, in this order. Moreover, each multiplicity l_j appears exactly k_j consecutive times.

We want to use this to give a convenient description of the resolution sequence. We define a total order on the following set of pairs associated to the euclidean sequence (4)

$$\Pi_{(m,n)} := \{(j,i), \quad \forall j = 1,\ldots,r, \quad i = 1,\ldots,k_j\}.$$

We will use the index i even though its range depends on j to ease the notation. We order $\Pi_{(m,n)}$ lexicographically as follows

$$(j',i') \ge (j,i) \text{ if } j' > j \text{ or } j' = j,\ i' \ge i.$$

If (j',i') is the minimum in $\Pi_{(m,n)}$ such that $(j',i') > (j,i)$, we say that (j',i') is next to (j,i), and write $(j',i') = \text{next}(j,i)$. Explicitly:

$$\text{next}(j,i) = \begin{cases} (j,i+1) & \text{if } i < k_j \\ (j+1,1) & \text{if } i = k_j. \end{cases}$$

We set $S_1^1 := \mathbb{P}^2$, $C_1^1 := C$, $q_1^1 := q$. Now, q_1^1 is the center of the first blow-up $\sigma_j^i : S_j^i \to S_1^1$ with $(j,i) = \text{next}(1,1)$. We denote by $C_j^i \subset S_j^i$ the proper transform of C, and by $q_j^i \in C_j^i$ the point lying over q. We generalize to all $(j,i) > (1,1)$. We denote

$$q_j^i \in C_j^i \subset S_j^i \xrightarrow{\sigma_j^i} S_{j'}^{i'} \longrightarrow \mathbb{P}^2$$

so that $(j,i) = \text{next}(j',i')$, the proper transform of C is C_j^i, and q_j^i is the only point of C_j^i lying over q. The map σ_j^i is the blow up at $q_{j'}^{i'}$, and we denote by $E_j^i \subset S_j^i$ its exceptional divisor. For convenience, we denote $L = E_1^1 \subset S_1^1 = \mathbb{P}^2$. The divisors E_j^1 and their proper transform play a special role, hence for every $j = 1,\ldots,r$ we denote by

$$F_j^i \subset S_j^i$$

the proper transform of E_j^1, so that $F_j^1 = E_j^1$. We abuse notation and denote by $\sigma_j^i : C_j^i \to C_{j'}^{i'}$ the restriction of σ_j^i to C_j^i.

Proposition 2.1.4. *Let $q \in C \subset \mathbb{P}^2$ be an (m, n)-point,* (4) *the euclidean sequence of (m, n), and $\nu_q : C_q^\nu \to C$ the desingularization of C at q. We have a chain of birational morphisms*

$$\beta : C_r^{k_r} \xrightarrow{\sigma_r^{k_r}} \ldots \to C_r^1 \xrightarrow{\sigma_r^1} C_{r-1}^{k_{r-1}} \to \ldots$$

$$\ldots \to C_j^{k_j} \xrightarrow{\sigma_j^{k_j}} \ldots \to C_j^i \xrightarrow{\sigma_j^i} C_j^{i-1} \to \ldots \to C_j^1 \xrightarrow{\sigma_j^1} C_{j-1}^{k_{j-1}} \to \ldots \to C_1^1 = C$$

such that the following holds.

(a) For every $j = 1, \ldots, r$ and $i = 1, \ldots k_j$ with $(j, i) \neq (r, k_r)$ the point $q_j^i \in C_j^i$ is an $(l_j, l_{j-1} - (i-1)l_j)$-point w.r.t. F_j^i.

(b) $C_r^{k_r}$ has a c-fold point in $q_r^{k_r}$ where it is transverse to $E_r^{k_r}$ and to $F_r^{k_r}$.

(c) If $c = 1$ then $C_q^\nu = C_r^1$, and q_r^1 is a $(1, l_{r-1})$-point w.r.t. F_r^1.

(d) If $c > 1$ then ν_q factors as follows $\nu_q : C_q^\nu \longrightarrow C_r^{k_r} \xrightarrow{\beta} C$.

Proof. The chain was defined before the statement. Of course, q is an (m, n)-point w.r.t. $L = E_1^1$. If $n/m \in \mathbb{N}$ then $r = 1$ and $c = m$, and the statement follows from Lemma 2.1.2, which is indeed a special case of this Proposition.

Assume $n/m \notin \mathbb{N}$, i.e. $k_1 \neq n/m$. By Lemma 2.1.2, for every $i = 1, \ldots, k_1$ the point $q_1^i \in C_1^i$ is an $(m, n-(i-1)m)$-point w.r.t. the proper transform, F_1^i, of E_1^1; moreover, $q_2^1 \in C_2^1$ is an (l_2, l_1)-point with respect to E_2^1. Now we continue inductively on j; assume the statement for the level $j-1$ and consider the j-th level of the chain, with $2 \leq j \leq r-1$. Then $q_j^1 \in C_j^1$ is an (l_j, l_{j-1})-point w.r.t. $E_j^1 = F_j^1$, and E_j^1 is a line locally at q_j^1. By the same lemma, for every $i = 1, \ldots, k_j$ the point $q_j^i \in C_j^i$ is an $(l_j, l_{j-1} - (i-1)l_j)$-point with respect to the proper transform, F_j^i, of E_j^1.

Consider the last level, $j = r$. Now $l_r = c$ and $l_{r-1}/l_r = k_r$. By Lemma 2.1.2, if $i = 1, \ldots, k_r - 1$ then $q_r^i \in C_r^i$ is an $(c, l_{r-1} - (i-1)c)$-point w.r.t. the proper transform, F_r^i, of E_r^1. Moreover, by part (b) of the same lemma, $C_r^{k_r}$ has a c-fold point in $q_r^{k_r}$, and it is transverse to $F_r^{k_r}$ and to the exceptional divisor. The rest of the statement is clear. ∎

We highlight the following

Remark 2.1.5. For every $j \geq 1$ the curve C_j^1 is tangent to E_j^1, whereas C_j^i meets E_j^i transversally for all $i \neq 1$.

2.2. **Normal crossings resolutions.** Let $q \in C \subset \mathbb{P}^2$ be an (m, n)-point and consider the sequence of maps in Proposition 2.1.4. We add the blow-up of the last surface, $S_r^{k_r}$, at $q_r^{k_r}$, written

$$\sigma_* : S_* \longrightarrow S_r^{k_r},$$

let $C_* \subset S_*$ be the proper transform of C and $q_* \in C_*$ the point lying over q. We set $\beta_* = \beta \circ \sigma_*$ so that we have, abusing notation,

$$\beta_* : C_* \xrightarrow{\sigma_*} C_r^{k_r} \longrightarrow C_r^{k_r-1} \longrightarrow \ldots \longrightarrow C_r^1 \longrightarrow \ldots \longrightarrow C.$$

The subscript "$_*$" indicates that, if $c > 1$, the type of singularity of q_* is not known a priori, in which case β_* is only a partial resolution of q. We write

$$D_* = D_*(q) := (\beta_*^{-1}(C \cup L))_{\text{red}} \tag{5}$$

for the reduced subscheme/divisor supporting $\beta_*^{-1}(C \cup L) \subset S_*$.

We write $A_j^i \subset S_*$ for the proper transform of E_j^i in S_*, so that A_1^1 is the proper transform of $L = E_1^1$. Then

$$D_* = E_* + C_* + \sum_{\substack{j=1,\ldots,r \\ i=1,\ldots,k_j}} A_j^i. \tag{6}$$

By construction, two components of D_* intersect in at most one point, and no three of them intersect in the same point.

Lemma 2.2.1. *Notation as above. Then*

(a) $C_* \cap A_j^i = \emptyset$ *for every* (j,i).

(b) $E_* \cap A_j^i \neq \emptyset$ *if and only if* $(j,i) \in \{(r,k_r),(1,k_r)\}$.

(c) For every $(j,i) \notin \{(r,k_r),(1,k_r)\}$ *there is a unique* $(j',i') > (j,i)$ *such that* $A_j^i \cap A_{j'}^{i'} \neq \emptyset$, *moreover*

$$(j',i') = \begin{cases} \text{next}(j,i) & \text{if } i \neq 1 \\ \text{next}(j+1,1) & \text{if } i = 1. \end{cases} \tag{7}$$

(d) The divisor $D_* - C_*$ *is normal crossings. If* $c = 1$ *and* C *is normal crossings away from* q, *then the divisor* D_* *is normal crossings.*

Proof. By Proposition 2.1.4, the curve $C_r^{k_r}$ has a unibranch c-fold point in $q_r^{k_r}$, where it meets transversally both the proper transform of E_r^1 and the exceptional divisor, $E_r^{k_r}$. Hence the exceptional divisor, E_*, of the blow-up in $q_r^{k_r}$, intersects C_* and the proper transforms of E_r^1 and of $E_r^{k_r}$ (i.e. A_r^1 and $A_r^{k_r}$) in three different points. Hence the first two statements follow,

To prove (c), let $(j',i') > (j,i)$; notice that $A_j^i \cap A_{j'}^{i'} \neq \emptyset$ if and only if for some $S = S_{j''}^{i''}$ with $(j'',i'') \geq (j',i')$ the proper transforms of E_j^i and $E_{j'}^{i'}$ in S intersect away from the proper transform of C. Suppose $i \neq 1$; Proposition 2.1.4 implies that E_j^i is transverse to C_j^i, hence in the next blow-up, the proper transform of E_j^i intersects the exceptional divisor away from the proper transform of C. Suppose $i = 1$; then E_j^1 is tangent to C_j^1, and its proper transform is tangent to C_j^i for every $i \leq k_j$. In S_{j+1}^1 the curve C_{j+1}^1 is tangent to E_{j+1}^1, hence the proper transform of E_j^1 intersects E_{j+1}^1 transversally in q_{j+1}^1. Therefore in the next blow-up, the proper transform of E_j^1 intersects the exceptional divisor away from the proper transform of C. (c) is proved.

All components of $D_* - C_*$ are smooth, intersect transversally, and only pairwise. Hence $D_* - C_*$ is normal crossings. If $c = 1$, then $q_r^{k_r}$ is a smooth point of $C_r^{k_r}$, hence D_* is normal crossings if so is C away from q. This proves (d) and concludes the proof. ■

3. Tropical curves of unibranch points

3.1. **Dual graphs of partial resolutions.** Let $C \subset \mathbb{P}^2$ have a unique (for simplicity) singular point in q, it is well known that there exists a unique surface S_ρ with a birational morphism $\rho : S_\rho \to \mathbb{P}^2$ such that the strict transform of C in S_ρ is smooth, the reduced scheme underlying $\rho^{-1}(C)$ is a divisor with normal crossings, written D_ρ, and S_ρ is minimal with respect to these properties. The dual graph, G_ρ, of D_ρ (whose vertices are the irreducible components of D_ρ, with an edge between two vertices if the two components intersect) is often called the resolution graph of q; [Wal04]. The graph G_ρ is not suitable here, and we will define a variant of it which will lead, in turn, to a tropical curve.

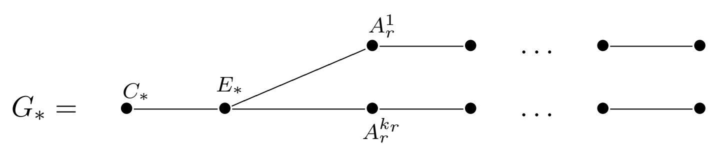
We consider the dual graph,

$$G_* = G(D_*(q))$$

of the divisor D_*, defined in Section 2.2. The vertices of G_* are the irreducible components of D_*, hence G_* has $2 + \sum_{i=1}^{r} k_i$ vertices, and there is an edge between two vertices if the two components intersect. For the next lemma, recall that a *leaf* of a graph is a vertex of degree 1.

Lemma 3.1.1. *Let $q \in C$ be an (m, n)-point.*

(a) The graph $G(D_*(q))$ is a tree with a unique vertex, E_*, of degree 3 and three leaves among which C_* and A_1^1, and has the following form



(b) $G(D_*(q))$ only depends on the pair (m, n), i.e. if $q' \in C' \subset S'$ is an (m, n)-point, then $G(D_*(q))$ is isomorphic to $G(D_*(q'))$ via an isomorphism that takes the vertex C_* to the vertex C'_*.

Proof. Write $G_* = G(D_*(q))$. (The fact that G_* is a tree should to be clear from its definition, but we give it a proof below.)

Lemma 2.2.1 implies that E_* is the only vertex of degree 3, and it is joined by an edge to C_*, $A_r^{k_r}$ and A_r^1, as shown in the picture. The same lemma implies that C_* and A_1^1 are leaves of G_* (and that the third leaf is A_2^1 if $k_1 = 1$, and A_1^2 if $k_1 > 1$).

Let ν_i be the number of vertices of degree i in G_*. Then $\nu_3 = 1$ and $\nu_1 \geq 2$. The number of edges of G_* is thus

$$|E(G_*)| = (\nu_1 + 2\nu_2 + 3\nu_3)/2 = (\nu_1 + 2\nu_2 + 3)/2$$

hence ν_1 must be odd, hence $\nu_1 \geq 3$. Since G_* is connected, its genus $g(G_*) = b_1(G_*)$ satisfies $g(G_*) = |E(G_*)| - |V(G_*)| + 1$. Moreover, since $|V(G_*)| = \nu_1 + \nu_2 + 1$, we have

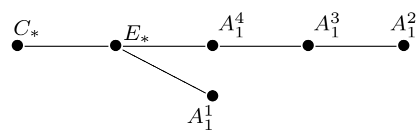
$$g(G_*) = (\nu_1 + 2\nu_2 + 3)/2 - (\nu_1 + \nu_2 + 1) + 1 = (3 - \nu_1)/2 \leq 0$$

with equality only if $\nu_1 = 3$. Since $g(G_*) \geq 0$ we get $\nu_1 = 3$ and $g(G_*) = 0$. (a) is proved. Part (b) is clear. ■

As $G(D_*(q))$ depends only on the pair (m, n), we set $G_*^{(m,n)} := G(D_*(q))$ for any (m, n)-point q on any curve C. We will show later (see Lemma 3.1.5) that it depends only on the euclidean sequence of (m, n).

Different pairs can give the same graph, as in the next example.

Example 3.1.2. Let q be a smooth point of type $(1, 4)$, hence $r = 1$ and $k_1 = 4$. The graph $G(D_*(q))$ is as follows



Let now q' be a $(2, 8)$-point, then q is singular, $r = 1$ and $k_1 = 4$, and the graph $G(D_*(q'))$ is the same as above.

Example 3.1.3. *Fibonacci singularities.* A *Fibonacci point* is an (m, n)-point such that (m, n) is a pair of consecutive integers in the Fibonacci sequence $\{f_n\}$ defined by $f_1 = 1$, $f_2 = 2$, and $f_n = f_{n-1} + f_{n-2}$:

$$1,\ 2,\ 3,\ 5,\ 8,\ 13\ ,21,\ \ldots$$

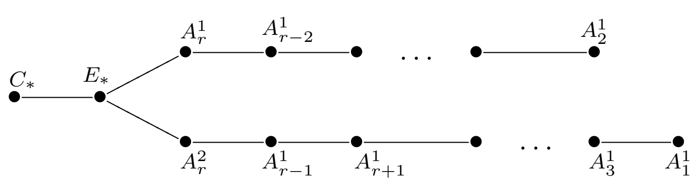
The "blow-up" of a Fibonacci point of type (f_n, f_{n-1}) is a Fibonacci point of type (f_{n-1}, f_{n-2}). The euclidean sequence for a Fibonacci (m, n)-point is equal to the reversed Fibonacci sequence truncated at n, for example for $(m, n) = (13, 21)$ we get

$$l_0 = 21 > l_1 = 13 > l_2 = 8 > l_3 = 5 > l_4 = 3 > l_5 = 2 > l_6 = 1.$$

An (m, n)-point is a Fibonacci point if and only if $k_r = 2$ and $k_j = 1$ for all $j = 1, \ldots, r - 1$. For such a point the chain of blow-ups in 2.1.4 is

$$C_* \longrightarrow C_r^2 \longrightarrow C_r^1 \longrightarrow C_{r-1}^1 \longrightarrow \ldots \longrightarrow C_2^1 \longrightarrow C$$

and the graph G_* is as follows, if r is even.



We ask under what conditions different pairs give the same graph.

Definition 3.1.4. Let (m,n) and (m',n') be two pairs of integers such that $n > m > 0$ and $n' > m' > 0$. The euclidean sequence for (m,n) is given in (4); let the euclidean sequence of (m',n') be

$$n' = l'_0 > m' = l'_1 > \dots l'_r = c' > l'_{r'+1} = 0 \tag{8}$$

where $c' = \gcd(m',n')$. Set $k'_j = \lfloor l_{j-1}/l_j \rfloor$ for $j = 1,\dots,r'$. We say that the two pairs are *equivalent*, and write

$$(m,n) \sim (m',n'),$$

if $r = r'$ and $k_j = k'_j$ for every $j = 1,\dots,r$.

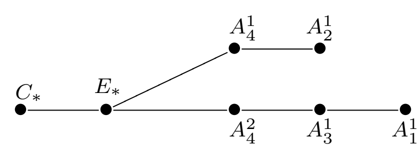
If $q \in C \subset \mathbb{P}^2$ is an (m,n)-point and $q' \in C' \subset \mathbb{P}^2$ is an (m',n')-point we say that q and q' are equivalent if $(m,n) \sim (m',n')$.

Lemma 3.1.5. *If $(m,n) \sim (m',n')$ then $G_*^{(m,n)} \cong G_*^{(m',n')}$.*

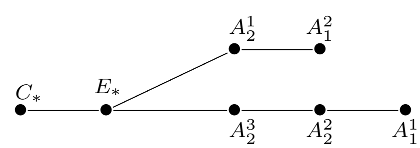
Proof. By Proposition 2.1.4, for any (m,n)-point q, the divisor $D_*(q)$ is completely determined by the integers r and $k_1,\dots,k_r$ of the euclidean sequence of (m,n), hence $G_*^{(m,n)}$ depends only on the equivalence class of (m,n). ■

The converse fails, as the following example shows.

Example 3.1.6. If $(m,n) = (5,8)$, then q is a Fibonacci point with $r = 4$ and the graph is as follows

$G_*^{(5,8)} =$ 

Now let $(m',n') = (3,7)$, then $r' = 2$ hence $(m,n) \not\sim (m',n')$. The graph below is clearly isomorphic to $G_*^{(5,8)}$

$G_*^{(3,7)} =$ 

It is well known that the condition $(m,n) \sim (m',n')$ can be expressed more directly as follows.

Remark 3.1.7. $(m,n) \sim (m',n')$ if and only if $m/n = m'/n'$.

See [HW08, Subsection 10.6 and Theorem 160], for example.

3.2. **The contact tropical curve of an (m,n)-point.** To improve upon Lemma 3.1.5 and give a geometric characterization for equivalence of pairs, we now introduce a tropical curve associated to an (m,n)-point. We first put on the graph $G_* = G(D_*(q))$ a length on its edges, $\ell_* : E(G_*) \to \mathbb{R}_+$, such that $\ell_*(e) = 1$ for every edge e of G_*. This defines a tropical curve $\Gamma_* := (G_*, \ell_*)$. We shall now define the *contact tropical curve* of the (m,n)-point q, written Γ_q. The word "contact" is motivated by Theorem 4.1.1. As for notation, set

$$\Gamma_q = (G_q, \ell_q), \qquad G_q = (V(G_q), E(G_q), L(G_q))$$

with G_q a graph whose vertex set, edge set, and leg set are written, respectively, $V(G_q)$, $E(G_q)$ and $L(G_q)$ (a leg is a half-edge attached to only one vertex), and ℓ_q is a function

$$\ell_q : E(G_q) \cup L(G_q) \longrightarrow \mathbb{R}_+.$$

We define Γ_q as the tropical curve obtained from Γ_* by removing every vertex A^i_j with $i > 1$ without disconnecting the graph and adding the corresponding lengths. Since the edges A^i_j have degree at most 2, this operation is well defined. Indeed, if A^i_j has degree 1, then the edge adjacent to it becomes a leg, or part of a leg, of Γ_q; if A^i_j has degree 2 the two edges adjacent to A^i_j are merged in a new edge or in a leg.

Remark 3.2.1. By Remark 2.1.5, we removed from G_* those vertices corresponding to exceptional divisors E^i_j that are not tangent to the proper transform C^i_j of C.

We have an obvious surjection from the edges of G_* to the edges and legs of G_q, written

$$\mu : E(G_*) \longrightarrow E(G_q) \cup L(G_q).$$

The length function ℓ_q is defined as follows

$$\ell_q(x) := \sum_{e \in \mu^{-1}(x)} \ell_*(e)$$

for any $x \in E(G_q) \cup L(G_q)$. The definition of the tropical curve Γ_q is now complete. Let us describe it more closely. Its vertex set is

$$V(G_q) = \{C_*,\ E_*,\ A^1_j, \quad j = 1, \ldots, r\}.$$

An edge of G_q can be written unambiguously as $e = vw$ with $v, w \in V(G_q)$, hence, by Lemma 2.2.1, we have

$$E(G_q) = \begin{cases} \{C_*E_*,\ E_*A^1_r,\ E_*A^1_{r-1},\ A^1_{j+1}A^1_{j-1},\ \forall j = 2, \ldots, r-1\} & \text{if } r > 1 \\ \{C_*E_*,\ E_*A^1_1\} & \text{if } r = 1 \end{cases}$$

and one easily checks the following

$$\ell_q(vw) := \begin{cases} 1 & \text{if } vw = E_*C_* \text{ or } vw = E_*A^1_r \\ k_r & \text{if } vw = E_*A^1_{r-1} \\ k_j & \text{if } vw = A^1_{j+1}A^1_{j-1}, \quad \forall j = 2, \ldots, r-2. \end{cases} \tag{9}$$

Finally, if $r > 1$ and $k_1 > 1$ then G_q has a leg attached to A_2^1 of length $k_1 - 1$. If $r = 1$ then G_q has a leg attached to E_* of length $k_r - 1$. In particular, Γ_q has at most one leg. Summarizing and applying Lemma 3.1.1:

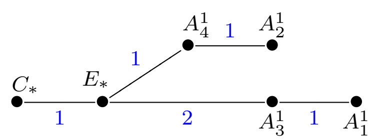
Lemma 3.2.2. *Let $q \in C \subset S$ be an (m, n)-point and $\Gamma_q = (G_q, \ell_q)$ its contact tropical curve.*

(a) G_q is a tree with $r + 2$ vertices, and a unique vertex, E_*, of degree 3.
The vertices C_* and A_1^1 are leaves of G_q (with no leg attached).
G_q has no legs if $k_1 = 1$, and a unique leg of length $k_1 - 1$ if $k_1 > 1$.
(b) If $q' \in C' \subset S'$ is an (m, n)-point, then the contact tropical curves Γ_q and $\Gamma_{q'}$ are isomorphic.

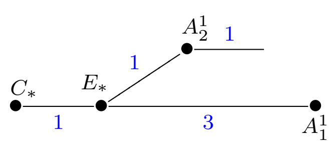
As Γ_q only depends on (m, n) we set, for any (m, n)-point $q \in C$,

$$\Gamma^{(m,n)} := \Gamma_q.$$

Example 3.2.3. Here are the tropical curves corresponding to Example 3.1.6.

$\Gamma^{(5,8)} =$ 

and

$\Gamma^{(3,7)} =$ 

These tropical curves are not isomorphic, while $G_*^{(5,8)} \cong G_*^{(3,7)}$.

We can now geometrically characterize equivalent pairs.

Proposition 3.2.4. $(m, n) \sim (m', n')$ *if and only if* $\Gamma^{(m,n)} \cong \Gamma^{(m',n')}$ *(i.e. $\Gamma^{(m,n)}$ and $\Gamma^{(m',n')}$ are isomorphic as tropical curves).*

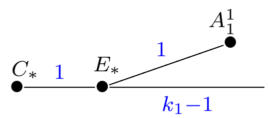
Proof. If $(m, n) \sim (m', n')$ then Lemma 3.1.5 yields $G_*^{(m,n)} \cong G_*^{(m',n')}$, hence $\Gamma^{(m,n)} \cong \Gamma^{(m',n')}$.

To prove the converse, write $\Gamma = \Gamma^{(m,n)}$ and $\Gamma' = \Gamma^{(m',n')}$ and assume there is an isomorphism of tropical curves $\phi : \Gamma \to \Gamma'$. Hence ϕ is a length-preserving isomorphism between the underlying graphs.

Therefore Γ and Γ' have the same number of vertices, equal to $r + 2$, hence $r = r'$. Moreover, Γ and Γ' have the same number of legs, equal to $\min\{1, k_1 - 1\} = \min\{1, k_1' - 1\}$; if this number is 0 then $k_1 = k_1' = 1$. If $\min\{1, k_1 - 1\} = 1$ then Γ and Γ' have each one leg of respective lengths $k_1 - 1$ and $k_1' - 1$. Since the length of the leg is preserved by the isomorphism ϕ, we get $k_1 = k_1'$. We write

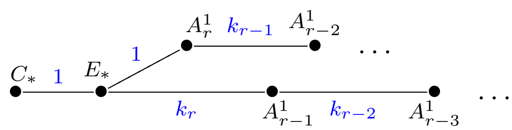
$$V(\Gamma') = \{C_*',\ E_*',\ (A_j^1)', \quad \forall j = 1, \dots, r\}.$$

Recall that Γ and Γ' have a unique vertex, namely E_* and E'_*, of degree 3, which must, of course, be preserved by an isomorphism. If $r = 1$ then $k_1 \geq 2$, hence Γ has a unique leg and looks as follows



Since we already proved that $k_1 = k'_1$, we are done.

If $r \geq 2$, Lemma 3.2.2 implies that Γ has the following form



and the same holds for Γ', mutatis mutandis. Since $k_r \geq 2$, the edges $E_*A_r^1$ and $E_*A_{r-1}^1$ have different lengths, equal to 1 and k_r. The same holds on Γ': as $k'_r \geq 2$ the edges $E'_*(A_r^1)'$ and $E'_*(A_{r-1}^1)'$ have lengths equal to 1 and k'_r. Hence an isomorphism from Γ to Γ' maps A_r^1 to $(A_r^1)'$ and A_{r-1}^1 to $(A_{r-1}^1)'$, hence $k_r = k'_r$. Hence the edge $A_r^1A_{r-2}^1$ is mapped to the edge $(A_r^1)'(A_{r-2}^1)'$, hence $k_{r-1} = k'_{r-1}$. Iterating, the edge $A_j^1A_{j-2}^1$ is mapped to the edge $(A_j^1)'(A_{j-2}^1)'$, hence $k_{j-1} = k'_{j-1}$ for all $j = 3, \ldots r$. The proof is complete. ■

4. Hypertangency

4.1. **Hypertangency at unibranch point.** Let $B, C \subset S$ be two integral curves and $q \in B \cap C$; let $\sigma : S' \to S$ be the blow up at q. If $B', C' \subset S'$ are the proper transforms of B and C, the following is well know and will be used often

$$(B' \cdot C') = (B \cdot C) - \text{mult}_q(B)\text{mult}_q(C). \tag{10}$$

For our next result, we shall assume that q is an (m, n)-point of C and an (m', n')-point of B. Recall, from Definition 3.1.4, that two pairs (m, n) and (m', n') are equivalent, written $(m, n) \sim (m', n')$, if $r = r'$ and $k_j = k'_j$ for every $j = 1, \ldots, r$, where the integers r, r', k_j, k'_j are determined by the euclidean sequences of the pairs.

Theorem 4.1.1. *Let $B, C \subset \mathbb{P}^2$ be integral curves of degree at least 2. Let q be an (m', n')-point for B and an (m, n)-point for C such that $B \cap C = \{q\}$. Then the following equivalent conditions hold*

(a) $\Gamma^{(m,n)} \cong \Gamma^{(m',n')}$.
(b) $m/n = m'/n'$.
(c) $(m', n') \sim (m, n)$.

Proof. The equivalence of the three conditions follows from Remark 3.1.7 and Proposition 3.2.4. We will prove $(m', n') \sim (m, n)$.

Set $b = \deg B$ and $d = \deg C$. By hypothesis $(B \cdot C)_q = (B \cdot C) = bd$. Notice that B and C are necessarily tangent in q. We can assume $r \leq r'$. Consider the chain of blow-ups described before Proposition 2.1.4 for the curve C. For every $j = 1, \ldots, r$ and $i = 1, \ldots, k_j$ we have a sequence of blow-ups, S_j^i, of $\mathbb{P}^2$, containing the proper transform, C_j^i, of C, and a point $q_j^i \in C_j^i$, which is the center of the blow up of S_j^i; we have $q = q_1^1$, $C = C_1^1$ and $\mathbb{P}^2 = S_1^1$. We consider also the curve C_* in the blow-up S_* of $S_r^{k_r}$ at $q_r^{k_r}$, defined before Lemma 2.2.1.

We denote by $B_{(j)}^{(i)} \subset S_j^i$ and $B_{(*)} \subset S_*$ the proper transforms of B. Since C and B meet only in q, if the curves C_j^i and $B_{(j)}^{(i)}$ intersect, they do so in the only point of C_j^i mapping to q, i.e. in q_j^i. Notice that if $(m,n) \sim (m',n')$ the chain of blow-ups for B described in Proposition 2.1.4 corresponds exactly to the one for C, hence $B_{(j)}^{(i)} = B_j^i$.

We prove $(m,n) \sim (m',n')$ by contradiction. Then, setting

$$h = \min\{j : k_j \neq k'_j,\ j = 1, \ldots, r\},$$

we have $h \leq r$. We can assume $k_h < k'_h$, the case $k_h > k'_h$ is similar. By the definition of h, we have $B_h^{k_h} = B_{(h)}^{(k_h)}$ and $C_h^{k_h} \cap B_h^{k_h} = \{q_h^{k_h}\}$.

Claim. *If $h < r$ then C_{h+1}^1 is not tangent to $B_{(h+1)}^{(1)}$; if $h = r$ then $C_r^{k_r}$ is not tangent to $B_{(r)}^{(k_r)}$.*

Recall that, by Proposition 2.1.4, for every (j,i) with $i > 1$, except the pair (r, k_r), the curve C_j^i is tangent to F_j^i (the proper transform in S_j^i of the exceptional divisor of S_j^1), and C_j^1 is tangent to the exceptional divisor $E_j^1 \subset S_j^1$. If $h < r$ both $C_h^{k_h}$ and $B_{(h)}^{(k_h)} = B_h^{k_h}$ are tangent to $F_h^{k_h}$, hence their proper transforms in the blow-up at $q_h^{k_h}$ intersect in q_{h+1}^1. Now, the proper transform of C is C_{h+1}^1 and the proper transform of B, in the notation of Proposition 2.1.4, is $B_h^{k_h+1}$ (as $k'_h > k_h$), and we have $B_h^{k_h+1} = B_{(h+1)}^{(1)}$. In q_{h+1}^1, the curve C_{h+1}^1 is tangent to E_{h+1}^1, whereas $B_h^{k_h+1}$ is tangent to the proper transform of $F_h^{k_h}$, hence C_{h+1}^1 is not tangent to $B_{(h+1)}^{(1)}$, as claimed.

If $h = r$, according to Proposition 2.1.4, the curve $C_r^{k_r}$ is not tangent to $F_r^{k_r}$ in $q_r^{k_r}$, while $B_r^{k_r}$ is, as $k'_h > k_h$. Since $B_r^{k_r} = B_{(r)}^{(k_r)}$ the claim is proved.

By the claim, we necessarily have

$$B_{(*)} \cap C_* = \emptyset. \tag{11}$$

If $h < r$ a stronger fact holds, to state which we depart momentarily from the notation in Proposition 2.1.4 and write $S_{k_h} \to S_{h+1}^1$ for the blow-up at q_{h+1}^1; denote by $C^{k_h}, B^{(k_h)} \subset S_{k_h}$ the proper transforms of B and C. By the claim, C_{h+1}^1 is not tangent to $B_{(h+1)}^{(1)}$, hence

$$C^{k_h} \cap B^{(k_h)} = \emptyset \tag{12}$$

(which is stronger than (11)).

We are ready to get a contradiction. Assume $h < r$; by iterating (10) we have

$$(C^{k_h} \cdot B^{(k_h)}) = bd - \sum_{j=1}^{h} k_j l_j l'_j + l_{h+1} l'_h. \tag{13}$$

Indeed, C^i_j has an l_j-fold point at q^i_j for $j = 1, \ldots, h$ and $i = 1, \ldots, k_h$, while C^1_{h+1} has an l_{h+1}-fold point. The curve $B^{(i)}_{(j)}$ has a l'_j-fold point at q^i_j for $j = 1, \ldots, h$ and $i = 1, \ldots, k_j$, and $B^{(1)}_{(h+1)}$ has an l'_h-fold point. Hence (13) follows. We combine (13) with Lemma 4.1.2 (which can be applied as $k_j = k'_j$ for all $j < h$) and we get

$$(C^{k_h} \cdot B^{(k_h)}) > bd - nn' \geq 0$$

(as $b \geq n'$ and $d \geq n$), contradicting (12).

Assume $h = r$, hence $k_j = k'_j$ for all $j < r$. Then for $j = 1, \ldots, r$ and $i = 1, \ldots, k_r$, at q^i_j the curve C^i_j has an l_j-fold point, and $B^{(i)}_{(j)}$ has l'_j-fold point. Therefore,

$$(C_* \cdot B_{(*)}) = bd - \sum_{j=1}^{r} k_j l_j l'_j. \tag{14}$$

Since $l_{r+1} = 0$ we get

$$\sum_{j=1}^{r} k_j l_j l'_j = \sum_{j=1}^{r} k_j l_j l'_j + l_{r+1} l'_r < nn'$$

by Lemma 4.1.2 with $h = r$. Hence

$$(C_* \cdot B_{(*)}) > bd - nn' \geq 0.$$

contradicting (11). The theorem is proved. ∎

To prove the theorem we used the following Lemma.

Lemma 4.1.2. *Let* (4) *and* (8) *be the euclidean sequences of* (m, n) *and* (m', n'). *Assume* $r \leq r'$ *and let* $h \leq r$. *If* $k_j = k'_j$ *for every* $j = 1, \ldots h - 1$ *then*

$$\sum_{j=1}^{h} k_j l_j l'_j + l_{h+1} l'_h < nn'.$$

Proof. The proof is based on the following identity:

$$\sum_{j=1}^{h} k_j l_j l'_j + l_{h+1} l'_h = \begin{cases} l_1 l'_0 & \text{if } h \text{ is even} \\ l_0 l'_1 & \text{if } h \text{ is odd.} \end{cases} \tag{15}$$

We have $m = l_1 < n = l_0$ and $m' = l'_1 < n = l'_0$, hence $l_1 l'_0 < nn'$ and $l'_1 l_0 < n'n$. Combining this with (15) the Lemma is proved.

We prove (15) by induction on h, applying a few times the basic identities:

$$l_{j-2} = k_{j-1} l_{j-1} + l_j, \qquad l'_{j-2} = k'_{j-1} l'_{j-1} + l'_j.$$

The base cases are $h = 1$ and $h = 2$. If $h = 1$ we get

$$k_1 l_1 l'_1 + l_2 l'_1 = (k_1 l_1 + l_2) l'_1 = l_0 l'_1.$$

If $h = 2$ then $k_1 = k'_1$, and we get

$$k_1 l_1 l'_1 + k_2 l_2 l'_2 + l_3 l'_2 = k'_1 l_1 l'_1 + (k_2 l_2 + l_3) l'_2 = k'_1 l_1 l'_1 + l_1 l'_2 = l_1 (k'_1 l'_1 + l'_2) = l_1 l'_0.$$

Let $h \geq 3$. If h is even we have

$$\begin{aligned} &\sum_{j=1}^{h} k_j l_j l'_j + l_{h+1} l'_h = \sum_{j=1}^{h-1} k_j l_j l'_j + k_h l_h l'_h + l_{h+1} l'_h = \\ &= \sum_{j=1}^{h-1} k_j l_j l'_j + (k_h l_h + l_{h+1}) l'_h = \sum_{j=1}^{h-1} k_j l_j l'_j + l_{h-1} l'_h = \\ &= \sum_{j=1}^{h-2} k_j l_j l'_j + l_{h-1} (k_{h-1} l'_{h-1} + l'_h) = \sum_{j=1}^{h-2} k_j l_j l'_j + l_{h-1} (k'_{h-1} l'_{h-1} + l'_h) = \\ &= \sum_{j=1}^{h-2} k_j l_j l'_j + l_{h-1} l'_{h-2} = l_1 l'_0 \end{aligned}$$

by the induction hypothesis ($h - 2$ is even). The case h odd follows in the same way. The Lemma is proved. ■

The following special case of the theorem is worth an explicit mention.

Corollary 4.1.3. *Hypotheses of Theorem 4.1.1.*

If $\gcd(m, n) = \gcd(m', n')$ *then* $(m, n) = (m', n')$*; in particular, if* B *and* C *are smooth at* q*, then* $n = n'$*.*

4.2. **Intersection multiplicity at equivalent points.** We proved in Theorem 4.1.1 that if two curves B and C are hypertangent in one point q, then $q \in B$ is equivalent to $q \in C$, i.e. their contact tropical curves are isomorphic. It is natural to pose the opposite problem, namely, what can be said about the intersection of two curves meeting at equivalent points.

Proposition 4.2.1. *Let* $B, C \subset \mathbb{P}^2$ *be integral curves and* $q \in B \cap C$*. Assume that* q *is an* (m', n')*-point of* B *and an* (m, n)*-point of* C*, with* $(m, n) \sim (m', n')$*; set* $k = \lfloor n/m \rfloor$*. If* $(B \cdot C)_q > m'm$ *(i.e.* B *and* C *are tangent in* q*), then*

$$(B \cdot C)_q \geq \begin{cases} km'm & \text{if } n/m \in \mathbb{N} \\ km'm + 1 + (n' - m')(n - m) & \text{if } n/m \notin \mathbb{N}. \end{cases} \tag{16}$$

Proof. Let L be the tangent line at q to C and B, and $S_1 \to \mathbb{P}^2$ the blow-up at q; denote by $L^1, B^1, C^1 \subset S_1$ the proper transforms of L, B, C, and by $q^1 \in L^1$ the point lying over q. Denote by $S_2 \to S_1$ the blow-up at q^1, by

$L^2, B^2, C^2 \subset S_2$ the proper transforms of L, B, C, and by $q^2 \in L^2$ the point lying over q^1. Iterating we get a sequence of blow-ups

$$S_{k+1} \to S_k \to S_{k-1} \to \dots S_{i+1} \to S_i \to \dots \to S_1 \to S_0 = \mathbb{P}^2$$

such that $L^i, B^i, C^i \subset S_i$ are the proper transforms of L, B, C and $S_{i+1} \to S_i$ is the blow-up at $q^i \in L^i$, with q^i lying over q; we write $E_i \subset S_i$ for the exceptional divisor.

By Lemma 2.1.2, for every $i \leq k-1$ the curve C^i, respectively B^i, has a point of multiplicity m, respectively m', lying over q. Therefore, setting

$$(C^i \cdot L^i)_{E_i} = \sum_{p \in E_i} (C^i \cdot L^i)_p$$

we have, as $k \leq n/m$

$$(C^i \cdot L^i)_{E_i} = (C \cdot L)_q - im = n - im \geq n-(k-1)m \geq n-(n/m-1)m = m \geq 1$$

hence, as $L^i \cap E_i = q^i$, we get $(C^i \cdot L^i)_{q^i} \geq 1$. An analogous argument applied to B gives $(B^i \cdot L^i)_{q^i} \geq 1$ for all $i \leq k-1$. Therefore $q^{k-1} \in B^{k-1} \cap C^{k-1}$ hence

$$(B^{k-1} \cdot C^{k-1})_{q^{k-1}} \geq m'm.$$

On the other hand

$$(B^{k-1} \cdot C^{k-1})_{q^{k-1}} = (B \cdot C)_q - (k-1)m'm$$

hence

$$(B \cdot C)_q \geq m'm + (k-1)m'm = km'm.$$

If $n/m \in \mathbb{N}$ there is nothing left to prove. Let $n/m \notin \mathbb{N}$, hence $k < n/m$. Then C^k has an $(n-m, m)$-point w.r.t. E_k, and B^k has an $(n'-m', m')$-point w.r.t. E_k (by Lemma 2.1.2); let us show that this point is q^k for both C^k and B^k. We have

$$(C^k \cdot L^k)_{q^k} = n - km > n - nm/m = 0$$

hence $q^k \in C^k$. The same argument shows that $q^k \in B^k$. Hence C^k and B^k intersect in q^k, as wanted. Now, since C^k and B^k are both tangent to E_k, their intersection is not transverse, hence their proper transforms in S_{k+1} satisfy

$$(B^{k+1} \cdot C^{k+1})_{\tilde{q}^{k+1}} \geq 1$$

with $\tilde{q}^{k+1}$ lying over q. On the other hand

$$(B^{k+1} \cdot C^{k+1})_{\tilde{q}^{k+1}} = (B \cdot C)_q - km'm - (n'-m')(n-m)$$

hence $(B \cdot C)_q \geq 1 + km'm + (n'-m')(n-m)$. The proof is complete. ■

5. Genus and dimension formulas

In this section we obtain bounds for the delta invariant (defined below) of a unibranch point of a curve $C \subset \mathbb{P}^2$, and for the codimension of the locus of plane curves with a (m, n) point.

5.1. **Genus computation.** Let $C \subset \mathbb{P}^2$ and $q \in C$; let $\deg C = d \geq 2$. The geometric genus, $g(C) = g(C^\nu)$ (with C^ν the normalization of C), satisfies

$$g(C) \geq (d-1)(d-2)/2 - \delta(q)$$

with equality if and only if q is the only singular point of C. The number $\delta(q)$ is called the *delta-invariant* of q. The following formula is well-known

$$\delta(q) = \sum m_p(m_p - 1)/2 \tag{17}$$

where p varies over all points infinitely near to q, and m_p is the multiplicity of p. For example, if q is an ordinary m-fold point (i.e. C has a m branches meeting transversally in q), then $\delta(q) = m(m-1)/2$.

Proposition 5.1.1. *Let $C \subset \mathbb{P}^2$ have degree $d \geq 2$ and let $q \in C$ be an (m, n)-point; set $\gcd(n, m) = c$. Then*

$$\delta(q) \geq (nm - n - m + c)/2$$

with equality if $c = 1$.

Proof. Proposition 2.1.4 considers the desingularization, $\nu_q : C^\nu_q \to C$, of C at q, and describes the points infinitely near to q appearing in the partial desingularization given by β. This, combined with (17), gives

$$\delta(q) \geq \sum_{j=1}^{r} k_j l_j (l_j - 1)/2. \tag{18}$$

Indeed, for every $j = 1, \ldots, r$ we have exactly k_j infinitely near points of multiplicity l_j, and each such point contributes by $l_j(l_j - 1)/2$. Note that if $c = 1$ the summand corresponding to $j = r$ vanishes, as $l_r = 1$. In this case the desingularization of C at q is actually included in β, hence there are no other infinitely near points, therefore in (18) we have equality.

The following Lemma 5.1.2 completes the proof. ■

Lemma 5.1.2. *Let $n > m > 0$ be two integers and consider the euclidean sequence* (4). *Then $k_j = (l_{j-1} - l_{j+1})/l_j$ and*

$$\sum_{j=1}^{r} k_j l_j (l_j - 1)/2 = (nm - n - m + c)/2. \tag{19}$$

Proof. The expression of k_j follows directly from its definition and the construction of the euclidean sequence. The second assertion follows from a special case of the following formula

$$\sum_{j=1}^{r} (l_{j-1} - l_{j+1})(l_j - 1) = l_0 l_1 - l_0 - l_1 + l_{r+1} + l_r - l_r l_{r+1}. \tag{20}$$

In our case $l_0 = n$, $l_1 = m$, $l_r = c$ and $l_{r+1} = 0$, from which (19) follows. To prove (20) we proceed by induction on r. If $r = 1$ then

$$(l_0 - l_2)(l_1 - 1) = l_0 l_1 - l_1 l_2 - l_0 + l_2,$$

as wanted. To prove the general case, the induction hypothesis reads as follows

$$\sum_{j=1}^{r-1}(l_{j-1}-l_{j+1})(l_j-1)=l_0l_1-l_0-l_1+l_{r-1}+l_r-l_rl_{r-1}.$$

Thus we obtain

$$\begin{aligned}\sum_{j=1}^{r}(l_{j-1}-l_{j+1})(l_j-1)=\\ l_0l_1-l_0-l_1+l_{r-1}+l_r-l_rl_{r-1}+(l_{r-1}-l_{r+1})(l_r-1)=\\ l_0l_1-l_0-l_1+l_{r-1}+l_r-l_rl_{r-1}+l_{r-1}l_r-l_rl_{r+1}-l_{r-1}+l_{r+1}=\\ l_0l_1-l_0-l_1+l_r-l_rl_{r+1}+l_{r+1}.\end{aligned}$$

The Lemma is proved. ■

5.2. **Dimension computation.** Let n, m be two integers with $n > m > 0$; let $L \subset \mathbb{P}^2$ be a line and $q \in L$ a point. We denote by

$$U_d^{(m,n)}(L;q) \subset \mathbb{P}^{d(d+3)/2}$$

or simply by $U_d^{(m,n)}$, the space of integral curves of degree $d \geq 2$ having a unibranch (m,n)-point at q with tangent line equal to L. As it will be clear (if it is not already), its closure, $\overline{U_d^{(m,n)}}$, is a linear subspace of $\mathbb{P}^{d(d+3)/2}$. We write $\operatorname{codim} U_d^{(m,n)} = \operatorname{codim} \overline{U_d^{(m,n)}}$ for the codimension in $\mathbb{P}^{d(d+3)/2}$.

Proposition 5.2.1. *Assume $d \geq n > m > 0$ and set $\gcd(n,m) = c$. Then*

$$\operatorname{codim} U_d^{(m,n)} \geq (nm+m+n+c-2)/2$$

with equality when $c = 1$.

Proof. We choose, as usual, affine coordinates (x, y) so that L has equation $y = 0$ and $q = (0,0)$. Let $C \in U_d^{(m,n)}(L;q)$ and let its affine equation be $f(x,y) = 0$ with

$$f(x,y) = \sum_{0\leq i+j\leq d} a_{i,j}x^iy^j.$$

By hypothesis $a_{n,0}, a_{0,m} \neq 0$.

As $(C\cdot L)_q = n$ we have $f(x,0) = x^n p(x)$ with $p(0) \neq 0$, hence the smallest power of x appearing in f is x^n. This gives the following n conditions

$$a_{i,0} = 0 \quad \forall i = 0, \dots, n-1. \tag{21}$$

Next, as q is an m-fold point we have $a_{i,j} = 0$ for all $i + j \leq m - 1$; these are $m(m+1)/2$ conditions, but exactly m of them are also in (21). Hence the number of new independent conditions on f is equal to

$$m(m+1)/2 - m = m(m-1)/2. \tag{22}$$

Next, C is unibranch at q, and its tangent line there is $y = 0$. Therefore the only summand of degree m is $a_{0,m}y^m$ and we get the following new $m-1$ conditions:

$$a_{i,j} = 0, \quad \forall i + j = m, \quad i, j \geq 1. \tag{23}$$

Now consider the blow-up of $\mathbb{P}^2$ at q, let C' and L' be the strict transforms of C and L, and $q' \in C'$ the point lying over q. We set $y = vx$ and use (x, v) as affine coordinates in the blow-up. The equation of C' is

$$f'(x, v) = a_{0,m}v^m + \sum_{m+1 \leq i+j \leq d} a_{i,j}x^{i+j-m}v^j = a_{0,m}v^m + \sum_{\substack{1 \leq h \leq d-m \\ m+1 \leq l \leq d}} a_{h-l+m,l}x^h v^l$$

with $j = l$ and $i = h - l + m$. The conditions imposed earlier imply exactly that f' has no summand of type x^h for all $h = 0, \ldots, d - m - 1$, and no summand of type v^l for all $l = 1, \ldots, m - 1$.

Let m' be the multiplicity of $q' \in C'$; the fact that C' is unibranch at q' yields new conditions; we claim that the number of them is

$$m'(m' - 1)/2.$$

If $n < 2m$ we have $n - m = m'$ and q' is a (m', m)-point of C' with respect to exceptional divisor, by Lemma 2.1.2. More precisely, the term of smallest degree of f' is $a_{n,0}x^{m'}$, and C' has a unibranch m'-fold point at q'. Hence every coefficient of a term of degree less than m', and of a term of degree equal to m' other than $x^{m'}$, must vanish. These are $m'(m' + 1)/2 + m'$ independent conditions, but some of them are already satisfied. Indeed, we know that f' has no term of type x^l for all $l = 0, \ldots, m' - 1$ (m' conditions), and no term of type v^n for all $n = 1, \ldots, m'$ (m' conditions). There are no other cases, as if $k, l > 0$ the coefficient of $x^k v^l$ is $a_{k-l+m,l}$ corresponding to a summand in f of degree $k + m \geq m + 1$, and no conditions had been imposed on such coefficients. Hence the number of new conditions is

$$m'(m' + 1)/2 + m' - m' - m' = m'(m' - 1)/2. \tag{24}$$

Let $n > 2m$. Then $m = m'$ and C' has a an $(m, n - m)$-point at q' with respect to L', by Lemma 2.1.2. Hence in f' there are no monomials of degree less than m, and no monomials of degree m, except v^m. These are $m(m+1)/2+m$ conditions, some of which are already satisfied, indeed f' has no term of type x^l for all $l = 0, \ldots, m$ ($m + 1$ conditions), and no summand of type v^n for all $n = 1, \ldots, m - 1$ ($m - 1$ conditions). Hence the number of new conditions is

$$m(m + 1)/2 + m - (m + 1) - (m - 1) = m(m - 1)/2. \tag{25}$$

Let $n = 2m$. Then C' has a an m-fold point at q'. Hence in f' there are no monomials of degree less than m. Moreover, as we saw in the proof of Lemma 2.1.2, the homogeneous part of degree m of f' has the form $(\alpha x + \beta v)^m$ with $\alpha^m = a_{2m,0}$ and $\beta^m = a_{0,m}$. This imposes $m-1$ conditions. We get a total of $m(m + 1)/2 + m - 1$ conditions, but, as in the earlier cases, some of them already hold. In fact f' has no term of type x^l for all $l = 0, \dots, m-1$ (m conditions), and no summand of type v^n for all $n = 1, \dots, m-1$ ($m-1$ conditions). Hence the number of new conditions is

$$m(m+1)/2 + m - 1 - m - (m-1) = m(m-1)/2. \tag{26}$$

The claim is proved.

Now we need to apply the claim to all the curves appearing in the chain described in Proposition 2.1.4, of which we use the notation. Proposition 2.1.4 gives a sequence of curves whose multiplicity at the point lying over q is equal to l_j, with $j = 1, \dots, r$. By the claim, each of these gives $l_j(l_j-1)/2$ new conditions. If $j \geq 2$ we get exactly k_j such curves, hence a total of $k_j l_j(l_j-1)/2$ conditions. For $j = 1$ we have only $k_1 - 1$ such blow-ups, hence $(k_1 - 1)m(m-1)/2$ conditions. But in the first part of the proof we had $m(m-1)/2$ conditions; see (22). Hence also for $j = 1$ we have $k_1 m(m-1)/2$ conditions, hence

$$\sum_{j=1}^{r} k_j l_j(l_j-1)/2$$

conditions. Adding these to the ones computed in (21) and (23) we get

$$\operatorname{codim} U_d^{(m,n)} = n + (m-1) + \sum_{j=1}^{r} k_j l_j(l_j-1)/2.$$

By applying Lemma 5.1.2 we are done. ■

References

[Che04] Xi Chen. On algebraic hyperbolicity of log varieties. *Commun. Contemp. Math.*, 6(4):513–559, 2004.

[CRY23] Xi Chen, Eric Riedl, and Wern Yeong. Algebraic hyperbolicity of complements of generic hypersurfaces in projective spaces. ArXiv preprint: 2208.07401, 2023.

[CT24] Lucia Caporaso and Amos Turchet. Hypertangency of plane curves and the algebraic exceptional set. ArXiv preprint: 2403.09886, 2024.

[CZ13] Pietro Corvaja and Umberto Zannier. Algebraic hyperbolicity of ramified covers of $\mathbb{G}_m^2$ (and integral points on affine subsets of $\mathbb{P}_2$). *J. Differential Geom.*, 93(3):355–377, 2013.

[EC85] Federigo Enriques and Oscar Chisini. *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. 2. Vol. III, IV*, volume 5 of *Collana di Matematica [Mathematics Collection]*. Zanichelli Editore S.p.A., Bologna, 1985. Reprint of the 1924 and 1934 editions.

[GNSW23] Ji Guo, Khoa D. Nguyen, Chia-Liang Sun, and Julie Tzu-Yueh Wang. Vojta's abc conjecture for algebraic tori and applications over function fields. ArXiv preprint: 2106.15881, 2023.

[HW08] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[MZ08] Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their Jacobians and theta functions. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 203–230. Amer. Math. Soc., Providence, RI, 2008.

[PR07] Gianluca Pacienza and Erwan Rousseau. On the logarithmic Kobayashi conjecture. *J. Reine Angew. Math.*, 611:221–235, 2007.

[Wal04] C. T. C. Wall. *Singular points of plane curves*, volume 63 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

(Caporaso) Dipartimento di Matematica e Fisica, Università Roma Tre, Largo San Leonardo Murialdo, I-00146 Roma, Italy
Email address: lucia.caporaso@uniroma3.it

(Turchet) Dipartimento di Matematica e Fisica, Università Roma Tre, Largo San Leonardo Murialdo, I-00146 Roma, Italy
Email address: amos.turchet@uniroma3.it