

Counting plane curves of any genus

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1. Introduction

In this paper we study the geometry of the *Severi varieties* parametrizing plane curves of given degree d and geometric genus g . As an application, we derive a recursive formula for their degrees. This is the formula in Theorem 1.1, which enumerates the number of curves in \mathbb{P}^2 of degree d with δ nodes that pass through the appropriate number of points; that result is actually more general, as we shall explain below.

Such a classical enumerative question has been an object of study by many other people. In 1989, Z. Ran described an inductive approach to the problem (cf. [R]), giving for the first time a recursive formula. More recently, interest in it has been revived by work on quantum cohomology and by the discovery by M. Kontsevich ([KM]) in 1993 of a beautiful recursion solving the problem for curves of genus 0. Another proof in the genus 0 case, using different techniques, was given independently by Ruan and Tian in [RT]. One aspect of the work of Kontsevich and Ruan-Tian is that they relate these numbers to the coefficients in the structure equations of an algebraic object, the quantum cohomology ring. It would be very interesting to see if

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any similar interpretation could be placed on the degrees of Severi varieties in positive genus. We don't at present know of any algebraic structure that generates these numbers. However, Ezra Getzler [G] has described a generating function having the degrees of the generalized Severi varieties as coefficients; Getzler observes that the recursion formula in Theorem 1.1 is equivalent to the fact that this function satisfies a certain differential equation.

Our approach is simple. We work over the complex numbers throughout. We denote by \mathbb{P}^N the projective space of all plane curves of degree d and by $V^{d,\delta} \subset \mathbb{P}^N$ the closure of the subset of \mathbb{P}^N corresponding to curves having exactly δ nodes as singularities. Also, for any point $p \in \mathbb{P}^2$ we let $H_p \subset \mathbb{P}^N$ be the hyperplane of curves containing the point p . Our procedure consists in intersecting the variety $V^{d,\delta}$ with a succession of hyperplanes of the form H_{p_i} , where the points p_i are general points on a fixed line $L \subset \mathbb{P}^2$. At each stage we are able to describe the irreducible components of the intersection; the point is, they all belong to a specific collection of varieties, which we call *generalized Severi varieties*. These parametrize plane curves of given degree and genus satisfying certain tangency conditions with respect to the line L . More generally, we can express the intersection of any generalized Severi variety with a hyperplane H_p corresponding to a general point $p \in L$ as a union of generalized Severi varieties of dimension one less; counting multiplicities correctly, this allows us to derive our recursive formula for their degrees.

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1.1. Notation and definitions

We now introduce the notation and precise definitions that will allow us to state our formula.

For any sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of nonnegative integers with all but finitely many α_i zero, set

$$\#\alpha = \#\{i : \alpha_i \neq 0\}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

$$I\alpha = \alpha_1 + 2\alpha_2 + \dots + n\alpha_n$$

and

$$I^\alpha = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \dots$$

We denote by $\text{lcm}(\alpha)$ the least common multiple of the set $\{\alpha_i \neq 0\}$.

We denote by e_k the sequence $(0, \dots, 0, 1, 0, \dots)$ that is zero except for a 1 in the k^{th} term (so that any sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ is expressible as $\alpha = \sum \alpha_k e_k$). By the inequality $\alpha \geq \alpha'$ we mean $\alpha_k \geq \alpha'_k$ for all k ; for such a pair of sequences we set

$$\binom{\alpha}{\alpha'} = \binom{\alpha_1}{\alpha'_1} \binom{\alpha_2}{\alpha'_2} \binom{\alpha_3}{\alpha'_3} \dots$$

We now define the main objects of study, the varieties $V^{d,\delta}(\alpha, \beta)$ parametrizing plane curves of given degree and geometric genus satisfying certain tangency conditions with respect to a line. Fix a line $L \subset \mathbb{P}^2$ and a collection

$$\Omega = \{p_{i,j}\}_{1 \leq j \leq \alpha_i} \subset L$$

of $|\alpha|$ general points on L . For any d, δ, α and β satisfying $I\alpha + I\beta = d$, we define the *generalized Severi variety* $V^{d,\delta}(\alpha, \beta)(\Omega)$ to be the closure of the locus of reduced plane curves X of degree d and geometric genus $g = \binom{d-1}{2} - \delta$, not containing L , with (informally) α_k “assigned” points of contact of order k and β_k “unassigned” points of contact of order k with L . Formally, we require that, if $v : X^v \rightarrow X$ is the normalization of X , then there exist $|\alpha|$ points $q_{i,j} \in X^v$, $j = 1, \dots, \alpha_i$, and $|\beta|$ points $r_{i,j} \in X^v$, $j = 1, \dots, \beta_i$, such that

$$v(q_{i,j}) = p_{i,j}$$

and

$$v^*(L) = \sum i \cdot q_{i,j} + \sum i \cdot r_{i,j} .$$

Where the dependency on the points $p_{i,j}$ is not relevant – for example, in discussions of the dimension or degrees of generalized Severi varieties – we will often suppress the Ω .

To take some simple cases, taking $\alpha = 0$ and $\beta = (d, 0, \dots)$ imposes no condition at all, that is, $V^{d,\delta}((0, 0, \dots), (d, 0, \dots))$ is simply the closure $V^{d,\delta}$ of the locus of plane curves of degree d with δ nodes. Taking $\alpha = (1, 0, \dots)$ and $\beta = (d - 1, 0, \dots)$ we get the closure of the locus of such curves passing through a single fixed point of L ; and taking $\alpha = 0$ and $\beta = (d - 2, 1, 0, \dots)$ we get the closure of the locus of such curves tangent to L at a smooth point of the curve.

Note that we do not require X to be irreducible. Classically, the term “Severi variety” means a variety parametrizing irreducible curves of given degree and genus, so we are somewhat at odds with traditional usage here; but we will find it much more convenient, in both the statement and proof of the results below, to include components of $V^{d,\delta}(\alpha, \beta)$ whose general member is reducible.

Let V be a possibly reducible variety. We will say that V has pure dimension if all irreducible components of V have the same dimension. Moreover whenever we make a statement about the general point of V , we mean that the statement holds for a general point of any irreducible component of V .

We will adopt the following convention: we will denote the various types of Severi varieties by the symbol “ V ” to which we will add certain decorations, and we will correspondingly use the symbol “ N ” with same decorations to denote the degree of V as a subvariety of \mathbb{P}^N ; for example, we define $N^{d,\delta}(\alpha, \beta) := \deg V^{d,\delta}(\alpha, \beta)$.

1.2. The formula for the degrees of generalized Severi varieties

As a result of the analysis of hyperplane sections of generalized Severi varieties we have the recursive formula

Theorem 1.1. *Let $N^{d,\delta}(\alpha, \beta)$ be the degree of $V^{d,\delta}(\alpha, \beta)$. Then*

$$N^{d,\delta}(\alpha, \beta) = \sum_{k:\beta_k>0} k \cdot N^{d,\delta}(\alpha + e_k, \beta - e_k) + \sum I^{\beta'-\beta} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N^{d-1,\delta'}(\alpha', \beta')$$

where the second sum is taken over all α', β' and $\delta' \geq 0$ satisfying

$$\begin{aligned} \alpha' &\leq \alpha \\ \beta' &\geq \beta \\ \delta' &\leq \delta \\ \delta - \delta' + |\beta' - \beta| &= d - 1 . \end{aligned}$$

Taking $\alpha = 0$ and $\beta = (d, 0, \dots)$, we get the degree of the closure of the variety of (not necessarily irreducible) plane curves of degree d with δ nodes. We can find the degree of the component parametrizing irreducible such curves – that is, the classical Severi variety – by subtracting off the degrees of the others, which we know recursively. Alternatively, we can give a recursion formula to calculate directly the degrees of the varieties parametrizing irreducible plane curves, and will do so in the last section of this chapter; but this formula is more complicated.

We now illustrate how the above formula works by computing the degree of the Severi variety of quartics with three nodes (we assume known the degrees of the generalized Severi varieties parametrizing cubics satisfying tangency conditions). To shorten notation, we write $(d, \delta, \alpha, \beta)$ for $N = N^{d,\delta}(\alpha, \beta)$, and suppress the zeroes at the end of sequences α and β and the parentheses around sequences α and β of length 1. The result of inter-

secting the variety $V^{4,3}(0,4)$ with five successive hyperplanes of the form H_p is then the following five equations (we denote the contribution of each component to the degree, where known, in angle brackets).

$$(4, 3, 0, 4) = (4, 3, 1, 3) = (4, 3, 2, 2) + (3, 0, 0, 3) \quad \langle \underline{1} \rangle$$

$$(4, 3, 2, 2) = (4, 3, 3, 1) + 3(3, 1, 0, 3) \quad \langle 3 \times 12 = \underline{36} \rangle + 2(3, 0, 1, 2) \quad \langle 2 \times 1 = \underline{2} \rangle$$

$$(4, 3, 3, 1) = (4, 3, 4, 0) + 3(3, 2, 0, 3) \quad \langle 3 \times 21 = \underline{63} \rangle + 2(3, 1, 0, (1, 1)) \quad \langle 2 \times 36 = \underline{72} \rangle + 6(3, 1, 1, 2) \quad \langle 6 \times 12 = \underline{72} \rangle + 3(3, 0, 2, 1) \quad \langle 3 \times 1 = \underline{3} \rangle$$

and finally

$$(4, 3, 4, 0) = (3, 3, 0, 3) \quad \langle \underline{15} \rangle + 4(3, 2, 1, 3) \quad \langle 4 \times 21 = \underline{84} \rangle + 2(3, 2, 0, (1, 1)) \quad \langle 2 \times 30 = \underline{60} \rangle + 6(3, 1, 2, 1) \quad \langle 6 \times 12 = \underline{72} \rangle + 8(3, 1, 1, (0, 1)) \quad \langle 8 \times 16 = \underline{128} \rangle + 3(3, 1, 0, (0, 0, 1)) \quad \langle 3 \times 21 = \underline{63} \rangle + 4(3, 0, 3, 0) \quad \langle 4 \times 1 = \underline{4} \rangle .$$

Adding it all up, we find that

$$(4, 3, 4, 0) = 15 + 84 + 60 + 72 + 128 + 63 + 4 = 426$$

$$(4, 3, 3, 1) = 426 + 63 + 72 + 72 + 3 = 636$$

$$(4, 3, 2, 2) = 636 + 36 + 2 = 674$$

and so

$$(4, 3, 0, 4) = (4, 3, 1, 3) = 674 + 1 = 675 .$$

Now, $V^{4,3}((0), (4))$ has two irreducible components of dimension 11, one coincides with the classical $V_{4,3}$ which parametrizes irreducible quartics with three nodes; the other parametrizes reducible curves that are the union of a line and a cubic and has degree $\binom{11}{2} = 55$ and so we conclude that the degree of the classical Severi variety is 620.

R. Vakil has checked the formula in all degrees up to 6, and D. Watabe in all degrees up to 10; and the results agree with those of Vainsencher for $\delta \leq 6$ and with those of Kontsevich-Manin for $g = 0$. We list below the numbers obtained (by applying this formula) for the degrees N of the Severi varieties $V^{d,\delta}(0, d)$ for $d = 5$ and 6 and all possible values of δ .

d	δ	g	N
5	0	6	1
	1	5	48
	2	4	882
	3	3	7915
	4	2	36975
	5	1	90027
	6	0	109781
	7	-1	65949
	8	-2	26136
	9	-3	6930
	10	-4	945
6	0	10	1
	1	9	75
	2	8	2370
	3	7	41310
	4	6	437517
	5	5	2931831
	6	4	12597900
	7	3	34602705
	8	2	59809860
	9	1	63338881
	10	0	40047888
	11	-1	15580020
	12	-2	4361721
	13	-3	918918
	14	-4	135135
15	-5	10395	

1.3. The main results

As we indicated at the outset, what we actually prove is an equality of cycles rather than of numbers: we show first that for $p \in L$ general the intersection $V^{d,\delta}(\alpha, \beta) \cap H_p$ is a union of varieties of the form $V^{d',\delta'}(\alpha', \beta')$, and say which ones occur; and then we will calculate the intersection multiplicity of $V^{d,\delta}(\alpha, \beta)$ and H_p along each such variety.

Remark. Throughout, we will identify the projective space of plane curves of degree $d - 1$ with the subspace of plane curves of degree d containing L . Thus, for example, by $V^{d-1,\delta'}(\alpha', \beta')(\Omega')$ we mean the closure in \mathbb{P}^N of the locus of curves $X_0 = X \cup L$ where X is a plane curve of degree $d - 1$, not containing L , having δ' nodes and α' assigned and β' unassigned points of contact with L .

Theorem 1.2. *The intersection $V^{d,\delta}(\alpha, \beta)(\Omega) \cap H_p$ is contained in a union of varieties (without common components) as follows:*

- a. *For each k such that $\beta_k > 0$, the variety*

$$V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p_{k,\alpha_k+1} = p\}) ;$$

and

- b. *For each $\alpha' \leq \alpha$, $\beta' \geq \beta$ and $\delta' \leq \delta$ with $\delta - \delta' + |\beta' - \beta| = d - 1$, the union of the varieties*

$$V^{d-1,\delta'}(\alpha', \beta')(\Omega')$$

where Ω' ranges over all subsets $\Omega' = \{p'_{i,j}\}_{1 \leq j \leq \alpha'_i} \subset \Omega$ such that $\{p'_{i,1}, \dots, p'_{i,\alpha'_i}\} \subset \{p_{i,1}, \dots, p_{i,\alpha_i}\}$ for each i .

Remarks. 1. We can express the last condition $\delta - \delta' + |\beta' - \beta| = d - 1$ on δ' and β' in terms of the geometric genera $g = \binom{d-1}{2} - \delta$ and $g' = \binom{d-2}{2} - \delta'$: we have

$$\begin{aligned} g - g' &= \left(\binom{d-1}{2} - \delta \right) - \left(\binom{d-2}{2} - \delta' \right) \\ &= d - 2 - (\delta - \delta') \\ &= |\beta' - \beta| - 1. \end{aligned}$$

2. Note that there are a total of $\binom{\alpha}{\alpha'}$ varieties of the form $V^{d-1,\delta'}(\alpha', \beta')$ in part (b) of this statement, which accounts for the factor $\binom{\alpha}{\alpha'}$ in the formula in Theorem 1.1. The remaining factors will be intersection multiplicities, as described in Theorem 1.3 below.

3. By the dimension counts of Sect. 2, all the varieties listed in the statement of Theorem 1.2 have pure dimension $\dim(V^{d,\delta}(\alpha, \beta)) - 1$; so it follows that the intersection $V^{d,\delta}(\alpha, \beta)(\Omega) \cap H_p$ will consist of the union of a subset of these. In fact the intersection is equal to the union of all of them, as will follow from the analysis of the local geometry of $V^{d,\delta}(\alpha, \beta)$ given in the proof of Theorem 1.3.

Having described the intersection set-theoretically, we now ask about the local geometry of the larger variety $V^{d,\delta}(\alpha, \beta)$ along each component of the intersection: how many branches it has, and with what multiplicity each intersects the hyperplane H_p . The answer to both (and hence the multiplicity with which each component of $V^{d,\delta}(\alpha, \beta) \cap H_p$ appears in the intersection cycle) is expressed in the following.

Theorem 1.3. a. *Let $V' = V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\})$ be as in part (a) of Theorem 1.2. Then $V' \subset V^{d,\delta}(\alpha, \beta)(\Omega) \cap H_p$, and at a general point of V' the variety $V^{d,\delta}(\alpha, \beta)$ is smooth and has intersection multiplicity k with H_p along V' .*

b. Let $V' = V^{d-1, \delta'}(\alpha', \beta')(\Omega')$ be as in part (b) of Theorem 1.2. At a general point of V' , the variety $V^{d, \delta}(\alpha, \beta)$ will have $\binom{\beta'}{\beta} I^{\beta' - \beta} / \text{lcm}(\beta' - \beta)$ branches, each of which will have intersection multiplicity $\text{lcm}(\beta' - \beta)$ with H_p along V' .

The recursive formula for the degrees $N^{d, \delta}(\alpha, \beta)$ of the generalized Severi varieties given in Theorem 1.1 follows directly from these two statements.

The proofs of Theorems 1.2 and 1.3 will be given in Chaps 3 and 4, following some preliminary deformation-theoretic arguments and dimension counts in Sect. 2. For the most part the proof of Theorem 1.2 follows the lines of that of Proposition 2.5 of [CH]. To prove it, we will use the technique of semistable reduction to analyze a family of curves $X \in V^{d, \delta}(\alpha, \beta)$ specializing to a curve X_0 containing L . This approach yields a number conditions that the curve X_0 corresponding to a general point of the intersection $V^{d, \delta}(\alpha, \beta) \cap H_p$ must satisfy, some of which are far from obvious from the point of view of the geometry of plane curves alone. We then compare these with the dimension estimates of Chap. 2, using the fact that X_0 is a general member of a family of dimension $\dim(V^{d, \delta}(\alpha, \beta)) - 1$, to obtain an exact description of the set-theoretic intersection $V^{d, \delta}(\alpha, \beta) \cap H_p$.

As for Theorem 1.3, this requires a deeper analysis of the local structure of the total space of such a family, based on the deformation theory of the tacnodes of X_0 . As in the case of Theorem 1.2, this is based on arguments in [CH]; but while the proof of Theorem 1.2 is largely parallel to the corresponding argument of [CH], the argument for Theorem 1.3 requires additional work. Briefly, the proof of Proposition 2.7 in [CH] rests on the description given there of the deformation space of a single tacnode; this suffices for the purposes of that paper. Here we do need to consider degenerations having more than one tacnode, hence we develop an analysis of the geometry of a product of deformation spaces of tacnodes, building on the description given in [CH] of the deformation space of a single tacnode. We should mention that some of the results on deformations of tacnodes have also been obtained by Ran in [R].

1.4. The formula for irreducible curves

We now give a formula for the degrees of the varieties parametrizing irreducible plane curves of given degree and genus satisfying tangency conditions.

Denote by $V_{d, \delta}(\alpha, \beta)$ the union of the components of $V^{d, \delta}(\alpha, \beta)$ whose general point $[X]$ corresponds to an irreducible curve $X \subset \mathbb{P}^2$, and by $N_{d, \delta}(\alpha, \beta)$ its degree. Now, consider the intersection of a variety $V_{d, \delta}(\alpha, \beta)$ with the hyperplane H_p , with $p \in L$, and let $[X_0]$ be a general point of a component V of the intersection. If $X_0 = X \cup L$ and X has irreducible components X_1, \dots, X_k of degrees d_1, \dots, d_k , then each component X_j will correspond to a general point of a variety $V_{d_j, \delta_j}(\alpha^j, \beta^j)$. The part of the

intersection $V_{d,\delta}(\alpha, \beta) \cap H_p$ corresponding to curves containing L will thus be a Segre image of a product of varieties $V_{d_j,\delta_j}(\alpha^j, \beta^j)$, and its degree will be the product of the degrees of the factors $V_{d_j,\delta_j}(\alpha^j, \beta^j)$, times a multinomial coming from the formula for the degrees of Segre images. Denote by γ_i^j the number of free points of i -fold tangency of X_j with L that are new, that is, that are *not* limits of free points of i -fold tangency of nearby curves X_λ with L as $[X_\lambda] \in V_{d,\delta}(\alpha, \beta)$ approaches $[X_0]$; so that for example we have

$$\sum |\beta^j| - \sum |\gamma^j| = |\beta| .$$

We arrive in this way at the formula

$$\begin{aligned} N_{d,\delta}(\alpha, \beta) &= \sum_{k:\beta_k>0} k \cdot N_{d,\delta}(\alpha + e_k, \beta - e_k) \\ &+ \sum \frac{1}{\sigma} \binom{2d + g - 2 + |\beta|}{2d_1 + g_1 - 1 + |\beta^1|, \dots, 2d_k + g_k - 1 + |\beta^k|} \cdot \binom{\alpha}{\alpha^1, \dots, \alpha^k} \\ &\cdot \prod_{j=1}^k \binom{\beta^j}{\beta^j - \gamma^j} \cdot \prod_{j=1}^k \Gamma^{\gamma^j} \cdot \prod_{j=1}^k N_{d_j,\delta_j}(\alpha^j, \beta^j) \end{aligned}$$

where the second sum is taken over all collections of integers d_1, \dots, d_k and $\delta_1, \dots, \delta_k$ and collections of sequences $\alpha^1, \dots, \alpha^k$, β^1, \dots, β^k and $\gamma^1, \dots, \gamma^k$ satisfying

$$\begin{aligned} \alpha' &= \alpha^1 + \dots + \alpha^k \leq \alpha \\ \beta' &= \beta^1 + \dots + \beta^k = \beta + \sum |\gamma^j| \\ &|\gamma^j| > 0 \\ d_1 + \dots + d_k &= d - 1 \quad \text{and} \\ \delta_1 + \dots + \delta_k &= \delta + \sum |\gamma^j| - \sum_{i<j} d_i d_j - d + 1 . \end{aligned}$$

Here by the symbol $\binom{n}{a_1, \dots, a_k}$ we mean the multinomial coefficient

$$\binom{n}{a_1, \dots, a_k} = \frac{n!}{a_1! \dots a_k! (n - a_1 - \dots - a_k)!}$$

and correspondingly for a collection of sequences α and $\alpha^1, \dots, \alpha^k$ we set

$$\binom{\alpha}{\alpha^1, \dots, \alpha^k} = \prod_i \binom{\alpha_i}{\alpha_i^1, \dots, \alpha_i^k} .$$

By g_j we mean $\binom{d_j-1}{2} - \delta_j$.

The symbol σ is 1 except in rare cases. It is the degree of the map from the union of the product of varieties of the form $V_{d_j,\delta_j}(\alpha^j, \beta^j)$ to its image in

\mathbb{P}^N : given the integers d_1, \dots, d_k and $\delta_1, \dots, \delta_k$ and collections of sequences $\alpha^1, \dots, \alpha^k, \beta^1, \dots, \beta^k$ and $\gamma^1, \dots, \gamma^k$ we define an equivalence relation on the set $\{1, 2, \dots, k\}$ by saying $i \sim j$ if $d_i = d_j, \delta_i = \delta_j, \alpha^i = \alpha^j, \beta^i = \beta^j$ and $\gamma^i = \gamma^j$ and define σ to be the product of the factorials of the cardinalities of the equivalence classes.

This formula follows from Theorems 1.2 and 1.3 in much the same way as Theorem 1.1. Note that we have here decomposed β into $\sum \beta^j$ and the difference $\beta' - \beta$ into $\sum \gamma^j$, and further specified that $|\gamma^j| > 0$. This is because (as we will see in Chap. 3) the “new” unassigned points of $X \cap L$ (that is, the points of $X \cap L$ that are not limits of points of intersection of nearby curves in $V_{d,\delta}(\alpha, \beta)$ with L) correspond to points of intersection of the normalizations of the components X_j with L in the nodal reduction of the family. Since we are only concerned here with curves X arising as limits in families of irreducible curves, their nodal reductions must be connected; and this corresponds to the requirement $|\gamma^j| > 0$.

Using this formula, Watabe has calculated the degrees $N_{d,\delta}(\alpha, \beta)$ for all $d \leq 7$; the results agree with known values in case $g = 0$.

2. Geometry of Severi varieties at a general point

2.1. Statement of results

In this section we will compute the dimensions of generalized Severi varieties V and we will describe the geometry of its general points.

A naive reasoning yields a lower bound for the dimension of V . Namely, if we impose no conditions on the intersections of our curves with the line L , the corresponding locus has codimension δ in the space \mathbb{P}^N . Requiring that a curve X have intersection multiplicity i with L at a specified point $p_{i,j}$ is i linear conditions on the coefficients of X , which we would expect to be independent; and if we don't specify the point, the codimension of the corresponding locus should be one less, that is, $i - 1$. In sum we have

$$\begin{aligned} \dim(V^{d,\delta}(\alpha, \beta)) &\geq \binom{d+2}{2} - 1 - \delta - I\alpha - (I\beta - |\beta|) \\ &= \binom{d+1}{2} - \delta + |\beta| . \end{aligned}$$

Or, in terms of the geometric genus of the curves involved,

$$\begin{aligned} \dim(V^{d,\delta}(\alpha, \beta)) &\geq 3d + g - 1 - I\alpha - (I\beta - |\beta|) \\ &= 2d + g - 1 + |\beta| . \end{aligned}$$

We shall prove that equality holds:

Proposition 2.1. $V^{d,\delta}(\alpha, \beta)$ has pure dimension $2d + g - 1 + |\beta|$.

Likewise, there are no surprises when it comes to the geometry of a general member X of a generalized Severi variety $V^{d,\delta}(\alpha, \beta)$. We would expect the curve X to have only nodes as singularities, to be smooth at its points of intersection with L , and so on; and this is indeed the case. We list the relevant facts in the following Proposition.

To do that, fix any curve $G \subset \mathbb{P}^2$ and any finite subset $\Gamma \subset \mathbb{P}^2$. Let $[X] \in V^{d,\delta}(\alpha, \beta)$ be a general point of a generalized Severi variety, $X \subset \mathbb{P}^2$ the corresponding curve, $v : X^v \rightarrow X \subset \mathbb{P}^2$ its normalization. Let $\{q_{i,j}\}$ and $\{r_{i,j}\} \subset X^v$ be such that $v(q_{i,j}) = p_{i,j}$ and

$$v^*L = \sum i \cdot q_{i,j} + \sum i \cdot r_{i,j}$$

and let $s_{i,j} = v(r_{i,j}) \in L$ be the image of $r_{i,j}$ (that is, $\{s_{i,j}\} \subset L$ will be the “unassigned points” of intersection of X with L). We have then the

Proposition 2.2. a. X has only nodes as singularities.

b. X is smooth along $X \cap L$.

c. The points $\{q_{i,j}\}$ and $\{r_{i,j}\} \subset X^v$ are all distinct.

d. The points $\{p_{i,j}\}$ and $\{s_{i,j}\} \subset L$ are all distinct.

e. X intersects G transversely (in particular, X is smooth along $X \cap G$) and is disjoint from Γ .

To prove these statements, we will need some results about deformations of maps, which will be the object of the next section.

2.2. Deformations of maps

Throughout, we will assume we are working over a field of characteristic zero, and will use the analytic topology where necessary.

We will be concerned with families of maps from a possibly variable smooth domain to a fixed smooth target space. In other words, we will consider a flat, smooth, proper family $f : \mathcal{X} \rightarrow B$ over a smooth connected base B , a smooth variety \mathcal{Y} and a morphism $\psi : \mathcal{X} \rightarrow B \times Y$ of B -schemes. For each $b \in B$, we let $\psi_b : X_b \rightarrow Y$ be the restriction of ψ to the fiber X_b of \mathcal{X} over b , and

$$d\psi_b : TX_b \rightarrow \psi_b^*TY$$

the differential of ψ_b . We let \mathcal{N}_b be the normal sheaf of ψ_b , that is, the cokernel of the morphism $d\psi_b$ of sheaves on X_b . Equivalently, if we let

$$d\psi : T\mathcal{X} \rightarrow \psi^*T(B \times Y)$$

be the differential of ψ and $\mathcal{N} = \text{Coker}(d\psi)$ the normal sheaf of ψ , then the normal sheaf \mathcal{N}_b of ψ_b is the restriction of \mathcal{N} to the fiber X_b , that is, $\mathcal{N}_b = \mathcal{N} \otimes \mathcal{O}_{X_b}$. Note that if ψ_b is an immersion, then \mathcal{N}_b will be locally free; more generally, if ψ_b is equidimensional onto its image then the sheaf \mathcal{N}_b will have a torsion subsheaf supported exactly on the locus where $d\psi_b$ fails to be an injective bundle map.

We now describe the *Horikawa map* of the family ψ of morphisms. This is a map $\kappa : T_b B \rightarrow H^0(X_b, \mathcal{N}_b)$ that associates to any tangent vector $v \in T_b B$ to B a global section $\sigma = \kappa(v)$ of the normal sheaf, in such a way that the family is trivial (that is, the family $\mathcal{X} \cong B \times X_b$ as B -schemes and the morphism $\psi = id_B \times \psi_b$) if and only if $\kappa(v) = 0$ for every v . To define it, let $\pi : B \times Y \rightarrow B$ be the projection. We have an inclusion of bundles

$$\pi^* T_B \hookrightarrow T_{B \times Y} .$$

We let $i : \psi^* \pi^* T_B \hookrightarrow \psi^* T(B \times Y)$ be the corresponding inclusion of pull-backs to \mathcal{X} , and let $\tilde{\kappa} : \psi^* \pi^* T_B \rightarrow \mathcal{N}$ be the composition of i with the surjection $\psi^* T_{B \times Y} \rightarrow \mathcal{N}$.

$$\begin{array}{ccccccc}
 & & & \psi^* \pi^* T_B & & & \\
 & & & \downarrow & \searrow \tilde{\kappa} & & \\
 0 & \longrightarrow & T_{\mathcal{X}} & \xrightarrow{d\psi} & \psi^* T_{B \times Y} & \longrightarrow & \mathcal{N} \longrightarrow 0
 \end{array}$$

Restricting to X_b and taking global sections, we get a map

$$\kappa_b : T_b B \hookrightarrow H^0(X_b, \psi^* \pi^* T_B) \longrightarrow H^0(X_b, \mathcal{N}_b)$$

which we will call the *Horikawa map* of the given family at b . Equivalently, we let κ be the pushforward of $\tilde{\kappa}$ to B , composed with the inclusion of T_B into $f_* \psi^* \pi^* T_B$: that is,

$$\kappa = f_* \tilde{\kappa} : T_B \hookrightarrow f_* \psi^* \pi^* T_B \longrightarrow f_* \mathcal{N} .$$

We will call κ the global Horikawa map of the family; the maps κ_b are then the composition of the induced maps $T_b B \rightarrow (f_* \mathcal{N})_b$ on stalks with the natural maps $(f_* \mathcal{N})_b \rightarrow H^0(X_b, \mathcal{N}_b)$.

The standard applications of this construction rest on two facts. The first is that if the family ψ of morphisms is nowhere isotrivial (that is, the restriction of ψ to the subfamily $\mathcal{X}_{B_0} = f^{-1}(B_0) \subset \mathcal{X}$ is not trivial for any analytic arc $B_0 \subset B$), then at a general point $b \in B$ the map κ_b must be injective, so that we have an a priori bound on the dimension of the family:

$$\dim(B) \leq h^0(X_b, \mathcal{N}_b) .$$

(If, for general $b \in B$, we had $\text{Ker}(\kappa_b) \neq 0$, we could in an analytic neighborhood of b restrict to a curve whose tangent space was contained in $\text{Ker}(\kappa_b)$ at each point.) Secondly, the chern classes of the normal sheaf are in general readily calculated, so that in many cases it may be possible to estimate $h^0(X_b, \mathcal{N}_b)$.

In the case of plane curves, for example, if X_b is a curve of genus g and $\psi_b : X_b \rightarrow \mathbb{P}^2$ is birational onto a plane curve of degree d , then \mathcal{N}_b is a rank one sheaf on the curve X_b , the degree of whose chern class is

$$\begin{aligned} \deg(c_1(\mathcal{N}_b)) &= \deg(c_1(\psi_b^*T_{\mathbb{P}^2})) - \deg(c_1(T_{X_b})) \\ &= 3d + 2g - 2 \\ &> 2g - 2 . \end{aligned}$$

We would thus expect that

$$\begin{aligned} \dim(B) &\leq h^0(X_b, \mathcal{N}_b) \\ &= \deg(c_1(\mathcal{N}_b)) - g + 1 \\ &= 3d + g - 1 . \end{aligned}$$

We cannot, however, conclude this yet. The difficulty arises from the possibility that ψ_b is not an immersion: if the differential $d\psi_b$ vanishes at points of X_b , the sheaf \mathcal{N}_b will have torsion there, and in this case the quotient $\mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}}$ (and hence \mathcal{N}_b itself) may well be special. In such a case, the dimension $h^0(X_b, \mathcal{N}_b)$ will indeed be larger than the naive estimate $3d + g - 1$ for the dimension of our family, and the method appears to fail.

Happily, there is a standard result, due to Arbarello and Cornalba [AC], that deals with this situation. The current version was worked out in conversations with Johan de Jong, to whom we are very grateful.

Let $\mathcal{X} \rightarrow B$ be as before and assume that $\psi : \mathcal{X} \rightarrow B \times Y$ is birational onto its image. Then we have

Lemma 2.3. *If $b \in B$ is a general point, then*

$$\text{Im}(\kappa_b) \cap H^0(X_b, (\mathcal{N}_b)_{\text{tors}}) = 0 .$$

Remarks. 1. If we do not assume the map ψ is birational onto its image, the conclusion of the Lemma may well be false. In fact, it will fail exactly when the map $\psi_b : X_b \rightarrow Y$ is multiple-to-one, with constant image but variable branch points.

2. While we will not introduce the definitions needed to make this precise, another way to express this Lemma is to say that “the first-order deformation of the map ψ_b corresponding to a torsion section of N_b can never be equisingular”. If $b \in B$ is general the first-order deformations of ψ_b

arising from the family $\psi : \mathcal{X} \rightarrow B \times Y$ are necessarily equisingular; it follows that they cannot be torsion.

Proof. Note first that, using the analytic topology, it is enough to prove the Lemma in case B is one-dimensional: if we had $\text{Im}(\kappa_b) \cap H^0(X_b, (\mathcal{N}_b)_{\text{tors}}) \neq 0$ at general $b \in B$ we could in an analytic neighborhood of b restrict to a curve whose tangent space was contained in $(\kappa_b)^{-1}(H^0(X_b, (\mathcal{N}_b)_{\text{tors}}))$ at each point.

We may thus assume that $\psi : \mathcal{X} \rightarrow B \times Y$ is a one-parameter family of maps, the image of whose Kodaira-Spencer map κ_b at a general point is contained in $H^0(X_b, (\mathcal{N}_b)_{\text{tors}})$. Let $Z = \psi(\mathcal{X}) \subset B \times Y$ be the image of \mathcal{X} , $p \in \mathcal{X}$ a general point with image $\psi(p) = (b, q) \in B \times Y$. We are assuming that for any $v \in T_b B$, the image $\kappa_b(v)$ vanishes at p ; that is, the tangent space $T_{(b,q)}Z$ is of the form

$$T_{(b,q)}Z = T_b B \times \Lambda_p$$

for some linear subspace $\Lambda_p \subset T_q Y$.

Now, let t be a local analytic coordinate on B near b , and (x, y_1, \dots, y_n) local coordinates on Y near q such that $\psi_b^* x$ is a local coordinate on X_b near p (so that the pair (t, x) give local coordinates on the surface \mathcal{X} near p). We can write the map ψ locally as

$$y_i = f_i(t, x), \quad i = 1, \dots, n .$$

The tangent space $T_{(b,q)}Z$ is then the zero locus of the linear forms

$$dy_i - \frac{\partial f_i}{\partial t} dt - \frac{\partial f_i}{\partial x} dx$$

and that statement that $T_{(b,q)}Z = T_b B \times \Lambda_p$ for some linear subspace $\Lambda_p \subset T_q Y$ says that $\frac{\partial f_i}{\partial t}$ vanishes identically near p . We deduce that the image of ψ_b is constant, i.e., that near (b, q) the image Z is equal to the product of a neighborhood of $b \in B$ with a neighborhood of $p \in X_b$.

This being true for general $p \in \mathcal{X}$, it follows that $Z = B \times \psi_b(X_b)$ everywhere. Finally, since the map ψ is assumed birational, it follows that \mathcal{X} is the normalization of Z ; thus it is likewise a product, the map $\psi = id_B \times \psi_b$ and the Kodaira-Spencer map identically zero. \square

We now introduce the map

$$\overline{\kappa}_b : T_b B \longrightarrow H^0(X_b, \mathcal{N}_b / (\mathcal{N}_b)_{\text{tors}})$$

defined to be the composition of κ_b with the natural map $H^0(\mathcal{N}_b) \rightarrow H^0(X_b, \mathcal{N}_b / (\mathcal{N}_b)_{\text{tors}})$. Notice that the Lemma implies that this map is an injection on the subspace $\text{Im}(\kappa_b)$.

As an application, we will fix the argument given above for plane curves. In fact, we will prove a slightly more general result. Let Y be a smooth surface and D an effective divisor on Y . Let V be an irreducible component of the Severi variety of curves of given geometric genus g that are linearly equivalent to D . There is a universal family $\mathcal{U} \subset V \times Y$ of curves over V . Consider the normalization \mathcal{U}^ν of \mathcal{U} and let $B \subset V$ be an open subset over which \mathcal{U}^ν is smooth. Let $\mathcal{X} \rightarrow B$ be the restriction of \mathcal{U}^ν to B and $\psi : \mathcal{X} \rightarrow B \times Y$ be as usual. Then we obtain the following well known result (cf. for example [K] and loc. cit.).

Corollary 2.4. *If, for general $b \in B$ we have $\deg_W(\psi_b^*\omega_Y) < 0$ on every component W of X_b , then*

$$\dim B \leq -\deg(\psi_b^*\omega_Y) + g - 1 .$$

In particular, $\dim V_{d,\delta} = 3d + g - 1$.

Proof. We have

$$\dim B \leq \dim(\text{Im}(\kappa_b)) \leq h^0(X_b, \mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}})$$

where the last inequality follows from the previous Lemma.

By the definition of \mathcal{N}_b , we obtain

$$\deg(c_1(\mathcal{N}_b)) = -\deg(\psi_b^*\omega_Y) + \deg(\omega_{X_b})$$

hence, by our hypothesis $\deg(c_1(\mathcal{N}_b)) > \deg(\omega_{X_b})$ on each component of X_b . We now state for both present and future use the following simple corollary of the Riemann-Roch theorem for curves:

Observation 2.5. *Let X be a smooth curve of genus g , and L any line bundle on X of degree d such that $L \otimes \omega_X^{-1}$ has positive degree on each component of X . If M is any line bundle on X such that $L \otimes M^{-1}$ has nonnegative degree on each component of X , then*

$$h^0(X, M) \leq h^0(X, L) = d - g + 1 .$$

If moreover the line bundle $L \otimes \omega_X^{-1}$ has degree 2 or more on each component of X , then $h^0(X, M) = d - g + 1$ only if $\deg M = \deg L$.

Applying this to $X = X_b$ and the line bundles $L = \psi_b^*\omega_Y^{-1} \otimes \omega_{X_b}$ and $M = \mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}}$, we have

$$\begin{aligned} \dim B &\leq h^0(X_b, \mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}}) \\ &\leq h^0(X_b, \psi_b^*\omega_Y^{-1} \otimes \omega_{X_b}) \\ &= -\deg(\psi_b^*\omega_Y) + p_a(X_b) - 1 . \end{aligned} \quad \square$$

We need now to consider deformations of a map $X_b \rightarrow Y$ that preserve tangency conditions with a fixed smooth curve $G \subset Y$. There are two cases, depending on whether we require tangency at a fixed point $p \in G$ or allow tangency at a variable point.

So, let Y be a smooth surface, $G \subset Y$ a curve and $p \in G$ a smooth point. Let $\mathcal{X} \rightarrow B$ be as above a smooth family of curves over a reduced base B , $\psi : \mathcal{X} \rightarrow B \times Y$ a morphism of B -schemes, and $Q \subset \mathcal{X}$ a section such that the pullback divisor $\psi^*(G)$ contains the section Q with multiplicity exactly m . Let $b \in B$ be a general point and $q = X_b \cap Q$; suppose $\psi(q) = p$. Let $v \in T_b B$ and let $\sigma = \overline{\kappa_b(v)} \in H^0(X_b, \mathcal{N}_b)$ the corresponding first-order deformation, and $\bar{\sigma} = \overline{\kappa_b(v)} \in H^0(X_b, \mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}})$. Suppose finally that the differential $d\psi_b$ vanishes to order $l - 1$ at q , so that the image $\psi_b(\Delta)$ of a small neighborhood Δ of $q \in X_b$ will have multiplicity l at $p = \psi(q)$. We have then the

Lemma 2.6. *$\bar{\sigma}$ vanishes to order at least $m - l$ at q , and cannot vanish to order exactly k for any k with $m - l < k < m$. Moreover, if we assume that $\psi(Q)$ is a point, $\bar{\sigma}$ vanishes to order at least m at q .*

Proof. It will be sufficient to do this in case B is one-dimensional. Next, since B is reduced and $b \in B$ is general, we may assume B smooth at b ; so that, restricting to an analytic neighborhood of $b \in B$ we may take b the origin in an open subset B of the affine line $\mathbb{A}^1 = \text{Spec } k[\epsilon]$. Finally, since again $b \in B$ is general we may assume that the divisor ψ_b^*G on X_b contains the point q with multiplicity exactly m as well.

Now, choose coordinates (x, y) in an analytic neighborhood of $p = \psi(q)$ so that the curve G is given simply as the zero locus of y . Let then $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ be the generators of the rank 2 bundle TY at p ; we will abuse notation and write $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ also for the corresponding sections of ψ_b^*TY .

The first thing we will show is that the image of $\frac{\partial}{\partial x}$ in $\mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}}$ vanishes to order $m - l$ at q .

We treat the case $l < m$ first for simplicity, and leave the case $l = m$ for later. Let t be an m^{th} root of ψ_b^*y in a neighborhood of $q \in X_b$, then t will be a local coordinate on X_b near q and the map ψ_b will be given as

$$\psi_b : t \mapsto (t^l + c_{l+1}t^{l+1} + \dots, t^m)$$

so that the differential $d\psi_b$ is given by

$$\begin{aligned} d\psi_b : \frac{\partial}{\partial t} &\mapsto (lt^{l-1} + (l+1)c_{l+1}t^l + \dots) \frac{\partial}{\partial x} + mt^{m-1} \frac{\partial}{\partial y} \\ &= t^{l-1} \left((l + (l+1)c_{l+1}t + \dots) \frac{\partial}{\partial x} + mt^{m-l} \frac{\partial}{\partial y} \right), \end{aligned}$$

Denote $\tau(t) := (l + (l + 1)c_{l+1}t + \dots) \frac{\partial}{\partial x} + mt^{m-l} \frac{\partial}{\partial y}$, we have that the torsion subsheaf $(\mathcal{N}_b)_{\text{tors}} \subset \mathcal{N}_b$ is isomorphic to $\mathcal{O}_{X_b}/\mathfrak{m}_q^{l-1}$, generated by the section $\tau(t)$. Moreover, the quotient

$$\mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}} = \mathcal{O}_{X_b} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} / \langle \tau \rangle$$

is generated for example by the image of the section $\frac{\partial}{\partial y}$. Note finally that modulo the subsheaf generated by τ ,

$$\frac{\partial}{\partial x} \sim \frac{mt^{m-l}}{l + (l + 1)c_{l+1}t + \dots} \cdot \frac{\partial}{\partial y}$$

so that the image of the section $\frac{\partial}{\partial x}$ in $\mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}}$ vanishes to order exactly $m - l$ at q .

Now, a general deformation ψ of the map ψ_b over the base $B \subset \mathbb{A}_\epsilon^1$ may be given in terms of coordinates t and ϵ on \mathcal{X} near q as

$$\begin{aligned} \psi(t, \epsilon) = & (\epsilon; t^l + c_{l+1}t^{l+1} + \dots + \epsilon(\alpha_0 + \alpha_1t + \dots) \\ & + (\epsilon)^2, t^m + \epsilon(\beta_0 + \beta_1t + \dots) + (\epsilon)^2) . \end{aligned}$$

The condition that the divisor $\psi^*((y)) = mQ$ near q says that we can take t to be an m^{th} root of the pullback ψ^*y not just on X_b , but in a neighborhood of q in \mathcal{X} . This means that a deformation satisfying the hypotheses of the lemma may be written as

$$\psi(t, \epsilon) = (\epsilon, t^l + c_{l+1}t^{l+1} + \dots + \epsilon(\alpha_0 + \alpha_1t + \dots) + (\epsilon)^2, t^m) .$$

From the definitions, the image $\kappa_b(\frac{\partial}{\partial \epsilon}) \in H^0(X_b, \mathcal{N}_b)$ of the tangent vector $\frac{\partial}{\partial \epsilon} \in T_bB$ under the Kodaira-Spencer map will be given as the image in \mathcal{N}_b of

$$\sigma := \kappa_b \left(\frac{\partial}{\partial \epsilon} \right) = (\alpha_0 + \alpha_1t + \dots) \frac{\partial}{\partial x} ,$$

whose image $\bar{\sigma}$ in $\mathcal{N}_b/(\mathcal{N}_b)_{\text{tors}}$, as we have seen, vanishes to order at least $m - l$ at q . Moreover, since $b \in B$ is general, the differential $d\psi_\epsilon$ will vanish to order $l - 1$ at $X_\epsilon \cap Q$ for all ϵ near b ; that is, $t^{l-1}|d\psi_\epsilon$. This implies that

$$\alpha_1 = \alpha_2 = \dots = \alpha_{l-1} = 0 ;$$

or in other words, $\bar{\sigma}$ cannot vanish to order exactly $m - l + 1, \dots, m - 1$ at p . To complete the proof in case $m > l$, the further condition that $\psi(Q) \equiv p \in G$ says that $\alpha_0 = 0$, which further implies that $\bar{\sigma}$ vanishes to order at least m at q .

The case $m = l$ is completely analogous. As before we write the map ψ_b as

$$\psi_b : t \mapsto (t^n + c_{n+1}t^{n+1} + \dots, t^m)$$

where now $n \geq m$. We leave it to the reader to check that the same argument yields that if $\psi(Q) \equiv p \in G$, the section $\bar{\sigma}$ vanishes to order at least m at q . □

2.3. Dimension counts and consequences

We will now use the general theory developed above to establish Propositions 2.1 and 2.2.

Proof of Proposition 2.1. To begin with, it follows from the naive dimension count at the beginning that the Severi variety $V^{d,\delta}(\alpha, \beta)$ has dimension at least $2d + g - 1 + |\beta|$.

We thus have to show that $\dim V^{d,\delta}(\alpha, \beta) \leq 2d + g - 1 + |\beta|$ everywhere. To do this, let $V \subset V^{d,\delta}(\alpha, \beta)$ be an irreducible component, $\mathcal{X} \subset V \times \mathbb{P}^2$ the universal family curve over V , and \mathcal{X}^v the normalization of the total space. We will actually restrict our attention to the open subset of V over which \mathcal{X}^v is smooth, which we will still call V .

Let $[X] \in V$ be a general point, so that the restriction $v = \psi_{[X]}$ of ψ to the fiber of \mathcal{X}^v over $[X]$ is the normalization $v : X^v \rightarrow X \subset \mathbb{P}^2$ of the corresponding curve $X \subset \mathbb{P}^2$; and let \mathcal{N} be the normal sheaf of the map v ; notice that this might appear as an abuse of notation, as we have already used the symbol \mathcal{N} with a different meaning, in the previous chapter; we hope that this will not create confusion. By the definition of $V^{d,\delta}(\alpha, \beta)$, we have in an analytic neighborhood of $[X]$ a collection of $|\alpha|$ and $|\beta|$ sections $\{Q_{i,j}\}$ and $\{R_{i,j}\} \subset \mathcal{X}^v$ such that

$$\psi(Q_{i,j}) = p_{i,j}$$

and

$$\psi^*(L) = \sum i \cdot Q_{i,j} + \sum i \cdot R_{i,j} .$$

Let $q_{i,j} = Q_{i,j} \cap X^v$ and $r_{i,j} = R_{i,j} \cap X^v$. Note that the points $\{q_{i,j}\}$ are necessarily distinct, since they have distinct images $p_{i,j} \in L \subset \mathbb{P}^2$. We may assume as well that the points $\{r_{i,j}\}$ are distinct, and disjoint from the $\{q_{i,j}\}$: if not, $[X]$ being general in V , V would be as well a component of a Severi variety $V^{d,\delta}(\alpha', \beta')$ for some (α', β') with $|\beta'| < |\beta|$, which we will show has dimension $2d + g - 1 + |\beta'| < 2d + g - 1 + |\beta|$.

We need to introduce one more bit of notation. We denote by $l_{i,j} - 1$ the order of vanishing of the differential dv at the point $r_{i,j}$. We then let D and $D_0 \in \text{Div}(X^v)$ be the divisors

$$D = \sum_{1 \leq j \leq \alpha_i} i \cdot q_{i,j} + \sum_{1 \leq j \leq \beta_i} (i - 1) \cdot r_{i,j} .$$

and

$$D_0 = \sum_{1 \leq j \leq \alpha_i} (l_{i,j} - 1) \cdot r_{i,j} .$$

Note that D is a divisor of degree

$$\begin{aligned} \text{deg}(D) &= I\alpha + I\beta - |\beta| \\ &= d - |\beta| . \end{aligned}$$

and that

$$\text{deg}((v^* \mathcal{O}_{\mathbb{P}^2}(1))(-D)) \geq 0 .$$

Note also that $\text{deg}(c_1(\mathcal{N}_{\text{tors}})) \geq \text{deg}(D_0)$, on every component of X^v , with equality holding if and only if v is an immersion away from $\{r_{i,j}\}$; so that

$$\text{deg}(c_1(\mathcal{N}/\mathcal{N}_{\text{tors}})) \leq \text{deg}(c_1(\mathcal{N})) - \text{deg}(D_0)$$

again with equality holding if and only if v is an immersion away from $\{r_{i,j}\}$.

Finally, let D_1 be the effective part of $D - D_0$.

Now, applying Lemmas 2.3 and 2.6, we see that

$$\dim V^{d,\delta}(\alpha, \beta) \leq h^0(X^v, (\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1)) .$$

We have

$$\text{deg}((\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1)) \leq \text{deg}(c_1(\mathcal{N})) - \text{deg}(D)$$

and since

$$c_1(\mathcal{N}) = v^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \omega_{X^v}$$

we see that the line bundle

$$(c_1(\mathcal{N})(-D)) \otimes \omega_{X^v}^{-1} = ((v^* \mathcal{O}_{\mathbb{P}^2}(1))(-D)) \otimes v^* \mathcal{O}_{\mathbb{P}^2}(2)$$

has strictly positive degree on each component of X^v . We may thus apply the simple Observation 2.5 to the line bundles $c_1(\mathcal{N})(-D)$ and $(\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1)$ to conclude that

$$\begin{aligned}
 \dim V^{d,\delta}(\alpha, \beta) &\leq h^0(X^v, (\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1)) \\
 &\leq \deg(c_1(\mathcal{N})(-D)) - g + 1 \\
 &= (3d + 2g - 2 - \deg(D)) - g + 1 \\
 &= 2d + g - 1 + |\beta| . \quad \square
 \end{aligned}$$

Remark. Notice that the argument above implies that the image of the Horikawa map can be identified as follows:

$$\text{Im}\bar{\kappa}_{[X]} = H^0(X^v, (\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1))$$

Proof of Proposition 2.2. We start by establishing what is perhaps the subtlest point: that the map v is indeed an immersion. In fact, much of this has already been accomplished in the proof of Proposition 2.1. Keeping the notations introduced there, we see that since the line bundle $(c_1(\mathcal{N})(-D) \otimes \omega_{X^v}^{-1})$ on X^v has degree at least 2 on any component of X^v , we may apply the second part of Observation 2.5 to deduce the equality

$$(\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1) = c_1(\mathcal{N})(-D)$$

so that $D_1 = D - D_0$ and

$$\mathcal{N}/\mathcal{N}_{\text{tors}} = c_1(\mathcal{N})(-D_0)$$

and hence v is an immersion away from $\{r_{i,j}\}$.

To see that v is an immersion at the point $r_{i,j}$, we may assume that the component X_0 of X^v containing $r_{i,j}$ does not map to a line, so that the line bundle $(c_1(\mathcal{N})(-D) \otimes \omega_{X^v}^{-1})$ has degree at least 4 on X_0 . It follows that there exists a section $\bar{\sigma}$ of $c_1(\mathcal{N})(-D) = (\mathcal{N}/\mathcal{N}_{\text{tors}})(-D_1)$ vanishing to order exactly 1 at $r_{i,j}$; and by the previous Remark, this section must be in the image of the Horikawa map

$$\bar{\kappa}_{[X]} : T_{[X]}V \longrightarrow H^0(X^v, (\mathcal{N}/\mathcal{N}_{\text{tors}})) .$$

But the multiplicity of $r_{i,j}$ in the divisor $D_1 = D - D_0$ is $(i - 1) - (l_{i,j} - 1) = i - l_{i,j}$, and it follows that $\bar{\sigma}$, viewed as a section of $\mathcal{N}/(\mathcal{N}_b)_{\text{tors}}$, vanishes to order exactly $i - l_{i,j} + 1$ at $r_{i,j}$. By Lemma 2.6, then, we must have $l_{i,j} = 1$; that is, v must be an immersion at $r_{i,j}$. Since we have already seen that v is an immersion away from $\{r_{i,j}\}$, we conclude that v is an immersion.

To show that X has only nodes as singularities, we have to show it has no triple points and that no two branches are tangent to each other. For the former, if $s, t, u \in X^v$ are points mapping to the same point $p \in X \subset \mathbb{P}^2$, it is

enough to show that there exists a section of $\mathcal{N}(-D)$ vanishing at s and t but not at u . This follows immediately from Riemann-Roch: if s and t and u all belong to the same component of X^v , that component must map to a plane curve of degree at least 4, so that $\mathcal{N} \otimes \omega_{X^v}^{-1}$ will have degree at least 8 there; while if two lie on the same component, $\mathcal{N}(-D) \otimes \omega_{X^v}^{-1}$ will have degree at least 6 there. Similarly, for the latter, it is enough to show that if $s, t \in X^v$ are points mapping to the same point $p \in X \subset \mathbb{P}^2$, there exists a section of the sheaf $\mathcal{N}(-D)$ vanishing at s but not at t , which follows from the same argument.

As for parts c and d of 2.2, we have already seen just from the dimension statement that the points $\{q_{i,j}\}$ and $\{r_{i,j}\}$ are all distinct, since otherwise V would be a component of a Severi variety $V^{d,\delta}(\alpha', \beta')$ for some (α', β') with $|\beta'| < |\beta|$; and the same logic implies that the points $\{s_{i,j}\}$ are disjoint from the points $\{p_{i,j}\}$. To see that the points $s_{i,j}$ are all distinct, on the other hand, it is sufficient to observe that, by the argument of the preceding paragraph, for any $(i', j') \neq (i, j)$ with $s_{i',j'} = s_{i,j}$, there is a section of the sheaf $\mathcal{N}(-D)$ vanishing at $r_{i,j}$ but not at $r_{i',j'}$; by Lemma 2.6 this will correspond to a deformation of X in which $s_{i',j'}$ moves but $s_{i,j}$ stays still.

Next, given that $v : X^v \rightarrow X$ is an immersion, part b follows from d; if v is one-to-one over points of L then X is smooth along L .

Finally, part e: if a branch of X corresponding to a point $s \in X^v$ were tangent to G , it would be enough to show that there exists a section of $\mathcal{N}(-D)$ not vanishing at s , which we know; and likewise if two points $s, t \in X^v$ mapped to the same point $p \in G$, it would be enough to show that there exists a section of $\mathcal{N}(-D)$ vanishing at s but not at t , which again we know. □

The following restatement of Proposition 2.1 will be useful in the applications in the next section. To set it up, fix a line $L \subset \mathbb{P}^2$ and a finite subset $\Omega \subset L$. Let $V \subset \mathbb{P}^N$ be any irreducible, locally closed subset of the space of plane curves of degree d , and $[X] \in V$ a general point. Let $\pi : W \rightarrow X \subset \mathbb{P}^2$ be any map not constant on any irreducible component of W , whose degree over each irreducible component X_i of X is equal to the multiplicity of X_i in X (so that in particular the pullback $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ has degree d). Let g be the geometric genus of W , and let e be the cardinality of the intersection $\#(X \cap (L \setminus \Omega))$. We have then

Corollary 2.7.

$$\dim V \leq 2d + g - 1 + e ;$$

and if equality holds and $\#(X \cap (L \setminus \Omega)) = \#\pi^{-1}(L \setminus \Omega)$ then V is a dense open subset of a generalized Severi variety $V^{d,\delta}(\alpha, \beta)$.

Proof. This follows readily from (2.1), after a few reductions. To begin with, it is enough to prove this in case W is smooth, since replacing W by its normalization only strengthens the inequality. Secondly, it is enough to do it

in case X is irreducible: applying the statement to the inverse image of each component of X in turn and adding the results yields the desired inequality in general. (This second reduction is not really essential, but will allow us to refer to the degree of the map $W \rightarrow X$ without confusion.)

Now, since W is smooth, the map $W \rightarrow X$ factors through the normalization $X^v \rightarrow X_{\text{red}}$, and we claim that it is enough to prove it in case $W = X^v$. To see this, assume the result proved in case $W = X^v$ and consider what happens if $W \rightarrow X^v$ is a finite map of degree $m > 1$. In this case the degree of X_{red} is d/m , and the genus h of X^v is related to the genus g of W by Riemann-Hurwitz:

$$g \geq mh - m + 1$$

Now, applying (2.1) directly to X_{red} , we have

$$\begin{aligned} \dim(V) &\leq 2\frac{d}{m} + h - 1 + e \\ &\leq 2\frac{d}{m} + \frac{g-1}{m} + e \\ &< 2d + g - 1 + e \end{aligned}$$

and we have a contradiction.

We may thus assume that $W = X^v$. Now, suppose first that $e = 0$. In this case, the statement we want to prove is exactly Proposition 2.1 and we are done. More generally, consider the map ϕ from a neighborhood of $[X] \in V$ to $|\mathcal{O}_L(e)|$ sending a point $[W] \in V$ to the reduced intersection $W \cap (L \setminus \Omega)$. Applying the $e = 0$ case of the statement of the Corollary to the fiber of ϕ over a general point $D \in |\mathcal{O}_L(e)|$ (replacing Ω by $\Omega' = \Omega \cup \text{supp}(D)$), we conclude that the fibers of ϕ have dimension at most $2d + g - 1$, and hence that $\dim(V) \leq 2d + g - 1 + e$. \square

Note that, in the case of equality, the map $\pi : W \rightarrow X$ is necessarily a birational isomorphism on each component of W .

2.4. Normal sheaves and normal bundles

Before going on, we should say a few words about the relationship between the treatment of Severi varieties given here and other possible approaches. This section will not be logically relevant to the rest of the paper, but should be of use in relating the results here to those of [CH, DH, H].

Briefly, there are two ways of analyzing the deformations of a plane curve X satisfying certain geometric conditions. In the approach taken here, which we may call the ‘‘parametric’’ approach, we look at deformations of the normalization map $v : X^v \rightarrow X \subset \mathbb{P}^2$; so that the tangent space to the space of deformations is a priori a subspace of the space of sections of the normal sheaf \mathcal{N} of the map. This has the virtue (at least, it is a virtue in our

present circumstances) of incorporating the condition that the geometric genus of X is preserved in the deformations. Moreover, the sheaf \mathcal{N} is a sheaf on a smooth curve. On the other hand, it has the defect that, until we know that v is an immersion, the sheaf \mathcal{N} may have torsion.

In the other approach, which we will call the ‘‘Cartesian’’ approach, we look instead at deformations of X as a subscheme of \mathbb{P}^2 ; so that the tangent space to the space of deformations is a priori a subspace of the space of sections of the normal bundle $N_{X/\mathbb{P}^2} \cong \mathcal{O}_X(d)$ of the divisor $X \subset \mathbb{P}^2$. This is in some ways more direct – all we are doing, after all, is practicing the time-honored tradition of varying the coefficients of the defining polynomial of X – and it is in particular useful when we want to intersect our family with other subvarieties of the space \mathbb{P}^N of plane curves of degree d . But it has the drawback that we have to impose extra conditions to ensure that the geometric genus of X stays constant. These conditions, moreover, sometimes interact badly with conditions such as tangency with a fixed curve.

What is the relationship between the two? In case $v : X^v \rightarrow X$ is an immersion, it is reasonably straightforward. To start with, let $\mathcal{I} \subset \mathcal{O}_X$ be the conductor ideal of X . This may be characterized in several equivalent ways:

- It is the annihilator of the sheaf $v_*\mathcal{O}_{X^v}/\mathcal{O}_X$;
- On an affine open subset of X with defining equation $f(x, y)$, it is the ideal of polynomials $g(x, y)$ such that the 1-form

$$v^*\left(\frac{g(x, y) dx}{\partial f/\partial y}\right)$$

is regular on X^v ; and

- More concretely, in case $v : X^v \rightarrow X$ is an immersion, it is the ideal in \mathcal{O}_X whose restriction to each branch Δ_i of X at each point $p \in X$ is equal to the restriction to that branch of the ideal of the union of all other branches of X through p . In other words, if $p_i \in X^v$ is the point lying over p in the branch Δ_i ,

$$v^*\mathcal{I} = \mathcal{O}_{X^v} \left(- \sum_i \left(\sum_{j \neq i} \text{mult}_p(\Delta_i \cdot \Delta_j) \right) \cdot p_i \right) .$$

However we characterize the conductor, it is not hard to see that, in case $v : X^v \rightarrow X$ is an immersion, the normal sheaf \mathcal{N} of the map v and the normal bundle $N_{X/\mathbb{P}^2} \cong \mathcal{O}_X(d)$ of the curve X are related by

$$\mathcal{N} = v^*(\mathcal{I} \otimes N_{X/\mathbb{P}^2}) .$$

This is perhaps most easily seen in terms of the last description of the conductor: if the local defining equation $f(x, y)$ of X at a point $p \in X$ factors in the completion of the local ring $\mathcal{O}_{X,p}$ as

$$f(x, y) = f_1(x, y)f_2(x, y) \cdots f_n(x, y)$$

then a general first-order deformation of the map will simply move each branch, resulting in a curve given by the equation

$$f_\epsilon(x, y) = (f_1(x, y) + \alpha_1\epsilon)(f_2(x, y) + \alpha_2\epsilon) \cdots (f_n(x, y) + \alpha_n\epsilon) .$$

As a deformation of the map, that is, as a section of \mathcal{N} , this will be nonzero at the point of X^v corresponding over the branch Δ_i given by $f_i(x, y) = 0$ if and only if the coefficient $\alpha_i \neq 0$. But the corresponding section of the normal bundle, that is, the restriction to X of the coefficient of ϵ in $f_\epsilon(x, y)$, on this branch is $\alpha_i \prod_{j \neq i} f_j(x, y)$, which vanishes to order $\sum_{j \neq i} \text{mult}_p(\Delta_j \cdot \Delta_i)$.

In any event, the conclusion is that the sections of the N_{X/\mathbb{P}^2} coming from deformations of the map are simply those lying in the conductor ideal (or, classically, “satisfying the adjoint conditions”, in view of the third characterization above). Moreover, if we impose further conditions of tangency with fixed curves, the allowed deformations of the map correspond to sections of \mathcal{N} vanishing to the appropriate order at the points of X^v lying over the points of tangency; and these sections, by the second characterization above, correspond to sections of $\mathcal{J} \otimes N_{X/\mathbb{P}^2}$ for a unique ideal sheaf $\mathcal{J} \subset \mathcal{I}$.

We thus have a very useful dictionary between the two languages, at least as long as v is an immersion. Otherwise the correspondence is more complicated. For example, if $[X]$ is a point on the variety of plane curves of given degree d and genus g , corresponding to a curve with a cusp and $\delta - 1 = \binom{d-1}{2} - g - 1$ nodes, then in a neighborhood of $[X]$ we cannot simultaneously normalize the fibers of the universal family $\mathcal{X} \subset V \times \mathbb{P}^2 \rightarrow V$; so deformations of X preserving the geometric genus do not correspond to deformations of the map.

3. Hyperplane sections of Severi varieties: set-theoretic description

We are now prepared to describe the hyperplane sections of the generalized Severi varieties. In this chapter we will prove Theorem 1.2, showing that the intersection $V^{d,\delta}(\alpha, \beta) \cap H_p$ is indeed a union of generalized Severi varieties of dimension one less, and saying which ones potentially occur. In the following chapter we will prove Theorem 1.3, establishing that all the generalized Severi varieties listed as possible components of $V^{d,\delta}(\alpha, \beta) \cap H_p$ do in fact occur, and describing the multiplicities with which they appear.

3.1. The basic setup

Let V' be any irreducible component of the intersection $V^{d,\delta}(\alpha, \beta) \cap H_p$ and $[X_0] \in V'$ a general point. If X_0 does not contain L it is easy to see that V' must be a component of one of the generalized Severi varieties listed in the first part of Theorem 1.2, so we will focus on the case $L \subset X_0$. We then consider a curve $\Gamma = \{[X_\gamma]\} \subset V^{d,\delta}(\alpha, \beta)$ passing through the point $[X_0]$, and the corresponding family of plane curves $\mathcal{X} \rightarrow \Gamma$. Applying a variant of

semistable reduction to this family in a neighborhood of $[X_0] \in \Gamma$ we arrive at a family $\mathcal{Y} \rightarrow B$ of nodal curves dominating the curves X_γ in our family. Analyzing this family, we find a number of geometric conditions that the curve X_0 must satisfy, which limit the number of its degrees of freedom. Playing these off against the fact that X_0 is a general member of a variety of dimension $\dim(V^{d,\delta}(\alpha, \beta)) - 1$ we are led to our characterization of $V^{d,\delta}(\alpha, \beta) \cap H_p$.

Although the approach is quite simple, the arguments tend to appear extremely complicated. In fact, there are a priori no restrictions on the number of components, for example, of the special fiber of the family $Y \rightarrow B$ or their configuration, the notation alone can be very cumbersome. We will therefore proceed in two steps: we will give the analysis first subject to a number of simplifying assumptions, which will make the logic of the argument relatively clear (all of these hypotheses, moreover, will in fact turn out to satisfied in reality). Then we will go back and prove the result without assumptions. Comparing the argument here with the first should make it clear why in fact the assumptions hold: if any of them were indeed violated, we could replace one of the inequalities of the first calculation with a strict inequality, and so arrive at a contradiction.

We start by defining the families $\mathcal{X} \rightarrow \Gamma$ and $\mathcal{Y} \rightarrow B$ that we will be working with, and establishing the relevant notation. As above, we let $[X_0] \in V'$ be a general point of a component V' of $V^{d,\delta}(\alpha, \beta) \cap H_p$, and $\Gamma \subset V^{d,\delta}(\alpha, \beta)$ a curve containing $[X_0]$. We will assume that the general point $[X_\gamma]$ of Γ is a general point of $V^{d,\delta}(\alpha, \beta)$; that is, it satisfies the conclusions of Proposition 2.1 above.

Now, let $\nu : \Gamma^\nu \rightarrow \Gamma$ be the normalization of Γ , and choose a point $b_0 \in \Gamma^\nu$ lying over $[X_0]$. Let \mathcal{X}^ν be the normalization of the total space of the pullback $X \times_\Gamma \Gamma^\nu$, so that the family $\mathcal{X}^\nu \rightarrow \Gamma^\nu$ has as general fiber a smooth curve of genus $g = \binom{d-1}{2} - \delta$.

Next, we want to carry out a nodal reduction of the family $\mathcal{X}^\nu \rightarrow \Gamma^\nu$ in a neighborhood of b_0 to arrive at a family of nodal curves $\mathcal{Y} \rightarrow B$ satisfying the requirements below. (This can be achieved after possibly further base change and blowing up of the nodal reduction.) Let $\mathcal{Y}^* = f^{-1}(B \setminus \{b_0\})$ be the complement of the special fiber $Y_0 = f^{-1}(b_0)$ of $\mathcal{Y} \rightarrow B$. Then we require that

- a. The total space \mathcal{Y} is smooth.
- b. The map carrying a general fiber Y_b of $\mathcal{Y} \rightarrow B$ to the corresponding plane curve $X_\gamma \subset \mathbb{P}^2$ extends to a regular morphism

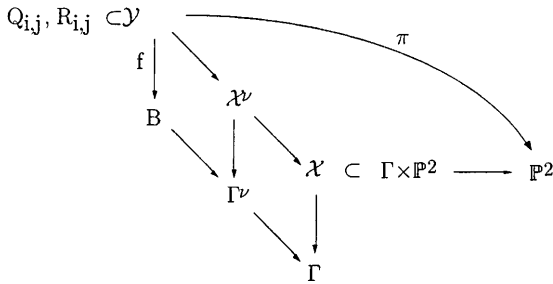
$$\pi : \mathcal{Y} \longrightarrow \mathbb{P}^2 .$$

- c. The inverse image $\pi^{-1}(L) \cap \mathcal{Y}^*$ consists of $|\alpha| + |\beta|$ disjoint sections $\{Q_{i,j}^*\}_{1 \leq j \leq \alpha_i}$ and $\{R_{i,j}^*\}_{1 \leq j \leq \beta_i}$, with $\pi(Q_{i,j}^*) \equiv p_{i,j}$ and $R_{i,j}^*$ intersecting the general fiber in a point $r_{i,j}$ of multiplicity i in the divisor $(\pi|_{Y_b})^*L$ – that is we have

$$\pi^*L \cap \mathcal{Y}^* = \sum i \cdot Q_{i,j}^* + \sum i \cdot R_{i,j}^* .$$

d. The closures $Q_{i,j}$ and $R_{i,j}$ in \mathcal{Y} of the sections $Q_{i,j}^*$ and $R_{i,j}^*$ are still disjoint and they do not pass through any of the singularities of Y_0 .

e. Finally, $\mathcal{Y} \rightarrow B$ is minimal with respect to these properties. In sum, we have the following diagram of objects and morphisms:



3.2. Some simplifying assumptions and some corollaries

In order to present as clearly as possible the actual picture of the families $\mathcal{X} \rightarrow \Gamma$ and $\mathcal{Y} \rightarrow B$, we will first carry out the analysis of the family $\mathcal{Y} \rightarrow B$ under three simplifying assumptions, all of which we will show in the last part of this section do in fact hold. We will also mention some interesting facts that will follow as consequences of the proof of Theorem 1.2 (in particular they will not be assumed in the course of the proof). Since they contribute to the picture of the family $\mathcal{Y} \rightarrow B$ we will state them here.

The first of our assumptions is perhaps the least obvious: it is that

Assumption (a). *The curve X_0 contains L with multiplicity 1; that is, $X_0 = X \cup L$, where X is a plane curve of degree $d - 1$ not containing L .*

(We should remark that this seems to be false in slightly more general situations, for example if we consider the variety V of plane curves of given degree and geometric genus having a triple point.)

Now, given (a), we see that the special fiber $Y_0 = f^{-1}(b_0)$ of $\mathcal{Y} \rightarrow B$ will contain a unique component \tilde{L} such that π maps \tilde{L} onto L , and indeed the map $\pi|_{\tilde{L}} : \tilde{L} \rightarrow L$ will be an isomorphism. We may then group together the remaining components of Y_0 into two sets: we will let $Y \subset Y_0$ be the union of the irreducible components mapping to X on which π is nonconstant, and $Z \subset Y_0$ the union of the irreducible components of Y_0 on which π is constant. In these terms, we will assume next that

Assumption (b). *The curve $Z \subset Y_0$ consists of a disjoint union of chains of rational curves joining \tilde{L} to Y .*

Assumption (c). *The sections $\{Q_{i,j}\}_{1 \leq j \leq \alpha_i}$ and $\{R_{i,j}\}_{1 \leq j \leq \beta_i}$ are disjoint from Z .*

Note that by the last statement, each such section meets either Y or \tilde{L} but not both. To keep track of how many of the sections $Q_{i,j}$ pass through each, we will introduce some more notation. First, we define two further sequences α' and α'' with $\alpha' + \alpha'' = \alpha$: we let α'_i be the number of the sections $\{Q_{i,j}\}_{j=1, \dots, \alpha_i}$ passing through Y , and α''_i the number passing through \tilde{L} . We will likewise label the sections passing through Y (respectively, \tilde{L}) as $\{Q'_{i,j}\}_{1 \leq j \leq \alpha'_i}$ (respectively, $\{Q''_{i,j}\}_{1 \leq j \leq \alpha''_i}$), their points of intersection with Y_0 as $\{q'_{i,j}\}_{1 \leq j \leq \alpha'_i}$ (respectively, $\{q''_{i,j}\}_{1 \leq j \leq \alpha''_i}$), and their image points in L as the subset $\mathcal{Q}' = \{p'_{i,j}\}_{1 \leq j \leq \alpha'_i}$ (respectively, $\{p''_{i,j}\}_{1 \leq j \leq \alpha''_i}$).

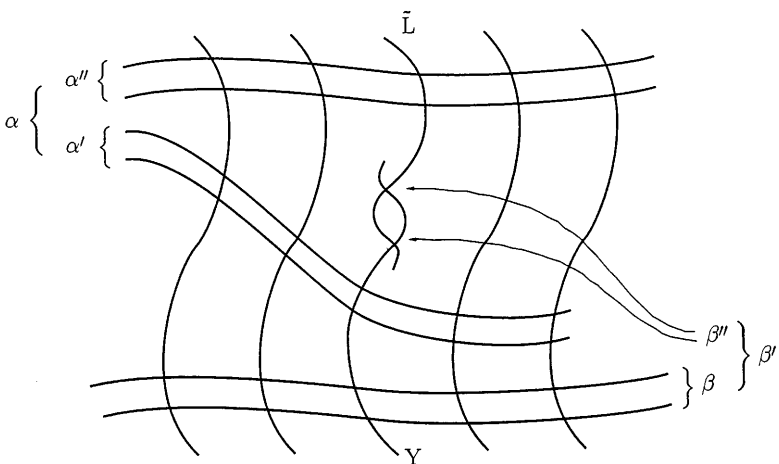
As for the sections $R_{i,j}$, the situation is a little different, by virtue of the first of our corollaries:

Consequence 1. *Every section $R_{i,j}$ passes through Y .*

Notice that this means that $\beta \leq \beta'$, so that this statement is actually part of Theorem 1.2. Thus, we do not introduce a new set of symbols. Rather, we will provisionally let β_i^0 be the number of the sections $\{R_{i,j}\}_{j=1, \dots, \beta_i}$ passing through Y , and suppose (after possibly relabeling) that the sections $R_{i,j}$ passing through Y are $\{R_{i,j}\}_{1 \leq j \leq \beta_i^0}$. We will let $r_{i,j}$ be the point of intersection of $R_{i,j}$ with Y_0 , and $s_{i,j} = \pi(r_{i,j}) \in L$ its image point.

Note also that, given (b), we may, at the expense of introducing rational double points into our surface \mathcal{Y} , collapse the connected components of Z to points, so that the fiber Y_0 consists simply of \tilde{L} and Y (we have done this in the diagram below for clarity). We will index the points of intersection of \tilde{L} with Y as follows: for each i , we let $r''_{i,1}, \dots, r''_{i,\beta''_i}$ be the points of $\tilde{L} \cap Y$ appearing with multiplicity i in the divisor $\pi^*L|_Y$. We will also let $s''_{i,j} = \pi(r''_{i,j}) \in L$ be the images of these points.

In sum, we have the following picture of the family $\mathcal{Y} \rightarrow B$ and its sections:



(Note that we have anticipated in this picture the statement (d) that every section $R_{i,j}$ passes through Y .)

We also want to underline two other corollaries of the proof:

Consequence 2. *The curve X is reduced; that is, the map $\pi|_Y : Y \rightarrow X$ is a birational isomorphism on each component of Y ; and*

Consequence 3. *Y is smooth (though not in general connected).*

3.3. Proof of simplified Theorem 1.2

We now prove Theorem 1.2 under assumptions (a), (b) and (c). We do that by comparing two different relations that the arithmetic genus g of Y has to satisfy

To begin with, note that the curves Y and \tilde{L} intersect in $|\beta''|$ points, which are nodes of Y_0 . Since the arithmetic genus of $\tilde{L} \cong L \cong \mathbb{P}^1$ is 0, we have the first relation

$$g = p_a(Y_0) = p_a(Y) + |\beta''| - 1 .$$

Now we apply Corollary 2.7 to obtain a second relation. We see that the dimension of the family in which X can move is at most $2(d-1) + g(Y) - 1 + |\beta'|$. But X is a general member of the V' , which has dimension

$$\dim(V') = \dim(V^{d,\delta}(\alpha, \beta)) - 1 = 2d + g - 2 + |\beta| .$$

Thus

$$\begin{aligned} 2d + g - 2 + |\beta| &\leq 2(d-1) + g(Y) - 1 + |\beta'| \\ &\leq 2(d-1) + p_a(Y) - 1 + |\beta'| \\ &= 2d - 2 + g - |\beta''| + |\beta'| \\ &= 2d - 2 + g + |\beta^0| \\ &\leq 2d + g - 2 + |\beta| . \end{aligned}$$

We conclude that equality holds throughout, and that V' is therefore a component of the Severi variety $V^{d-1,\delta'}(\alpha', \beta')(\Omega')$ satisfying the equality

$$g' = g(X) = g - |\beta' - \beta| + 1$$

or equivalently

$$\begin{aligned} \delta - \delta' &= \left(\binom{d-1}{2} - g \right) - \left(\binom{d-2}{2} - g' \right) \\ &= d - 2 + g' - g \\ &= d - 1 - |\beta' - \beta| . \end{aligned}$$

Another way to view this situation is via the pullback $\pi^*(L)$ of L to \mathscr{Y} , and in particular its restriction to Y . We have

$$\pi^*L = m \cdot \tilde{L} + D + \sum_{1 \leq j \leq \alpha_i} i \cdot Q_{i,j} + \sum_{1 \leq j \leq \beta_i} i \cdot R_{i,j}$$

where D is supported on Z and m is some integer. Restricting to Y we have

$$\pi^*L|_Y = \sum_{1 \leq j \leq \alpha'_i} i \cdot q'_{i,j} + \sum_{1 \leq j \leq \beta_i^0} i \cdot r_{i,j} + \sum_{1 \leq j \leq \beta''_i} i \cdot r''_{i,j} .$$

In particular, if we set $\beta' = \beta^0 + \beta''$, the degree

$$d - 1 = \text{deg}(\pi^*L|_Y) = I\alpha' + I\beta' .$$

The point is, of the $d - 1$ points of intersection of X with L , $I\alpha'$ will occur at the assigned points $p'_{i,j}$, while the remaining $I\beta' = I\beta^0 + I\beta''$ will occur at the $|\beta'|$ unassigned points $s_{i,j}$ and $s''_{i,j}$. The greatest degree of freedom would thus seem to be attained when α' is as small as possible and $\beta' = \beta^0 + \beta''$ as large as possible. But β^0 is bounded above by β , and there is a penalty for taking β'' large: this will decrease the geometric genus of the curve X , which will drop the dimension of the family which it can move.

Note some additional consequences of this analysis. First, we see from the fact that equality holds in the last of the series of inequalities above that $\beta^0 = \beta$, so that every section $R_{i,j}$ passes through Y , as stated, in other words, $\beta \leq \beta'$. Statements (e) and (f) above likewise follow from the proof: the fact that X is reduced is a consequence of the application of Corollary 2.7; while if Y were singular, we would have $g(Y) < p_a(Y)$, giving rise to a strict inequality in the second of series of inequalities above. This completes the proof of Theorem 1.2 subject to hypotheses (a)–(c).

3.4. The local picture of the degeneration

Based on the above analysis, we have a complete picture of the behavior of the family of plane curves X_γ as they degenerate to X_0 . Away from L , there is no apparent degeneration; the family is equisingular. We will describe the family near each of the relevant points of L :

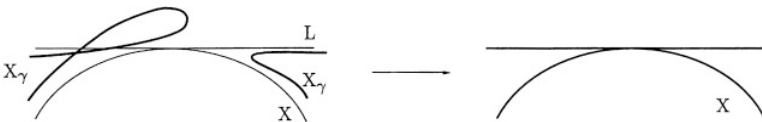
- At a point $p''_{i,j}$ – that is, a point of $\Omega \setminus \Omega'$ – the curve X_0 is smooth, since X does not pass through $p''_{i,j}$. Thus, in a neighborhood of $p''_{i,j}$ we see the curves X_γ simply “flatten out” to the line L .

- At a point $p'_{i,j} \in \Omega'$, the curve X_0 has an i^{th} order tacnode, since X is smooth at $p'_{i,j}$ and has contact of order i with L there. On the other hand, the inverse image of $p'_{i,j}$ in Y_0 has two distinct points, one in Y and one in \tilde{L} ; in other words, the map $Y_0 \rightarrow X_0$ factors through the normalization of X_0 at $p_{i,j}$. Thus, in an analytic neighborhood U of $p'_{i,j}$ the curves X_γ will have two branches, one tending to a neighborhood of $p'_{i,j}$ in L and one to a neighborhood of $p'_{i,j}$ in X . Moreover, these two branches of X_γ will have i points of intersection in U , merging to form the one point of intersection multiplicity i ; thus, as the curves X_γ approach X_0 we see i nodes of the curves X_γ approach the point $p'_{i,j}$ and coalesce to form the tacnode of X_0 . Note finally that, of the two branches of X_γ near $p'_{i,j}$, it is the one tending to a neighborhood of $p'_{i,j}$ in X that has contact of order i with L at $p'_{i,j}$.



- The picture at a point $s_{i,j} \in L$ – that is, a limit of an unassigned point of intersection of X_γ with L – is exactly the same as the picture above near a point $p'_{i,j}$: the nearby curves X_γ will have two branches in a neighborhood of $s_{i,j}$, one tending to L and one to X ; and correspondingly we will see i nodes of the curve X_γ merge to form the i^{th} -order tacnode of X_0 at $s_{i,j}$. The only difference, in fact, is that where in the preceding case the curves X_γ all had a fixed point of contact of order i with L at $p'_{i,j}$, in this case the curves X_g have a point of intersection multiplicity i with L at a variable point tending to $s_{i,j}$. Note that as in the preceding case, of the two branches of X_γ near $s_{i,j}$, it is the one tending to X that has contact of order i with L at $s_{i,j}$.

- The picture near the “new” tacnodes of X_0 – that is, the points $s''_{i,j}$ of intersection of X with L that are not limits of points of intersection of X_γ with L – is the most interesting. Here, the inverse image of $s''_{i,j}$ in Y_0 is a single point, which is an ordinary node of Y_0 . It follows that in an analytic neighborhood of $s''_{i,j}$ the curves X_γ are irreducible, with $i - 1$ nodes tending to the i^{th} -order tacnode $s_{i,j}$ of X_0 . Local equations for (and another picture of) such a family will be given in Sect 4.1 below, when we discuss the geometry of deformations of a tacnode.



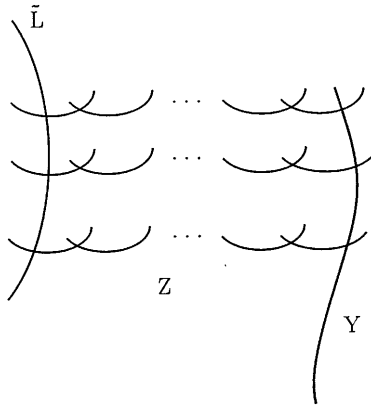
Note that we can now account for all the nodes of the general curve X_γ of our family as X_γ tends to X_0 . First, δ' of the nodes stay away from L , and

become simply nodes of $X \subset X_0$. At each point $p'_{i,j}$, we see i nodes of X_γ absorbed into the tacnode formed by X and L ; and similarly at $s_{i,j}$. Finally, we see $i - 1$ nodes absorbed into each point $s''_{i,j}$. In particular, we see once more that the number δ' of nodes of X is

$$\begin{aligned} \delta' &= \delta - I\alpha' - I\beta - (I\beta'' - |\beta''|) = \delta - I\alpha' - I\beta' + |\beta''| \\ &= \delta - (d - 1) + |\beta' - \beta| . \end{aligned}$$

We can also use the above analysis to give a complete description of the components of the curve Z . Let us suppose that the curve \tilde{L} appears with multiplicity m in the pullback divisor π^*L . Since the restriction of π^*L to Y has multiplicity i at each point $r''_{i,j}$, if $i < m$ it cannot meet \tilde{L} itself: rather, it must meet a component Z_1 of Z that appears with multiplicity i in π^*L . Since the degree of the restriction $\pi^*L|_{Z_1}$ is zero and Z_1 has self-intersection -2 , we see that Z_1 must meet another component of Y_0 that appears with multiplicity $2i$ in π^*L , and so on. Ultimately, we see that i must divide m , and that Z will include a chain of $m/i - 1$ rational curves joining $r''_{i,j}$ to \tilde{L} , which appear with multiplicities $i, 2i, 3i, \dots, m - 2i, m - i$ in π^*L .

The picture of the actual fiber Y_0 of $\mathcal{Y} \rightarrow B$ is thus:



where the length of the chain of rational curves joining \tilde{L} to a point $r''_{i,j} \in Y$ is $m/i - 1$. Note that the multiplicity m with which curve \tilde{L} appears in the pullback divisor π^*L must therefore be a multiple of $\text{lcm}(\beta'')$. We will see in the following section that for suitably general Γ , we have $m = \text{lcm}(\beta'')$.

3.5. Verifying the assumptions

To complete the proof of Theorem 1.2, we will now go back and justify the assumptions (a)–(c). This amounts to setting up the same calculation without any of the hypotheses, and then observing that if any of them were violated we would have a strict inequality in one of the series of inequalities above.

To begin with, we extend the definitions of \tilde{L} , $|\beta^0|$, $|\beta''|$ and $|\beta'|$ to the general setting. First, we take \tilde{L} to be simply the union of the components of the fiber Y_0 dominating L . Then, we define β_i^0 to be the number of the sections $\{R_{i,j}\}_{j=1,\dots,\beta_i}$ that meet either Y or a connected component of Z meeting Y – again, if we consider the limits as $\gamma \rightarrow 0$ of the $|\beta|$ unassigned points of intersection of X_γ with L , $|\beta^0|$ will be simply the number that lie on X . Similarly, we let $|\beta''|$ be the number of points of Y meeting \tilde{L} , plus the number of connected components of Z meeting both \tilde{L} and Y , and set $|\beta'| = |\beta^0| + |\beta''|$.

A key observation is that no connected component of $\pi^{-1}(L)$ can be contained in Z , since otherwise we could contract it to obtain an isolated point in the inverse image of L . In other words, every connected component of Z mapping to a point of L must meet \tilde{L} . Thus, with these definitions, we have as before

$$\#(X \cap (L \setminus \Omega)) \leq |\beta'| .$$

Moreover, the genus satisfies

$$g = p_a(Y_0) \geq p_a(\tilde{L}) + p_a(Y) + |\beta''| - 1 .$$

Now consider the hypothesis (a) that X_0 contains L simply. If this were not the case, that is, $X_0 = m \cdot L \cup X$, where X is now a plane curve of degree $d - m$. Since the map $\pi|_{\tilde{L}} : \tilde{L} \rightarrow L$ has degree e , we have to replace the equality $p_a(\tilde{L}) = 0$ with the inequality

$$p_a(\tilde{L}) \geq -m + 1$$

and correspondingly in place of $p_a(Y) = g - |\beta''| + 1$ we have

$$p_a(Y) \leq g - |\beta''| + m .$$

This appears to work against us. But at the same time, the degree of the image X of Y is now $d - m$ rather than $d - 1$, so that the dimension of the family in which it moves is

$$\begin{aligned} 2(d - m) + p_a(Y) + |\beta'| - 1 &\leq 2(d - m) + g - |\beta''| + |\beta'| + m - 1 \\ &= 2d + g + |\beta| - m - 1 \\ &< 2d + g + |\beta| - 2 \end{aligned}$$

but our hypothesis is that X moves in V' that has dimension $2d + g + |\beta| - 2$, hence we have a contradiction.

Assumption (b) is more intuitively clear, if slightly more cumbersome to check. The point is, our basic inequality on the genus of Y was based on the fact that

$$g = p_a(Y_0) = p_a(Y) + p_a(\tilde{L}) + |\beta''| - 1 .$$

Now, if any connected component of Z met L twice, or met Y twice, or itself had strictly positive arithmetic genus, we would have a strict inequality

$$g > p_a(Y) + p_a(\tilde{L}) + |\beta''| - 1$$

and so would arrive at a contradiction. Thus each connected component Z_0 of Z is a tree of rational curves meeting each of \tilde{L} and Y at most once.

To verify (b), then, we simply have to check that every leaf of this tree (that is, every irreducible component W of Z meeting at most one other component of Z) meets either Y or \tilde{L} . But, if it did not, by the minimality of \mathscr{Y} at least two of the sections $Q_{i,j}$ and $R_{i,j}$ would have to meet it; otherwise we could blow down W in \mathscr{Y} and still satisfy all the conditions imposed on \mathscr{Y} . Now, clearly no two of the sections $Q_{i,j}$ can meet the same component of Z , since they have distinct images in \mathbb{P}^2 . On the other hand, our basic estimate on the dimension of V' is based on the fact that the curve $X = \pi(Y)$ can have at most $|\beta'| = |\beta| + |\beta''|$ points of intersection with L outside of Ω , corresponding to the points of intersection of Y with the sections $R_{i,j}$ and the points of intersection of Y with the connected components of Z meeting \tilde{L} . If a section $Q_{i,j}$ and a section $R_{i,j}$ met the same component of Z , the point $s_{i,j} = \pi(r_{i,j})$ would lie in Ω , and there would be strictly fewer than $|\beta'|$ points of intersection of X with $L \setminus \Omega$, and by Corollary (2.7) the dimension of V' would be strictly less than $2d + g + |\beta| - 2$. Likewise, if two of the sections $R_{i,j}$ met the same component of Z , two of the points $s_{i,j}$ would coincide, and X would again have strictly fewer than $|\beta'|$ points of intersection with L outside of Ω .

Finally, the verification of (c) follows the same pattern as that of (b): given that Z consists simply of chains joining \tilde{L} to the points $r_{i,j} \in Y$, if any of the sections $Q_{i,j}$ met Z at all, we would have $s_{i,j} \in \Omega$; while if any of the sections $R_{i,j}$ met Z at all, we would have $s_{i,j} = s''_{i',j'}$ for some i, j, i', j' and once more X would again have strictly fewer than $|\beta'|$ points of intersection with L outside of Ω .

4. Hyperplane sections of Severi varieties: local geometry

In this section we describe the geometry of $V^{d,\delta}(\alpha, \beta)$ in a neighborhood of a general point of a component of its intersection with a hyperplane H_p . As might be expected, this is relatively straightforward for $V^{d,\delta}(\alpha + e_k, \beta - e_k)$ (the first possibility described in Theorem 1.2), and substantially more complex in the case of a component of $V^{d-1,\delta'}(\alpha', \beta')$. In the analysis of the latter case, we will rely heavily on the description given in [CH] of various loci in the versal deformation space of an m th order tacnode. The arguments

in this section are based on conversations with Ravi Vakil, and appear as well in [V].

4.1. Deformation spaces of tacnodes

We start by recalling the relevant results from [CH] (the reader is referred to Sect. 2.4 of [CH] for details). To begin with, let (C, p) be an m^{th} order tacnode, that is, a curve singularity analytically equivalent to the origin in the plane curve given by the equation

$$y(y - x^m) = 0 .$$

The versal deformation space of (C, p) is then the family $\pi : \mathcal{S} \rightarrow \Delta$, where $\Delta \cong \mathbb{A}^{2m-1}$ with coordinates $(a_{m-2}, \dots, a_0, b_{m-1}, \dots, b_0)$, \mathcal{S} is the subscheme of $\Delta \times \mathbb{A}^2$ given by the equation

$$\begin{aligned} f(x, y, a_{m-2}, \dots, a_0, b_{m-1}, \dots, b_0) \\ = y^2 + (x^m + a_{m-2}x^{m-2} + \dots + a_1x + a_0)y + b_{m-1}x^{m-1} + \dots + b_1x + b_0 \\ = 0 \end{aligned}$$

and $\pi : \mathcal{S} \rightarrow \Delta$ is the projection $\mathcal{S} \subset \Delta \times \mathbb{A}^2 \rightarrow \Delta$.

As is clear from the description of the generalized Severi varieties in the preceding section, we are primarily interested in two loci in the base Δ of the versal deformation: the locus $\Delta_m \subset \Delta$ of points $(a, b) \in \Delta$ over which the fiber $\mathcal{S}_{a,b}$ of $\mathcal{S} \rightarrow \Delta$ is reducible – equivalently, the closure of the locus of (a, b) such that $\mathcal{S}_{a,b}$ has m nodes – and the closure Δ_{m-1} of the locus of (a, b) such that $\mathcal{S}_{a,b}$ has $m - 1$ nodes. Since the defining equation $f(x, y, a, b)$ for \mathcal{S} exhibits \mathcal{S} as a double cover of the x -line $\Delta \times \mathbb{A}_x^1$ over Δ , we can describe these two loci in terms of the branch divisor of the cover: the discriminant $\delta = \delta_{a,b}(x)$ of the equation $f(x, y, a, b)$ above as a quadratic polynomial in y is given by

$$\delta_{a,b}(x) = (x^m + a_{m-2}x^{m-2} + \dots + a_1x + a_0)^2 - 4(b_{m-1}x^{m-1} + \dots + b_1x + b_0) ,$$

and the loci Δ_m and $\Delta_{m-1} \subset \Delta$ are the closures of the loci of (a, b) such that $\delta_{a,b}$ has m double roots and such that $\delta_{a,b}$ has $m - 1$ double roots, respectively (Δ_m could also be characterized as the locus of squares). In particular, we see that Δ_m is simply the $(m - 1)$ -plane $\Delta_m \cong \mathbb{A}^{m-1} \subset \Delta$ given by the equations $b_{m-1} = \dots = b_1 = b_0 = 0$; and that Δ_{m-1} is an m -dimensional subvariety of Δ containing Δ_m (and having multiplicity m at a general point of Δ_m).

In these terms, we can now state the results of [CH] that we will use here:

Lemma 4.1. *Let $m \geq 2$, and let $W \subset \Delta$ be any smooth, m -dimensional subvariety containing the $(m - 1)$ -plane Δ_m , and suppose only that its tangent plane at the origin is not contained in the hyperplane $H \subset \Delta$ given by $b_0 = 0$. Then we have*

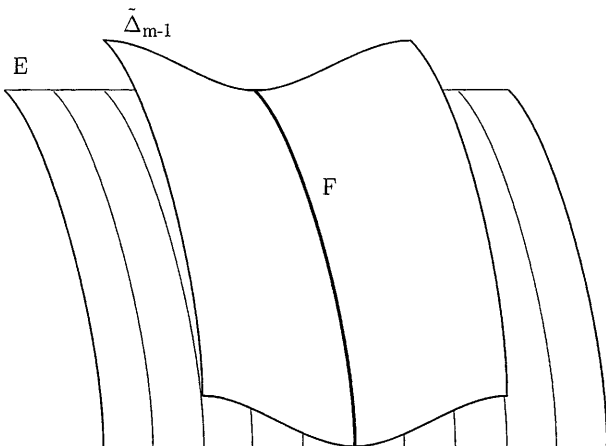
$$W \cap \Delta_{m-1} = \Delta_m \cup \Gamma$$

where Γ is a smooth curve having contact of order exactly m with Δ_m at the origin.

There is an alternative way to express this lemma, which is very useful in both its proof and the present application; it may also provide more geometric insight. Let $\tilde{\Delta}$ be the blow-up of Δ along the $(m - 1)$ -plane Δ_m ; let $E \subset \tilde{\Delta}$ be the exceptional divisor and $F \cong \mathbb{P}^{m-1} \subset E$ the fiber of E over the origin in Δ . Let $\tilde{\Delta}_{m-1}$ be the proper transform of Δ_{m-1} in $\tilde{\Delta}$. Then we can state the last result as the

Lemma 4.2. *The intersection $\tilde{\Delta}_{m-1} \cap E$ contains F as a component of multiplicity m . Moreover, $\tilde{\Delta}_{m-1}$ is smooth at any point of F not contained in the proper transform \tilde{H} in $\tilde{\Delta}$ of the hyperplane $H \subset \Delta$ given by $b_0 = 0$.*

To see the equivalence of the two statements, note that by the second the tangent plane to $\tilde{\Delta}_{m-1}$ at any point of F not contained in the proper transform of the hyperplane $b_0 = 0$ must contain the tangent plane to F and be contained in the tangent plane to E . In particular, the proper transform \tilde{W} of any subvariety $W \subset \Delta$ satisfying the hypotheses of the first statement must intersect $\tilde{\Delta}_{m-1}$ transversely in a smooth curve $\tilde{\Gamma}$ having intersection multiplicity m with E at its point of intersection with F ; and the first statement follows. Conversely, if we apply the first statement just to the m -planes W_b in Δ containing Δ_m , we see that $\tilde{\Delta}_{m-1}$ contains F and (away from the proper transform of the hyperplane $b_0 = 0$) is swept out by the smooth curves \tilde{W}_b ; the second statement follows.



It will be helpful also to recall the steps in the proof of these lemmas in [CH]. Briefly, the statement of Lemma 4.1 is first verified by direct calculation in case W is the specific m -plane $b_{m-1} = b_{m-2} = \dots = b_1 = 0$. Then the homogeneity of $F \setminus (F \cap \tilde{H})$ under the action of the automorphism group of the singularity (C, p) allows us to deduce it for W any m -plane containing Δ_m and not contained in H . This, as noted, is all we need to deduce the statement of Lemma 4.2, and then Lemma 4.1 in general follows as well.

4.2. Products of deformation spaces of tacnodes

What we have to do now is to use this information to develop an analogous picture in the product of deformation spaces associated to a collection of higher-order tacnodes. For what follows, then, we will let m_1, m_2, \dots be any sequence of integers $m_j \geq 2$, and (C_j, p_j) be an $(m_j)^{\text{th}}$ order tacnode. We will denote the versal deformation of (C_j, p_j) by $\pi_j : \mathcal{S}_j \rightarrow \Delta_j$, and let $(a_{j,m_j-2}, \dots, a_{j,0}, b_{j,m_j-1}, \dots, b_{j,0})$ be coordinates on Δ_j as above. For each j , we will let Δ_{j,m_j} and $\Delta_{j,m_j-1} \subset \Delta_j$ be as above the closures of the loci in Δ_j over which the fibers of π_j have m_j and $m_j - 1$ nodes respectively. Finally, we set

$$\Delta = \Delta_1 \times \Delta_2 \times \dots ,$$

$$\Delta_m = \Delta_{1,m_1} \times \Delta_{2,m_2} \times \dots ,$$

and

$$\Delta_{m-1} = \Delta_{1,m_1-1} \times \Delta_{2,m_2-1} \times \dots .$$

Note that Δ , Δ_m and Δ_{m-1} have dimensions $\sum 2m_j - 1$, $\sum m_j - 1$ and $\sum m_j$ respectively.

Our goal now is to describe how a smooth subvariety $W \subset \Delta$ of dimension $\sum(m_j - 1) + 1$, containing Δ_m , will intersect Δ_{m-1} , again with some hypothesis on its tangent plane at the origin. Specifically, let $H \subset \Delta$ be the union of the hyperplanes $(b_{j,0} = 0) \subset \Delta$, and suppose that the tangent plane to W is not contained in H . By the dimension count, we would expect W to intersect Δ_{m-1} in the union of Δ_m and a residual curve Γ ; we will show that this is indeed the case, and that the intersection number of Γ with Δ_m at the origin in W is $\prod m_j$. (What will be different from the single-tacnode case is the local geometry of Γ : as we will see, it may have many branches, each of which may be singular at the origin.) To make the full statement, let λ the the least common multiple of the m_j , let $\mu = \prod m_j$ and set $\kappa = \mu/\lambda$. We will prove the

Lemma 4.3. *With the hypotheses above, in an étale neighborhood of the origin in Δ the intersection*

$$W \cap \Delta_{m-1} = \Delta_m \cup \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_\kappa$$

where $\Gamma_1, \dots, \Gamma_\kappa \subset W$ are distinct reduced unibranch curves having intersection multiplicity exactly λ with Δ_m at the origin. The curves Γ_α will all be smooth if $\lambda = m_j$ for some j ; otherwise they will all be singular, with multiplicity $\lambda / \max_j \{m_j\}$.

As before, it will be helpful to express this in terms of the geometry of a blow-up: we let $\tilde{\Delta}$ be the blow-up of Δ along the plane Δ_m , $E \subset \tilde{\Delta}$ the exceptional divisor, $F \subset E$ the fiber of E over the origin in Δ . Let $\tilde{\Delta}_{m-1}$ be the proper transform of Δ_{m-1} in $\tilde{\Delta}$. Then we can state the last result as the

Lemma 4.4. *The intersection $\tilde{\Delta}_{m-1} \cap E$ contains F as a component of multiplicity m . Moreover, in an étale neighborhood of any point $p \in F$ not contained in the proper transform \tilde{H} of H in $\tilde{\Delta}$, $\tilde{\Delta}_{m-1}$ consists of κ reduced branches, each having multiplicity $\lambda / \max_j \{m_j\}$, intersection number λ with E along F , and tangent cone at p supported on a linear space contained in E .*

Proof. We will prove this in two stages: first, we will verify the statement of Lemma 4.3 directly in case W is any linear subspace of Δ ; from this we will deduce the full statement of Lemma 4.4, and hence Lemma 4.3 for arbitrary W . Note that we can not do this as in the single-tacnode case by giving an explicit calculation in the case of a particularly simple plane W and then invoking homogeneity, for one reason: in the single-tacnode case we have an a priori lower bound of m for the intersection multiplicity of Γ with Δ_m , so that if we verify that Γ is a smooth curve with contact of order m with Δ_m for one plane W , we may deduce it for any plane W' such that W lies in the closure of the orbit of $[W'] \in G(m, 2m - 1)$ under the action of the automorphism group of the deformation. Here we do not have the analogous lower bound $(\Delta_m \cdot \Gamma) \geq \mu$, and so we have to deal first with an arbitrary linear space W .

So: let $W \subset \Delta$ be any plane of dimension $\sum(m_j - 1) + 1$ containing Δ_m and not contained in H ; W will be spanned by Δ_m and one additional vector $v \in \Delta$. Let t be a nonzero linear function on W vanishing on the hyperplane Δ_m . Let $\rho_j : \Delta \rightarrow \Delta_j$ be the projection, and $v_j = \rho_j(v)$, so that the image $W_j = \rho_j(W) \subset \Delta_j$ of W in Δ_j will be the plane spanned by the subspace Δ_{j,m_j} and the vector v_j . Since by hypothesis v_j does not lie in the hyperplane $b_{j,0} = 0$, by Lemma 2.10 of [CH], we may write the intersection $W_j \cap \Delta_{j,m_j}$ as the union of Δ_{j,m_j} and a smooth curve Γ_j having contact of order m_j with Δ_{j,m_j} at the origin. It follows that in some étale neighborhood of the origin in W_j we may choose coordinates $(x_{j,0}, x_{j,1}, \dots, x_{j,m-2}, t_j)$ so that

$$(\rho_j)^* t_j = t ;$$

and the hyperplane Δ_{j,m_j} is given by $t_j = 0$ and the curve Γ_j by the equations

$$x_{j,1} = x_{j,2} = \cdots = x_{j,m-2} = 0$$

and

$$(t_j)^{m_j} = x_{j,0} .$$

We may then take the collection of functions $\{y_{j,i} = (\rho_j)^* x_{j,i}\}_{1 \leq i \leq m_j-2}$ and t as local coordinates in an étale neighborhood of the origin in W , in terms of which the hyperplane Δ_m is given by $t = 0$ and the residual intersection Γ of W with Δ_{m-1} by the equations

$$y_{j,i} = 0 , \forall j \text{ and } i : 1 \leq i \leq m_j - 2$$

and

$$t^{m_j} = y_{j,0} , \quad \forall j .$$

Γ is a curve, since specifying the value of the coordinate t at a point of Γ determines the value of the coordinates $y_{j,0}$ (and hence all the coordinates $y_{j,i}$) up to a choice of an $(m_j)^{\text{th}}$ root. More explicitly, suppose that we choose for each j an $(m_j)^{\text{th}}$ root ζ_j . Then we may parametrize a branch Γ_ζ of the curve Γ by

$$t = z^\lambda ,$$

$$y_{j,0} = \frac{z^{\lambda/m_j}}{\zeta_j}$$

and of course $y_{j,i} = 0 \forall i > 0$. This parametrization is one-to-one, since the powers of z appearing have no common factor; the multiplicity of the image at the origin is the smallest power of z appearing, which is $\lambda / \max\{m_j\}$; and since the pullback of t is z^λ , the intersection multiplicity of the image with the hyperplane $\Delta_m \subset W$ defined by $t = 0$ is λ . Moreover, all branches of Γ are parametrized in this fashion; and two collections of roots ζ_j and η_j will give rise to the same branch if, and only if, for some λ^{th} root of unity ϵ we have

$$\zeta_j = \epsilon^{m_j} \eta_j$$

for all j . Since $\zeta_j = \epsilon^{m_j} \zeta_j$ for all j only if $\epsilon = 1$, the number of such branches is the number μ of collections of roots ζ_j divided by λ , that is, κ . Thus the statement of Lemma 4.3 is established for any linear space W .

The remainder of the argument for Lemmas 4.3 and 4.4 is straightforward: exactly as before, the statement of Lemma 4.3 for a linear space W (satisfying the hypotheses of the Lemma) implies the statement of Lemma 4.4, which in turn implies 4.3 in general. To carry this out, let

$U = \mathbb{P}(\Delta/\Delta_m) \setminus \mathbb{P}(H)$ be the complement of the projectivizations of the hyperplanes $b_{j,0} = 0$ in the projectivization of the quotient Δ/Δ_m , so that we have a morphism

$$\tau : V = \tilde{\Delta} \setminus \tilde{H} \longrightarrow U$$

expressing the complement V of the proper transform \tilde{H} of H in the blow up $\tilde{\Delta}$ as a projective bundle (with fiber dimension $\sum m_j - 1$) over U . Let $\tilde{\Delta}_{m-1}^0 = \Delta_{m-1} \cap V$ be the intersection of the proper transform $\tilde{\Delta}_{m-1}$ with this open subset of $\tilde{\Delta}$, and let $\sigma : \tilde{\Delta}_{m-1}^0 \rightarrow U$ be the restriction of τ to $\tilde{\Delta}_{m-1}^0$. The statement of Lemma 4.3 for linear spaces W says precisely that the fibers of σ are curves consisting of κ reduced branches, each having multiplicity $\lambda/\max_j\{m_j\}$, intersection number λ with E at its unique point p of intersection with F , and tangent cone at p supported on a linear space contained in the tangent space to E . It follows that exactly the same is true of $\tilde{\Delta}_{m-1}$ in a neighborhood of any point $p \in F$ not in \tilde{H} : it has κ reduced branches, each having multiplicity $\lambda/\max_j\{m_j\}$ and intersection multiplicity λ with E along F , and tangent cone supported on a linear space contained in the tangent space to E . In other words, we have proved Lemma 4.4; and as before Lemma 4.3 follows for an arbitrary smooth m -dimensional subvariety $W \subset \Delta$ satisfying the hypotheses of the lemma. □

4.3. *The local geometry around irreducible curves*

In the remaining two parts of this section we will complete the proof of Theorem 1.3 by analyzing the geometry of a generalized Severi variety $V^{d,\delta}(\alpha, \beta)$ in a neighborhood of a general point $[X_0]$ of one of the generalized Severi varieties V' listed in Theorem 1.2. We will start in this subsection with the (relatively) simple case of a general point of $V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\})$; in the following one we will apply the preceding results to carry out the analysis at a general point of $V^{d-1,\delta'}(\alpha', \beta')(\Omega')$.

So: assume that $\beta_k > 0$, and let $[X_0]$ be a general point of $V' = V^{d,\delta}(\alpha + e_k, \beta - e_k)(\Omega \cup \{p\})$. We have then the

Proposition 4.5. *The variety $V^{d,\delta}(\alpha, \beta)$ contains $[X_0]$; it is smooth there and has intersection multiplicity k with H_p along V' .*

Proof. This follows directly from an analogous statement about the linear series of divisors on $L \cong \mathbb{P}^1$. To set this up, consider the rational map

$$\pi : |\mathcal{O}_{\mathbb{P}^2}(d)| \cong \mathbb{P}^N \longrightarrow |\mathcal{O}_L(d)| \cong \mathbb{P}^d$$

given by restriction (note that this is a linear projection, with vertex the subspace in \mathbb{P}^N of curves containing L). Inside the target space $|\mathcal{O}_L(d)|$, we consider three loci: we let $H = \{D : D - p \geq 0\}$ be the hyperplane of divisors containing the point p ; and we set

$$\Phi = \left\{ D \in |\mathcal{O}_L(d)| : D = \sum_{1 \leq j \leq \alpha_i} i \cdot p_{i,j} + \sum_{1 \leq j \leq \beta_i} i \cdot p'_{i,j} \text{ for some } p'_{i,j} \in L \right\}$$

and similarly, setting $\beta' = \beta - e_k$,

$$\Psi = \left\{ D \in |\mathcal{O}_L(d)| : D = k \cdot p + \sum_{1 \leq j \leq \alpha_i} i \cdot p_{i,j} + \sum_{1 \leq j \leq \beta'_i} i \cdot p'_{i,j} \text{ for some } p'_{i,j} \in L \right\} .$$

Now, let $V = V^{d,\delta} \subset \mathbb{P}^N$ be the ordinary Severi variety. We have

$$H_p = \pi^{-1}(H) ;$$

and if we let

$$\sigma = \pi|_V : V \longrightarrow |\mathcal{O}_L(d)|$$

be the restriction of π to V ,

$$V^{d,\delta}(\alpha, \beta) = \sigma^{-1}(\Phi)$$

and

$$V^{d,\delta}(\alpha + e_k, \beta - e_k) = \sigma^{-1}(\Psi) .$$

Proposition 4.5 will thus follow from the combination of the two Lemmas

Lemma 4.6. *The differential $d\sigma$ of $\sigma : V \rightarrow |\mathcal{O}_L(d)|$ is surjective at $[X]$.*

and

Lemma 4.7. *In a neighborhood of the point $D_0 = X \cdot L \in |\mathcal{O}_L(d)|$, the variety Φ is smooth and has intersection multiplicity k with the hyperplane H along Ψ .*

Proof of Lemma 4.6. Since the map $\pi : |\mathcal{O}_{\mathbb{P}^2}(d)| \cong \mathbb{P}^N \rightarrow |\mathcal{O}_L(d)| \cong \mathbb{P}^d$ is a linear projection, we need only check that the projective tangent plane $PT_{[X]}V \subset \mathbb{P}^N$ to V at $[X]$ intersects the vertex $L + |\mathcal{O}_{\mathbb{P}^2}(d-1)| \cong \mathbb{P}^{N-d-1}$ of π transversely, that is, in codimension $d+1$ in $PT_{[X]}V$. But now the projective tangent space $PT_{[X]}V$ is simply the linear series of curves of degree d passing through the δ nodes of X , and its intersection with the vertex the linear series of curves of degree $d-1$ passing through the nodes; and since the nodes impose independent conditions on curves of degree at least $d-2$ these will have dimensions $\frac{d(d+3)}{2} - \delta$ and $\frac{(d-1)(d+2)}{2} - \delta = \frac{d(d+3)}{2} - \delta - (d+1)$ respectively. □

Proof of Lemma 4.7. We can do this simply in coordinates: let x be an affine coordinate on $L \cong \mathbb{P}^1$ such that the point p is given by $x = 0$; let the point $p_{i,j}$ have coordinate $\lambda_{i,j}$ and suppose that the β'_i points other than p and $\{p_{i,j}\}$ at which the divisor D_0 has multiplicity i have coordinates $\mu_{i,j}$ (note that by Lemma 2.2 the $\lambda_{i,j}$ and the $\mu_{i,j}$ are all distinct). Then we can parametrize a neighborhood of $[D_0]$ in Φ by

$$(\epsilon, \epsilon_{i,j}) \mapsto [f(x)] = [(x - \epsilon)^k \prod (x - \lambda_{i,j})^i \prod (x - \mu_{i,j} - \epsilon_{i,j})^i]$$

from which we see in particular that Φ is smooth at the point $[D_0]$. Now, writing a point $[f(x)] \in |\mathcal{O}_L(d)|$ as

$$f(x) = x^d + b_{d-1}x^{d-1} + \dots + b_1x + b_0$$

the defining equation of the hyperplane $H \subset |\mathcal{O}_L(d)|$ is simply $b_0 = 0$, which pulls back via this parametrization to ϵ^k times a polynomial in the $\epsilon_{i,j}$ nonzero at the origin; it follows that in a neighborhood of $[D_0]$, the divisor cut on Φ by H is simply k times Ψ . □

4.4. The local geometry around reducible curves

It remains to describe the local geometry of a generalized Severi variety $V^{d,\delta}(\alpha, \beta)$ in a neighborhood of a point $[X_0]$, where $X_0 = X \cup L$ and $[X]$ is a general point of a generalized Severi variety $V' = V^{d-1,\delta'}(\alpha', \beta')(\Omega')$. Thus, we will suppose that $\Omega'_i = \{p'_{i,j}\}_{1 \leq j \leq \alpha'_i}$ is any subset of cardinality $|\alpha'|$ of $\Omega_i = \{p_{i,j}\}_{1 \leq j \leq \alpha_i}$ such that $\{p'_{i,1}, \dots, p'_{i,\alpha'_i}\} \subset \{p_{i,1}, \dots, p_{i,\alpha_i}\}$ for each i ; and that there are points $\{q'_{i,j}\}_{1 \leq j \leq \alpha'_i}$ and $\{r'_{i,j}\}_{1 \leq j \leq \beta'_i}$ in the normalization $v: \tilde{X} \rightarrow X \subset \mathbb{P}^2$ such that $v(q'_{i,j}) = p'_{i,j}$ and the pullback $v^*(L) = \sum iq_{i,j} + \sum ir_{i,j}$. With this said, our basic result is the

Proposition 4.8. *In a neighborhood of $[X_0]$, the variety $V^{d,\delta}(\alpha, \beta)$ will have*

$$\binom{\beta'}{\beta} \frac{I^{\beta' - \beta}}{\text{lcm}(\beta' - \beta)}$$

branches, each of which will have intersection multiplicity $\text{lcm}(\beta' - \beta)$ with H_p along V' .

Proof. As in the case of Lemma 4.5, we want to deduce this from a local calculation, in this case, Lemma 4.3. Before we can do this, we have to specify which of the points $r_{i,j} \in \tilde{X}$ will be limits of points of unassigned tangency on nearby curves in the family; each such specification will determine a collection of branches of $V^{d,\delta}(\alpha, \beta)$. So: to start with, choose any subset $\Lambda = \{r_{i,j}\}_{1 \leq j \leq \beta_i}$ of the set $\{r'_{i,j}\}_{1 \leq j \leq \beta'_i}$ such that $\{r_{i,1}, \dots, r_{i,\beta_i}\}$

$\subset \{r'_{i,1}, \dots, r'_{i,\beta'_i}\}$ for each i . By way of notation, let $\beta'' = \beta' - \beta$, and label the complement of the subset $\{r_{i,j}\} \subset \{r'_{i,j}\}$ as $\{r''_{i,j}\}_{1 \leq j \leq \beta''_i}$; and let $s_{i,j} \subset L \subset \mathbb{P}^2$ (respectively, $s'_{i,j}, s''_{i,j}$) be the image of the point $r_{i,j}$ (respectively, $r'_{i,j}, r''_{i,j}$). Similarly, set $\alpha'' = \alpha - \alpha'$, and label the complement of the subset $\{p'_{i,j}\} \subset \{p_{i,j}\}$ as $\{p''_{i,j}\}_{1 \leq j \leq \alpha''_i}$.

Now, in an analytic neighborhood of the point $[X_0] = [X + L] \in \mathbb{P}^N$, we will define the relaxed local Severi variety W_Λ to be the closure of the locus of curves X_t satisfying the following six conditions:

- i) X_t preserves the δ' nodes of X ; that is, for every node of X_0 away from L , X_t will have a node nearby.
- ii) At each point $p''_{i,j}$, X_t has contact of order i with L .
- iii) In a neighborhood of each point $p'_{i,j}$, X_t has i nodes
- iv) In a neighborhood of each point $s_{i,j}$, X_t has i nodes

To specify the remaining two conditions we need to make one remark. Conditions iii) requires that, in an étale or analytic neighborhood of a point $p'_{i,j}$, the deformation X_t of X will be reducible; that is, it will continue to have two branches, one a deformation of a neighborhood of the point $p'_{i,j}$ in L and the other a deformation of a neighborhood of $p'_{i,j}$ in X . Similarly, a deformation X_t of X_0 satisfying condition iv) will have two branches near each point $s_{i,j}$, deformations of the two branches of X_0 at $s_{i,j}$. In these terms, we make the further requirements that:

- v) In a neighborhood of each point $p'_{i,j}$, the branch of X_t that is a deformation of a neighborhood of $p'_{i,j}$ in X has contact of order i with L at $p'_{i,j}$
- vi) In a neighborhood of each point $s_{i,j}$, the branch of X_t that is a deformation of a neighborhood of $s_{i,j}$ in X has a point of contact of order i with L .

Remarks. 1. We are being colloquial here in the definition of the relaxed Severi variety, using terms like “nearby” and “in a neighborhood of” each point $p'_{i,j}$ or $s_{i,j}$. This is to avoid introducing yet more notation. The definition may be made precise, for example, by specifying in \mathbb{P}^2 disjoint analytic neighborhoods $U_{i,j}, V_{i,j}$ and $W_1, \dots, W_{\delta'}$ of the points $p'_{i,j}$ and $s_{i,j}$, and the nodes $u_1, \dots, u_{\delta'}$ of X ; or by considering deformations $\{X_t\}$ of X_0 having nodes at deformations $\{u_i(t)\}$ of the points u_i , etc.

2. We remark again that the relaxed local Severi variety W_Λ depends on the choice of subset $\{r_{i,j}\} \subset \{r'_{i,j}\}$; there are thus $\binom{\beta'}{\beta}$ such varieties W_Λ in a neighborhood of $[X_0]$.

Note that we are at this point making no requirements about the deformations X_t in a neighborhood of a point $s''_{i,j}$, even though we have seen that a family of curves X_t in $V^{d,\delta}(\alpha, \beta)$ tending to X_0 will have $i - 1$ nodes tending to each point $s''_{i,j}$ (hence the name “relaxed”). Thus, in particular, a general point $[X_t] \in W_\Lambda$ will correspond to a curve X_t with only $\delta'' = \delta - (I\beta'' - |\beta''|)$ nodes – in other words, W_Λ will be an open subset of the variety $V^{d,\delta''}(\alpha, \beta)$. In fact, our strategy is exactly this: to consider first the conditions a curve X_t must satisfy away from the points $s''_{i,j}$ in order to

belong to a component of $V^{d,\delta}(\alpha, \beta)$ containing $[X_0]$ in its closure; and then secondly the conditions around the points $s''_{i,j}$.

The point is, the conditions on the curve X_t at the points $p'_{i,j}$ and $s'_{i,j}$ and the nodes u_1, \dots, u_δ of X are all essentially linear conditions, and well behaved. Thus, omitting any requirements on the behavior of the curves X_t around $s''_{i,j}$ will result in a parameter space W_Λ that is smooth with identifiable tangent space at the point $[X_0]$. Once we have described this space, we will then consider the map ϕ_Λ from W_Λ to the product Δ of the deformation spaces of the tacnodes of X_0 at the points $s''_{i,j}$. In a neighborhood of $[X_0]$, the Severi variety $V^{d,\delta}(\alpha, \beta)$ will be the union, over all Λ , of the closures of the inverse images $\phi_\Lambda^{-1}(\Delta_{m-1} \setminus \Delta_m)$. Once we have shown that for each Λ the image $\phi_\Lambda(W_\Lambda)$ satisfies the hypotheses of Lemma 4.3, then, Theorem 1.3 will follow.

The first step in carrying out this plan is thus the identification of the tangent space to W_Λ at $[X_0]$ (from which it will follow that W_Λ is indeed smooth at $[X_0]$, once we estimate its dimension). This tangent space, viewed as a subspace of the tangent space $T_{[X_0]}\mathbb{P}^N = H^0(X_0, \mathcal{O}(d))$, is the subspace $H^0(X_0, \mathcal{I}(d))$ determined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{X_0}$, which we will describe in the following lemma.

To do this, we have to introduce some local ideals. Specifically, for each i and j with $1 \leq j \leq \alpha'_i$, we let $\mathcal{I}'_{i,j} \subset \mathcal{O}_{X_0}$ be the sheaf of regular functions in a neighborhood of $p'_{i,j} \in X_0$ whose restriction to $L \subset X_0$ vanishes to order i at $p'_{i,j}$ and whose restriction to $X \subset X_0$ vanishes to order $2i$ at $p'_{i,j}$. Similarly, for each i and j with $1 \leq j \leq \beta_i$, we let $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_0}$ be the sheaf of regular functions in a neighborhood of $s_{i,j} \in X_0$ whose restriction to $L \subset X_0$ vanishes to order i at $s_{i,j}$ and whose restriction to $X \subset X_0$ vanishes to order $2i - 1$ at $s_{i,j}$. We have then the

Lemma 4.9. *The variety W_Λ is smooth at $[X_0]$, and its tangent space is given by the linear series*

$$T_{[X_0]}W_\Lambda = H^0(X_0, \mathcal{I}(d)) \subset T_{[X_0]}\mathbb{P}^N = H^0(X_0, \mathcal{O}(d))$$

where the ideal sheaf \mathcal{I} is the product

$$\mathcal{I} = \prod_{i=1}^{\delta'} \mathfrak{m}_{u_i} \cdot \prod_{1 \leq j \leq \alpha'_i} \mathfrak{m}_{p'_{i,j}}^i \cdot \prod_{1 \leq j \leq \alpha'_i} \mathcal{I}'_{i,j} \cdot \prod_{1 \leq j \leq \beta_i} \mathcal{I}_{i,j}$$

Proof. We will first show that the tangent space to W_Λ at $[X_0]$ is contained in the series $H^0(X_0, \mathcal{I}(d))$. We will then argue that the ideal \mathcal{I} imposes independent conditions on the series $H^0(X_0, \mathcal{O}(d))$, so that we can calculate the dimension $h^0(X_0, \mathcal{I}(d))$; comparing this with the actual dimension of W_Λ we deduce the smoothness of W_Λ at $[X_0]$ and the identification of the tangent space with $H^0(X_0, \mathcal{I}(d))$.

For both parts, it will be useful to introduce a partial normalization \tilde{X}_0 of X_0 : specifically, we let $\mu : \tilde{X}_0 \rightarrow X_0$ be the normalization of X_0 at the points $p'_{i,j}$ and $s_{i,j}$, and the nodes $u_1, \dots, u_{g'}$ of X , but not at the points $s''_{i,j}$. Note that the normalization \tilde{X} of X is actually a closed subscheme of \tilde{X}_0 . We will abuse notation slightly and denote by $q'_{i,j}$ and $r_{i,j}$ the points of $\tilde{X} \subset \tilde{X}_0$ lying over $p'_{i,j}$ and $s_{i,j}$. We will also denote by $q''_{i,j} \in \tilde{X}_0$ the (unique) point of \tilde{X}_0 lying over $p''_{i,j}$.

Note that since \mathcal{F} is contained in the conductor $\mathcal{G} \subset \mathcal{O}_{X_0}$ of the map $\tilde{X}_0 \rightarrow X_0$, the pullback $\mu^*\mathcal{F}$ is locally free on \tilde{X}_0 ; specifically, it is the sheaf

$$\mu^*\mathcal{F} = \mu^*\mathcal{G} \otimes \mathcal{O}_{\tilde{X}_0} \left(-\sum i \cdot q'_{i,j} - \sum i \cdot q''_{i,j} - \sum (i-1) \cdot r_{i,j} \right) .$$

Also, because \mathcal{F} is contained in the conductor \mathcal{G} of the map $\tilde{X}_0 \rightarrow X_0$, the space of sections $H^0(\tilde{X}_0, \mu^*\mathcal{F}(d)) = \mu^*H^0(X_0, \mathcal{F}(d))$.

Now, to prove the inclusion $T_{[X_0]}W_\Lambda \subset H^0(X_0, \mathcal{F}(d))$, we observe that by conditions i), iii) and iv) in the definition of W_Λ any deformation of X_0 in W_Λ arises from a deformation of the composite map $\bar{\mu} : \tilde{X}_0 \rightarrow X_0 \rightarrow \mathbb{P}^2$. Thus, any tangent vector to W_Λ at $[X_0]$ must lie in the subspace

$$H^0(X_0, \mathcal{F}(d)) \subset H^0(X_0, \mathcal{O}(d))$$

which we may identify in turn with $H^0(\tilde{X}_0, \mu^*\mathcal{F}(d))$. Now, as we observed in Sect 2 above, we may further identify $\mu^*\mathcal{F}(d)$ with the normal sheaf $N = N_{\tilde{\mu}}$ of the map $\bar{\mu}$; and in terms of these identifications conditions ii), v) and vi) of the definition of W_Λ amount to the assertion that the tangent space to W_Λ at $[X_0]$ satisfies

$$\begin{aligned} T_{[X_0]}W_\Lambda &\subset H^0\left(\tilde{X}_0, N\left(-\sum i \cdot q'_{i,j} - \sum i \cdot q''_{i,j} - \sum (i-1) \cdot r_{i,j}\right)\right) \\ &= H^0\left(\tilde{X}_0, \mu^*\mathcal{F}(d)\left(-\sum i \cdot q'_{i,j} - \sum i \cdot q''_{i,j} - \sum (i-1) \cdot r_{i,j}\right)\right) \\ &= H^0(\tilde{X}_0, \mu^*\mathcal{F}(d)) \\ &= H^0(X_0, \mathcal{F}(d)) \end{aligned}$$

For the second part, to estimate the dimension $h^0(X_0, \mathcal{F}(d))$, we will equate this with the dimension $h^0(\tilde{X}_0, \mu^*\mathcal{F}(d))$ of the space of sections of the pullback, and apply Riemann-Roch on \tilde{X}_0 . A key fact is that the line bundle $\mu^*\mathcal{F}(d)$ is nonspecial. In the sequel we will need as well the fact that for $p \in L$ general the bundle $\mu^*\mathcal{F}(d)(-p)$ is nonspecial as well; we will state these as the

Lemma 4.10. *For $p \in L$ general,*

$$h^1(\tilde{X}_0, \mu^*\mathcal{F}(d)) = h^1(\tilde{X}_0, \mu^*\mathcal{F}(d)(-p)) = 0 .$$

Proof. First, note that the dualizing sheaf of \tilde{X}_0 is given by

$$\omega_{\tilde{X}_0} = \mu^*(\mathcal{I}(d-3)) .$$

Thus, we may write

$$\mu^*\mathcal{I}(d) = \mu^*\mathcal{O}(3) \otimes \omega_{\tilde{X}_0} \left(-\sum i \cdot q'_{i,j} - \sum i \cdot q''_{i,j} - \sum (i-1) \cdot r_{i,j} \right) .$$

and correspondingly

$$\omega_{\tilde{X}_0} \otimes (\mu^*\mathcal{I}(d))^{-1} = \mu^*\mathcal{O}(-3) \left(\sum i \cdot q'_{i,j} + \sum i \cdot q''_{i,j} + \sum (i-1) \cdot r_{i,j} \right) .$$

Now, the restriction of this line bundle to \tilde{X} has degree

$$\begin{aligned} \deg \left(\omega_{\tilde{X}_0} \otimes (\mu^*\mathcal{I}(d))^{-1} \otimes \mathcal{O}_{\tilde{X}} \right) &= -3(d-1) + I\alpha' + I\beta \\ &\leq -3(d-1) + I\alpha' + I\beta' \\ &= -2(d-1) \\ &< 0 \end{aligned}$$

and so every global section $\sigma \in H^0(\omega_{\tilde{X}_0} \otimes (\mu^*\mathcal{I}(d))^{-1})$ must vanish identically on \tilde{X} . The restriction of σ to the component \tilde{L} of \tilde{X}_0 lying over L is then a section of the bundle

$$\omega_{\tilde{X}_0} \otimes (\mu^*\mathcal{I}(d))^{-1} \otimes \mathcal{I}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{L}}$$

and since the restriction $\mathcal{I}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{L}}$ to \tilde{L} of the ideal sheaf of \tilde{X} has degree $-I\beta''$, we have

$$\begin{aligned} \deg \left(\omega_{\tilde{X}_0} \otimes (\mu^*\mathcal{I}(d))^{-1} \otimes \mathcal{I}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{L}} \right) &= I\alpha'' - I\beta'' - 3 \\ &= I\alpha - I\alpha' - (I\beta' - I\beta) - 3 \\ &= (I\alpha + I\beta) - (I\alpha' + I\beta') - 3 \\ &= d - (d-1) - 3 \\ &= -2 . \end{aligned}$$

Thus σ must vanish identically on \tilde{L} as well, and hence

$$h^1(\tilde{X}_0, \mu^*\mathcal{I}(d)) = h^0(\tilde{X}_0, \omega_{\tilde{X}_0} \otimes (\mu^*\mathcal{I}(d))^{-1}) = 0 .$$

Finally, if we had started with $\mu^*\mathcal{I}(d)(-p)$ in place of $\mu^*\mathcal{I}(d)$, we would have wound up with

$$\deg\left(\omega_{\tilde{X}_0} \otimes (\mu^* \mathcal{I}(d)(-p))^{-1} \otimes \mathcal{I}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{L}}\right) = -1$$

and we would conclude as before that the line bundle $\mu^* \mathcal{I}(d)(-p)$ is nonspecial. \square

We may now apply the Riemann-Roch formula on \tilde{X}_0 to complete the proof of Lemma 4.9. By Lemma 4.10, we have

$$\begin{aligned} h^0(X_0, \mathcal{I}(d)) &= h^0(\tilde{X}_0, \mu^* \mathcal{I}(d)) \\ &= \deg(\mu^* \mathcal{I}(d)) - p_a(\tilde{X}_0) + 1 \end{aligned}$$

Since

$$\begin{aligned} \deg(\mu^* \mathcal{I}(d)) &= \deg\left(\mu^* \mathcal{O}(3) \otimes \omega_{\tilde{X}_0} \left(-\sum i \cdot q'_{i,j} - \sum i \cdot q''_{i,j} - \sum (i-1) \cdot r_{i,j}\right)\right) \\ &= 3d + 2p_a(\tilde{X}_0) - 2 - I\alpha - (I\beta - |\beta|) \end{aligned}$$

we can rewrite this as

$$h^0(X_0, \mathcal{I}(d)) = 3d + p_a(\tilde{X}_0) - 1 - I\alpha - (I\beta - |\beta|) .$$

Now, the arithmetic genus of \tilde{X}_0 is simply the genus of a general member X_t of W ; thus

$$p_a(\tilde{X}_0) = \binom{d-1}{2} - \delta'' .$$

Equivalently, we could arrive at this by observing that the arithmetic genus of \tilde{X}_0 is simply the genus of \tilde{X} , plus the degree of the intersection of $\tilde{X} \subset \tilde{X}_0$ with the component \tilde{L} of \tilde{X}_0 lying over the line $L \subset \mathbb{P}^2$, minus 1; thus

$$\begin{aligned} p_a(\tilde{X}_0) &= \binom{d-2}{2} - \delta' + I\beta'' - 1 \\ &= \binom{d-2}{2} - \delta + d - 1 - |\beta' - \beta| + I\beta'' - 1 \\ &= \binom{d-1}{2} - \delta + I\beta'' - |\beta''| \\ &= \binom{d-1}{2} - \delta'' \end{aligned}$$

Either way, we have

$$\begin{aligned}
 \dim(T_{[X_0]}W_\Lambda) &\leq h^0(X_0, \mathcal{I}(d)) \\
 &= 3d + p_a(\tilde{X}_0) - 1 - I\alpha - (I\beta - |\beta|) \\
 &= 3d + \binom{d-1}{2} - \delta'' - 1 - I\alpha - (I\beta - |\beta|) \\
 &= \binom{d+2}{2} - 1 - \delta'' - I\alpha - (I\beta - |\beta|) \\
 &= \dim(V^{d,\delta''}(\alpha, \beta)) \\
 &= \dim(W_\Lambda) \ ;
 \end{aligned}$$

so W_Λ must be smooth, with tangent space equal to $H^0(X_0, \mathcal{I}(d))$. □

All that remains to complete the proof of Proposition 4.8 is to consider the map ϕ_Λ from W_Λ to the product Δ of the deformation spaces of the tacnodes of X_0 at the points $s'_{i,j}$. Note first that the image of ϕ_Λ contains the locus $\Delta_m \subset \Delta$, and that the inverse image $W_0 = (\phi_\Lambda)^{-1}(\Delta_m)$ of this locus is simply the set of points $[X_t] \in W_\Lambda$ corresponding to curves X_t containing L . This has codimension at most one in W_Λ : if we choose any point $p \in L$ not among the points p_{ij} or $s_{i,j}$, then any curve X_t in W_Λ containing p will have a total of

$$I\alpha + I\beta + 1 = d + 1$$

points of intersection with L , and so will contain L .

On the other hand, under the differential

$$d\phi_\Lambda : T_{[X_0]}W_\Lambda \longrightarrow T_0\Delta$$

of the map ϕ_Λ , the inverse image of the tangent space $T_0\Delta_m \subset T_0\Delta$ is simply the subspace of $H^0(X_0, \mathcal{I}(d))$ of sections vanishing on L . Now, the restriction of $\mathcal{I}(d)$ to L has degree $d - I\alpha - I\beta = 0$ – that is, it is trivial – and by Lemma 4.10 not every global section of $\mathcal{I}(d)$ vanishes on L . Thus

$$\dim((d\phi_\Lambda)^{-1}(T_0\Delta_m)) = \dim(\phi^{-1}(\Delta_m)) = \dim(W_\Lambda) - 1 .$$

We may conclude that the image of ϕ_Λ is smooth of dimension

$$\dim(\phi_\Lambda(W_\Lambda)) = \dim(\Delta_m) + 1$$

with tangent space the image of $d\phi_\Lambda$. Since again the linear system $H^0(X_0, \mathcal{I}(d))$ has no base points on L , the image $W = \phi_\Lambda(W_\Lambda)$ satisfies the hypotheses of Lemma 4.3. Thus we may apply Lemma 4.3 to conclude that the closure of the inverse image $\phi_\Lambda^{-1}(\Delta_{m-1} \setminus \Delta_m)$ will have $\kappa = I\beta' - \beta / \text{lcm}(\beta' - \beta)$ reduced branches, each having intersection multi-

plicity $\text{lcm}(\beta' - \beta)$ with W_0 and hence with the hyperplane H_p . Since in a neighborhood of $[X_0]$ the Severi variety

$$V^{d,\delta}(\alpha, \beta) = \bigcup_{\Lambda} \overline{\phi_{\Lambda}^{-1}(\Delta_{m-1} \setminus \Delta_m)}$$

we conclude finally that near $[X_0]$, $V^{d,\delta}(\alpha, \beta)$ will have $\binom{\beta'}{\beta} I^{\beta' - \beta} / \text{lcm}(\beta' - \beta)$ branches, each of which will have intersection multiplicity $\text{lcm}(\beta' - \beta)$ with H_p along V' . This completes the proof of Proposition 4.8 and thereby of Theorem 1.3. \square

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