HYPERTANGENCY OF PLANE CURVES AND THE ALGEBRAIC EXCEPTIONAL SET

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Abstract. We investigate plane curves intersecting in at most two unibranch points to study the algebraic exceptional set appearing in standard conjectures of diophantine and hyperbolic geometry. Our first result compares the local geometry of two hypertangent curves, i.e. curves having maximal contact at one unibranch point. This is applied to fully describe the exceptional set and, more generally, the hyper-bitangency set, of a plane curve with three components.

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1. Introduction

The goal of this paper is to study complex plane curves that intersect a given curve in few points. The main motivation is the quasi-projective analogue of a well-known conjecture of Lang predicting that on a projective surface of general type there are only finitely many rational and elliptic curves. The analogue for the complement of a curve in $\mathbb{P}^2$ is the following.

Conjecture 1. *Let $B \subset \mathbb{P}^2$ be a reduced curve with at most normal crossing singularities (i.e. nodes). If $\deg B \geq 4$, then the set of rational curves $C \subset \mathbb{P}^2$ such that $|\nu_C^{-1}(C \cap B)| \leq 2$ is finite, where $\nu_C : C^\nu \to C$ is the normalization.*

The set in the conjecture is often referred to as the *algebraic exceptional set* (see for example [Lan86]); we denote it by $\mathcal{E}(B)$, i.e.

$$\mathcal{E}(B) := \{C \subset \mathbb{P}^2 : \ C \text{ rational}, \ \ |\nu_C^{-1}(C \cap B)| \leq 2\}. \tag{1}$$

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The curves $C \setminus B$ with $C \in \mathcal{E}(B)$ are quasi-projective analogues of rational and elliptic curves, both arithmetically (they have a potentially infinite set of integral points when defined over a number field) and geometrically (see for example [KM99]).

Conjecture 1 can be deduced by several conjectures of Lang, Vojta, Demailly and Campana, as in [Lan91, Conj. 3.7], [Voj87, Conj. 3.4.3] [Dem97] and [Cam04, Conj. 9.2 and 9.20] (see [BG06, HS00, AT20, DT15] for introductions and discussions), on the distribution of integral and rational points on quasi-projective surfaces defined over number fields and function fields. From the arithmetic point of view, $\mathcal{E}(B)$ is conjecturally made of the curves that contain all potentially infinite families of integral points in the affine surface $\mathbb{P}^2 \setminus B$.

Historically, describing the set $\mathcal{E}(B)$, has proven harder the fewer the irreducible components of the curve B, and Conjecture 1 is open if B has at most two components. When B has at least three components, Corvaja and Zannier proved the function field version of Vojta's Conjecture for $\mathbb{P}^2 \setminus B$, in [CZ08] and [CZ13], which implies that curves in $\mathcal{E}(B)$ have bounded degree (the same result in the so-called non-split case was proved in [CT22], building upon [Tur17]). More recently [GNSW23] showed (as a special case of their Theorem 4) that the set $\mathcal{E}(B)$ corresponds to a closed subset of $\mathbb{P}^2$ and described a closed set containing it. If two of the three components are lines, this can be deduced from an earlier result in [CZ00], which covers also some non normal-crossing B, see Subsection 4.4 for further details.

A different line of investigation assumes that the curve B is very general. In [CRY23] the authors showed that when B is very general of degree 4 the set $\mathcal{E}(B)$ consists only of the bitangents and the flex lines, answering a question of Lang. On the other hand, when B is very general of degree at least 5 the set $\mathcal{E}(B)$ is empty: this has been proven independently in [Che04] and in [PR07]. No similar result is known without the "very general" assumption.

In the present article we extend our analysis from rational curves to curves of arbitrary genus. This enables us, in particular, to give an explicit description of the set $\mathcal{E}(B)$ when B has (at least) three irreducible components. To state our results we introduce, generalizing (1), the set of curves "hyper-bitangent" to B

$$\mathrm{Hyp}(B,2) := \{C \subset \mathbb{P}^2 : \ C \text{ integral}, \quad |\nu_C^{-1}(C \cap B)| \leq 2\}$$

so that, of course, $\mathcal{E}(B) \subset \mathrm{Hyp}(B,2)$. We now summarize in one statement our main results; see Proposition 3.2.1, and Theorems 3.3.1, 3.3.2, and 3.3.4 for more precise and stronger statements.

Theorem. *Let $B \subset \mathbb{P}^2$ be a reduced curve with three irreducible components and only nodal singularities. If $\deg B \geq 4$ then*

(a) $\mathcal{E}(B) = \mathrm{Hyp}(B,2)$*;*
(b) $\mathrm{Hyp}(B,2)$ *is finite and effectively bounded;*

(c) if $\deg B \geq 5$ *and* B *is general, then* $\mathrm{Hyp}(B, 2)$ *is empty.*

The above effective bounds on $|\mathrm{Hyp}(B, 2)|$ depend only on the degrees of the irreducible components of B. The phrase "B is general" means that, for every fixed triple of degrees summing to d, the curve B varies in an open dense subset of the space of triples of curves of fixed degrees.

Proposition 3.2.1 gives the bound for the number of hyper-bitangent lines, and Theorem 3.3.4 gives the bound for the number of hyper-bitangent curves of degree at least 2.

The proof of these facts relies on a geometric result of independent interest proved at the beginning of the paper, Theorem 2.2.1, which analyzes two plane curves, B and C, meeting in only one point. The precise statement requires some technical preliminaries; informally speaking, it establishes that the local geometries of B and C at their intersection point are closely related.

Our results are related to the hyperbolic properties of $\mathbb{P}^2 \setminus B$. Demailly in [Dem97] introduced an algebraic analogue of hyperbolicity for projective varieties, which was extended to quasi projective varieties in [Che04]. Using this analogue, if B has degree at least 4 one expects the existence of a positive constant A such that, for every integral curve $C \subset \mathbb{P}^2$ not contained in the exceptional set $\mathcal{E}(B)$, the following bound holds:

$$\deg C \leq A \cdot \Big(2g(C^\nu) - 2 + |\nu_C^{-1}(C \cap B)|\Big). \tag{2}$$

Equivalently one expects that $\mathbb{P}^2 \setminus B$ is "algebraically hyperbolic" modulo $\mathcal{E}(B)$; see [Jav20, Section 9] for further references and discussions on algebraic hyperbolicity.

To see the link with our results, curves in $\mathrm{Hyp}(B, 2)$ correspond exactly to curves that satisfy $|\nu_C^{-1}(C \cap B)| \leq 2$. Hence our Theorem proves that, when B has at least three irreducible components and $|\nu_C^{-1}(C \cap B)| = 2$, the bound (2) holds. More precisely, we show that the degree of a curve in $\mathrm{Hyp}(B, 2)$ is bounded uniformly independently of the genus, and (2) holds for curves in $\mathrm{Hyp}(B, 2)$ with $A = \deg B - 2$; see Corollary 3.3.3. This strengthens, in this particular case, [CZ13, Theorem 1] which proves (weak) algebraic hyperbolicity of $\mathbb{P}^2 \setminus B$ when B has at least three components. We point out that our method is completely different and uses purely geometric techniques.

In a parallel direction, the logarithmic Green-Griffiths-Lang Conjecture predicts that, when $\deg B \geq 4$, there exists an *analytic exceptional set*, i.e. a Zariski proper closed subset of $\mathbb{P}^2 \setminus B$ that contains all the images of the non constant holomorphic maps $\mathbb{C} \to \mathbb{P}^2 \setminus B$, i.e. $\mathbb{P}^2 \setminus B$ is Brody hyperbolic modulo such analytic exceptional set. In general, the analytic exceptional set always contains the algebraic one, and they are conjectured to coincide, by Lang. In our setting the two coincide, and hence our results provide the description of the analytic exceptional set for $\mathbb{P}^2 \setminus B$, in the case where B has at least three irreducible components. This is only possible since algebraic degeneracy (i.e. every holomorphic map $\mathbb{C} \to \mathbb{P}^2 \setminus B$ has non dense image) was already proven in [NWY08].

Our arguments do not use any of the known results we mentioned in this introduction. Therefore we expect our techniques to extend to log surfaces where boundedness is not known.

Outline of the paper. In Section 2 we establish our principal geometric tool, Theorem 2.2.1, on the local geometry of two plane curves meeting in only one unibranch point. In Section 3 we prove our main results describing the sets $\mathrm{Hyp}(B, 2)$ for curves with three components, thus proving the previously stated Theorem. Finally, Section 4 collects various special cases and examples related to the earlier topics.

Notation and terminology. We work over $\mathbb{C}$. We denote by C an integral (i.e. reduced and irreducible) projective curve lying in a smooth projective surface S, by $\nu_C : C^\nu \to C$ its normalization, by $p_a(C)$ its arithmetic genus, and by $g(C) = p_a(C^\nu)$ its geometric genus. A *rational* curve is an integral curve of geometric genus zero. Given a point $p \in C$, we write $\mathrm{mult}_p(C)$ for the multiplicity of C at p. We say that p is *unibranched* if $|\nu_C^{-1}(p)| = 1$. We denote by C_p^ν the partial normalization of C at p; the *δ-invariant*, $\delta_C(p)$, of p is $\delta_C(p) = p_a(C) - p_a(C_p^\nu)$. The following formula is well-known

$$\delta_C(p) = \sum m_q(m_q - 1)/2 \tag{3}$$

where q varies over all points infinitely near to p (including p), and m_q is the multiplicity of q.

If $C, B \subset S$ are reduced curves with no components in common, and $p \in C \cap B$, we write $(C \cdot B)_p$ for their multiplicity of intersection at p; we say that C and B are transverse at p if $(C \cdot B)_p = \mathrm{mult}_p(B)\mathrm{mult}_p(C)$.

We say that C is *hypertangent* to B if $|\nu_C^{-1}(B \cap C)| = 1$.

Let $S = \mathbb{P}^2$, fix a reduced curve $B \subset \mathbb{P}^2$ and a point $q \in B$. The set of integral curves hypertangent to B at q is

$$\mathrm{Hyp}(B; q) := \{C \subset \mathbb{P}^2 : \ C \text{ integral}, \ C \cap B = \{q\}, \ |\nu_C^{-1}(q)| = 1\},$$

and the set of all integral curves hypertangent to B is

$$\mathrm{Hyp}(B, 1) := \cup_{q \in B} \mathrm{Hyp}(B; q).$$

For a positive integer d we write $\mathrm{Hyp}_d(B; q) \subset \mathrm{Hyp}(B; q)$ and $\mathrm{Hyp}_d(B, 1) \subset \mathrm{Hyp}(B, 1)$, for the subsets parametrizing curves of degree d; we view both of them as subspaces of the projective space $\mathbb{P}^{d(d+3)/2}$ parametrizing plane curves of degree d.

If $C \in \mathrm{Hyp}_d(B; q)$ and C is singular at q, then it necessarily has a unibranch singularity. We set for $m \geq 1$

$$\mathrm{Hyp}^m(B; q) = \{C \in \mathrm{Hyp}(B; q) : \ \mathrm{mult}_q(C) = m\}$$

and we define $\mathrm{Hyp}_d^m(B; q) \subset \mathrm{Hyp}_d(B; q)$ analogously.

Now we extend to double intersections. As we said before, C is *hyper-bitangent* to B if $\nu_C^{-1}(C \cap B) \leq 2$ and we denote by $\mathrm{Hyp}(B,2)$ the set of curves hyper-bitangent to B. We have $\mathrm{Hyp}(B,1) \subset \mathrm{Hyp}(B,2)$. We set

$$\mathrm{Hyp}_d(B,2) := \{C \in \mathrm{Hyp}(B,2) : \ \deg C = d\}$$

so that $\mathrm{Hyp}_d(B,2) \subset \mathbb{P}^{d(d+3)/2}$.

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2. Hypertangency

2.1. **Preliminaries on unibranch points.** We here state some basic facts about unibranch points of curves; we refer to [Wal04] for an exhaustive treatise. Let q be a unibranch point of an integral curve $C \subset \mathbb{P}^2$; the tangent line to C at q is well defined, we denote it by L. When $L \neq C$ we say that q is an *(m,n)-point* if $\mathrm{mult}_q(C) = m$ and if $(C \cdot L)_q = n$.

If $m = 1$ and $n \geq 3$ we say that q is a *flex* of C. We choose local coordinates, x, y, so that $q = (0,0)$, the line L has equation $y = 0$, and C has equation $f(x,y) = 0$ with $\deg f = d \geq n$ with

$$f(x,y) = a_{0,m}y^m + \sum_{m+1 \leq i+j \leq d} a_{i,j}x^i y^j \tag{4}$$

such that $a_{0,m} \neq 0$ and the smallest power of x appearing in f is x^n, i.e. $a_{i,0} = 0$ for $i < n$ and $a_{n,0} \neq 0$. Notice that, as will be clear in the sequel, having local equation of type (4) is not sufficient for q to be unibranched.

More generally, let S be any smooth surface and $q \in C \subset S$ a unibranch point of multiplicity m; let $D \subset S$ be a smooth integral curve, $C \neq D$. If $n := (C \cdot D)_q > m$ we say that D is tangent to C at q, and that q is an *(m,n)-point of C with respect to D*.

We are interested in the case where the surface S is an iterated blow-up of $\mathbb{P}^2$ over a fixed point $q^0 \in \mathbb{P}^2$, i.e. we have a finite chain of blow-ups

$$S \longrightarrow S^i \longrightarrow \ldots \longrightarrow S^0 = \mathbb{P}^2$$

all centered at a point lying over q^0. Let $C \subset S$ be an integral curve with a unibranch point $q \in C$. Let $L \subset S$ be the strict transform of a line in $\mathbb{P}^2$ such that q is an (m,n)-point with respect to L. Consider the blow-up, $S' \to S$, at q, let $E \subset S'$ be the exceptional divisor and C' and L' the strict transforms of C and L. The map $\sigma : C' \to C$ induced by this blow-up is bijective; set $q' := \sigma^{-1}(q)$. With this set-up, we have the following

Lemma 2.1.1. *Let $0 < m < n$ and let $q \in C \subset S$ be an (m,n)-point with respect to L*

(a) If $n < 2m$ then q' is an $(n-m, m)$-point of C' with respect to E.
(b) If $n > 2m$ then q' is an $(m, n-m)$-point of C' with respect to L'.
(c) If $n = 2m$ then q' is an m-fold point of C', and C' is neither tangent to E nor to L' at q'.

Proof. We can work locally and use on S affine coordinates so that the set-up described for (4) holds. To blow-up at q we set $y = vx$ and use (x, v) as local coordinates in the blow-up at $q' = (0,0)$, the local equation of the exceptional divisor E is $x = 0$, and the local equation of L' is $v = 0$. Let $f'(x, v) = 0$ be the affine equation of C' obtained from (4):

$$f'(x,v) = x^{-m} f(x, vx) = a_{0,m} v^m + \sum_{m+1 \leq i+j \leq d} a_{i,j} x^{i+j-m} v^j.$$

The smallest power of x appearing as a summand of f' is x^{n-m}, and the smallest power of v is v^m. We have three cases.

Case (a) $n < 2m$. Hence $n - m < m$ and we set $r = n - m$. Since C' is unibranch at q', and f' contains the summand x^r but not the summand v^r (as $r < m$), all terms in f' of degree at most r divisible by xv must vanish (for otherwise the lowest homogeneous part of f' will be reducible, contradicting the fact that q' is a unibranch point of C'). Hence the tangent line to C' at q' has local equation $x = 0$, hence C' is tangent to E. Since the smallest power of v in f' is v^m, we get that q' is an (r, m)-point with respect to E.

Case (b) $n > 2m$. The smallest power of v appearing in f' is v^m and, arguing as in the previous case, f' has no other term of degree at most m. Hence the tangent line to C' at q' has local equation $v = 0$, so that C' is tangent to L. As the smallest power of x is x^{n-m}, we get that q' is an $(m, n-m)$-point with respect to L'.

Case (c) $n = 2m$. We have both $x^{n-m} = x^m$ and v^m appearing in f' as smallest powers of x and v. Since C' is unibranch at q', this is possible only if the homogeneous part of degree m of f', has form $f'_m = (\alpha v + \beta x)^m$, with $\alpha, \beta \neq 0$. Hence q' is an m-fold point such that neither E nor L' are tangent to C at q'. ∎

We will frequently use the following well known facts.

Remark 2.1.2. Let $B, C \subset S$ be two integral curves and $q \in B \cap C$. Suppose that B is smooth at q, and that C has a unibranch m_C-fold point at q. If $S' \to S$ is the blow up at q and $C', B' \subset S'$ are the strict transforms of C and B, then $(C' \cdot B') = (C \cdot B) - m_C$.

Moreover, $(C \cdot B)_q > m_C$ if and only if C and B are tangent at q.

2.2. **Hypertangency at unibranch points of plane curves.** We describe plane curves hypertangent to a curve at a smooth point. See Subsection 4.1 for some examples. Recall that a point q of a plane curve $C \subset \mathbb{P}^2$ of degree at least 2 is an (m,n)-point if it is unibranch, $m = \operatorname{mult}_q(C)$ and $(L \cdot C)_q = n$, where L is the tangent line to C at q.

Theorem 2.2.1. *Let $B, C \subset \mathbb{P}^2$ be integral curves of degree at least 2 such that $B \cap C = \{q\}$; assume that q is a $(1,l)$-point for B and an (m,n)-point for C. Then $n = lm$ and the δ-invariant of q on C satisfies*

$$\delta_C(q) \geq (m-1)(\deg B \deg C - m)/2.$$

Proof. Set $b = \deg B$, $d = \deg C$, and let q be an (m,n) point of C. Hence $l \leq b$ and $1 \leq m < n \leq d$. By hypothesis,

$$(C \cdot B)_q = (C \cdot B) = bd > m$$

therefore C and B must have the same tangent line at q (by Remark 2.1.2); we write L for this line. Let C^1, B^1 and L^1 be the strict transforms of C, B and L in the blow-up, S^1, of $\mathbb{P}^2$ at q, and let $q^1 \in C^1$ be the point lying over q. By hypothesis C^1 and B^1 meet only in q^1, hence (as $d > m$, $b \geq 2$)

$$(C^1 \cdot B^1)_{q^1} = (C^1 \cdot B^1) = bd - m > bm - m \geq 2m - m = m.$$

Now, B^1 is smooth at q^1 and $\operatorname{mult}_{q^1}(C^1) \leq \operatorname{mult}_q(C) = m$, hence

$$(C^1 \cdot B^1)_{q^1} > m \geq \operatorname{mult}_{q^1}(C^1)\operatorname{mult}_{q^1}(B^1),$$

hence C^1 and B^1 are tangent in q^1.

We prove that C has an (m, lm)-point at q. Suppose $l = 2$; Lemma 2.1.1 implies that B^1 is neither tangent to L^1, nor to the exceptional divisor, hence the same holds for C^1. Hence, by the same Lemma, q is a $(m, 2m)$ point for C, and we are done.

To set-up an inductive argument we introduce a sequence of blow-ups as follows. We already considered the blow-up, S^1, of $S^0 = \mathbb{P}^2$ at $q^0 = q$. For $i \geq 1$ we let S^i be the blow-up of S^{i-1} at the unique point $q^{i-1} \in C^{i-1}$ lying over q. We denote by C^i, B^i and L^i the strict transforms of C, B and L. Notice that the multiplicity of C^i at q^i is at most m.

Claim. *Let $l \geq 3$; for every $1 \leq i \leq l-2$ the point q^i is an $(m, n-im)$-point with respect to L^i for C^i, moreover C^i and B^i are tangent at q^i.*

We prove the claim by induction. If $i = 1$ then q is a $(1, l \geq 3)$-point for B, hence Lemma 2.1.1 gives that B^1 is tangent to L^1 over q. We proved earlier that B^1 and C^1 are tangent in q^1, hence C^1 is tangent to L^1 at q^1; hence, by Lemma 2.1.1, q is an (m,n)-point for C with $n > 2m$, and $q^1 \in C^1$ is an $(m, n-m)$-point with respect to L^1. The proof of the base is complete.

Suppose C^{i-1} has a $(m, m-(i-1)n)$-point with respect to L^{i-1} at q^{i-1}, and B^{i-1} and C^{i-1} are tangent to L^{i-1} at q^{i-1}. Now B^{i-1} has a $(1, l-(i-1))$-point in q^{i-1}; as $i \leq l-2$ we have $l-(i-1) \geq l-l+3 = 3$, hence B^i is tangent to L^i. Now, $l \leq b$, hence $i \leq l-2 \leq b-2$. Therefore

$$(C^i \cdot B^i)_{q^i} = (C^i \cdot B^i) = bd - im > bm - im \geq bm - (b-2)m = 2m > m.$$

Hence C^i and B^i are tangent in q^i at L^i. Therefore C^i is tangent to L^i and case (b) of Lemma 2.1.1 must occur for C^{i-1}, i.e. $m-(i-1)n > 2m$ and C^i has a $(m, n-im)$-point with respect to L^i. The claim is proved.

Thus C^{l-2} has an $(m, n-(l-2)m)$-point with respect to L^{l-2} at q^{l-2} , and B^{l-2} and C^{l-2} are tangent to L^{l-2} at q^{l-2}. Now, B^{l-2} has a $(1,2)$-point at q^{l-2}, hence its strict transform, B^{l-1}, is neither tangent to L^{l-1} nor to the exceptional divisor. We have, as $m<d$ and $l \leq b$

$$(C^{l-1} \cdot B^{l-1})_{q^{l-1}} = (C^{l-1} \cdot B^{l-1}) = bd-(l-1)m > bm-(b-1)m = m,$$

hence C^{l-1} and B^{l-1} are tangent in q^{l-1}, hence C^{l-1} is neither tangent to L^{l-1} nor to the exceptional divisor. Now case (c) of Lemma 2.1.1 occurs for C^{l-2}, i.e. q^{l-2} is a $(m,2m)$-point. Therefore $2m = n-(l-2)m$, hence $n = lm$, as stated.

Let us now study the δ-invariant for q as a point of C. Set

$$h = \lceil (bd-m)/m \rceil.$$

We prove, by induction on i, that q^i is an m-fold point of C^i for every $i \leq h-1$, and B^i and C^i are tangent in q^i. We already proved this for every $i \leq l-1$, hence the base case is settled and we assume $i \geq l$. The argument is similar to the one used in the previous part. Assume $\mathrm{mult}_{q^{i-1}}(C^{i-1}) = m$, and B^{i-1} tangent to C^{i-1} in q^{i-1}. Then, as $i \leq h-1$, we have

$$\begin{gathered}(C^i \cdot B^i)_{q^i} = (C^i \cdot B^i) = bd - im \geq bd-(h-1)m = \\ bd - (\lceil (bd-m)/m \rceil - 1)m > bd - (bd/m - 1)m = bd - bd + m = m\end{gathered}$$

(as $\lceil (bd-m)/m \rceil < bd/m$). Hence $(C^i \cdot B^i)_{q^i} > m$, hence C^i and B^i are tangent at q^i. Now, the curve B^{i-1} has a smooth point at q^{i-1}, hence B^i is not tangent to the exceptional divisor in q^i. Hence the same holds for C^i. Hence case (a) of Lemma 2.1.1 does not occur for C^{i-1}, hence q^i is an m-fold point of C^i. So we are done.

Since C^i has an m-fold point at q^i for every $i = 0, \dots, h-1$, by (3) the δ-invariant of q satisfies

$$\begin{aligned}\delta_C(q) &\geq hm(m-1)/2 \\ &= \lceil (bd-m)/m \rceil m(m-1)/2 \\ &\geq ((bd-m)/m)m(m-1)/2 \\ &= (bd-m)(m-1)/2.\end{aligned}$$

The proof is now complete. ∎

3. Hyper-bitangency for 3C-curves

3.1. **Definition and simple cases.** We study hyper-bitangent curves to a curve $B \subset \mathbb{P}^2$ which is the transverse union of three integral curves.

Definition 3.1.1. A *3C-curve* is a reduced plane curve $B = B_1 \cup B_2 \cup B_3$, with B_i integral of degree $b_i \geq 1$, such that every point in $B_i \cap B_j$ is a node of B for all $i \neq j$. We always assume $b_1 \leq b_2 \leq b_3$. We set

$$B_i \cap B_j = \{p^t_{i,j}, \quad t = 1, \dots, b_i b_j\}$$

with $p^t_{i,j} = p^t_{j,i}$; we often omit the superscript t. Notice that the components of B meet only pairwise, and transversally. We write $N := \cup_{i \neq j} B_i \cap B_j$.

We begin with the case $b_1 = b_2 = b_3 = 1$. This is a particularly simple curve which can be easily handled.

Proposition 3.1.2. *Let $B = B_1 \cup B_2 \cup B_3$ be a 3C-curve of degree 3. Then $\dim \mathrm{Hyp}_d(B, 2) \geq 1$ for every $d \geq 1$.*

Proof. The curve B is the union of three lines; set $B_i \cap B_j = \{p_{i,j}\}$ so that $N = \{p_{1,2}, p_{1,3}, p_{2,3}, \}$.

Suppose $d = 1$; then the one-dimensional space of lines through $\{p_{i,j}\}$, with B_i and B_j removed, lies in $\mathrm{Hyp}_1(B, 2)$, and these are the only elements of $\mathrm{Hyp}_1(B, 2)$. Hence $\dim \mathrm{Hyp}_1(B, 2) = 1$.

Let $d \geq 2$ and $C \in \mathrm{Hyp}_d(B, 2)$. Obviously $C \cap N \neq \emptyset$, say $p_{1,2} \in C$. Then the tangent line to C at $p_{1,2}$ (which is well defined as C is unibranched at $p_{1,2}$) must be different from at least one of the (different) tangent lines to B_1 and B_2 at $p_{1,2}$. Hence C must be transverse to at least one between B_1 and B_2, say C transverse to B_1, hence C meets B_1 in a further point. Since C must also meet B_3 we get $p_{1,3} \in C$, and C cannot be transverse to B_3. We derive that C is hypertangent to B_2 and B_3 respectively at $p_{1,2}$ and $p_{1,3}$. Now, setting $m = m_{p_{1,2}}(C)$ and $n = m_{p_{1,3}}(C)$, we have

$$d = (C \cdot B_1) = (C \cdot B_1)_{p_{1,2}} + (C \cdot B_1)_{p_{1,3}} = m + n \leq d$$

hence $d = n + m$. Interchanging the three components, we derive

$$\mathrm{Hyp}_d(B, 2) = \bigcup_{m=1}^{d-1} \bigcup_{\substack{i,j,h=1,2,3 \\ i \neq j}} \mathrm{Hyp}^m_d(B_i; p_{h,i}) \cap \mathrm{Hyp}^{d-m}_d(B_j; p_{h,j}).$$

Consider $m = 1$ and the subspace $\mathrm{Hyp}^1_d(B_i; p_{h,i}) \cap \mathrm{Hyp}^{d-1}_d(B_j; p_{h,j})$. Now, $\mathrm{Hyp}^1_d(B_i; p_{h,i})$ is the space of integral degree-d curves passing through $p_{h,i}$, and meeting the line B_i with multiplicity d at $p_{h,i}$. It is easy to see that the closure of this space in $\mathbb{P}^{d(d+3)/2}$ is a linear subspace of codimension d.

Next, $\mathrm{Hyp}^{d-1}_d(B_j; p_{h,j})$ is the space of integral degree-d curves having a $(d-1, d)$-point at $p_{h,j}$ with tangent line equal to B_j. To compute the codimension of its closure in $\mathbb{P}^{d(d+3)/2}$ we can assume that $p_{h,j}$ is the origin and the line B_j has equation $y = 0$. Then a curve $C \in \overline{\mathrm{Hyp}^{d-1}_d(B_j; p_{h,j})}$ has equation $\sum_{d-1\leq i+j\leq d} a_{i,j}x^iy^j = 0$ with $a_{i,j} = 0$ for every $i + j = d - 1$ and $i \neq 0$. One easily checks that such polynomials form a linear subspace of codimension equal to $d(d-1)/2 + d - 1 = (d^2 + d - 2)/2$. Therefore

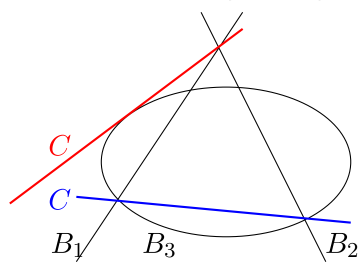
$$\dim \overline{\mathrm{Hyp}^1_d(B_i; p_{h,i})} \cap \overline{\mathrm{Hyp}^{d-1}_d(B_j; p_{h,j})} \geq d(d+3)/2 - d - (d^2 + d - 2)/2 = 1.$$

To conclude the proof it suffices to prove that a general point of the above intersection parametrizes an integral curve, i.e. an element in $\mathrm{Hyp}_d(B, 2)$. To do this, it suffices to prove that the above intersection contains one integral curve, i.e. that there exists an integral curve of degree d meeting B_i with multiplicity d at $p_{h,i}$ and with a $(d-1, d)$-point at $p_{h,j}$ with B_j as tangent line. By choosing projective coordinates x, y, z such that B_i and B_j have respective equations $z = 0$ and $y = 0$, with $p_{h,i} = (0 : 0 : 1)$ and $p_{h,j} = (0 : 1 : 0)$, we have the curve of equation $y^{d-1}z = x^d$ (see subsection 4.4 for the properties of such a curve). ■

3.2. **Hyper-bitangent lines.** We now describe hyper-bitangent lines to a 3C-curve of degree at least 4. This is quite elementary, possibly part of it already known. We include it for completeness and lack of references.

Proposition 3.2.1. *Let $B = B_1 \cup B_2 \cup B_3$ be a 3C-curve of degree $b \geq 4$.*

(a) If $b = 4$ then $|\mathrm{Hyp}_1(B, 2)| = 6$.
More precisely, $C \in \mathrm{Hyp}_1(B, 2)$ if and only if C is one of the four lines through $B_3 \cap (B_1 \cup B_2)$ different from B_1 and B_2, or C is one of the two lines tangent to B_3 and passing through the point $B_1 \cap B_2$.



(b) If $b \geq 5$ then $\mathrm{Hyp}_1(B, 2)$ is finite, and it is empty if B is general.
More precisely, if $b_1 = b_2 = 1$ then $|\mathrm{Hyp}_1(B, 2)| \leq 3b_3(b_3 - 2)$, and $|\mathrm{Hyp}_1(B, 2)| \leq 2|N|$ otherwise (recall $N = \cup_{i\neq j} B_i \cap B_j$).

Proof. If $b = 4$ then $b_1 = b_2 = 1$ and $b_3 = 2$; set $B_1 \cap B_2 = \{p_{1,2}\}$. We first look at the lines through two points of N. If C is a line through $p^t_{1,3}$ and $p^{t'}_{2,3}$ it clearly lies in $\mathrm{Hyp}_1(B, 2)$. If C is a line through $p^t_{i,3}$ and $p^{t'}_{i,3}$ then it is equal to B_i which is not possible. Now suppose that $C \in \mathrm{Hyp}_1(B, 2)$ is not one of these lines; then we must have $p_{1,2} \in C$ and C must meet B_3 in a unique point, hence C must be tangent to B_3. Part (a) is proved.

Let $b \geq 5$ and let B a general curve; we can make the following assumptions. If $b_3 \geq 3$ then B_3 has finitely many flexes, hence finitely many flex lines, and finitely many bitangent lines; we assume that B_1 and B_2 do not pass through any flex of B_3 or through any point where B_3 meets its bitangent lines, and that $B_1 \cap B_2$ intersects no line hyper-bitangent to B_3.

If $b_3 = 2$ then $b_2 = 2$, we assume that B_1 does not intersect $B_2 \cup B_3$ at any point where $B_2 \cup B_3$ meets its bitangent lines.

Finally, given any $p_{2,3} \in B_2 \cap B_3$, there are finitely many lines through $p_{2,3}$ that are tangent to B_3, hence there are finitely many points $r \in B_3$ such that the tangent line to B_3 at r passes through $B_2 \cap B_3$; we assume that B_1 does not intersect B_3 in any of these points. Also, if $b_2 \neq 1$, we assume that B_1 does not intersect B_2 in any point lying on some tangent line to B_2 or B_3 passing through $p_{2,3}$.

By contradiction, let $C \in \mathrm{Hyp}_1(B, 2)$. Suppose $|C \cap N| = 2$.

If $b_3 \geq 3$ then, as N contains no flex of B_3, we have $C \cap B = \{p_{1,3}, p_{2,3}\}$ and, as C cannot be a bitangent to B_3, we have

$$b_3 = (C \cdot B_3)_{p_{1,3}} + (C \cdot B_3)_{p_{2,3}} \leq 3$$

hence $b_3 = 3$. Now C must be tangent to B_3 in one of the points, $p_{i,3}$. Hence C is a line through $B_2 \cap B_3$ tangent to B_3 and intersecting $B_1 \cap B_3$; we excluded the existence of such lines, so we are done.

Let $b_3 = 2$, hence $b_2 = 2$. If $b_1 = 1$, up to switching B_2 and B_3 we have only the case $C \cap B = \{p_{1,2}, p_{2,3}\}$. Then C is tangent to B_3 at $p_{2,3}$ and intersects $B_1 \cap B_2$, which is excluded.

Let $b_1 = 2$. Our generality assumptions prevent us from switching B_1 with B_2 or B_3, so we have more cases. If $C \cap B = \{p_{1,2}, p_{1,3}\}$ then C is a bitangent of $B_2 \cup B_3$, which is excluded (as before). If $C \cap B = \{p_{1,2}, p_{2,3}\}$ then C is tangent to B_3 at $p_{2,3}$ and intersects $B_1 \cap B_2$, which is excluded. As we can switch B_2 with B_3 we are done. We thus proved that $|C \cap N| = 1$.

Suppose $C \cap N = \{p_{1,2}\}$. Then C meets B_3 in a point $r \notin N$, and it is hypertangent to B_3 at r. If $b_3 \geq 3$ then r is a flex and, by our assumptions, the flex line does not pass through $B_1 \cap B_2$. If $b_3 = 2$ then $b_2 = 2$ and C is a bitangent of $B_2 \cup B_3$, which is also excluded.

Suppose $C \cap N = \{p_{1,3}\}$, then either $b_3 \geq 3$ and C is a flex line of B_3 which is excluded, or $b_3 = b_3 = 2$ and C is a bitangent of $B_2 \cup B_3$, which is excluded.

Suppose $C \cap N = \{p_{2,3}\}$. Now, C cannot be tangent to both B_2 and B_3, hence $b_2 = 1$, hence $b_1 = 1$, hence $b_3 \geq 3$, hence $p_{2,3}$ is a flex of B_3. A contradiction. We thus proved that $\mathrm{Hyp}_1(B, 2)$ is empty for B general.

If B is an arbitrary curve, the proof shows that for a curve $C \in \mathrm{Hyp}_1(B, 2)$ only two cases can occur. First case: C is tangent to some component of B in a point of N; since at each point of N there are two such tangent lines we have at most $2|N|$ possibilities for such a C.

Second case: C meets all components of B transversally along N. Then one easily checks that $b_1 = b_2 = 1$, moreover C passes through $B_1 \cap B_2$ and is hypertangent to B_3 in a flex (or rather a hyper-flex) or in a unibranch singular point. It is well known that the number of flexes of B_3 is at most equal to $3b_3(b_3-2)$; hence in the second case we have at most $3b_3(b_3-2)$ possibilities for C.

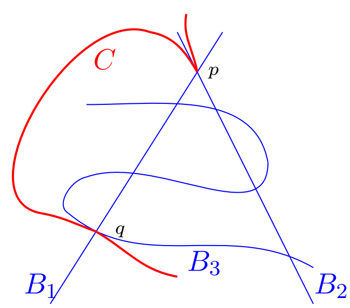
If $b_1 = b_2 = 1$ the first case occurs only with C hypertangent B_3, hence the bound $\mathrm{Hyp}_1(B,2) \leq 3b_3(b_3-2)$ holds. ■

3.3. **Hyper-bitangent curves of higher degree.** We begin with a geometric description of hyper-bitangent curves.

Theorem 3.3.1. *Let $d \geq 2$. Let $B = B_1 \cup B_2 \cup B_3$ be a 3C-curve of degree $b \geq 4$ such that $\mathrm{Hyp}_d(B,2)$ is not empty. Then $b_1 = 1$ and the following occur.*

(a) If $b_2 = 1$, setting $B_1 \cap B_2 = \{p\}$, we have

$$\mathrm{Hyp}_d(B,2) = \bigcup_{\substack{i=1,2 \\ q \in B_i \cap B_3}} \mathrm{Hyp}_d^{d-1}(B_i;p) \cap \mathrm{Hyp}_d(B_3;q).$$



(b) If $b_2 \geq 2$ then $\mathrm{Hyp}_d(B,2)$ is empty for $d \geq 3$ and

$$\mathrm{Hyp}_2(B,2) = \bigcup_{\substack{p \in B_1 \cap B_2 \\ q \in B_1 \cap B_3}} \mathrm{Hyp}_2(B_2,p) \cap \mathrm{Hyp}_2(B_3,q).$$

(c) Every $C \in \mathrm{Hyp}_d(B,2)$ is rational (hence $\mathcal{E}(B) = \mathrm{Hyp}(B,2)$, with $\mathcal{E}(B)$ defined in (1)).

Proof. Since $b \geq 4$ we have $b_3 \geq 2$. Let $C \in \mathrm{Hyp}_d(B,2)$, then $|C \cap B| \leq 2$; on the other hand $B_1 \cap B_2 \cap B_3 = \emptyset$, hence $|C \cap B| = 2$.

Let us prove that $C \cap B_3 \subset N$ and $|C \cap B_3| = 1$. By contradiction, suppose $C \cap B_3$ contains a point not in N. Hence C must intersect $B_1 \cup B_2$ in exactly one point, $p_{1,2} \in B_1 \cap B_2$. Now, $p_{1,2}$ is a unibranch n-fold point of C, for some $n < d$ (as $d > 1$). Since B_1 and B_2 meet transversally, C is tangent to one of them and transverse to the other, say C is transverse to B_i with $i < 3$. Therefore

$$b_i d = (C \cdot B_i) = (C \cdot B_i)_{p_{1,2}} = n < d$$

a contradiction. Hence $C \cap B_3 \subset N$.

By contradiction, suppose $|C \cap B_3| = 2$; as C must intersect B_1 and B_2 we have $C \cap B_3 = \{p_{1,3}, p_{2,3}\} \subset N$, with $p_{i,3} \in B_i \cap B_3$. Since B_3 and B_i meet transversally, C must be transverse to either B_i or B_3 at $p_{i,3}$. If C is transverse to B_i then, arguing as above, we get $(C\cdot B_i) < d$, a contradiction. Hence C is transverse to B_3 at both $p_{1,3}$ and $p_{2,3}$. Set $m_i = \mathrm{mult}_{p_{i,3}}(C)$, so that $m_1 + m_2 \leq d$. Then

$$b_3 d = (C \cdot B_3) = (C \cdot B_3)_{p_{1,3}} + (C \cdot B_3)_{p_{2,3}} = m_1 + m_2 \leq d$$

which is impossible, as $b_3 \geq 2$.

We thus proved that $C \cap B_3 = \{p_{i,3}\}$ for one $i \in \{1, 2\}$. Hence C is hypertangent to B_3 at $p_{i,3}$. As C is not transverse to B_3 a $p_{i,3}$, it must be transverse to B_i, hence it must meet B_i in a further point. As C must meet the other component, B_j with $j \neq i, 3$, there exists a point $p_{1,2} \in B_1 \cap B_2$ such that $p_{1,2} \in C$. We obtain $C \cap B = \{p_{1,2}, p_{i,3}\}$ and $C \cap B_j = \{p_{1,2}\}$. Now B_j and C meet only at $p_{1,2}$, hence C cannot be transverse to B_j at this point, hence it must be transverse to B_i. Therefore

$$b_i d = (C \cdot B_i) = (C \cdot B_i)_{p_{1,2}} + (C \cdot B_i)_{p_{i,3}} = \mathrm{mult}_{p_{1,2}}(C) + \mathrm{mult}_{p_{i,3}}(C) \leq d$$

hence $b_i = 1$ and equality holds, i.e. $\mathrm{mult}_{p_{1,2}}(C)+\mathrm{mult}_{p_{i,3}}(C) = d$. Therefore $b_1 = 1$ and $C \in \mathrm{Hyp}_d^{d-m}(B_2, p_{1,2}) \cap \mathrm{Hyp}_d^m(B_3, p_{1,3})$, with $m = \mathrm{mult}_{p_{1,3}}(C)$.

We now show that $m = 1$. Since B_3 has degree at least 2 and is smooth at $p_{1,3}$, it has a $(1, l)$-point there for some $l \geq 2$. We can apply Theorem 2.2.1 to B_3 and C, getting that $p_{1,3}$ is an (m, lm) point for C. Suppose $m \geq 2$; the same theorem yields $\delta_C(p_{1,3}) \geq (b_3 d - m)(m - 1)/2$. On the other hand C has also a $(d-m)$-fold point at $p_{1,2}$, hence $\delta_C(p_{1,2}) \geq (d-m)(d-m-1)/2$. Therefore

$$\begin{aligned}
g(C) &\leq \binom{d-1}{2} - (b_3 d - m)(m-1)/2 - (d-m)(d-m-1)/2 \\
&= \binom{d-1}{2} - (d^2 - d - 2md + b_3 md - b_3 d + 2m)/2 \\
&= \binom{d-1}{2} - (d^2 + d(-1 + m(b_3 - 2) - b_3) + 2m)/2 \\
&\leq \binom{d-1}{2} - (d^2 - 3d + 4)/2 \\
&= (d^2 - 3d + 2)/2 - (d^2 - 3d + 4)/2 \\
&< 0,
\end{aligned}$$

as $m \geq 2$ and $b_3 \geq 2$. This is impossible. Hence $m = 1$ and

$$C \in \mathrm{Hyp}_d^{d-1}(B_2, p_{1,2}) \cap \mathrm{Hyp}_d(B_3, p_{1,3}). \tag{5}$$

If $b_2 = 1$ we can switch roles between B_1 and B_2; part (a) is proved.

Assume $b_2 \geq 2$. Then, as B_2 is smooth at $p_{1,2}$, we can apply Theorem 2.2.1, which gives that C has an (h, lh) point at $p_{1,2}$, for some $h \geq 1$. Now (5) implies that $p_{1,2}$ is a $(d-1, d)$-point of C. Therefore $(h, lh) =$

$(d-1,d)$, hence $h=1$ and $d=2$. Therefore $\mathrm{Hyp}_d(B,2)$ is empty if $d \geq 3$, and the proof of part (b) is complete.

A curve of degree d having a $(d-1)$-fold point is necessarily rational, hence part (c) follows from the previous parts. ■

If $d \geq 2$, Theorem 3.3.1 implies that $\mathrm{Hyp}_d(B,2) = \emptyset$ whenever $b_1 > 1$, or $b_2 \geq 2$ and $d \geq 3$. We now treat the remaining cases. We denote by $\mathrm{Hyp}_{\geq 2}(B,2)$ the set of curves of degree at least 2 that are hyper-bitangent to B, i.e. $\mathrm{Hyp}_{\geq 2}(B,2) := \cup_{d\geq 2} \mathrm{Hyp}_d(B,2)$.

Theorem 3.3.2. *Let B be a 3C-curve of degree $b \geq 4$ such that $b_1 = 1$. Then*
(a) $|\mathrm{Hyp}_{\geq 2}(B,2)| \leq b_3 \max\{2, b_2\}$;
(b) $|\mathrm{Hyp}_{\geq 2}(B,2)| = 0$ if B is general.

Proof. Let $C \in \mathrm{Hyp}_d(B,2)$ with $d \geq 2$; by Theorem 3.3.1, up to switching B_1 and B_2 when $b_2 = 1$, we have $C \in \mathrm{Hyp}_d^{d-1}(B_2;p) \cap \mathrm{Hyp}_d(B_3;q)$ with $p \in B_1 \cap B_2$ and $q \in B_1 \cap B_3$. Now, C has a $(d-1,d)$-fold point at p where it is tangent to B_2, and a smooth point at q where it is tangent to B_3. We can choose homogeneous coordinates (X,Y,Z) in $\mathbb{P}^2$ so that $p = (0:1:0)$ and the tangent line to C at p has equation $z = 0$. Therefore in the open subset where $Y \neq 0$ the curve C has affine equation $z^{d-1} = \sum_{i=0}^{d} c_i x^i z^{d-i}$. We can assume that $q = (0:0:1)$ and the tangent line to B_3 has equation $y = 0$. Hence $c_0 = c_1 = 0$ and the affine equation of C where $Z \neq 0$ is

$$y = g(x) \qquad \text{where} \qquad g(x) := c_d x^d + c_{d-1}x^{d-1} + \ldots + c_2 x^2.$$

Claim. $g(x) = c_d x^d$.

We can assume $d \geq 3$. We provide two proofs of the claim, giving different insights.

First proof. We follow the proof of [Kól24, Lm. 42]. The tangent line, L, to C at q has equation $y = 0$, hence it suffices to prove that $(L \cdot C)_q = d$. Denote by $(B_3)_C$ and L_C the (Cartier) divisors cut by B_3 and L on C. Set $n := b_3$ to simplify; we have $(B_3)_C = ndq$ and, of course, $(B_3)_C \sim nL_C$, hence $n(dq - L_C) \sim 0$. On the other hand, C is a rational curve whose unique singular point, p, is unibranch, hence $\mathrm{Pic}\, C$ has no torsion; see [Har77, Ex. 6.11.4]. Therefore $dq \sim L_C$, hence (as curves of degree d cut on C a complete linear series) $dq = L_C$, and the claim is proved.

Second proof. Let $f(x,y) = 0$ be the affine equation of B_3; as B_3 is smooth at q and tangent to $y = 0$, we have

$$f(x,y) = y + \sum_{2 \leq i+j \leq n} a_{i,j} x^i y^j \tag{6}$$

with $n = b_3$. We have $(C \cdot B_3) = (C \cdot B_3)_q = dn$, therefore $f(x, g(x)) = \lambda x^{dn}$ for some $\lambda \neq 0$. Hence the following holds

$$g(x) + \sum_{2 \leq i+j \leq n} a_{i,j} x^i g(x)^j = \lambda x^{dn}. \tag{7}$$

Now, for every i, j as above, the product $x^i g(x)^j$ is a sum of monomials in x whose degrees range in the set $I(i,j)$, where

$$I(i,j) = [i+2j, i+jd] \cap \mathbb{N}.$$

We have

$$i + dj \leq d(i+j) \leq dn$$

with equality if and only if $j = n$ and $i = 0$. We have

$$I(0,n) = [2n, dn], \qquad I(1, n-1) = [2n-1, dn-d+1],$$

therefore in the left side of (7), the following monomials (up to scalar)

$$x^{dn},\ x^{dn-1}, \ldots, x^{dn-d+2}$$

appear only in $g(x)^n$. For (7) to hold, in its left side

(a) the coefficient of x^{dn} is non zero, hence $a_{0,n} \neq 0$ and $c_d \neq 0$;

(b) the coefficients of $x^{dn-1}, \ldots, x^{dn-(d-2)}$ are zero.

We now show, by induction on h, that (a) and (b) imply that $c_{d-h} = 0$ for $h = 1, \ldots, d-2$. We write $g(x)^n$ as follows

$$g(x)^n = \sum_{k=1}^{d-2} \left(\sum_{\substack{2 \leq i_1 < \ldots < i_k \leq d \\ \sum_1^k n_j = n,\ n_j \geq 1}} \mu_{n_1,\ldots,n_k} c_{i_1}^{n_1} \cdot \ldots \cdot c_{i_k}^{n_k} x^{n_1 i_1 + \ldots + n_k i_k} \right)$$

where $\mu_{n_1,\ldots,n_k}$ are positive integers which we can ignore. Since $i_k \leq d$ and $i_j < i_{j+1}$ we have $i_j \leq i_k - (k-j) \leq d - (k-j)$, hence the exponent of x above satisfies the following

$$\sum_{j=1}^k n_j i_j \leq \sum_{j=1}^k n_j (d - (k-j)) = n_1(d-(k-1)) + \ldots + n_{k-1}(d-1) + n_k d$$
$$= d \sum_{j=1}^k n_j - \sum_{j=1}^{k-1} n_j (k-j) \leq dn - (k-1)k/2$$

where in the second inequality we used $n_j \geq 1$ for all $j \leq k-1$. Hence we have equality if and only if $i_j = d - (k-j)$ for $j \leq k$ and $n_j = 1$ for $j \leq k-1$. If these conditions are satisfied, we furthermore have

$$\sum_{j=1}^k n_j i_j = dn - 1 \quad \text{ if and only if } \quad k = 2,$$

i.e. $i_1 = d-1$, $i_2 = d$, $n_1 = 1$, $n_2 = n-1$. Therefore the term x^{dn-1} appears in $g(x)^n$ only once, with coefficient equal to (a positive integer multiple of) $c_d^{n-1} c_{d-1}$. Hence x^{nd-1} appears in the left of (7) with coefficient $a_{0,n} c_{d-1} c_d^{n-1}$; as this coefficient must be zero and $a_{0,n} c_d \neq 0$, we get $c_{d-1} = 0$. The induction base is proved.

Assume $c_{d-1} = c_{d-2} = \ldots = c_{d-h+1} = 0$. We have a non-zero coefficient of $x^{\sum_1^k n_j i_j}$ only if $i_j \notin \{i_{d-1}, i_{d-2}, \ldots, i_{d-h+1}\}$, in which case we have

$$\sum_{j=1}^{k} n_j i_j \leq n_1(d-h-k+2) + \ldots + n_{k-2}(d-h-1) + n_{k-1}(d-h) + n_k d$$
$$= dn - h\sum_{j=1}^{k-1} n_j - \sum_{j=1}^{k-2} j \leq dn - h(k-1) - (k-1)(k-2)/2.$$

The first inequality is an equality if and only if $i_k = d$, and $i_{k-1} = d-h$, and $i_j = i_{j+1} - 1$ for $j < k-1$; the second inequality is an equality if and only if $n_j = 1$ for all $j \leq k-1$. If these conditions hold, we furthermore have $\sum_{j=1}^{k} n_j i_j = dn - h$ if and only if $k = 2$. Therefore, arguing as above, x^{dn-h} appears in $g(x)^n$ with coefficient $c_{d-h}c_d^{n-1}$. Hence it appears in the left of (7) with coefficient $a_{0,n}c_{d-h}c_d^{n-1}$; as this must be zero, we get $c_{d-h} = 0$. The claim is proved.

Thus C has equation $y = c_d x^d$, hence q is a $(1,d)$-point of C. But C is hypertangent to B_3 at q, hence Theorem 2.2.1 implies that B_3 also has a $(1,d)$-point at q.

Assume $d \geq 3$. If B is general we can assume that no point in $B_3 \cap (B_1 \cup B_2)$ is a flex of B_3. Hence q is a $(1,2)$-point of B_3 and we get a contradiction. Therefore $\mathrm{Hyp}_d(B,2) = \emptyset$ if B is general.

If, instead, B_3 has a $(1,l)$-point in q for some l with $3 \leq l \leq b_3$, we get $d = l$, hence d is determined by B_3, and $d \leq b_3$. Moreover no term of type x^i with $i < d$ can appear in the equation of B_3 (for q is a $(1,d)$-point), hence $a_{i,0} = 0$ for all $i < d$, and from (7) we derive

$$c_d x^d + a_{d,0}x^d + (\text{terms of degree } > d) = \lambda x^{dn}.$$

As $n \geq 2$ we get $c_d = -a_{d,0}$. This proves that C (if it exists) is uniquely determined by B and q.

Assume $d = 2$, then q is a $(1,2)$-point for C and B_3. Now (7) gives

$$c_2x^2 + a_{2,0}x^2 + a_{1,1}c_2x^3 + a_{3,0}x^3 + (\text{terms of degree } > 3) = \lambda x^{2n}.$$

We obtain $a_{2,0} = -c_2$ and $a_{1,1}c_2 = -a_{3,0}$, hence $a_{3,0} = a_{1,1}a_{2,0}$. Hence B_3 is not a general curve of degree n (for its equation, (6), must satisfy $a_{3,0} = a_{1,1}a_{2,0}$). Hence $\mathrm{Hyp}_2(B,2) = \emptyset$ if B is general. (b) is proved.

If B_3 is not general then, as before, C is determined by the condition $c_2 = -a_{2,0}$, hence it is determined by B and q.

Summarizing, for all $d \geq 2$ we proved that, for every $q \in B_1 \cap B_3$ there exists at most one curve $C \in \mathrm{Hyp}_d(B_2;p) \cap \mathrm{Hyp}_d(B_3;q)$; as q varies in $B_1 \cap B_3$ we get at most b_3 curves in $\mathrm{Hyp}(B,2)$.

If $b_2 = 1$ the same argument applies by taking $q \in B_2 \cap B_3$ hence we might have b_3 new elements in $\mathrm{Hyp}(B,2)$. Hence $|\mathrm{Hyp}_{\geq 2}(B,2)| \leq 2b_3$.

If $b_2 \geq 2$ we have b_2 choices for the point $p \in B_1 \cap B_2$, hence we obtain $|\operatorname{Hyp}_{\geq 2}(B,2)| \leq b_2 b_3$. ■

For an example where the bound in the theorem is attained, see subsection 4.2.

Corollary 3.3.3 (of the proof)**.** *If B is a 3C-curve of degree at least 4, and $C \in \operatorname{Hyp}(B,2)$, then $\deg C \leq b_3$ (hence $\deg C \leq \deg B - 2$).*

It is worth stating the following simple consequence of our earlier results.

Theorem 3.3.4. *The set $\operatorname{Hyp}(B,2)$ of a 3C-curve B of degree at least 5 is finite, and it is empty if B is general.*

Proof. If $d = 1$ this follows from Proposition 3.2.1. Assume $d \geq 2$; then this follows from Theorem 3.3.1 if $b_1 > 1$, and from Theorem 3.3.2 if $b_1 = 1$. ■

4. Examples

We collect here some examples and special cases related to our results.

4.1. **Examples of hypertangent curves.** Let $B, C \subset \mathbb{P}^2$ be two integral curves as in Theorem 2.2.1, i.e. they have degree at least 2 and $B \cap C = \{q\}$ with q a unibranch point for B and C. The following examples show that there are cases in which q is a $(1,l)$-point for B and q is a singular point of C, i.e. is a (m, lm) point with $m \geq 2$. In particular, in this setting, the type of the point q for B is different from its type as a point of C.

(1) Consider the following two curves:

$$B : y = x^2 \qquad C : (y - x^2)^3 + y^7.$$

One easily checks that $C \in \operatorname{Hyp}(B; q)$ where q is the origin, and that B has a $(1,2)$ point while C has a $(3,6)$ point, so that $m = 3$ and $l = 2$ in our notation. More generally, the following pair of curves,

$$B : y = x^2 \qquad C : (y - x^2)^c + y^{2c+1},$$

gives similar examples where B has a $(1,2)$ point and C has a $(c, 2c)$ point for every $c \geq 3$.

(2) This second example shows that a similar situation can happen when l is not 2. Consider the following two curves:

$$B : y = x^3 \qquad C : (y - x^3)^3 + y^9.$$

In this case one checks that q is a $(1,3)$-point for B while it is a $(3,9)$-point for C; as before $C \in \operatorname{Hyp}(B; q)$.

We stress that both these examples do not yield examples of curves in $\operatorname{Hyp}(\tilde{B}, 2)$ for a 3C-curve $\tilde{B}$ containing B. In fact, the proof of Theorem 3.3.2 shows that the curves C described above will have to meet the curve $\tilde{B}$ in more than two points.

4.2. **Explicit examples of hyper-bitangent conics.** We show that the bounds in Theorem 3.3.2 are sharp, in the sense that there are examples in which the set $\mathrm{Hyp}_{\geq 2}(B, 2)$ consists of exactly $2b_3$ curves.

For an explicit example consider the following three components of a 3C-curve $B = B_1 \cup B_2 \cup B_3$ with $(b_1, b_2, b_3) = (1, 1, 2)$:

$$B_1 : x = 0, \qquad B_2 : z = 0, \qquad B_3 : zy = x^2 - y^2.$$

Let $\{q_1, q_2, q_3, q_4\} = (B_1 \cap B_3) \cup (B_2 \cap B_3)$, where

$$q_0 = (0 : 0 : 1), \quad q_1 = (0 : -1 : 1), \quad q_2 = (1 : 1 : 0), \quad q_3 = (1 : -1 : 0).$$

Then for $i = 0, 1, 2, 3$ the following conics C_i satisfy $C_i \in \mathrm{Hyp}_2(B_3; q_i)$:

$$C_0 : zy = x^2, \qquad C_1 : zy = -z^2 - x^2,$$
$$C_2 : 4xz + 8xy - 8x^2 = z^2, \qquad C_3 : 4xz + 8xy + 8x^2 = -z^2.$$

Moreover, $C_0, C_1 \in \mathrm{Hyp}_2(B_2; p)$ and $C_2, C_3 \in \mathrm{Hyp}_2(B_1; p)$ where $p = (0 : 1 : 0)$. In other words B_2 is the tangent line to C_0 and C_1 at p, while B_1 is the tangent line to C_2 and C_3 at p. Therefore, for every $i = 0, 1, 2, 3$, $C_i \in \mathrm{Hyp}_2(B; p, q_i)$ and hence $|\mathrm{Hyp}_2(B, 2)| = 4$ reaching the upper bound of (a) in the statement of Theorem 3.3.2. A picture of these conics in the affine patch $y = 1$ can be seen in figure 1.

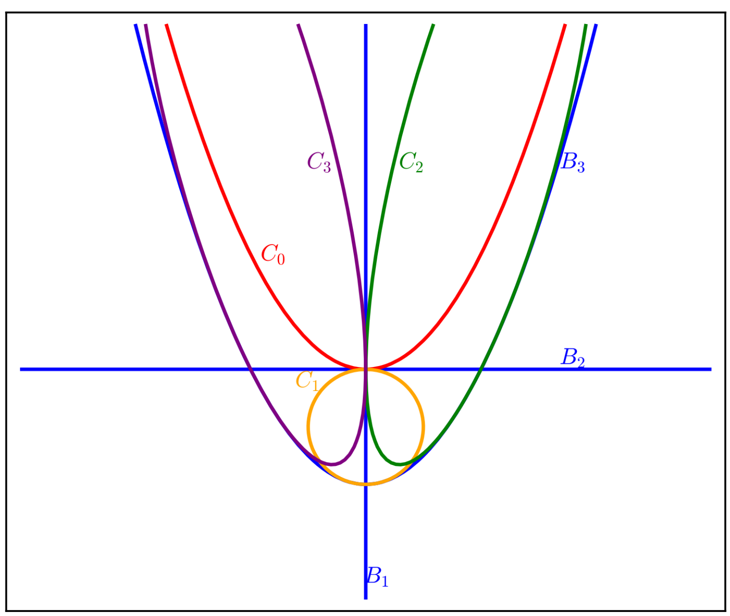


FIGURE 1. Example of 4 conics in $\mathrm{Hyp}_2(B, 2)$

4.3. **Curves with many components.** If a curve B has more than three components, it is not hard to prove that the only hyper-bitangent curves to B are lines, and describe such lines effectively. We include the analysis of this case for completeness.

Proposition 4.3.1. *Let B be a reduced curve of degree b with $c \geq 4$ irreducible components, such that every point in the intersection of two components is a node of B.*

(a) If $c \geq 5$ then $\mathrm{Hyp}(B, 2) = \emptyset$.
(b) If $d = 1$ then $\mathrm{Hyp}_1(B, 2)$ is finite; moreover
  (i) if $b = 4$ then $|\mathrm{Hyp}_1(B, 2)| = 3$;
  (ii) if B does not contain two lines, then $\mathrm{Hyp}_1(B, 2) = \emptyset$;
  (iii) if $b \geq 5$ and B is general then $\mathrm{Hyp}_1(B, 2) = \emptyset$.
(c) If $d \geq 2$ then $\mathrm{Hyp}_d(B, 2) = \emptyset$.

Proof. Write $B = B_1 \cup \ldots \cup B_c$ with $c \geq 4$; by hypothesis through every node of B there pass at most two components. Suppose there exists a curve $C \in \mathrm{Hyp}(B, 2)$. Then C meets B in at most two points, and must intersect all components of B. Hence $C \cap B = \{p, q\}$ and p, q belong to exactly two components, say $p \in B_1 \cap B_2$ and $q \in B_3 \cap B_4$. In particular, B has only 4 components, proving (a).

Let $d = 1$ and $C \in \mathrm{Hyp}_1(B, 2)$. As we said, $C \cap B = \{p, q\}$. Since B has finitely many (intersection) nodes, $\mathrm{Hyp}_1(B, 2)$ is finite.

If $b = 4$ then B is the union of 4 lines. It is clear that $\mathrm{Hyp}_1(B, 2)$ is made of the three bitangent lines of B not contained in B (namely the three lines not in B and joining a pair of nodes of B).

Let $b \geq 5$. We can assume that one component, B_4, has degree ≥ 2. Now, as B_3 and B_4 meet transversally in q and $C \cap B_4 = \{q\}$, the line C is necessarily hypertangent to B_4 at q; also, B_3 must have degree 1, for it meets C transversally in only one point. Now, if such a line C exists it is unique and has to pass through the point p as well. Arguing in the same way for B_1 and B_2 we have that at least one between B_1 and B_2 has degree 1. Hence for $\mathrm{Hyp}_1(B, 2)$ to be non-empty at least two components of B have degree 1.

Now, if the curve B is general, we can assume that no such line exists, i.e. we can assume that for every point $q \in B_i \cap B_j$ the tangent lines to B_i and B_j in q do not pass through any other intersection node of B. This proves that $\mathrm{Hyp}_1(B)$ is empty if B is general. (b) is proved.

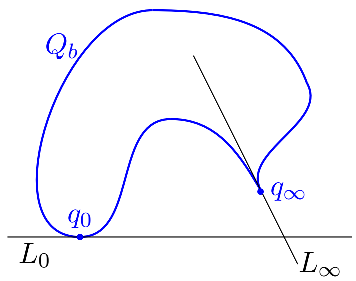
Let $d \geq 2$. By contradiction, let $C \in \mathrm{Hyp}_d(B, 2)$. As before, we assume $C \cap B = \{p, q\}$ with $p \in B_1 \cap B_2$ and $q \in B_3 \cap B_4$. Now, B_1 and B_2 meet transversally, hence C in p must be transverse to at least one between B_1 and B_2; say C is transverse to B_1. Hence C must intersect B_1 in a further point, and this point must be q, which is impossible as q cannot belong to three components of B. ∎

4.4. **Hypertangency of rational curves.** For an integral curve $B \subset \mathbb{P}^2$ of degree $b \geq 4$ having at most nodal singularities, Conjecture 1 predicts that there exist only finitely many rational curves hyper-bitangent to B, i.e. the set $\mathcal{E}(B)$ is finite. We will provide an example showing the necessity, in the conjecture, that B have only nodal singularities. This example was already considered by Vojta in [Voj87, Example 3.5.3].

For every integer $b \geq 3$ we denote by $Q_b \subset \mathbb{P}^2$ the curve given by the homogeneous equation

$$z^{b-1}y = x^b.$$

The curve Q_b is smooth at $q_0 = (0:0:1)$, with tangent line, L_0, of equation $y = 0$. We have $Q_b \cap L_0 = \{q_0\}$, so L_0 is hypertangent to Q_b. Next, Q_b has an $(b-1)$-fold unibranch point at $q_\infty = (0:1:0)$ with tangent line L_∞ of equation $z = 0$. We have $Q_b \cap L_\infty = \{q_\infty\}$ so that L_∞ is hypertangent and q_∞ is an $(b-1, b)$-point.



One checks easily that Q_b is integral, has no other singular point, and is a rational curve; it is, of course, not a nodal curve. Let us look at the set $\mathcal{E}_d(Q_b) \subset \mathbb{P}^{d(d+3)/2}$ of rational curves of degree d hyper-bitangent to Q_b. If $d = 1$ this has dimension 1, as it contains all lines through q_∞. We have

Proposition 4.4.1. *For every $d \geq b$ and $b \geq 4$ we have*

$$\dim \mathrm{Hyp}_d(Q_b, 1) \geq 1 \tag{8}$$

and

$$\dim \mathcal{E}_b(Q_b) \geq 1 \tag{9}$$

Proof. Consider the curve C_t of equation $y - x^b + ty^d = 0$ with $t \in \mathbb{C}$. It is clear that C_t lies in $\mathrm{Hyp}_d(Q_b; q_0)$ for every $t \neq 0$ hence (8) follows. Notice that C_t is smooth, hence not rational, for $t \neq 0$.

To prove (9) we will exhibit a one-dimensional family of curves in $\mathcal{E}_b(B)$. For every $t \neq 0, 1$ let R_t be the curve having equation $z^{b-1}y = tx^b$. It is easy to check that R_t is integral, and its singular locus consists of q_∞ which is a $(b-1, b)$-singular point, hence R_t is rational. Moreover, we have

$$R_t \cdot Q_b = bq_0 + b(b-1)q_\infty$$

hence $R_t \in \mathrm{Hyp}(Q_b; q_0, q_\infty)$. As t varies in $\mathbb{C}$ the curves R_t form a family with a non-integral member, for $t = 0$, hence the family is not constant. Therefore the R_t's give a one-dimensional subspace of $\mathcal{E}_b(Q_b)$. ■

We stress that the condition that the integral curve $B \subset \mathbb{P}^2$ has at most nodal singularities, although sufficient is not necessary. In other words there exist curves B, with worse singularities, for which $\mathcal{E}(B)$ is still finite. A series of examples was constructed in [CZ00], see in particular the Proposition in page 2. To see the link with our setting, fixing x, y, z as coordinates of $\mathbb{P}^2$, the curve B is given in op. cit. by the union of the line at infinity, the line $x = 0$, and a third component of affine equation $f(x, y) = 0$.

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