

EXCEPTIONAL LOCI IN ALGEBRAIC SURFACES

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ABSTRACT. We study the algebraic exceptional set for surfaces (S, B) of log general type, when B has at least three irreducible components; we prove that in most cases it is finite or empty.

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1. INTRODUCTION

We work over \mathbb{C} . Let S be a smooth connected projective surface, and let $B \subset S$ be a reduced projective curve of the form $B = B_1 \cup \dots \cup B_n$, with B_i integral curves such that for all $i \neq j$ every point in $B_i \cap B_j$ is a node of B . The main focus of this article is Lang's (algebraic) exceptional set, i.e.

$$(1) \quad \mathcal{E}(B) := \{C \subset S : C \text{ integral curve, } 2g(C) - 2 + |\nu_C^{-1}(C \cap B)| \leq 0\},$$

where $\nu : C^\nu \rightarrow C$ is the normalization and C is projective.

Remark 1.1. When B is ample, as it will be the case for most of our applications, then $C \cap B$ is not empty. Hence if $C \in \mathcal{E}(B)$ then C is necessarily rational.

According to Lang and Vojta conjectures (see for example [Lan86, Voj87]), the union of the curves in $\mathcal{E}(B)$ should coincide with the arithmetic exceptional set (the largest set containing all but finitely many integral points for every finite extension of the base field, when S is defined over a number field), and with the analytic exceptional set (the Zariski closure of the union of all the images of non constant holomorphic maps $\mathbb{C} \rightarrow S \setminus B$). Moreover, if the pair (S, B) is of log general type, the set $\mathcal{E}(B)$ is conjectured to be finite.

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The main result of this article, confirming the above conjecture, is the description of the set $\mathcal{E}(B)$ when B has at least three very ample irreducible components, generalizing some results obtained in [CT25] for $S = \mathbb{P}^2$. The case of Hirzebruch surfaces $S = \mathbb{F}_n$ is treated in [Che25].

The following is a summary of our results, with the case $n = 3$ being the most interesting one.

Theorem 1.2. *Let S and $B = \cup_{i=1}^n B_i$ be as above. The following hold:*

- (1) *if $n \geq 5$ then $\mathcal{E}(B)$ is empty;*
- (2) *if $n = 4$ and B_i is very ample for every i , then $\mathcal{E}(B)$ is finite, and is empty if at least three components of B are not primitive;*
- (3) *if $n = 3$, B_i is very ample for every i , and at least two components of B are not primitive, then $\mathcal{E}(B)$ is finite, and is empty if all components of B are not primitive.*

In the statement by “primitive” we mean not multiple of very ample. The hypotheses of the theorem ensure that the pair (S, B) is indeed of log general; cf. Remark 4.5.

For the cases $n = 4, 5$ see Proposition 4.1. For $n = 3$ see Theorems 4.2 and 4.4. To prove our results we drop the genus assumption and consider the following larger set

$$(2) \quad \text{Hyp}(B, 2) := \{C \subset S : C \text{ integral curve, } |\nu^{-1}(C \cap B)| \leq 2\} \supset \mathcal{E}(B).$$

We then prove a stronger statement, namely we prove that Theorem 1.2 holds with $\text{Hyp}(B, 2)$ instead of $\mathcal{E}(B)$. We thus prove the finiteness of $\text{Hyp}(B, 2)$; by contrast, if B has only two irreducible components, we have examples where $\text{Hyp}(B, 2)$ is infinite and strictly larger than $\mathcal{E}(B)$.

The subsets of S made by the curves in $\text{Hyp}(B, 2)$ and $\mathcal{E}(B)$ will be denoted as follows

$$[\text{Hyp}(B, 2)] := \bigcup_{C \in \text{Hyp}(B, 2)} C \subset S, \quad [\mathcal{E}(B)] := \bigcup_{C \in \mathcal{E}(B)} C \subset S.$$

We abuse terminology and refer to $[\mathcal{E}(B)]$ as the (algebraic) exceptional set.

The exceptional set, particularly in connection with Vojta conjecture [Voj87, Conj. 3.4.3], has been considered in several articles; see for instance [CZ13],[GNSW25],[CRY23], and references therein. These papers establish the geometric form of Vojta conjecture and, as a consequence, deduce various properties of the exceptional set. Our Theorem 1.2 applies in some situations which are not covered by such earlier results; it applies to surfaces that are not ramified covers of \mathbb{G}_m^2 , and we impose no generality assumptions on the surface S or on the divisor B . On the other hand, our normal crossings and primitivity assumptions prevent Theorem 1.2 from applying to various cases (as in the above papers) where the geometric Vojta conjecture is known to hold.

1.1. Brody Hyperbolicity. Theorem 1.2 also provides many examples of Brody hyperbolic surfaces where the boundary divisor has (at least) three irreducible components.

Recall that a complex analytic space Y is called *Brody hyperbolic* if every holomorphic map $\mathbb{C} \rightarrow Y$ is constant. The Green-Griffiths-Lang Conjecture predicts that, if X is a complex algebraic variety of log general type, there exists a proper closed subset Z , such that if $f : \mathbb{C} \rightarrow X$ is a non-constant holomorphic map, then $f(\mathbb{C}) \subset Z$. In this case we say that X is *Brody hyperbolic modulo Z* . In particular, X is Brody hyperbolic when X is hyperbolic modulo $Z = \emptyset$.

In our setting the space under consideration is the complement $S \setminus B$, for S a smooth complex projective surface and $B = B_1 \cup \cdots \cup B_n$ a reduced projective curve such that every point of $B_i \cap B_j$ is a node of B , for every $i \neq j$. The irreducible components B_i are called *numerically parallel* if there exist $a_i \in \mathbb{Z}$, $i = 1, \dots, n$, such that $a_i B_i$ is numerically equivalent to $a_j B_j$, for every $i \neq j$.

If $n \geq 3$, the pair (S, B) is of log general type, and the components B_i are numerically parallel, then [RW25, Theorem 1] of Ru and Wang (extending work of Noguchi Winkelman and Yamanoi in [NWY08]) implies that the image of every non constant holomorphic map $\mathbb{C} \rightarrow S \setminus B$ is an algebraic curve. This implies that the analytic exceptional set coincides with the algebraic exceptional set $[\mathcal{E}(B)]$. Therefore we obtain the following consequence of our Theorem 1.2.

Theorem 1.3. *Let S and $B = \cup_{i=1}^n B_i$ be as above. Assume that the irreducible components B_i of B are numerically parallel and very ample. The following hold:*

- (1) *if $n \geq 5$ then $S \setminus B$ is Brody hyperbolic;*
- (2) *if $n = 4$ then $S \setminus B$ is Brody hyperbolic modulo $[\mathcal{E}(B)]$ and if at least 3 of the components of B are not primitive then $S \setminus B$ is Brody hyperbolic;*
- (3) *if $n = 3$ and at least two among the components of B are not primitive then $S \setminus B$ is Brody hyperbolic modulo $[\mathcal{E}(B)]$, and if all components of B are not primitive then $S \setminus B$ is Brody hyperbolic.*

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2. PRELIMINARIES

In our paper the word “curve” stands for “projective curve”.

As before S denotes a smooth connected projective surface, C denotes an integral curve contained in S , and $\nu : C^\nu \rightarrow C$ the normalization.

For a point $q \in C \subset S$, we denote by $\sigma : S' \rightarrow S$ be the blow-up at q , by E the exceptional divisor and by C' the strict transform of C . If q is a unibranch point of C (i.e. $|\nu^{-1}(q)| = 1$), we denote by $q' \in C'$ the point lying over q .

Let $B, C \subset S$ be two integral curves and $q \in B \cap C$. We denote by $(C \cdot B)_q$ the intersection multiplicity at q . We say that B and C are *transverse at q* if $B' \cap C' \cap E = \emptyset$. If q is unibranch for both B and C , we say that B and C are *tangent at q* if they are not transverse at q .

We will frequently use the following elementary lemmas (whose proofs are included for completeness).

Lemma 2.1. *Let $B, C \subset S$ be two integral curves and $q \in B \cap C$. Then*

$$(C \cdot B)_q \geq \text{mult}_q(B)\text{mult}_q(C)$$

with equality if and only if B and C are transverse at q .

Let $A, B, C \subset S$ be integral curves with a unibranch point at q . If A and B are tangent at q , and B and C are tangent at q , then A and C are tangent at q .

Proof. Let $\sigma : S' \rightarrow S$ be the blow-up at q and $A', B', C' \subset S'$ the strict transforms of A, B, C . Then $C' = \sigma^*C - \text{mult}_q(C)E$ and $B' = \sigma^*B - \text{mult}_q(B)E$, hence

$$(C' \cdot B') = (C \cdot B) - \text{mult}_q(B)\text{mult}_q(C) = (C \cdot B)_q + r - \text{mult}_q(B)\text{mult}_q(C)$$

where $r = (C \cdot B) - (C \cdot B)_q \geq 0$. On the other hand

$$(C' \cdot B') = (C' \cdot B')_E + r$$

where $(C' \cdot B')_E$ denotes the intersection multiplicity along points on the exceptional divisor, E . Combining the two identities we obtain

$$(C \cdot B)_q - \text{mult}_q(B)\text{mult}_q(C) = (C' \cdot B')_E.$$

As $(C' \cdot B')_E \geq 0$ we get $(C \cdot B)_q \geq \text{mult}_q(B)\text{mult}_q(C)$, with equality if and only if $(C' \cdot B')_E = 0$, i.e. if and only if C and B are transverse at q .

Assume q is a unibranch point for A, B , and C . Let $q' \in B'$ be the unique point lying over q . If A and B are tangent at q , then A' and B' intersect along E , hence $q' \in A'$; similarly, if C and B are tangent at q , then $q' \in C'$. Hence $q' \in C' \cap A'$, hence C and A are not transverse at q . ■

Lemma 2.2. *Let q be a unibranch point of $C \subset S$. Let $C' \subset S'$ be the strict transform of C under the blow-up at q and $q' \in C'$ the point lying over q .*

Then $\text{mult}_{q'}(C') \leq \text{mult}_q(C)$ with equality if and only if C' is transverse to E (at q').

Proof. We have

$$\text{mult}_q(C) = (C' \cdot E)_{q'} \geq \text{mult}_{q'}(C')\text{mult}_{q'}(E) = \text{mult}_{q'}(C'),$$

by Lemma 2.1, equality holds if and only if C' is transverse to E . ■

Remark 2.3. By Lemma 2.1, we have $\text{mult}_q(C) \leq (C \cdot B)$ for any integral curve B passing through q . If $\text{mult}_q(C) \geq 2$ and $S = \mathbb{P}^2$ then strict inequality always holds, but this does not generalize to other surfaces. For example, let S be the blow-up of \mathbb{P}^2 in one point, p . Let $C_0 \subset \mathbb{P}^2$ be a cubic containing p , having a cusp in $q_0 \neq p$, and let B_0 be the line through q_0 and p . Let C and B be the strict transforms of B_0 and C_0 in S , and let q be the point lying over q_0 . Since $(C_0 \cdot B_0)_{q_0} = 2$ and $(C_0 \cdot B_0)_p = 1$ we have $2 = \text{mult}_q(C) = (C \cdot B)_q = (C \cdot B)$.

Let H be a very ample line divisor on a surface S ; we say that H is *primitive* if it is not a nontrivial multiple of a very ample divisor. Note that if H satisfies $(C \cdot H) = 1$ for some curve $C \subset S$, then H is primitive.

Lemma 2.4. *Let $C, H \subset S$ be two curves with H very ample; set $d = (C \cdot H)$. Then the following hold.*

- (a) *For any $q \in C$ either $d > \text{mult}_q(C)$, or $d = \text{mult}_q(C) = 1$, hence H is primitive.*
- (b) *If $d \geq 2$ then for any $q_1, q_2 \in C$, we have $d \geq \text{mult}_{q_1}(C) + \text{mult}_{q_2}(C)$, and if equality holds, then H is primitive.*

Proof. We view $S \subset \mathbb{P}^r$ embedded by the linear system $|H|$, so that $r = h^0(H) - 1$ and $\deg C = d$. The linear subspace spanned by C is $\langle C \rangle \cong \mathbb{P}^l$ with $d \leq l \leq r$. If $l = 1$ then C is a line, hence $d = 1$ and $\text{mult}_q(C) = 1$. Assume $d \geq 2$. If $l = 2$ then $\langle C \rangle \cong \mathbb{P}^2$ where all statements are well known. We prove (a) by induction on l . Let $l \geq 3$. Then the general point $p \in \langle C \rangle \setminus C$ does not lie on a line passing through q and through another point of C (indeed, the closure of the union of all such lines has dimension 2). Fix such a point p and let

$$\pi : C \longrightarrow \mathbb{P}^{l-1}$$

be the projection from p . Set $\bar{C} = \pi(C)$ and $\bar{q} = \pi(q)$. By construction, $\deg \bar{C} = d$ and $\text{mult}_{\bar{q}}(\bar{C}) \geq \text{mult}_q(C)$. By the induction hypothesis, either $\deg \bar{C} > \text{mult}_{\bar{q}}(\bar{C})$, hence $d > \text{mult}_q(C)$, or $\deg \bar{C} = \text{mult}_{\bar{q}}(\bar{C}) = 1$, hence $d = \text{mult}_q(C) = 1$. (a) is proved.

For (b), set $m_i = \text{mult}_{q_i}(C)$ for $i = 1, 2$. As H is very ample, up to replacing it with a linearly equivalent effective divisor we can assume $q_1, q_2 \in H$. Hence

$$d = (H \cdot C) \geq (H \cdot C)_{q_1} + (H \cdot C)_{q_2} \geq m_1 + m_2$$

as claimed. If $m_1 + m_2 = (H \cdot C)$ and $H = nA$ with A very ample, then

$$m_1 + m_2 = (nA \cdot C) = n(A \cdot C) \geq n((A \cdot C)_{q_1} + (A \cdot C)_{q_2}) \geq n(m_1 + m_2)$$

hence $n = 1$. This shows that H is primitive. \blacksquare

Lemma 2.5. *Let $B, C \subset S$ be integral curves such that B is very ample and $(C \cdot B) \geq 2$. If $C \cap B = \{q\}$ and B is smooth at q , then B is tangent to C at q .*

Proof. Set $d = (B \cdot C)$. By contradiction, suppose that the proper transforms, B' and C' in the blow-up of S at q do not intersect over q . Hence they do not intersect at all, by hypothesis, i.e. $(B' \cdot C') = 0$. Now

$$0 = (B' \cdot C') = (B \cdot C)_q - \text{mult}_q(B)\text{mult}_q(C) = d - \text{mult}_q(C).$$

Hence $d = \text{mult}_q(C)$, which, as $d \geq 2$, contradicts Lemma 2.4(a). \blacksquare

3. HYPERTANGENCY

Let $C \subset S$ and $q \in C$. The arithmetic genus, $p_a(C)$, of C is computed by adjunction, as follows: $p_a(C) = (C \cdot (K_S + C))/2 + 1$. The geometric genus, $g(C)$, satisfies the following inequality

$$g(C) \geq p_a(C) - \delta_C(q)$$

with equality if and only if q is the only singular point of C . The following extends the second part of Theorem 2.2.1 in [CT25].

Proposition 3.1. *Let $B, C \subset S$ be two integral curves such that B is very ample and not primitive. Assume $B \cap C = \{q\}$, where q is smooth for B , and unibranch of multiplicity m for C . Then*

$$\delta_C(q) \geq (m-1)((B \cdot C) - m)/2.$$

Proof. By hypothesis there exists a very ample H such that $B \sim aH$ with $a \geq 2$.

If $(C \cdot H) = 1$ then C is necessarily smooth (it is a line in the embedding determined by H). Hence we assume $d := (C \cdot H) \geq 2$.

We denote by $S_1 \rightarrow S$ the blow-up at q , by C^1, B^1 and E_1 the proper transforms of C, B and the exceptional divisor; we write $q^1 \in C^1$ for the point lying over q . Now we define inductively, for $i \geq 2$, the blow-up of $S_i \rightarrow S_{i-1}$ at the unique point $q^{i-1} \in C^{i-1}$, let $C^i, B^i, E_i \subset S_i$ be the proper transforms of C^{i-1} and B^{i-1} and the exceptional divisor, and $q^i \in C^i$ the point lying over q^{i-1} . We set

$$h = \lceil ((B \cdot C) - m)/m \rceil.$$

We claim that for every $1 \leq i \leq h-1$ the curves C^i and B^i are tangent in q^i , and C^i has an m -fold point at q^i .

As H is very ample and $d \geq 2$, we have

$$d > m.$$

Let us treat the induction base. Since $B \cap C = \{q\}$ we have

$$(B^1 \cdot C^1)_{q^1} = (B^1 \cdot C^1) = (B \cdot C) - m$$

hence

$$(B^1 \cdot C^1)_{q^1} = (aH \cdot C) - m = ad - m > am - m = (a-1)m \geq m$$

(as $a \geq 2$). Since B^1 is smooth at q^1 , Lemma 2.1 gives that C^1 and B^1 are tangent in q^1 . On the other hand B^1 is not tangent at E_1 , hence neither is C^1 . By Lemma 2.2 we get $\text{mult}_{q^1}(C^1) = m$. The induction base is proved.

Now suppose C^{i-1} has an m -fold point at q^{i-1} , and B^{i-1} and C^{i-1} are tangent in q^{i-1} . Therefore

$$(B^i \cdot C^i)_{q^i} = (B^i \cdot C^i) = (B \cdot C) - im \geq (B \cdot C) - (h-1)m.$$

Now,

$$h = \lceil ((B \cdot C) - m)/m \rceil = \lceil (B \cdot C)/m - 1 \rceil < (B \cdot C)/m$$

hence

$$(B^i \cdot C^i)_{q^i} > (B \cdot C) - ((B \cdot C)/m - 1)/m = m$$

Hence C^i and B^i are tangent in q^i . Hence C^i is not tangent to the exceptional divisor E_i , because B^i is not. Hence $\text{mult}_{q^i}(C^i) = m$, by Lemma 2.2. The claim is proved.

Since C^i has an m -fold point at q^i for every $i = 0, \dots, h-1$, we have

$$\begin{aligned} \delta_C(q) &\geq hm(m-1)/2 = \lceil ((B \cdot C) - m)/m \rceil m(m-1)/2 \\ &\geq \left(\lceil ((B \cdot C) - m)/m \rceil m(m-1)/2 \right) = ((B \cdot C) - m)(m-1)/2. \end{aligned}$$

■

For a curve $B \subset S$ the set of curves hyperbitangent to B was defined in the introduction:

$$\text{Hyp}(B, 2) = \{C \subset S : C \text{ integral, } |\nu^{-1}(B)| \leq 2\};$$

for $d \in \mathbb{N}$ we set

$$\text{Hyp}_d(B, 2) = \{C \in \text{Hyp}(B, 2) : (C \cdot B) = d\}.$$

Given $q, q' \in B$, we denote

$$\text{Hyp}(B; q) := \{C \in \text{Hyp}(B, 2) : C \cap B = \{q\}\}$$

$$\text{Hyp}(B; q, q') := \{C \in \text{Hyp}(B, 2) : C \cap B = \{q, q'\}\},$$

and

$$\text{Hyp}_d(B; q) := \{C \in \text{Hyp}(B; q) : (C \cdot B) = d\}.$$

We write $\text{Hyp}_{\geq d}$ when the degree condition “ $(C \cdot B) = d$ ” is replaced by “ $(C \cdot B) \geq d$ ”.

If $B = \cup_{i=1}^n B_i$, for any $\underline{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$ we set

$$\text{Hyp}_{\underline{d}}(B, 2) := \{C \in \text{Hyp}(B, 2) : (C \cdot B_i) = d_i, \forall i = 1, \dots, n\}$$

and $\text{Hyp}_{\underline{d}}(B, q) := \{C \in \text{Hyp}(B; q) : (C \cdot B_i) = d_i, \forall i = 1, \dots, n\}$.

Definition 3.2. An nC -curve is a reduced curve $B = B_1 \cup \dots \cup B_n$, with B_i integral such that for all $i \neq j$ every point in $B_i \cap B_j$ is a node of B . We write $N := \cup_{i \neq j} B_i \cap B_j$.

Recall that a singular point q on a curve B is called a node if a local analytic neighborhood of q is isomorphic to a neighborhood of the origin of the curve in \mathbb{A}^2 of equation $xy = 0$.

Remark 3.3. By definition, the components of an nC -curve are smooth at each of the intersection points in N , where they meet only pairwise and transversally.

Proposition 3.4. *Let $B = B_1 \cup B_2$ be a $2C$ -curve with B_1 and B_2 very ample; let $q \in B_1 \cap B_2$.*

- (a) *If B_1 and B_2 are not primitive, then $\text{Hyp}(B; q) = \emptyset$.*
- (b) *If B_1 is primitive and B_2 is not primitive, then $|\text{Hyp}(B; q)| \leq 1$. More precisely, there exists at most one $l_2 \in \mathbb{N}$ such that $|\text{Hyp}_{(1, l_2)}(B; q)| = 1$ and $\text{Hyp}_{(d_1, d_2)}(B; q) = \emptyset$ for every $(d_1, d_2) \neq (1, l_2)$.*
- (c) *If B_1 and B_2 are both primitive, then $|\text{Hyp}(B, q) \setminus \text{Hyp}_{(1,1)}(B, q)| \leq 2$.*

Proof. Clearly,

$$\text{Hyp}_{(d_1, d_2)}(B; q) = \text{Hyp}_{d_1}(B_1; q) \cap \text{Hyp}_{d_2}(B_2; q).$$

Let $C \in \text{Hyp}_{(d_1, d_2)}(B; q)$. Let $S' \rightarrow S$ be the blow-up at q , E the exceptional divisor, C' and B'_i the proper transforms of C and B_i with $i = 1, 2$. As B_1 and B_2 are transverse at q , we have $E \cap B'_i = \{q_i\}$ with $q_1 \neq q_2$. Since $C' \cap E$ consists of only one point, up to exchanging B_1 and B_2 we have $q_2 \in C'$ and $q_1 \notin C'$. Therefore $(C' \cdot B'_1) = 0$; now, as B_1 is smooth at q , we have

$$0 = (C' \cdot B'_1) = (C' \cdot B'_1)_{q_1} = (C \cdot B_1)_q - \text{mult}_q(C) = (C \cdot B_1) - \text{mult}_q(C).$$

Hence $(C \cdot B_1) = \text{mult}_q(C)$; Lemma 2.4(a) implies that B_1 is primitive and $d_1 = (B_1 \cdot C) = 1$. This proves (a).

Assume that B_1 is primitive; we just proved that if $\text{Hyp}_{(d_1, d_2)}(B; q)$ is non empty then $d_1 = 1$. In the projective embedding of S determined by B_1 , the curve C is a line passing through q and such that $(C \cdot B_2)_q = (C \cdot B_2) = d_2$. If $d_2 \geq 2$ then C is the line of S hypertangent to B_2 at q ; it is clear that such a line, if it exists, is uniquely determined by B_2 and q (and by the embedding of S given by B_1), and so is the intersection number $(C \cdot B_2)$. If B_2 is not primitive, we necessarily have $d_2 \geq 2$, hence part (b) is proved with $l_2 = (C \cdot B_2)$.

If B_2 is primitive we can have $d_2 = 1$ (and $\text{Hyp}_{(1,1)}(B, q)$ can be infinite, see the next remark). The set $\text{Hyp}(B, q) \setminus \text{Hyp}_{(1,1)}(B, q)$ may contain two curves, the one in $\text{Hyp}_{(1, l_2)}(B, q)$ described above, and the one (applying the previous argument to B_2 in place of B_1) in $\text{Hyp}_{(l_1, l_2)}(B, q)$ for a unique $l_1 \in \mathbb{N}$. This concludes the proof of part (c). \blacksquare

Remark 3.5. In case B_1 and B_2 are both primitive, the set $\text{Hyp}_{(1,1)}(B, q)$ can easily be infinite. For example, if B_1 and B_2 are two lines in \mathbb{P}^2 meeting at the point q , then $\text{Hyp}_{(1,1)}(B, q)$ is the set of all lines through q .

4. HYPERBITANGENCY

Proposition 4.1. *Let $B = \cup_{i=1}^n B_i \subset S$ be an nC -curve.*

- (1) *If $n \geq 5$ then $\text{Hyp}(B, 2)$ is empty.*

- (2) If $n = 4$ and B_i is very ample for every $i = 1, \dots, 4$, then $\text{Hyp}(B, 2)$ is finite, and it is empty if least three among the components of B are not primitive.

Proof. The first part is an obvious consequence of the fact that every point of B belongs to at most two irreducible components.

Suppose $n = 4$. If $C \in \text{Hyp}(B, 2)$ then we have $C \in \text{Hyp}(B; p_{i,j}, p_{h,k})$ with $p_{i,j} \in B_i \cap B_j$ and $\{i, j, h, k\} = \{1, 2, 3, 4\}$. Therefore

$$C \in \text{Hyp}(B_i \cup B_j, p_{i,j}) \cap \text{Hyp}(B_h \cup B_k, p_{h,k}).$$

By Proposition 3.4, the set $\text{Hyp}(B_i \cup B_j, p_{i,j})$ is finite, and empty if B_i and B_j are not primitive. Hence $\text{Hyp}(B; p_{i,j}, p_{h,k})$ is finite, and if it is non-empty then at least one between B_i and B_j , and at least one between B_h and B_k , must be primitive. \blacksquare

Theorem 4.2. *Let $B = B_1 \cup B_2 \cup B_3 \subset S$ be a 3C-curve such that B_i is very ample and not primitive for $i = 1, 2, 3$. Then $\text{Hyp}(B, 2)$ is empty.*

Proof. By contradiction, let $C \in \text{Hyp}(B, 2)$. As B_i is not primitive, we have $d_i := (C \cdot B_i) \geq 2$. Since the components of B intersect only pairwise, we necessarily have $|C \cap B| = 2$. Suppose that C intersects one of the three curves, say B_3 , away from N . Then C must intersect B_1 and B_2 in a point $q \in B_1 \cap B_2$, and

$$C \in \text{Hyp}_{(d_1, d_2)}(B_1 \cup B_2; q).$$

Hence, by Proposition 3.4, one between B_1 and B_2 is primitive, which is impossible. Therefore $C \cap B \subset N$. We can assume $C \cap B = \{p_{1,2}, p_{1,3}\}$ with $p_{i,j} \in B_i \cap B_j$. Let us show that C is tangent to B_2 at $p_{1,2}$ and tangent to B_3 at $p_{1,3}$. We have

$$C \in \text{Hyp}(B_2; p_{1,2}).$$

Since $(C \cdot B_2) \geq 2$, Lemma 2.5 implies that B_2 is tangent to C at $p_{1,2}$. By a similar reasoning, B_3 is tangent to C at $p_{1,3}$.

As C is tangent to B_i at $p_{1,i}$ for $i = 2, 3$, by Lemma 2.1 it is transverse to B_1 at both points. Hence, as B_1 is smooth at both points, we have

$$d_1 = (C \cdot B_1) = (C \cdot B_1)_{p_{1,2}} + (C \cdot B_1)_{p_{1,3}} = \text{mult}_{p_{1,2}}(C) + \text{mult}_{p_{1,3}}(C).$$

Lemma 2.4(b) implies that B_1 is primitive, which is a contradiction. \blacksquare

For an integer $m \geq 1$ we denote by $\text{Hyp}_*^m(B; q) \subset \text{Hyp}_*(B; q)$ the set of curves in $\text{Hyp}_*(B; q)$ having a point of multiplicity exactly m at q .

Proposition 4.3. *Let $B = B_1 \cup B_2 \cup B_3 \subset S$ be a 3C-curve with B_i very ample for $i = 1, 2, 3$. Assume B_1 primitive and B_3 not primitive. Then*

- (a) $\text{Hyp}(B, 2) = \mathcal{E}(B)$.
 (b) Assume $\underline{d} = (d_1, d_2, d_3) \in \mathbb{Z}^3$ with $d_i \geq 2$ for all i .
 (i) If B_2 is not primitive then $d_1 = 2$ and

$$\text{Hyp}_{\underline{d}}(B, 2) = \bigcup_{\substack{p \in B_1 \cap B_2 \\ q \in B_1 \cap B_3}} \text{Hyp}_2(B_1; p, q) \cap \text{Hyp}_{d_2}(B_2; p) \cap \text{Hyp}_{d_3}(B_3; q).$$

(ii) If B_2 is primitive, then

$$\text{Hyp}_{\underline{d}}(B, 2) = \bigcup_{\{i,j\}=\{1,2\}} \bigcup_{\substack{p \in B_i \cap B_2 \\ q \in B_i \cap B_3}} \text{Hyp}_{d_i}(B_i; p, q) \cap \text{Hyp}_{d_j}^{d_i-1}(B_j; p) \cap \text{Hyp}_{d_3}^1(B_3; q).$$

Proof. Let us show that (a) follows from (b). Let $C \in \text{Hyp}_{\underline{d}}(B, 2)$. If $d_i = 1$ for some i , then C is a line in the embedding in projective space determined by B_i , hence C is rational. If $d_i \geq 2$ for all i , we can apply (b). If B_2 is not primitive, in the projective embedding determined by B_1 , the curve C is a conic, hence is rational. If B_2 is primitive, then for some $i \in \{1, 2\}$ the embedding of C given by B_i is a curve of degree d_i with a point of multiplicity $d_i - 1$, so it is rational.

To prove (b), we first observe that for all $i \neq j$ and $p_{i,j} \in B_i \cap B_j$ we have

$$\text{Hyp}_{\geq 2}(B_i; p_{i,j}) \cap \text{Hyp}_{\geq 2}(B_j; p_{i,j}) = \emptyset.$$

Indeed, by Lemma 2.5, any curve in the above intersection would have to be tangent to both B_i and B_j at $p_{i,j}$, which is not possible as B_i and B_j meet transversally.

Let $C \in \text{Hyp}_{\underline{d}}(B, 2)$. The preceding observation implies $C \cap B \subset N$. We claim that $|C \cap B_3| = 1$. Indeed, if $|C \cap B_3| = 2$ then $C \cap B = \{p_{1,3}, p_{2,3}\}$ with $p_{i,j} \in B_i \cap B_j$. Then $C \in \text{Hyp}_{\geq 2}(B_1; p_{1,3})$, hence C is not transverse to B_1 at $p_{1,3}$, hence it is transverse to B_3 at $p_{1,3}$; the same argument switching B_1 with B_2 gives that C is transverse to B_3 at $p_{2,3}$. Hence

$$(C \cdot B_3) = \text{mult}_{p_{1,3}}(C) + \text{mult}_{p_{2,3}}(C),$$

now Lemma 2.4(b) implies that B_3 is primitive, which is not possible.

This argument shows that either $C \cap B = \{p_{1,2}, p_{1,3}\}$, and

$$(3) \quad d_1 = \text{mult}_{p_{1,2}}(C) + \text{mult}_{p_{1,3}}(C)$$

or $C \cap B = \{p_{1,2}, p_{2,3}\}$, with B_2 primitive, and

$$(4) \quad d_2 = \text{mult}_{p_{1,2}}(C) + \text{mult}_{p_{2,3}}(C).$$

Assume (3) holds. Set $m = \text{mult}_{p_{1,3}}(C)$ so that $\text{mult}_{p_{1,2}}(C) = d_1 - m$. We claim that $m = 1$. As B_3 is not primitive, we have $B_3 = aH$ for some primitive, very ample H , and $a \geq 2$. If $(C \cdot H) = 1$, then C is a line with respect to H , hence C is smooth and $m = 1$. If $(C \cdot H) \geq 2$ we apply Proposition 3.1 to B_3 and C , getting

$$\delta_C(p_{1,3}) \geq (m-1)((B_3 \cdot C) - m)/2 = (m-1)(a(C \cdot H) - m)/2.$$

Now set

$$e = \min\{(C \cdot H), d_1\},$$

so that

$$(5) \quad \delta_C(p_{1,3}) \geq (m-1)(ae - m)/2.$$

Now, as e is the degree of some embedding of C in projective space, the arithmetic genus of C satisfies

$$(6) \quad p_a(C) \leq \binom{e-1}{2}.$$

Moreover, as $p_{1,2}$ has multiplicity $d_1 - m$, we have

$$(7) \quad \delta_C(p_{1,2}) \geq (d_1 - m)(d_1 - m - 1)/2 \geq (e - m)(e - m - 1)/2.$$

Assume, by contradiction, $m \geq 2$. From the above inequalities we obtain

$$\begin{aligned} g(C) &\leq p_a(C) - \delta_C(p_{1,2}) - \delta_C(p_{1,3}) \leq \\ &\binom{e-1}{2} - (e - m)(e - m - 1)/2 - (m - 1)(ae - m)/2 = \\ &\binom{e-1}{2} - (e^2 - e - 2me + ame - ae + 2m)/2 = \\ &\binom{e-1}{2} - (e^2 + e(-1 + m(a - 2) - a) + 2m)/2 \leq \\ &\binom{e-1}{2} - (e^2 - 3e + 4)/2 = \\ &(e^2 - 3e + 2)/2 - (e^2 - 3e + 4)/2 < 0 \end{aligned}$$

as $m \geq 2$ and $a \geq 2$. This is impossible. Hence $m = 1$. If B_2 is not primitive, then (3) necessarily holds. Moreover, we can apply the same argument with B_2 in place of B_3 and with $\text{mult}_{p_{1,2}}(C)$ in place of $m = \text{mult}_{p_{1,3}}(C)$. We thus get $\text{mult}_{p_{1,2}}(C) = 1 = \text{mult}_{p_{1,3}}(C)$, hence by (3), we obtain $d_1 = 1 + 1 = 2$. This proves the first part of (b).

For the last part of (b), assume that B_2 is primitive; then either (3) or (4) holds. Hence, by the above argument, for some $i \in \{1, 2\}$ we have that C has degree d_i in the embedding given by B_i , and a point of multiplicity $d_i - 1$ in some $p_{i,3} \in B_i \cap B_3$. The proof is complete. \blacksquare

Theorem 4.4. *Let $B = B_1 \cup B_2 \cup B_3 \subset S$ be a 3C-curve with B_i very ample for $i = 1, 2, 3$, and such that B_1 is primitive and B_2, B_3 are not primitive. Then $\mathcal{E}(B)$ is finite, and $(C \cdot B_1) \leq 2$ for every $C \in \mathcal{E}(B)$,*

Proof. By Proposition 4.3 we have $\mathcal{E}(B) = \text{Hyp}(B, 2)$. Let $C \in \text{Hyp}_{\underline{d}}(B, 2)$, with, as usual, $\underline{d} = (d_1, d_2, d_3)$ and $d_i = (C \cdot B_i)$.

We view S in projective space embedded by the linear system $|B_1|$.

Suppose $d_1 = 1$, then C is a line in our embedding. If $C \cap B \subset N$ then C is one of the finitely many lines through two points of N . More exactly, since $d_1 = 1$ we have only two cases, either $C \cap B = \{p_{1,2}, p_{2,3}\}$ or $C \cap B = \{p_{1,3}, p_{2,3}\}$ (with the usual convention $p_{i,j} \in B_i \cap B_j$). Hence we have at most $(B_1 \cdot B_2)(B_2 \cdot B_3) + (B_1 \cdot B_3)(B_2 \cdot B_3)$ such lines.

If $C \cap B \not\subset N$ then there are two indices, i, j and $p_{i,j} \in B_i \cap B_j$ such that

$$C \in \text{Hyp}_{(d_i, d_j)}(B_i \cup B_j, p_{i,j}).$$

By Proposition 3.4 the set $\text{Hyp}_{(d_i, d_j)}(B_i \cup B_j, p_{i,j})$ is not empty only if one between B_i and B_j is primitive, therefore we must have $i = 1$. Assume, with no loss of generality, $j = 2$; then C is hypertangent to B_2 at $p_{1,2}$ and hypertangent to B_3 in a point not lying on N ; since B_2 and B_3 are not primitive, they are not lines in our embedding. Therefore such a C , if it exists, is the unique line in S tangent to B_2 at $p_{1,2}$. Since we have $(B_1 \cdot B_2)$ choices for $p_{1,2}$, and we can argue in the same way if $j = 3$, we get at most $(B_1 \cdot B_2) + (B_1 \cdot B_3)$ possibilities for such a C .

Now let $d_1 = 2$. By Proposition 4.3 the curve C is a conic hypertangent to B_2 at $p_{1,2}$ and to B_3 at $p_{1,3}$. Therefore (as B_i and C lie in the smooth surface S) the tangent lines of C and B_i at $p_{1,i}$ coincide for $i = 2, 3$, in symbols, $T_{p_{1,2}}C = T_{p_{1,2}}B_2$ and $T_{p_{1,3}}C = T_{p_{1,3}}B_3$. Let $P \cong \mathbb{P}^2$ be the plane containing C ; as P contains the tangent lines to C , it contains the lines $T_{p_{1,2}}B_2$ and $T_{p_{1,3}}B_3$, which therefore have to be incident. But then P is uniquely determined by $T_{p_{1,2}}B_2$ and $T_{p_{1,3}}B_3$. Since the intersection of P with S contains only finitely many conics, we have finitely many curves such that $d_1 = 2$.

By Proposition 4.3 (a) the set $\text{Hyp}_d(B, 2)$ is empty if $d_1 \geq 3$, so we are done \blacksquare

Remark 4.5. The hypotheses in Theorem 4.4 imply that (S, B) is of log general type. This follows from Fujita's Conjecture for surfaces (a well-known consequence of results of Reider [Rei88], see [Laz04, Section 10.4]). Indeed, as at least one component of B is not primitive, we have $K_S + B = K_S + \sum_{i=1}^4 H_i$ with H_i very ample for all i . Now Fujita's Conjecture states that $K_S + 4H_i$ is very ample, hence $4(K_S + B)$ is also very ample, hence $K_S + B$ is ample.

The same holds for Theorems 1.2 and 1.3, since this argument works if we replace "4" with $n \geq 4$.

Comparing with the results obtained in [CT25] for $S = \mathbb{P}^2$ it is natural to consider the following questions for further investigation.

Question 1. *Let $B = B_1 \cup B_2 \cup B_3 \subset S$ be a 3C-curve with B_i very ample for all $i = 1, 2, 3$, B_1 and B_2 primitive, and B_3 not primitive. Is $\mathcal{E}(B)$ finite?*

If $S = \mathbb{P}^2$ the answer is yes, see [CT25, Proposition 3.2.1 and Theorem 3.3.2]. Moreover, on a slightly different vein, [CT25, Theorem 3.3.4] shows that the answer to the next question is affirmative for \mathbb{P}^2 .

Question 2. *Let $B = B_1 \cup B_2 \cup B_3 \subset S$ be a 3C-curve with B_i very ample for all $i = 1, 2, 3$, B_1 primitive, and B_2, B_3 not primitive. If the curve B is general, is $\mathcal{E}(B)$ empty?*

Here general means that the components B_1, B_2, B_3 vary in a (Zariski) open subset of $|B_1| \times |B_2| \times |B_3|$, where $|B_i|$ is the linear system associated to B_i .

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