

# TROPICAL CURVES OF UNIBRANCH POINTS AND HYPERTANGENCY

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ABSTRACT. We study plane curves meeting at a single unibranch point and show that they must satisfy two equivalent conditions. A geometric one: the tropical curves associated to the contact point are isomorphic. An arithmetic one: the local invariants at the contact point satisfy a simple relation. Furthermore, we prove closed formulas for the delta-invariant of a unibranch singularity, and for the dimension of the locus of curves with an assigned unibranch point. Our work is motivated by interest in the Lang's exceptional set.

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## 1. INTRODUCTION

**1.1. Results.** In this paper we study unibranch points of curves, i.e. points which have a single preimage in the normalization of the curve. Our first goal is to study under what conditions two projective plane curves of degree at least 2 intersect in exactly one point, and this point is unibranch for both of them. This problem is also relevant for the study of the Lang exceptional set appearing in some conjectures related to arithmetic and hyperbolic geometry, as we shall explain later in this introduction.

To a unibranch point  $q$  of a plane curve  $C \subset \mathbb{P}^2$  of degree at least 2 we associate a type made of a pair of integers  $(m, n)$ , where  $m = \text{mult}_q(C)$  is the multiplicity of  $C$  at  $q$ , and  $n$  is the multiplicity of the intersection of  $C$  with its tangent line  $L$  at  $q$ , i.e.  $(L \cdot C)_q = n$ ; we say that  $q$  is a  $(m, n)$ -point for  $C$ . Then, in Section 3, we associate a tropical curve, denoted by  $\Gamma^{(m,n)}$ ,

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to a unibranch point of type  $(m, n)$ . Our main result answering the above problem can be stated as follows.

**Theorem 1.1.1.** *Let  $B, C \subset \mathbb{P}^2$  be integral curves of degree at least 2 such that  $B \cap C = \{q\}$ ; assume that  $q$  is an  $(m', n')$ -point for  $B$  and an  $(m, n)$ -point for  $C$ . Then there is an isomorphism of tropical curves between  $\Gamma^{(m, n)}$  and  $\Gamma^{(m', n')}$ ; equivalently,  $m'/n' = m/n$ .*

See Theorem 3.2.4 and Theorem 4.1.1.

Tropical curves are metric graphs, i.e. graphs enriched by the assignment of a length (a positive real number) to each edge; see [MZ08]. In the classical literature non-metric graphs, such as Enriques diagrams, have been widely used to study singularities of plane curves; see [EC85] or [Wal04]. Even though such graphs do not quite fit our purpose, they inspired our construction of the tropical curve  $\Gamma^{(m, n)}$ ; see Subsection 3.1 for more details.

In the second part of Section 4 we investigate the opposite problem, describing the intersection of two curves at points of the same (or, more generally, equivalent) type.

Finally, our techniques enable us to obtain closed formulas for the delta-invariant of a unibranch singularity, in Proposition 5.1.1, and for the dimension of the space of curves with an assigned unibranch point, in Proposition 5.2.1. The proof of these formulas occupy the last section of this article.

**1.2. Context.** Our work in this paper is motivated by the algebro-geometric counterparts of some outstanding conjectures from diophantine and hyperbolic geometry. To introduce them, let  $B \subset \mathbb{P}^2$  be a complex projective curve of degree at least 3, having at most nodal singularities. The so-called algebraic exceptional set of  $B$ , introduced by Lang, is denoted as follows

$$(1) \quad \mathcal{E}(B) = \{C \subset \mathbb{P}^2 : C \text{ rational, } |\nu_C^{-1}(C \cap B)| \leq 2\}$$

where  $\nu_C : C^\nu \rightarrow C$  is the normalization, so that  $C^\nu \cong \mathbb{P}^1$  if  $C$  is rational. One of the main ideas behind the circle of conjectures of Lang, Vojta, Demailly and Campana, is that  $\mathcal{E}(B)$  should contain all the infinite families of integral points in the quasi-projective surface  $\mathbb{P}^2 \setminus B$ , and all the images of nonconstant holomorphic maps  $\mathbb{C} \rightarrow \mathbb{P}^2 \setminus B$ .

Conjecturally,  $\mathcal{E}(B)$  is finite as soon as  $\deg B \geq 4$ , which corresponds to the pairs  $(\mathbb{P}^2, B)$  being of log general type. More generally, Lang conjectured that the fact that  $\mathcal{E}(B)$  is a proper closed subset of  $\mathbb{P}^2$  should be equivalent to the fact that  $(\mathbb{P}^2, B)$  is of log general type (and this should hold more in general for projective varieties  $X$  with a normal crossing divisor  $B$ ). For  $\mathbb{P}^2$  this conjecture is known only for  $B$  reducible with at least three components; the finiteness follows from [CZ13, GNSW25], and we refer to [CT25] for further references and explicit bounds. It remains open when  $B$  is irreducible or has only two irreducible components, although [Che04, PR07, CRY23, ATY24] show that it holds for a very general  $B$ .

If  $C \in \mathcal{E}(B)$  and  $C \cap B = \{q, q'\}$ , then it is clear that  $q$  and  $q'$  must be unibranch points of  $C$ . Therefore one is naturally led to investigate unibranch points, and curves having high order of contact at such points. In particular when  $B$  has (at least) three components, any  $C \in \mathcal{E}(B)$  will intersect at least one of the components of  $B$  in a single unibranch point. This is precisely the setting of Theorem 1.1.1.

As an example of application of this type of results we mention that, in our previous paper [CT25], we used as a crucial ingredient a special case of Theorem 1.1.1 (Theorem 2.2.1 in loc.cit. which assumes  $B$  is smooth at  $q$ ) to show that  $\mathcal{E}(B)$  is empty for a general  $B$  with three irreducible components of total degree at least 5.

**1.3. Notation.** We work over  $\mathbb{C}$ . Throughout the paper,  $S$  will be a smooth projective surface endowed with a birational morphism to  $\mathbb{P}^2$ , and  $C \subset S$  an integral projective curve.

Given a point  $p \in C$ , we write  $\text{mult}_p(C)$  for the multiplicity of  $C$  at  $p$ ; if  $m = \text{mult}_p(C)$  we say that  $p$  is an  $m$ -fold point. We denote by  $\nu_C : C^\nu \rightarrow C$  the normalization. If  $|\nu_C^{-1}(p)| = 1$  we say that  $p$  is *unibranch*.

If  $C, B \subset S$  are reduced curves with no components in common, and  $p \in C \cap B$ , we write  $(C \cdot B)_p$  for their multiplicity of intersection at  $p$ .

If  $(C \cdot B)_p = \text{mult}_p(C)\text{mult}_p(B)$  we say that  $B$  and  $C$  are *transverse* at  $p$ .

If  $|\nu_C^{-1}(B \cap C)| = 1$ , we say that  $C$  is *hypertangent* to  $B$ .

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## 2. UNIBRANCH POINTS

This section is devoted to foundational concepts on unibranch points of curves on surfaces.

### 2.1. Resolving unibranch points.

**Definition 2.1.1.** Let  $q \in C \subset S$  be a unibranch point; set  $m = \text{mult}_q(C)$ . Let  $F \subset S$  be a smooth integral curve,  $F \neq C$ ; set  $n = (C \cdot F)_q$ . If  $n > m$  we say that  $q$  is an  $(m, n)$ -point of  $C$  with respect to  $F$ .

For brevity we write: “ $q \in C$  is an  $(m, n)$ -point w.r.t.  $F$ ”.

If  $S = \mathbb{P}^2$  and  $F$  is the tangent line to  $C$  at  $q$ , we simply say that  $q$  is an  $(m, n)$ -point of  $C$ .

For example, a  $(1, 2)$ -point of a curve  $C \subset \mathbb{P}^2$  is a smooth point which is not a flex.

By definition, if  $q \in C$  is an  $(m, n)$ -point with respect to  $F$ , then  $F$  and  $C$  are not transverse at  $q$ .

Our surface  $S$  is an iterated blow-up of  $\mathbb{P}^2$ , hence every point  $q \in S$  has an open neighborhood isomorphic to  $\mathbb{A}^2$ ; we will always choose coordinates in such a neighborhood so that  $q$  is the origin. We say that a curve  $F \subset S$  such that  $q \in F$  is a *line locally at  $q$*  if there is an open neighborhood  $U \cong \mathbb{A}^2$  of  $q$  such that the equation of  $F \cap U$  has degree 1. In particular, if  $E \subset S$  is an exceptional divisor, then  $E$  is a line locally at any of its points.

For a unibranch point  $q \in C \subset S$  we introduce an iterated sequence of blow-ups. Set  $S_1 = S$ , let  $S_2 \rightarrow S_1$  be the blow-up at  $q$ , and  $E_2 \subset S_2$  the exceptional divisor. We denote by  $C^2 \subset S_2$  the proper transform of  $C$  and by  $q^2 \in C^2$  the point lying over  $q$ , i.e.  $q^2 = C^2 \cap E_2$ . We use superscripts for strict transforms, and subscripts otherwise, but we denote  $C = C^1$  and  $q = q^1$  for convenience. We iterate to get a sequence of arbitrary length

$$\dots C^{h+1} \longrightarrow C^h \longrightarrow C^{h-1} \longrightarrow \dots \longrightarrow C^1 = C$$

where  $C^h \subset S_h$  is the proper transform of  $C^{h-1}$  under the blow-up of  $S_{h-1}$  at  $q^{h-1} \in C^{h-1}$ , and  $q^h = C^h \cap E_h$ , where  $E_h$  is the exceptional divisor of  $S_h \rightarrow S_{h-1}$ . If  $F \subset S$  is a curve, we write  $F^h \subset S_h$  for its proper transform.

We need the following

**Lemma 2.1.2.** *Let  $q \in C$  be an  $(m, n)$ -point w.r.t.  $L$ , where  $L$  is a line locally at  $q$ ; set  $k = \lfloor n/m \rfloor$ . Then  $q^h \in C^h$  is an  $(m, n - (h-1)m)$ -point w.r.t.  $L^h$  for every  $h \leq k-1$ , and  $q^k$  is an  $m$ -fold point. Moreover*

- (a) *if  $n/m \neq k$  then  $q^k \in C^k$  is an  $(m, n - (k-1)m)$ -point w.r.t.  $L^k$ , and  $q^{k+1} \in C^{k+1}$  is an  $(n - km, m)$ -point w.r.t.  $E_{k+1}$ ;*
- (b) *if  $n/m = k$  then  $C^k$  is transverse to both  $E_k$  and  $L^k$ .*

*Proof.* We proceed as in the proof of [CT25, Lemma 2.1.1], of which the present lemma is a generalization. We write, as above,  $q^1 = q \in C^1 = C \subset S^1 = S$ . We use local coordinates at  $q^1 \in U \cong \mathbb{A}_{x_1, y_1}^2$  so that  $q^1 = (0, 0)$  and  $L = L^1$  has equation  $y_1 = 0$ . As  $q^1 \in C^1$  is an  $(m, n)$ -point w.r.t.  $L^1$ , the defining polynomial,  $f_1(x_1, y_1)$ , of  $C^1$  is

$$f_1(x_1, y_1) = a_{0,m} y_1^m + \sum_{m+1 \leq i+j \leq d} a_{i,j} x_1^i y_1^j$$

with

$$(2) \quad a_{0,m} \neq 0, \quad a_{n,0} \neq 0, \quad a_{i,0} = 0 \quad \forall i < n.$$

We blow-up at  $q^1$  and use local coordinates  $(x_2, y_2)$  at  $q^2 = (0, 0)$ , with  $x_1 = x_2$  and  $y_1 = y_2 x_2$ . The local equation of the exceptional divisor  $E_2$  is  $x_2 = 0$ , and the local equation of  $L^2$  is  $y_2 = 0$ . Let  $f_2(x_2, y_2) = 0$  be the

local equation of  $C^2$ , so that

$$f_2(x_2, y_2) = a_{0,m}y_2^m + \sum_{m+1 \leq i+j \leq d} a_{i,j}x_2^{i+j-m}y_2^j.$$

We iterate: for every  $h \leq k$  we use local coordinates  $(x_h, y_h)$  at  $q^h = (0, 0)$ , with  $x_{h-1} = x_h$  and  $y_{h-1} = y_h x_h$ , so that the local equation of  $E_h$  is  $x_h = 0$  and the local equation of  $L^h$  is  $y_h = 0$ . The defining polynomial of  $C^h$  is

$$f_h(x_h, y_h) = a_{0,m}y_h^m + \sum_{m+1 \leq i+j \leq d} a_{i,j}x_h^{i+j-(h-1)m}y_h^j.$$

By (2), the smallest power of  $y_h$  appearing as a summand above is  $y_h^m$ , and the smallest power of  $x_h$  is  $x_h^{n-(h-1)m}$ . We have

$$(3) \quad n - (h-1)m = n - hm + m \geq n - km + m \geq n - n + m = m$$

using  $h \leq k$  in the first inequality and  $k \leq n/m$  in the second.

If  $h \leq k-1$ , the first inequality is strict, hence  $n - (h-1)m > m$ . Therefore  $f_h(x_h, y_h)$  contains the summand  $y_h^m$  but not the summand  $x_h^m$ . Hence, as  $q^h \in C^h$  is unibranch, all terms of degree at most  $m$  divisible by  $x_h y_h$  must vanish. Hence  $\text{mult}_{q^h}(C^h) = m$  and, as  $y_h = 0$  is the local equation of  $L^h$ , we get  $(L^h \cdot C^h)_{q^h} = n - (h-1)m$ , hence  $q^h \in C^h$  is an  $(m, n - (h-1)m)$ -point w.r.t.  $L^h$ , as claimed.

Let  $h = k$ . If  $n/m \neq k$  then  $k < n/m$  and the second inequality of (3) is strict. Hence, by the above argument,  $q^k \in C^k$  is an  $(m, n - (k-1)m)$ -point w.r.t.  $L^k$ . Now look at  $q^{k+1} \in C^{k+1}$ , we have

$$f_{k+1}(x_{k+1}, y_{k+1}) = a_{0,m}y_{k+1}^m + \sum_{m+1 \leq i+j \leq d} a_{i,j}x_{k+1}^{i+j-km}y_{k+1}^j.$$

As before,  $y_{k+1}^m$  is the smallest power of  $y_{k+1}$  and  $x_{k+1}^{n-km}$  the smallest power of  $x_{k+1}$ , but now  $n - km < m$ . Since  $q^{k+1}$  is unibranch, by the same argument,  $\text{mult}_{q^{k+1}}(C^{k+1}) = n - km$  and, as  $x_{k+1} = 0$  is the local equation of  $E_{k+1}$ , we conclude that  $q^{k+1} \in C^{k+1}$  is an  $(n - km, m)$ -point w.r.t.  $E_{k+1}$ .

Assume  $n/m = k$ . Then in (3) we have equality. Therefore we have both  $x_k^{n-km} = x_k^m$  and  $y_k^m$  appearing in  $f_k$ . Since  $q^k$  is unibranch for  $C^k$ , this is possible only if the homogeneous part,  $(f_k)_m$ , of degree  $m$  of  $f_k$ , has form  $(f_k)_m = (\alpha y_k + \beta x_k)^m$  with  $\alpha, \beta \neq 0$ . Hence  $\text{mult}_{q^k}(C^k) = m$ , and  $C^k$  is transverse to both  $E_k$  and  $L^k$  at  $q^k$ .  $\blacksquare$

**Remark 2.1.3.** If  $k = 1$  the first part of the statement is vacuous and the only interesting part is (a), which states that  $q^2 \in C^2$  is an  $(n - m, m)$ -point with respect to the exceptional divisor  $E_2$

If  $q \in C \subset \mathbb{P}^2$  is a singular unibranch point, hence an  $(m, n)$ -point, one can resolve the singularity by a finite sequence of blow-ups of  $\mathbb{P}^2$ , each centered at a point lying over  $q$ , such as the one described before Lemma 2.1.2. This process leads to consider the euclidean algorithm for the pair  $(m, n)$ . This

connection is known; see [Wal04], for example. Our setting here is different, as we need to extend our analysis to smooth points of curves and to tangent lines.

To a pair of integers,  $(m, n)$ , with  $n > m > 0$ , the euclidean algorithm associates a chain of integers determining their greatest common divisor,  $c := \gcd(m, n)$ , as follows

$$(4) \quad l_0 = n > l_1 = m > l_2 > \dots > l_{r-1} > l_r = c > l_{r+1} = 0$$

where  $l_j$  is defined inductively for  $2 \leq j \leq r+1$

$$l_j := l_{j-2} - l_{j-1} \lfloor l_{j-2}/l_{j-1} \rfloor.$$

We refer to (4) as the *euclidean sequence* of the pair  $(m, n)$ . We set

$$k_j = \lfloor l_{j-1}/l_j \rfloor, \quad j = 1, \dots, r.$$

As we shall see in Proposition 2.1.4, in the resolution sequence resolving the  $(m, n)$ -point  $q$ , the first multiplicities that appear over  $q$  are  $l_1, l_2, \dots, l_r$ , in this order. Moreover, each multiplicity  $l_j$  appears exactly  $k_j$  consecutive times.

We want to use this to give a more convenient notation for the resolution sequence. We define a total order on the following set of pairs associated to the euclidean sequence (4)

$$\Pi_{(m,n)} := \{(j, i), \quad \forall j = 1, \dots, r, \quad i = 1, \dots, k_j\}.$$

We will use the index  $i$  even though its range depends on  $j$  to ease the notation. We order  $\Pi_{(m,n)}$  lexicographically as follows

$$(j', i') \geq (j, i) \quad \text{if } j' > j \text{ or } j' = j, i' \geq i.$$

If  $(j', i')$  is the minimum in  $\Pi_{(m,n)}$  such that  $(j', i') > (j, i)$ , we say that  $(j', i')$  is *next* to  $(j, i)$ , and write  $(j', i') = \text{next}(j, i)$ . Explicitly:

$$\text{next}(j, i) = \begin{cases} (j, i+1) & \text{if } i < k_j \\ (j+1, 1) & \text{if } i = k_j. \end{cases}$$

We set  $S_1^1 := \mathbb{P}^2$ ,  $C_1^1 := C$ ,  $q_1^1 := q$ . Now,  $q_1^1$  is the center of the first blow-up  $\sigma_j^i : S_j^i \rightarrow S_1^1$  with  $(j, i) = \text{next}(1, 1)$ . We denote by  $C_j^i \subset S_j^i$  the proper transform of  $C$ , and by  $q_j^i \in C_j^i$  the point lying over  $q$ . We generalize to all  $(j, i) > (1, 1)$ . We denote

$$q_j^i \in C_j^i \subset S_j^i \xrightarrow{\sigma_j^i} S_{j'}^{i'} \longrightarrow \mathbb{P}^2$$

so that  $(j, i) = \text{next}(j', i')$ , the proper transform of  $C$  is  $C_{j'}^{i'}$ , and  $q_{j'}^{i'}$  is the only point of  $C_{j'}^{i'}$  lying over  $q$ . The map  $\sigma_j^i$  is the blow up at  $q_{j'}^{i'}$ , and we denote by  $E_j^i \subset S_j^i$  its exceptional divisor. For convenience, we denote  $L = E_1^1 \subset S_1^1 = \mathbb{P}^2$ . The divisors  $E_j^1$  and their proper transform play a special role, hence for every  $j = 1, \dots, r$  we denote by

$$F_j^i \subset S_j^i$$

the proper transform of  $E_j^1$ , so that  $F_j^1 = E_j^1$ . We abuse notation and denote by  $\sigma_j^i : C_j^i \rightarrow C_j^{i'}$  the restriction of  $\sigma_j^i$  to  $C_j^i$ .

**Proposition 2.1.4.** *Let  $q \in C \subset \mathbb{P}^2$  be an  $(m, n)$ -point, (4) the euclidean sequence of  $(m, n)$ , and  $\nu_q : C_q^\nu \rightarrow C$  the desingularization of  $C$  at  $q$ . We have a chain  $\beta$  of birational morphisms*

$$\begin{aligned} \beta : C_r^{k_r} \xrightarrow{\sigma_r^{k_r}} \dots \rightarrow C_r^1 \xrightarrow{\sigma_r^1} C_{r-1}^{k_{r-1}} \rightarrow \dots \\ \dots \rightarrow C_j^{k_j} \xrightarrow{\sigma_j^{k_j}} \dots \rightarrow C_j^i \xrightarrow{\sigma_j^i} C_j^{i-1} \rightarrow \dots \rightarrow C_j^1 \xrightarrow{\sigma_j^1} C_{j-1}^{k_{j-1}} \rightarrow \dots \rightarrow C_1^1 = C \end{aligned}$$

such that the following holds.

- (a) For every  $j = 1, \dots, r$  and  $i = 1, \dots, k_j$  with  $(j, i) \neq (r, k_r)$  the point  $q_j^i \in C_j^i$  is an  $(l_j, l_{j-1} - (i-1)l_j)$ -point w.r.t.  $F_j^i$ .
- (b)  $C_r^{k_r}$  has a  $c$ -fold point in  $q_r^{k_r}$  where it is transverse to  $E_r^{k_r}$  and to  $F_r^{k_r}$ .
- (c) If  $c = 1$  then  $C_q^\nu = C_r^1$ , and  $q_r^1$  is a  $(1, l_{r-1})$ -point w.r.t.  $F_r^1$ .
- (d) If  $c > 1$  then  $\nu_q$  factors as follows  $\nu_q : C_q^\nu \xrightarrow{\beta} C_r^{k_r} \xrightarrow{\beta} C$ .

*Proof.* The chain was defined before the statement. Of course,  $q$  is an  $(m, n)$ -point w.r.t.  $L = E_1^1$ . If  $n/m \in \mathbb{N}$  then  $r = 1$  and  $c = m$ , and the statement follows from Lemma 2.1.2, which is indeed a special case of this Proposition.

Assume  $n/m \notin \mathbb{N}$ , i.e.  $k_1 \neq n/m$ . By Lemma 2.1.2, for every  $i = 1, \dots, k_1$  the point  $q_1^i \in C_1^i$  is an  $(m, n - (i-1)m)$ -point w.r.t. the proper transform,  $F_1^i$ , of  $E_1^1$ ; moreover,  $q_2^1 \in C_2^1$  is an  $(l_2, l_1)$ -point with respect to  $E_2^1$ . Now we continue inductively on  $j$ ; assume the statement for the level  $j-1$  and consider the  $j$ -th level of the chain, with  $2 \leq j \leq r-1$ . Then  $q_j^1 \in C_j^1$  is an  $(l_j, l_{j-1})$ -point w.r.t.  $E_j^1 = F_j^1$ , and  $E_j^1$  is a line locally at  $q_j^1$ . By the same lemma, for every  $i = 1, \dots, k_j$  the point  $q_j^i \in C_j^i$  is an  $(l_j, l_{j-1} - (i-1)l_j)$ -point with respect to the proper transform,  $F_j^i$ , of  $E_j^1$ .

Consider the last level,  $j = r$ . Now  $l_r = c$  and  $l_{r-1}/l_r = k_r$ . By Lemma 2.1.2, if  $i = 1, \dots, k_r - 1$  then  $q_r^i \in C_r^i$  is an  $(c, l_{r-1} - (i-1)c)$ -point w.r.t. the proper transform,  $F_r^i$ , of  $E_r^1$ . Moreover, by part (b) of the same lemma,  $C_r^{k_r}$  has a  $c$ -fold point in  $q_r^{k_r}$ , and it is transverse to  $F_r^{k_r}$  and to the exceptional divisor. The rest of the statement is clear.  $\blacksquare$

We highlight the following

**Remark 2.1.5.** For every  $j \geq 1$  the curve  $C_j^1$  is tangent to  $E_j^1$ , whereas  $C_j^i$  meets  $E_j^i$  transversally for all  $i \neq 1$ .

**2.2. Normal crossings resolutions.** Let  $q \in C \subset \mathbb{P}^2$  be an  $(m, n)$ -point and consider the sequence of maps in Proposition 2.1.4. We now add the blow-up of the last surface,  $S_r^{k_r}$ , at  $q_r^{k_r}$ , written

$$\sigma_* : S_* \longrightarrow S_r^{k_r}.$$

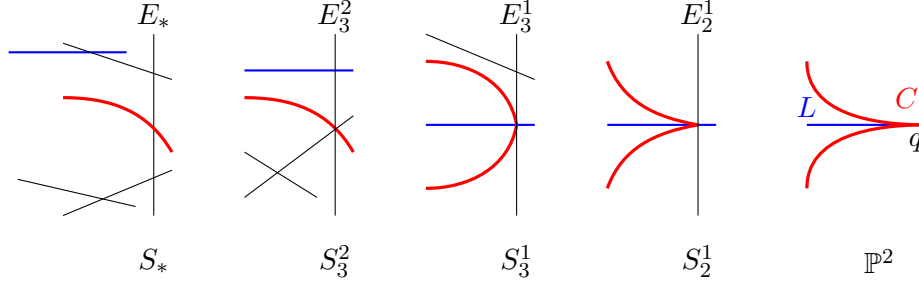
Its exceptional divisor will be written  $E_*$ ; let  $C_* \subset S_*$  be the proper transform of  $C$  and  $q_* \in C_*$  the point lying over  $q$ . We set  $\beta_* = \beta \circ \sigma_*$  so that we have, abusing notation,

$$\beta_* : C_* \xrightarrow{\sigma_*} C_r^{k_r} \longrightarrow C_r^{k_r-1} \longrightarrow \dots \longrightarrow C_r^1 \longrightarrow \dots \longrightarrow C.$$

The subscript “ $*$ ” indicates that, if  $c = \gcd(m, n) > 1$ , the type of singularity of  $q_*$  is not known a priori (we know it is a  $(c, h)$ -point for some  $h$ ) in which case, by part (d) of Proposition 2.1.4,  $\beta_*$  is only a partial resolution of  $q$ .

**Example 2.2.1.** Consider a curve  $C \subset \mathbb{P}^2$  with a  $(3, 5)$  point,  $q$ , with respect to the line  $L$ . The euclidean sequence is  $5 > 3 > 2 > 1$  hence  $r = 3$ , and  $k_1 = k_2 = 1$ , and  $k_3 = 2$ . The picture below represents the preimage of  $C \cup L$  under four consecutive blow-ups. The first three blow-ups are those described in Proposition 2.1.4, the last, introduced above, is such that the resulting curve has normal crossings singularities. The first blow-up gives a  $(2, 3)$ -point whose tangent is the proper transform of  $L$ . The second blow-up gives a  $(1, 2)$ -point whose tangent is the exceptional divisor.

For better clarity, the curve  $C$  and its proper transforms are curvilinear and red. The line  $L$  and its proper transforms are horizontal and blue. The exceptional divisor of each blow-up is a vertical segment, and its proper transforms are segments (neither vertical nor horizontal).



This example is a Fibonacci point, described in general in example 3.1.3.

We write

$$(5) \quad D_* = D_*(q) := (\beta_*^{-1}(C \cup L))_{\text{red}} \subset S_*$$

for the reduced subscheme/divisor supporting  $\beta_*^{-1}(C \cup L)$ . In the above picture,  $D_*$  is the first from the left.

We write  $A_j^i \subset S_*$  for the proper transform of  $E_j^i$  in  $S_*$ , so that  $A_1^1$  is the proper transform of  $L = E_1^1$ . Then

$$(6) \quad D_* = C_* + E_* + \sum_{\substack{j=1, \dots, r \\ i=1, \dots, k_j}} A_j^i.$$

By construction, two components of  $D_*$  intersect in at most one point, and no three of them intersect in the same point.

**Lemma 2.2.2.** *Notation as above. Then*

- (a)  $C_* \cap A_j^i = \emptyset$  for every  $(j, i)$ .  
 (b)  $E_* \cap A_j^i \neq \emptyset$  if and only if  $(j, i) \in \{(r, k_r), (r, 1)\}$ .  
 (c) For every  $(j, i) \notin \{(r, k_r), (r, 1)\}$  there is a unique  $(j', i') > (j, i)$  such that  $A_j^i \cap A_{j'}^{i'} \neq \emptyset$ , moreover

$$(7) \quad (j', i') = \begin{cases} \text{next}(j, i) & \text{if } i \neq 1 \\ \text{next}(j+1, 1) & \text{if } i = 1. \end{cases}$$

- (d) The divisor  $D_* - C_*$  is normal crossings. If  $c = 1$  and  $C$  is normal crossings away from  $q$ , then the divisor  $D_*$  is normal crossings.

*Proof.* By Proposition 2.1.4, the curve  $C_r^{k_r}$  has a unibranch  $c$ -fold point in  $q_r^{k_r}$ , where it meets transversally both the proper transform of  $E_r^1$  and the exceptional divisor,  $E_r^{k_r}$ . Hence the exceptional divisor,  $E_*$ , of the blow-up in  $q_r^{k_r}$ , intersects  $C_*$  and the proper transforms of  $E_r^1$  and of  $E_r^{k_r}$  (i.e.  $A_r^1$  and  $A_r^{k_r}$ ) in three different points. Hence the first two statements follow,

To prove (c), let  $(j', i') > (j, i)$ ; notice that  $A_j^i \cap A_{j'}^{i'} \neq \emptyset$  if and only if for some  $S = S_{j''}^{i''}$  with  $(j'', i'') \geq (j', i')$  the proper transforms of  $E_j^i$  and  $E_{j'}^{i'}$  in  $S$  intersect away from the proper transform of  $C$ . Suppose  $i \neq 1$ ; Proposition 2.1.4 implies that  $E_j^i$  is transverse to  $C_j^i$ , hence in the next blow-up, the proper transform of  $E_j^i$  intersects the exceptional divisor away from the proper transform of  $C$ . Suppose  $i = 1$ ; then  $E_j^1$  is tangent to  $C_j^1$ , and its proper transform is tangent to  $C_j^i$  for every  $i \leq k_j$ . In  $S_{j+1}^1$  the curve  $C_{j+1}^1$  is tangent to  $E_{j+1}^1$ , hence the proper transform of  $E_j^1$  intersects  $E_{j+1}^1$  transversally in  $q_{j+1}^1$ . Therefore in the next blow-up, the proper transform of  $E_j^1$  intersects the exceptional divisor away from the proper transform of  $C$ . (c) is proved.

All components of  $D_* - C_*$  are smooth, intersect transversally, and only pairwise. Hence  $D_* - C_*$  is normal crossings. If  $c = 1$ , then  $q_r^{k_r}$  is a smooth point of  $C_r^{k_r}$ , hence  $D_*$  is normal crossings if so is  $C$  away from  $q$ . This proves (d) and concludes the proof.  $\blacksquare$

### 3. TROPICAL CURVES OF UNIBRANCH POINTS

In this section we introduce the ‘‘contact’’ tropical curve associated to a unibranch point of a curve.

**3.1. Dual graphs of partial resolutions.** Let  $C \subset \mathbb{P}^2$  have a unique (for simplicity) singular point in  $q$ . It is well known that there exists a unique surface  $S_\rho$  with a birational morphism  $\rho : S_\rho \rightarrow \mathbb{P}^2$  such that the strict transform of  $C$  in  $S_\rho$  is smooth, the reduced scheme underlying  $\rho^{-1}(C)$  is a divisor with normal crossings, written  $D_\rho$ , and  $S_\rho$  is minimal with respect to these properties. The dual graph,  $G_\rho$ , of  $D_\rho$  (whose vertices are the irreducible components of  $D_\rho$ , with an edge between two vertices if the two components intersect) is often called the resolution graph of  $q$ ; [Wal04]. The

graph  $G_\rho$  is not suitable here, and we will define a variant of it which will lead, in turn, to a tropical curve.

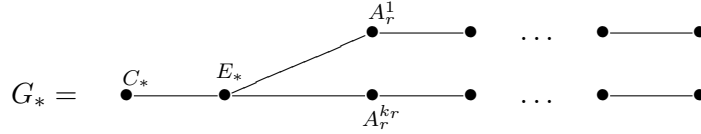
We consider the dual graph,

$$G_* = G(D_*(q))$$

of the divisor  $D_*$ , defined in Section 2.2. The vertices of  $G_*$  are the irreducible components of  $D_*$ , hence  $G_*$  has  $2 + \sum_{i=1}^r k_i$  vertices, and there is an edge between two vertices if the two components intersect. We shall abuse notation and denote the vertices and the components they represent by the same symbol. For the next lemma, recall that a *leaf* of a graph is a vertex of degree 1 (and the degree, or valency, of a vertex is the number of half-edges adjacent to it).

**Lemma 3.1.1.** *Let  $q \in C$  be an  $(m, n)$ -point.*

(a) *The graph  $G(D_*(q))$  is a tree with a unique vertex,  $E_*$ , of degree 3 and three leaves among which  $C_*$  and  $A_1^1$ , and has the following form*



(b)  *$G(D_*(q))$  only depends on the pair  $(m, n)$ , i.e. if  $q' \in C' \subset S'$  is an  $(m, n)$ -point, then  $G(D_*(q))$  is isomorphic to  $G(D_*(q'))$  via an isomorphism that takes the vertex  $C_*$  to the vertex  $C'_*$ .*

*Proof.* Write  $G_* = G(D_*(q))$ . (The fact that  $G_*$  is a tree should be clear from its definition, but we give it a proof below.)

Lemma 2.2.2 implies that  $E_*$  is the only vertex of degree 3, and it is joined by an edge to  $C_*$ ,  $A_r^{k_r}$  and  $A_r^1$ , as shown in the picture. The same lemma implies that  $C_*$  and  $A_1^1$  are leaves of  $G_*$  (and that the third leaf is  $A_2^1$  if  $k_1 = 1$ , and  $A_1^2$  if  $k_1 > 1$ ).

Let  $\nu_i$  be the number of vertices of degree  $i$  in  $G_*$ . Then  $\nu_3 = 1$  and  $\nu_1 \geq 2$ . The number of edges of  $G_*$  is thus

$$|E(G_*)| = (\nu_1 + 2\nu_2 + 3\nu_3)/2 = (\nu_1 + 2\nu_2 + 3)/2$$

hence  $\nu_1$  must be odd, hence  $\nu_1 \geq 3$ . Since  $G_*$  is connected, its genus  $g(G_*) = b_1(G_*)$  satisfies  $g(G_*) = |E(G_*)| - |V(G_*)| + 1$ . Moreover, since  $|V(G_*)| = \nu_1 + \nu_2 + 1$ , we have

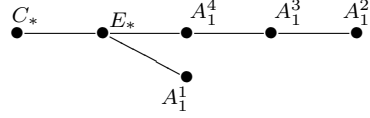
$$g(G_*) = (\nu_1 + 2\nu_2 + 3)/2 - (\nu_1 + \nu_2 + 1) + 1 = (3 - \nu_1)/2 \leq 0$$

with equality only if  $\nu_1 = 3$ . Since  $g(G_*) \geq 0$  we get  $\nu_1 = 3$  and  $g(G_*) = 0$ . (a) is proved. Part (b) is clear.  $\blacksquare$

As  $G(D_*(q))$  depends only on the pair  $(m, n)$ , we set  $G_*^{(m, n)} := G(D_*(q))$  for any  $(m, n)$ -point  $q$  on any curve  $C$ . We will show later (see Lemma 3.1.5) that it depends only on the euclidean sequence of  $(m, n)$ .

Different pairs can give the same graph, as in the next example.

**Example 3.1.2.** Let  $q$  be a smooth point of type  $(1, 4)$ , hence  $r = 1$  and  $k_1 = 4$ . The graph  $G(D_*(q))$  is as follows



Let now  $q'$  be a  $(2, 8)$ -point, then  $q'$  is singular,  $r = 1$  and  $k_1 = 4$ , and the graph  $G(D_*(q'))$  is the same as above.

**Example 3.1.3. Fibonacci singularities.** A *Fibonacci point* is an  $(m, n)$ -point such that  $(m, n)$  is a pair of consecutive integers in the Fibonacci sequence  $\{f_n\}$  defined by  $f_1 = 1, f_2 = 2$ , and  $f_n = f_{n-1} + f_{n-2}$ :

$$1, 2, 3, 5, 8, 13, 21, \dots$$

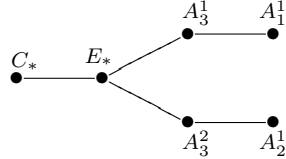
The “blow-up” of a Fibonacci point of type  $(f_n, f_{n-1})$  is a Fibonacci point of type  $(f_{n-1}, f_{n-2})$ . The euclidean sequence for a Fibonacci  $(m, n)$ -point is equal to the reversed Fibonacci sequence truncated at  $n$ , for example for  $(m, n) = (13, 21)$  we get

$$l_0 = 21 > l_1 = 13 > l_2 = 8 > l_3 = 5 > l_4 = 3 > l_5 = 2 > l_6 = 1.$$

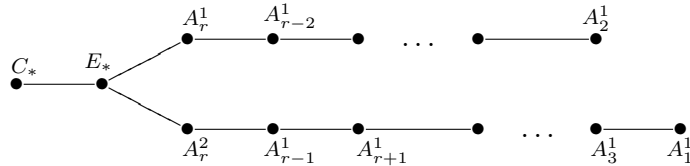
An  $(m, n)$ -point is a Fibonacci point if and only if  $k_r = 2$  and  $k_j = 1$  for all  $j = 1, \dots, r - 1$ . For such a point the chain of blow-ups in 2.1.4 is

$$C_* \longrightarrow C_r^2 \longrightarrow C_r^1 \longrightarrow C_{r-1}^1 \longrightarrow \dots \longrightarrow C_2^1 \longrightarrow C.$$

The graph  $G_*$  for Example 2.2.1 is the following



If  $r$  is even,  $G_*$  is as follows



We now study under what conditions different pairs give the same graph.

**Definition 3.1.4.** Let  $(m, n)$  and  $(m', n')$  be two pairs of integers such that  $n > m > 0$  and  $n' > m' > 0$ . The euclidean sequence for  $(m, n)$  is given in (4); let the euclidean sequence of  $(m', n')$  be

$$(8) \quad n' = l'_0 > m' = l'_1 > \dots l'_r = c' > l'_{r+1} = 0$$

where  $c' = \gcd(m', n')$ . Set  $k'_j = \lfloor l_{j-1}/l_j \rfloor$  for  $j = 1, \dots, r'$ . We say that the two pairs are *equivalent*, and write

$$(m, n) \sim (m', n'),$$

if  $r = r'$  and  $k_j = k'_j$  for every  $j = 1, \dots, r$ .

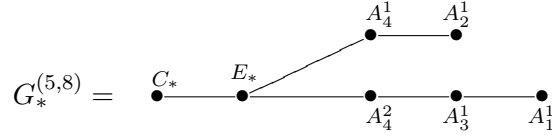
If  $q \in C \subset \mathbb{P}^2$  is an  $(m, n)$ -point and  $q' \in C' \subset \mathbb{P}^2$  is an  $(m', n')$ -point we say that  $q$  and  $q'$  are equivalent if  $(m, n) \sim (m', n')$ .

**Lemma 3.1.5.** *If  $(m, n) \sim (m', n')$  then  $G_*^{(m,n)} \cong G_*^{(m',n')}$ .*

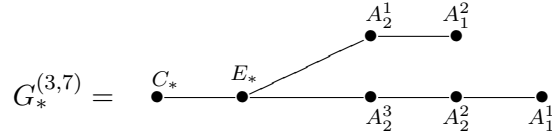
*Proof.* By Proposition 2.1.4, for any  $(m, n)$ -point  $q$ , the divisor  $D_*(q)$  is completely determined by the integers  $r$  and  $k_1, \dots, k_r$  of the euclidean sequence of  $(m, n)$ , hence  $G_*^{(m,n)}$  depends only on the equivalence class of  $(m, n)$ . ■

The converse fails, as the following example shows.

**Example 3.1.6.** If  $(m, n) = (5, 8)$ , then  $q$  is a Fibonacci point with  $r = 4$  and the graph is as follows



Now let  $(m', n') = (3, 7)$ , then  $r' = 2$  hence  $(m, n) \not\sim (m', n')$ . The graph below is clearly isomorphic to  $G_*^{(5,8)}$



It is well known that the condition  $(m, n) \sim (m', n')$  can be expressed more directly as follows.

**Remark 3.1.7.**  $(m, n) \sim (m', n')$  if and only if  $m/n = m'/n'$ .

See [HW08, Subsection 10.6 and Theorem 160], for example.

**3.2. The contact tropical curve of an  $(m, n)$ -point.** To improve upon Lemma 3.1.5 and give a geometric characterization for equivalence of pairs, we now introduce a tropical curve associated to an  $(m, n)$ -point. We first put on the graph  $G_* = G(D_*(q))$  a length on its edges,  $\ell_* : E(G_*) \rightarrow \mathbb{R}_+$ , such that  $\ell_*(e) = 1$  for every edge  $e$  of  $G_*$ . This defines a tropical curve  $\Gamma_* := (G_*, \ell_*)$ . We shall now define the *contact tropical curve* of the  $(m, n)$ -point  $q$ , written  $\Gamma_q$ . The word “contact” is motivated by Theorem 4.1.1. As for notation, set

$$\Gamma_q = (G_q, \ell_q), \quad G_q = (V(G_q), E(G_q), L(G_q))$$

with  $G_q$  a graph whose vertex set, edge set, and leg set are written, respectively,  $V(G_q)$ ,  $E(G_q)$  and  $L(G_q)$  (a leg is a half-edge attached to only one vertex), and  $\ell_q$  is a function

$$\ell_q : E(G_q) \cup L(G_q) \longrightarrow \mathbb{R}_+.$$

We define  $\Gamma_q$  as the tropical curve obtained from  $\Gamma_*$  by removing every vertex  $A_j^i$  with  $i > 1$  without disconnecting the graph and adding the corresponding lengths. Since the edges  $A_j^i$  have degree at most 2, this operation is well defined. Indeed, if  $A_j^i$  has degree 1, then the edge adjacent to it becomes a leg, or part of a leg, of  $\Gamma_q$ ; if  $A_j^i$  has degree 2 the two edges adjacent to  $A_j^i$  are merged in a new edge or in a leg.

**Remark 3.2.1.** By Remark 2.1.5, we removed from  $G_*$  those vertices corresponding to exceptional divisors  $E_j^i$  that are not tangent to the proper transform  $C_j^i$  of  $C$ .

We have an obvious surjection from the edges of  $G_*$  to the edges and legs of  $G_q$ , written

$$\mu : E(G_*) \longrightarrow E(G_q) \cup L(G_q).$$

The length function  $\ell_q$  is defined as follows

$$\ell_q(x) := \sum_{e \in \mu^{-1}(x)} \ell_*(e)$$

for any  $x \in E(G_q) \cup L(G_q)$ . The definition of the tropical curve  $\Gamma_q$  is now complete. Let us describe it more closely. Its vertex set is

$$V(G_q) = \{C_*, E_*, A_j^1, \quad j = 1, \dots, r\}.$$

An edge of  $G_q$  can be written unambiguously as  $e = vw$  with  $v, w \in V(G_q)$ , hence, by Lemma 2.2.2, we have

$$E(G_q) = \begin{cases} \{C_*E_*, E_*A_r^1, E_*A_{r-1}^1, A_{j+1}^1A_{j-1}^1, \forall j = 2, \dots, r-1\} & \text{if } r > 1 \\ \{C_*E_*, E_*A_1^1\} & \text{if } r = 1 \end{cases}$$

and one easily checks the following

$$(9) \quad \ell_q(vw) := \begin{cases} 1 & \text{if } vw = E_*C_* \text{ or } vw = E_*A_r^1 \\ k_r & \text{if } vw = E_*A_{r-1}^1 \\ k_j & \text{if } vw = A_{j+1}^1A_{j-1}^1, \quad \forall j = 2, \dots, r-2. \end{cases}$$

Finally, if  $r > 1$  and  $k_1 > 1$  then  $G_q$  has a leg attached to  $A_2^1$  of length  $k_1 - 1$ . If  $r = 1$  then  $G_q$  has a leg attached to  $E_*$  of length  $k_r - 1$ . In particular,  $\Gamma_q$  has at most one leg. Summarizing and applying Lemma 3.1.1:

**Lemma 3.2.2.** *Let  $q \in C \subset S$  be an  $(m, n)$ -point and  $\Gamma_q = (G_q, \ell_q)$  its contact tropical curve.*

(a)  $G_q$  is a tree with  $r + 2$  vertices, and a unique vertex,  $E_*$ , of degree 3.

The vertices  $C_*$  and  $A_1^1$  are leaves of  $G_q$  (with no leg attached).

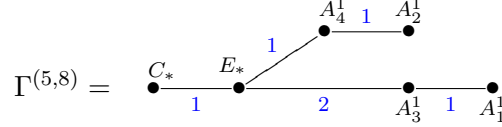
$G_q$  has no legs if  $k_1 = 1$ , and a unique leg of length  $k_1 - 1$  if  $k_1 > 1$ .

(b) If  $q' \in C' \subset S'$  is an  $(m, n)$ -point, then the contact tropical curves  $\Gamma_q$  and  $\Gamma_{q'}$  are isomorphic.

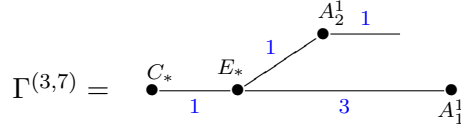
As  $\Gamma_q$  only depends on  $(m, n)$  we set, for any  $(m, n)$ -point  $q \in C$ ,

$$\Gamma^{(m,n)} := \Gamma_q.$$

**Example 3.2.3.** Here are the tropical curves corresponding to Example 3.1.6.



and



These tropical curves are not isomorphic, while  $G_*^{(5,8)} \cong G_*^{(3,7)}$ .

We can now geometrically characterize equivalent pairs.

**Theorem 3.2.4.**  $(m, n) \sim (m', n')$  if and only if  $\Gamma^{(m,n)} \cong \Gamma^{(m',n')}$  (i.e.  $\Gamma^{(m,n)}$  and  $\Gamma^{(m',n')}$  are isomorphic as tropical curves).

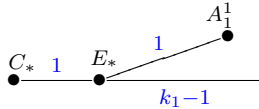
*Proof.* If  $(m, n) \sim (m', n')$  then Lemma 3.1.5 yields  $G_*^{(m,n)} \cong G_*^{(m',n')}$ , hence, by what we saw above,  $\Gamma^{(m,n)} \cong \Gamma^{(m',n')}$ .

To prove the converse, write  $\Gamma = \Gamma^{(m,n)}$  and  $\Gamma' = \Gamma^{(m',n')}$  and assume that there is an isomorphism of tropical curves  $\phi : \Gamma \rightarrow \Gamma'$ . Hence  $\phi$  is a length-preserving isomorphism between the underlying graphs.

Therefore  $\Gamma$  and  $\Gamma'$  have the same number of vertices, equal to  $r + 2$ , hence  $r = r'$ . Moreover,  $\Gamma$  and  $\Gamma'$  have the same number of legs, equal to  $\min\{1, k_1 - 1\} = \min\{1, k'_1 - 1\}$ ; if this number is 0 then  $k_1 = k'_1 = 1$ . If  $\min\{1, k_1 - 1\} = 1$  then  $\Gamma$  and  $\Gamma'$  have each one leg of respective lengths  $k_1 - 1$  and  $k'_1 - 1$ . Since the length of the leg is preserved by the isomorphism  $\phi$ , we get  $k_1 = k'_1$ . We write

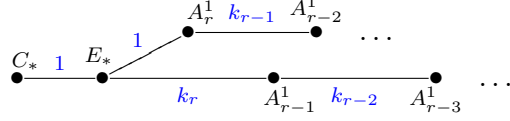
$$V(\Gamma') = \{C'_*, E'_*, (A_j^1)'\}, \quad \forall j = 1, \dots, r\}.$$

Recall that  $\Gamma$  and  $\Gamma'$  have each a unique vertex, respectively  $E_*$  and  $E'_*$ , of degree 3. Therefore the isomorphism  $\phi$  has to map the vertex  $E_*$  to the vertex  $E'_*$ . If  $r = 1$  then  $k_1 \geq 2$ , hence  $\Gamma$  has a unique leg and looks as follows



Since we already proved that  $k_1 = k'_1$ , we are done.

If  $r \geq 2$ , Lemma 3.2.2 implies that  $\Gamma$  has the following form



and the same holds for  $\Gamma'$ , mutatis mutandis. Since  $k_r \geq 2$ , the edges  $E_*A_r^1$  and  $E_*A_{r-1}^1$  have different lengths, equal to 1 and  $k_r$ . The same holds on  $\Gamma'$ : as  $k'_r \geq 2$  the edges  $E'_*(A_r^1)'$  and  $E'_*(A_{r-1}^1)'$  have lengths equal to 1 and  $k'_r$ . Hence an isomorphism from  $\Gamma$  to  $\Gamma'$  maps  $A_r^1$  to  $(A_r^1)'$  and  $A_{r-1}^1$  to  $(A_{r-1}^1)'$ , hence  $k_r = k'_r$ . Hence the edge  $A_r^1A_{r-1}^1$  is mapped to the edge  $(A_r^1)'(A_{r-1}^1)'$ , hence  $k_{r-1} = k'_{r-1}$ . Iterating, the edge  $A_j^1A_{j-2}^1$  is mapped to the edge  $(A_j^1)'(A_{j-2}^1)'$ , hence  $k_{j-1} = k'_{j-1}$  for all  $j = 3, \dots, r$ . The proof is complete.  $\blacksquare$

#### 4. HYPERTANGENCY

**4.1. Hypertangency at unibranch point.** Let  $B, C \subset S$  be two integral curves and  $q \in B \cap C$ ; let  $\sigma : S' \rightarrow S$  be the blow up at  $q$ . If  $B', C' \subset S'$  are the proper transforms of  $B$  and  $C$ , the following is well know and will be used often

$$(10) \quad (B' \cdot C') = (B \cdot C) - \text{mult}_q(B)\text{mult}_q(C).$$

For our next result, we shall assume that  $q$  is an  $(m, n)$ -point of  $C$  and an  $(m', n')$ -point of  $B$ . Recall, from Definition 3.1.4, that two pairs  $(m, n)$  and  $(m', n')$  are equivalent, written  $(m, n) \sim (m', n')$ , if  $r = r'$  and  $k_j = k'_j$  for every  $j = 1, \dots, r$ , where the integers  $r, r', k_j, k'_j$  are determined by the euclidean sequences of the pairs.

The proof of the next statement combines our earlier results with results in [GBP15].

**Theorem 4.1.1.** *Let  $B, C \subset \mathbb{P}^2$  be integral curves of degree at least 2. Let  $q$  be an  $(m', n')$ -point for  $B$  and an  $(m, n)$ -point for  $C$  such that  $B \cap C = \{q\}$ . Then the following equivalent conditions hold*

- (a)  $\Gamma^{(m,n)} \cong \Gamma^{(m',n')}$ .
- (b)  $m/n = m'/n'$ .
- (c)  $(m', n') \sim (m, n)$ .

*Proof.* The equivalence of the three conditions follows from Remark 3.1.7 and Theorem 3.2.4.

Set  $b = \deg B$  and  $d = \deg C$ . By hypothesis  $(B \cdot C)_q = (B \cdot C) = bd$ . The assumption implies that  $B$  and  $C$  have the same tangent line, written  $L$ , at  $q$ , so that  $(C \cdot L)_q = n$  and  $(B \cdot L)_q = n'$ .

To prove that  $m/n = m'/n'$  we apply the results in subsection 2.3 of [GBP15], in particular the ‘‘Strong Triangle Inequality’’ in Theorem 2.8 and

Corollary 2.9. They imply that the smallest among the following three numbers are equal

$$(11) \quad \frac{(B \cdot C)_q}{\text{mult}_q(B)\text{mult}_q(C)}, \quad \frac{(B \cdot L)_q}{\text{mult}_q(B)\text{mult}_q(L)}, \quad \frac{(C \cdot L)_q}{\text{mult}_q(C)\text{mult}_q(L)}.$$

We have,

$$\frac{(B \cdot C)_q}{\text{mult}_q(B)\text{mult}_q(C)} = \frac{bd}{m'm} \geq \frac{bn}{m'm} > \frac{m'n}{m'm} = \frac{n}{m} = \frac{(C \cdot L)_q}{\text{mult}_q(C)\text{mult}_q(L)}$$

(using  $d \geq n$  and  $b > m'$  for the two inequalities). Similarly

$$\frac{(B \cdot C)_q}{\text{mult}_q(B)\text{mult}_q(C)} > \frac{n'}{m'} = \frac{(L \cdot B)_q}{\text{mult}_q(L)\text{mult}_q(B)}.$$

Hence the second and third number in (11) are the smallest, hence they are equal, hence  $n/m = n'/m'$  and we are done.  $\blacksquare$

We now give a different, self-contained proof which applies the material of the previous sections.

*Alternative proof.* We shall prove  $(m', n') \sim (m, n)$ . We can assume  $r \leq r'$ . Consider the chain of blow-ups described before Proposition 2.1.4 for the curve  $C$ . For every  $j = 1, \dots, r$  and  $i = 1, \dots, k_j$  we have a sequence of blow-ups,  $S_j^i$ , of  $\mathbb{P}^2$ , containing the proper transform,  $C_j^i$ , of  $C$ , and a point  $q_j^i \in C_j^i$ , which is the center of the blow up of  $S_j^i$ ; we have  $q = q_1^1$ ,  $C = C_1^1$  and  $\mathbb{P}^2 = S_1^1$ . We consider also the curve  $C_*$  in the blow-up  $S_*$  of  $S_r^{k_r}$  at  $q_r^{k_r}$ , defined before Lemma 2.2.2.

We denote by  $B_{(j)}^{(i)} \subset S_j^i$  and  $B_{(*)} \subset S_*$  the proper transforms of  $B$ . Since  $C$  and  $B$  meet only in  $q$ , if the curves  $C_j^i$  and  $B_{(j)}^{(i)}$  intersect, they do so in the only point of  $C_j^i$  mapping to  $q$ , i.e. in  $q_j^i$ . Notice that if  $(m, n) \sim (m', n')$  the chain of blow-ups for  $B$  described in Proposition 2.1.4 corresponds exactly to the one for  $C$ , hence  $B_{(j)}^{(i)} = B_j^i$ .

We prove  $(m, n) \sim (m', n')$  by contradiction. Then, setting

$$h = \min\{j : k_j \neq k'_j, j = 1, \dots, r\},$$

we have  $h \leq r$ . We can assume  $k_h < k'_h$ , the case  $k_h > k'_h$  is similar. By the definition of  $h$ , we have  $B_h^{k_h} = B_{(h)}^{(k_h)}$  and  $C_h^{k_h} \cap B_h^{k_h} = \{q_h^{k_h}\}$ .

**Claim.** *If  $h < r$  then  $C_{h+1}^1$  is not tangent to  $B_{(h+1)}^{(1)}$ ; if  $h = r$  then  $C_r^{k_r}$  is not tangent to  $B_{(r)}^{(k_r)}$ .*

Recall that, by Proposition 2.1.4, for every  $(j, i)$  with  $i > 1$ , except the pair  $(r, k_r)$ , the curve  $C_j^i$  is tangent to  $F_j^i$  (the proper transform in  $S_j^i$  of the exceptional divisor of  $S_j^1$ ), and  $C_j^1$  is tangent to the exceptional divisor  $E_j^1 \subset S_j^1$ . If  $h < r$  both  $C_h^{k_h}$  and  $B_{(h)}^{(k_h)} = B_h^{k_h}$  are tangent to  $F_h^{k_h}$ , hence their proper transforms in the blow-up at  $q_h^{k_h}$  intersect in  $q_{h+1}^1$ . Now, the proper

transform of  $C$  is  $C_{h+1}^1$  and the proper transform of  $B$ , in the notation of Proposition 2.1.4, is  $B_h^{k_h+1}$  (as  $k'_h > k_h$ ), and we have  $B_h^{k_h+1} = B_{(h+1)}^{(1)}$ . In  $q_{h+1}^1$ , the curve  $C_{h+1}^1$  is tangent to  $E_{h+1}^1$ , whereas  $B_h^{k_h+1}$  is tangent to the proper transform of  $F_h^{k_h}$ , hence  $C_{h+1}^1$  is not tangent to  $B_{(h+1)}^{(1)}$ , as claimed.

If  $h = r$ , according to Proposition 2.1.4, the curve  $C_r^{k_r}$  is not tangent to  $F_r^{k_r}$  in  $q_r^{k_r}$ , while  $B_r^{k_r}$  is, as  $k'_h > k_h$ . Since  $B_r^{k_r} = B_{(r)}^{(k_r)}$  the claim is proved.

By the claim, we necessarily have

$$(12) \quad B_{(*)} \cap C_* = \emptyset.$$

If  $h < r$  a stronger fact holds, to state which we depart momentarily from the notation in Proposition 2.1.4 and write  $S_{k_h} \rightarrow S_{h+1}^1$  for the blow-up at  $q_{h+1}^1$ ; denote by  $C^{k_h}, B^{(k_h)} \subset S_{k_h}$  the proper transforms of  $B$  and  $C$ . By the claim,  $C_{h+1}^1$  is not tangent to  $B_{(h+1)}^{(1)}$ , hence

$$(13) \quad C^{k_h} \cap B^{(k_h)} = \emptyset$$

(which is stronger than (12)).

We are ready to get a contradiction. Assume  $h < r$ ; by iterating (10) we have

$$(14) \quad (C^{k_h} \cdot B^{(k_h)}) = bd - \sum_{j=1}^h k_j l_j l'_j + l_{h+1} l'_h.$$

Indeed,  $C_j^i$  has an  $l_j$ -fold point at  $q_j^i$  for  $j = 1, \dots, h$  and  $i = 1, \dots, k_h$ , while  $C_{h+1}^1$  has an  $l_{h+1}$ -fold point. The curve  $B_{(j)}^{(i)}$  has a  $l'_j$ -fold point at  $q_j^i$  for  $j = 1, \dots, h$  and  $i = 1, \dots, k_j$ , and  $B_{(h+1)}^{(1)}$  has an  $l'_h$ -fold point. Hence (14) follows. We combine (14) with Lemma 4.1.2 (which can be applied as  $k_j = k'_j$  for all  $j < h$ ) and we get

$$(C^{k_h} \cdot B^{(k_h)}) > bd - nn' \geq 0$$

(as  $b \geq n'$  and  $d \geq n$ ), contradicting (13).

Assume  $h = r$ , hence  $k_j = k'_j$  for all  $j < r$ . Then for  $j = 1, \dots, r$  and  $i = 1, \dots, k_r$ , at  $q_j^i$  the curve  $C_j^i$  has an  $l_j$ -fold point, and  $B_{(j)}^{(i)}$  has  $l'_j$ -fold point. Therefore,

$$(15) \quad (C_* \cdot B_{(*)}) = bd - \sum_{j=1}^r k_j l_j l'_j.$$

Since  $l_{r+1} = 0$  we get

$$\sum_{j=1}^r k_j l_j l'_j = \sum_{j=1}^r k_j l_j l'_j + l_{r+1} l'_r < nn'$$

by Lemma 4.1.2 with  $h = r$ . Hence

$$(C_* \cdot B_{(*)}) > bd - nn' \geq 0.$$

contradicting (12). The theorem is proved.  $\blacksquare$

We used the following elementary Lemma.

**Lemma 4.1.2.** *Let (4) and (8) be the euclidean sequences of  $(m, n)$  and  $(m', n')$ . Assume  $r \leq r'$  and let  $h \leq r$ . If  $k_j = k'_j$  for every  $j = 1, \dots, h-1$  then*

$$\sum_{j=1}^h k_j l_j l'_j + l_{h+1} l'_h < n n'.$$

*Proof.* The proof is based on the following identity:

$$(16) \quad \sum_{j=1}^h k_j l_j l'_j + l_{h+1} l'_h = \begin{cases} l_1 l'_0 & \text{if } h \text{ is even} \\ l_0 l'_1 & \text{if } h \text{ is odd.} \end{cases}$$

We have  $m = l_1 < n = l_0$  and  $m' = l'_1 < n = l'_0$ , hence  $l_1 l'_0 < n n'$  and  $l'_1 l_0 < n' n$ . Combining this with (16) the Lemma is proved.

We prove (16) by induction on  $h$ , applying a few times the basic identities:

$$l_{j-2} = k_{j-1} l_{j-1} + l_j, \quad l'_{j-2} = k'_{j-1} l'_{j-1} + l'_j.$$

The base cases are  $h = 1$  and  $h = 2$ . If  $h = 1$  we get

$$k_1 l_1 l'_1 + l_2 l'_1 = (k_1 l_1 + l_2) l'_1 = l_0 l'_1.$$

If  $h = 2$  then  $k_1 = k'_1$ , and we get

$$k_1 l_1 l'_1 + k_2 l_2 l'_2 + l_3 l'_2 = k'_1 l_1 l'_1 + (k_2 l_2 + l_3) l'_2 = k'_1 l_1 l'_1 + l_1 l'_2 = l_1 (k'_1 l'_1 + l'_2) = l_1 l'_0.$$

Let  $h \geq 3$ . If  $h$  is even we have

$$\begin{aligned} \sum_{j=1}^h k_j l_j l'_j + l_{h+1} l'_h &= \sum_{j=1}^{h-1} k_j l_j l'_j + k_h l_h l'_h + l_{h+1} l'_h = \\ &= \sum_{j=1}^{h-1} k_j l_j l'_j + (k_h l_h + l_{h+1}) l'_h = \sum_{j=1}^{h-1} k_j l_j l'_j + l_{h-1} l'_h = \\ &= \sum_{j=1}^{h-2} k_j l_j l'_j + l_{h-1} (k_{h-1} l'_{h-1} + l'_h) = \sum_{j=1}^{h-2} k_j l_j l'_j + l_{h-1} (k'_{h-1} l'_{h-1} + l'_h) = \\ &= \sum_{j=1}^{h-2} k_j l_j l'_j + l_{h-1} l'_{h-2} = l_1 l'_0 \end{aligned}$$

by the induction hypothesis ( $h-2$  is even). The case  $h$  odd follows in the same way. The Lemma is proved.  $\blacksquare$

The following special case of the theorem is worth an explicit mention.

**Corollary 4.1.3.** *Hypotheses of Theorem 4.1.1.*

*If  $\gcd(m, n) = \gcd(m', n')$  then  $(m, n) = (m', n')$ ; in particular, if  $B$  and  $C$  are smooth at  $q$ , then  $n = n'$ .*

**Example 4.1.4.** Let  $B \subset \mathbb{P}^2$  be the projective conic of affine equation  $y = x^2$ , and  $C \subset \mathbb{P}^2$  be the projective cubic of affine equation  $y = x^2 + y^3$ . A simple computation shows that  $B$  and  $C$  intersect only at  $q = (0, 0)$ . The theorem implies that  $q$  is not a flex of  $C$  (which is easy to check in this case).

**4.2. Intersection multiplicity at equivalent points.** We proved in Theorem 4.1.1 that if two curves  $B$  and  $C$  are hypertangent in one point  $q$ , then  $q \in B$  is equivalent to  $q \in C$ , i.e. their contact tropical curves are isomorphic. It is natural to pose the opposite problem, namely, what can be said about the intersection of two curves at equivalent points.

**Proposition 4.2.1.** *Let  $B, C \subset \mathbb{P}^2$  be integral curves and  $q \in B \cap C$ . Assume that  $q$  is an  $(m', n')$ -point of  $B$  and an  $(m, n)$ -point of  $C$ , with  $(m, n) \sim (m', n')$ ; set  $k = \lfloor n/m \rfloor$ . If  $(B \cdot C)_q > m'm$  (i.e.  $B$  and  $C$  are tangent in  $q$ ), then*

$$(17) \quad (B \cdot C)_q \geq \begin{cases} km'm & \text{if } n/m \in \mathbb{N} \\ km'm + 1 + (n' - m')(n - m) & \text{if } n/m \notin \mathbb{N} \end{cases}$$

*Proof.* Let  $L$  be the tangent line at  $q$  to  $C$  and  $B$ , and  $S_1 \rightarrow \mathbb{P}^2$  the blow-up at  $q$ ; denote by  $L^1, B^1, C^1 \subset S_1$  the proper transforms of  $L, B, C$ , and by  $q^1 \in L^1$  the point lying over  $q$ . Denote by  $S_2 \rightarrow S_1$  the blow-up at  $q^1$ , by  $L^2, B^2, C^2 \subset S_2$  the proper transforms of  $L, B, C$ , and by  $q^2 \in L^2$  the point lying over  $q^1$ . Iterating we get a sequence of blow-ups

$$S_{k+1} \rightarrow S_k \rightarrow S_{k-1} \rightarrow \dots \rightarrow S_{i+1} \rightarrow S_i \rightarrow \dots \rightarrow S_1 \rightarrow S_0 = \mathbb{P}^2$$

such that  $L^i, B^i, C^i \subset S_i$  are the proper transforms of  $L, B, C$  and  $S_{i+1} \rightarrow S_i$  is the blow-up at  $q^i \in L^i$ , with  $q^i$  lying over  $q$ ; we write  $E_i \subset S_i$  for the exceptional divisor.

By Lemma 2.1.2, for every  $i \leq k-1$  the curve  $C^i$ , respectively  $B^i$ , has a point of multiplicity  $m$ , respectively  $m'$ , lying over  $q$ . Therefore, setting

$$(C^i \cdot L^i)_{E_i} = \sum_{p \in E_i} (C^i \cdot L^i)_p$$

we have, as  $k \leq n/m$

$$(C^i \cdot L^i)_{E_i} = (C \cdot L)_q - im = n - im \geq n - (k-1)m \geq n - (n/m - 1)m = m \geq 1$$

hence, as  $L^i \cap E_i = q^i$ , we get  $(C^i \cdot L^i)_{q^i} \geq 1$ . An analogous argument applied to  $B$  gives  $(B^i \cdot L^i)_{q^i} \geq 1$  for all  $i \leq k-1$ . Therefore  $q^{k-1} \in B^{k-1} \cap C^{k-1}$  hence

$$(B^{k-1} \cdot C^{k-1})_{q^{k-1}} \geq m'm.$$

On the other hand

$$(B^{k-1} \cdot C^{k-1})_{q^{k-1}} = (B \cdot C)_q - (k-1)m'm$$

hence

$$(B \cdot C)_q \geq m'm + (k-1)m'm = km'm.$$

If  $n/m \in \mathbb{N}$  there is nothing left to prove. Let  $n/m \notin \mathbb{N}$ , hence  $k < n/m$ . Then  $C^k$  has an  $(n - m, m)$ -point w.r.t.  $E_k$ , and  $B^k$  has an  $(n' - m', m')$ -point w.r.t.  $E_k$  (by Lemma 2.1.2); let us show that this point is  $q^k$  for both  $C^k$  and  $B^k$ . We have

$$(C^k \cdot L^k)_{q^k} = n - km > n - nm/m = 0$$

hence  $q^k \in C^k$ . The same argument shows that  $q^k \in B^k$ . Hence  $C^k$  and  $B^k$  intersect in  $q^k$ , as wanted. Now, since  $C^k$  and  $B^k$  are both tangent to  $E_k$ , their intersection is not transverse, hence their proper transforms in  $S_{k+1}$  satisfy

$$(B^{k+1} \cdot C^{k+1})_{\tilde{q}^{k+1}} \geq 1$$

with  $\tilde{q}^{k+1}$  lying over  $q$ . On the other hand

$$(B^{k+1} \cdot C^{k+1})_{\tilde{q}^{k+1}} = (B \cdot C)_q - km'm - (n' - m')(n - m)$$

hence  $(B \cdot C)_q \geq 1 + km'm + (n' - m')(n - m)$ . The proof is complete. ■

## 5. GENUS AND DIMENSION FORMULAS

In this section we obtain bounds for the delta invariant (defined below) of a unibranch point of a curve  $C \subset \mathbb{P}^2$ , and for the codimension of the locus of plane curves with a  $(m, n)$  point.

**5.1. Genus computation.** Let  $C \subset \mathbb{P}^2$  and  $q \in C$ ; let  $\deg C = d \geq 2$ . The geometric genus,  $g(C) = g(C^\nu)$  (with  $C^\nu$  the normalization of  $C$ ), satisfies

$$g(C) \geq (d - 1)(d - 2)/2 - \delta(q)$$

with equality if and only if  $q$  is the only singular point of  $C$ . The number  $\delta(q)$  is called the *delta-invariant* of  $q$ . The following formula is well-known

$$(18) \quad \delta(q) = \sum m_p(m_p - 1)/2$$

where  $p$  varies over all points infinitely near to  $q$  (the points added over  $q$  in the proper transforms of the blow-ups), and  $m_p$  is the multiplicity of  $p$ . For example, if  $q$  is an ordinary  $m$ -fold point (i.e.  $C$  has  $m$  branches meeting transversally in  $q$ ), then  $\delta(q) = m(m - 1)/2$ .

**Proposition 5.1.1.** *Let  $C \subset \mathbb{P}^2$  have degree  $d \geq 2$  and let  $q \in C$  be an  $(m, n)$ -point; set  $\gcd(n, m) = c$ . Then*

$$\delta(q) \geq (nm - n - m + c)/2$$

*with equality if  $c = 1$ .*

*Proof.* Proposition 2.1.4 considers the desingularization,  $\nu_q : C_q^\nu \rightarrow C$ , of  $C$  at  $q$ , and describes the points infinitely near to  $q$  appearing in the partial desingularization given by  $\beta$ . This, combined with (18), gives

$$(19) \quad \delta(q) \geq \sum_{j=1}^r k_j l_j (l_j - 1)/2.$$

Indeed, for every  $j = 1, \dots, r$  we have exactly  $k_j$  infinitely near points of multiplicity  $l_j$ , and each such point contributes by  $l_j(l_j - 1)/2$ . Note that if  $c = 1$  the summand corresponding to  $j = r$  vanishes, as  $l_r = 1$ . In this case the desingularization of  $C$  at  $q$  is actually included in  $\beta$ , hence there are no other infinitely near points, therefore in (19) we have equality.

The following Lemma 5.1.2 completes the proof.  $\blacksquare$

**Lemma 5.1.2.** *Let  $n > m > 0$  be two integers and consider the euclidean sequence (4). Then  $k_j = (l_{j-1} - l_{j+1})/l_j$  and*

$$(20) \quad \sum_{j=1}^r k_j l_j (l_j - 1)/2 = (nm - n - m + c)/2.$$

*Proof.* The expression of  $k_j$  follows directly from its definition and the construction of the euclidean sequence. The second assertion follows from a special case of the following formula

$$(21) \quad \sum_{j=1}^r (l_{j-1} - l_{j+1})(l_j - 1) = l_0 l_1 - l_0 - l_1 + l_{r+1} + l_r - l_r l_{r+1}.$$

In our case  $l_0 = n$ ,  $l_1 = m$ ,  $l_r = c$  and  $l_{r+1} = 0$ , from which (20) follows. To prove (21) we proceed by induction on  $r$ . If  $r = 1$  then

$$(l_0 - l_2)(l_1 - 1) = l_0 l_1 - l_1 l_2 - l_0 + l_2,$$

as wanted. To prove the general case, the induction hypothesis reads as follows

$$\sum_{j=1}^{r-1} (l_{j-1} - l_{j+1})(l_j - 1) = l_0 l_1 - l_0 - l_1 + l_{r-1} + l_r - l_r l_{r-1}.$$

Thus we obtain

$$\begin{aligned} & \sum_{j=1}^r (l_{j-1} - l_{j+1})(l_j - 1) = \\ & l_0 l_1 - l_0 - l_1 + l_{r-1} + l_r - l_r l_{r-1} + (l_{r-1} - l_{r+1})(l_r - 1) = \\ & l_0 l_1 - l_0 - l_1 + l_{r-1} + l_r - l_r l_{r-1} + l_{r-1} l_r - l_r l_{r+1} - l_{r-1} + l_{r+1} = \\ & l_0 l_1 - l_0 - l_1 + l_r - l_r l_{r+1} + l_{r+1}. \end{aligned}$$

The Lemma is proved.  $\blacksquare$

**5.2. Dimension computation.** Let  $n, m$  be two integers with  $n > m > 0$ ; let  $L \subset \mathbb{P}^2$  be a line and  $q \in L$  a point. We denote by

$$U_d^{(m,n)}(L; q) \subset \mathbb{P}^{d(d+3)/2}$$

or simply by  $U_d^{(m,n)}$ , the space of integral curves of degree  $d \geq 2$  having a unibranch  $(m, n)$ -point at  $q$  with tangent line equal to  $L$ . As it will be clear (if it is not already), its closure,  $\overline{U_d^{(m,n)}}$ , is a linear subspace of  $\mathbb{P}^{d(d+3)/2}$ . We write  $\text{codim } U_d^{(m,n)} = \text{codim } \overline{U_d^{(m,n)}}$  for the codimension in  $\mathbb{P}^{d(d+3)/2}$ .

**Proposition 5.2.1.** *Assume  $d \geq n > m > 0$  and set  $\gcd(n, m) = c$ . Then*

$$\text{codim } U_d^{(m,n)} \geq (nm + m + n + c - 2)/2$$

with equality when  $c = 1$ .

*Proof.* We choose, as usual, affine coordinates  $(x, y)$  so that  $L$  has equation  $y = 0$  and  $q = (0, 0)$ . Let  $C \in U_d^{(m,n)}(L; q)$  and let its affine equation be  $f(x, y) = 0$  with

$$f(x, y) = \sum_{0 \leq i+j \leq d} a_{i,j} x^i y^j.$$

By hypothesis  $a_{n,0}, a_{0,m} \neq 0$ .

As  $(C \cdot L)_q = n$  we have  $f(x, 0) = x^n p(x)$  with  $p(0) \neq 0$ , hence the smallest power of  $x$  appearing in  $f$  is  $x^n$ . This gives the following  $n$  conditions

$$(22) \quad a_{i,0} = 0 \quad \forall i = 0, \dots, n-1.$$

Next, as  $q$  is an  $m$ -fold point we have  $a_{i,j} = 0$  for all  $i + j \leq m - 1$ ; these are  $m(m+1)/2$  conditions, but exactly  $m$  of them are also in (22). Hence the number of new independent conditions on  $f$  is equal to

$$(23) \quad m(m+1)/2 - m = m(m-1)/2.$$

Next,  $C$  is unibranch at  $q$ , and its tangent line there is  $y = 0$ . Therefore the only summand of degree  $m$  is  $a_{0,m} y^m$  and we get the following new  $m-1$  conditions:

$$(24) \quad a_{i,j} = 0, \quad \forall i + j = m, \quad i, j \geq 1.$$

Now consider the blow-up of  $\mathbb{P}^2$  at  $q$ , let  $C'$  and  $L'$  be the strict transforms of  $C$  and  $L$ , and  $q' \in C'$  the point lying over  $q$ . We set  $y = vx$  and use  $(x, v)$  as affine coordinates in the blow-up. The equation of  $C'$  is

$$f'(x, v) = a_{0,m} v^m + \sum_{m+1 \leq i+j \leq d} a_{i,j} x^{i+j-m} v^j = a_{0,m} v^m + \sum_{\substack{1 \leq h \leq d-m \\ m+1 \leq l \leq d}} a_{h-l+m, l} x^h v^l$$

with  $j = l$  and  $i = h - l + m$ . The conditions imposed earlier imply exactly that  $f'$  has no summand of type  $x^h$  for all  $h = 0, \dots, d - m - 1$ , and no summand of type  $v^l$  for all  $l = 1, \dots, m - 1$ .

Let  $m'$  be the multiplicity of  $q' \in C'$ ; the fact that  $C'$  is unibranch at  $q'$  yields new conditions; we claim that the number of them is

$$m'(m' - 1)/2.$$

If  $n < 2m$  we have  $n - m = m'$  and  $q'$  is a  $(m', m)$ -point of  $C'$  with respect to exceptional divisor, by Lemma 2.1.2. More precisely, the term of smallest degree of  $f'$  is  $a_{n,0} x^{m'}$ , and  $C'$  has a unibranch  $m'$ -fold point at  $q'$ . Hence every coefficient of a term of degree less than  $m'$ , and of a term of degree equal to  $m'$  other than  $x^{m'}$ , must vanish. These are  $m'(m'+1)/2 + m'$  independent conditions, but some of them are already satisfied. Indeed, we know that  $f'$  has no term of type  $x^l$  for all  $l = 0, \dots, m' - 1$  ( $m'$  conditions), and no term of type  $v^n$  for all  $n = 1, \dots, m'$  ( $m'$  conditions). There are no

other cases, as if  $k, l > 0$  the coefficient of  $x^k v^l$  is  $a_{k-l+m, l}$  corresponding to a summand in  $f$  of degree  $k + m \geq m + 1$ , and no conditions had been imposed on such coefficients. Hence the number of new conditions is

$$(25) \quad m'(m' + 1)/2 + m' - m' - m' = m'(m' - 1)/2.$$

Let  $n > 2m$ . Then  $m = m'$  and  $C'$  has a an  $(m, n - m)$ -point at  $q'$  with respect to  $L'$ , by Lemma 2.1.2. Hence in  $f'$  there are no monomials of degree less than  $m$ , and no monomials of degree  $m$ , except  $v^m$ . These are  $m(m+1)/2 + m$  conditions, some of which are already satisfied, indeed  $f'$  has no term of type  $x^l$  for all  $l = 0, \dots, m$  ( $m + 1$  conditions), and no summand of type  $v^n$  for all  $n = 1, \dots, m - 1$  ( $m - 1$  conditions). Hence the number of new conditions is

$$(26) \quad m(m + 1)/2 + m - (m + 1) - (m - 1) = m(m - 1)/2.$$

Let  $n = 2m$ . Then  $C'$  has a an  $m$ -fold point at  $q'$ . Hence in  $f'$  there are no monomials of degree less than  $m$ . Moreover, as we saw in the proof of Lemma 2.1.2, the homogeneous part of degree  $m$  of  $f'$  has the form  $(\alpha x + \beta v)^m$  with  $\alpha^m = a_{2m, 0}$  and  $\beta^m = a_{0, m}$ . This imposes  $m - 1$  conditions. We get a total of  $m(m + 1)/2 + m - 1$  conditions, but, as in the earlier cases, some of them already hold. In fact  $f'$  has no term of type  $x^l$  for all  $l = 0, \dots, m - 1$  ( $m$  conditions), and no summand of type  $v^n$  for all  $n = 1, \dots, m - 1$  ( $m - 1$  conditions). Hence the number of new conditions is

$$(27) \quad m(m + 1)/2 + m - 1 - m - (m - 1) = m(m - 1)/2.$$

The claim is proved.

Now we need to apply the claim to all the curves appearing in the chain described in Proposition 2.1.4, of which we use the notation. Proposition 2.1.4 gives a sequence of curves whose multiplicity at the point lying over  $q$  is equal to  $l_j$ , with  $j = 1, \dots, r$ . By the claim, each of these gives  $l_j(l_j - 1)/2$  new conditions. If  $j \geq 2$  we get exactly  $k_j$  such curves, hence a total of  $k_j l_j(l_j - 1)/2$  conditions. For  $j = 1$  we have only  $k_1 - 1$  such blow-ups, hence  $(k_1 - 1)m(m - 1)/2$  conditions. But in the first part of the proof we had  $m(m - 1)/2$  conditions; see (23). Hence also for  $j = 1$  we have  $k_1 m(m - 1)/2$  conditions, hence

$$\sum_{j=1}^r k_j l_j(l_j - 1)/2$$

conditions. Adding these to the ones computed in (22) and (24) we get

$$\text{codim } U_d^{(m, n)} = n + (m - 1) + \sum_{j=1}^r k_j l_j(l_j - 1)/2.$$

By applying Lemma 5.1.2 we are done. ■

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