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Linear Series on Semistable Curves

Lucia Caporaso
Dipartimento di Matematica, Università Roma Tre, Largo S.L. Murialdo, 00146 Rome, Italy

Correspondence to be sent to: e-mail: caporaso@mat.uniroma3.it

For a semistable curve X of genus g, the number $h^0(X, L)$ is studied for line bundles L of degree d parameterized by the compactified Picard scheme. The theorem of Riemann is shown to hold. The theorem of Clifford is shown to hold in the following cases: X has two components; X is any semistable curve, and $d=0$ or $d=2g-2$; X is stable, free from separating nodes, and $d \leq 4$. These results are shown to be sharp. Applications to the Clifford index, to the combinatorial description of hyperelliptic curves, and to plane quintics are given.

1 Introduction and Preliminaries

The dimension of complete linear series on singular curves is, in general, quite difficult to control. This is one of the reasons why several interesting degeneration problems about line bundles and linear series remain unsolved. For singular curves, the Riemann–Roch theorem does not yield as strong information as for smooth curves, and several other classical theorems fail, as we shall illustrate.

On the other hand, it is well known that the Picard scheme of a singular curve tends to be too large, so that any good compactification of the generalized Jacobian parameterizes only a distinguished subset of line bundles. At present, the

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geometric and functorial properties of the compactified Picard scheme are rather well understood, making it a natural place to study limits of line bundles and related problems.

This is the main theme of this paper, which investigates the dimension of complete linear series parameterized by the compactified Picard scheme of stable curves. They correspond to the so-called “balanced” line bundles on semistable curves (defined in Section 2.1.1).

There exist other approaches to this type of questions. Some of them are by now considered classical, such as the theory of admissible covers, of Harris and Mumford [12], and the theory of limit linear series, of Eisenbud and Harris [9]. Although these techniques have been successfully applied by their creators to solve important problems, and they have been further studied by others ([3, 10, 14] for example), several questions, some considered in the present paper, remain open. Our method, applied also in [6], is different as it departs from the compactified Picard scheme and does not use degeneration techniques.

We proceed in analogy with the classical theory of Riemann surfaces. Our first result is Theorem 2.3, generalizing a theorem of Riemann, computing $h^0(X, L)$ for a balanced line bundle L of large degree on a semistable curve X. Although this theorem fails on infinitely many components of the Picard scheme of a reducible curve (see Example 2.6), we prove that, quite pleasingly, it does hold for every balanced line bundle, that is for every element of the compactified Picard scheme of X.

We then turn to study the theorem of Clifford. The situation is much more complex, as this theorem turns out to fail, even for balanced line bundles, in certain situations. Nonetheless, we prove that Clifford’s theorem does hold in several cases. Namely, it holds for all degrees on curves with two components (Theorem 3.3). Also it holds for all stable curves if the degree is 0 or $2g-2$ (Theorems 4.2 and 4.4). Finally, it holds for degree at most 4, for all stable curves free from separating nodes (Theorem 4.11). Some counterexamples are exhibited to show that the result is sharp: the Clifford inequality fails for all positive degrees for curves with separating nodes; furthermore, if $d \geq 5$, then it fails even for curves free from separating nodes (see Example 4.17).

The last section is devoted to applications. For curves with two components Clifford’s theorem is valid, it is thus interesting to study their (suitably defined) Clifford’s index and its connection with the gonality; we do that in Proposition 5.4, stating that a curve is weakly hyperelliptic (i.e., it admits a balanced g^1_2) if and only if its Clifford index is 0. Next, we focus on weakly hyperelliptic curves, give a combinatorial characterization of them (Theorem 5.9) and use it to describe the combinatorics of hyperelliptic curves

(Proposition 5.11). We conclude the paper with a classification of g^2_5's on two-component curves of genus 6 (Theorem 5.12).

1.1 Conventions

We work over any algebraically closed field. The following notation and terminology will be used throughout the paper. The word "curve" stands for reduced projective scheme of pure dimension 1. X is a connected curve, having at most nodes as singularities. g is the arithmetic genus of X. The irreducible component decomposition of X is written $X=\bigcup_{i=1}^{\gamma} C_i$, and g_i is the arithmetic genus of C_i. We usually denote by Z a (complete, reduced, of pure dimension 1) subcurve of X, by g_Z its arithmetic genus, and by $Z^c = \overline{X \smallsetminus Z}$ its complementary curve.

Given a line bundle $L \in \operatorname{Pic} X$ we denote by L_Z its restriction to $Z \subset X$.

Given two subcurves Z and Z' of X with no components in common, we denote

$$Z \cdot Z' := \# Z \cap Z' \quad \text{and} \quad \delta_Z := Z \cdot Z^c = \# Z \cap Z^c. \tag{1}$$

We often write $Z \cap Z' \subset Z$ to denote the set of points where Z intersects Z' and $\mathcal{O}_Z(Z \cap Z') \in \operatorname{Pic} Z$ to denote the corresponding line bundle.

The formula $g = g_Z + g_{Z^c} + \delta_Z - 1$ will be used several times.

Whenever we decompose a curve as a union of subcurves, for example, $X = Z \cup Y$, it will be understood that Z and Y have no components in common.

$\underline{d} = (d_1, \ldots, d_\gamma)$ will always be an element of $\mathbb{Z}^\gamma$ and $|\underline{d}| = \sum_1^\gamma d_i$. By $\underline{d} \le 0$ (resp. $\underline{d} \ge 0$) we mean that $d_i \le 0$ (resp. $d_i \ge 0$) for every i. We denote by $\operatorname{Pic}^{\underline{d}} X$, the set of line bundles L on X having multidegree $d_i = \deg_{C_i} L$ for $i = 1, \ldots, \gamma$, and, for any integer $r \ge 0$ we set $W^r_{\underline{d}}(X) := \{L \in \operatorname{Pic}^{\underline{d}} X : h^0(L) \ge r+1\}$.

1.2 Gluing global sections

In this section, we collect several technical results needed in the sequel.

1.2.1

Let $\nu : Y \to X$ be some partial (possibly total) normalization of X; consider the (surjective) morphism $\nu^* : \operatorname{Pic} X \to \operatorname{Pic} Y$. For every $M \in \operatorname{Pic} Y$ we will denote the fiber of ν^* over M as follows:

$$F_M(X) := \{L \in \operatorname{Pic} X : \nu^* L = M\}. \tag{2}$$

Let δ be the number of nodes normalized by $\nu: Y \to X$. For each of such node, n_i, let $\{p_i, q_i\} = \nu^{-1}(n_i)$ be its branches. We represent the above data by the self-explanatory notation

$$Y \longrightarrow X = Y/_{\{p_i = q_i,\ i=1,\dots,\delta\}}. \tag{3}$$

Fix $M \in \operatorname{Pic} Y$ such that $h^0(Y, M) \neq 0$. Pick $L \in F_M(X)$; then (cf. [5, 2.1.1])

$$h^0(Y, M) - \delta \leq h^0(X, L) \leq h^0(Y, M). \tag{4}$$

To study when $h^0(X, L) = h^0(Y, M)$ we introduce a convenient notation.

Definition 1.1. Let Y be a curve, $M \in \operatorname{Pic} Y$ and p and q nonsingular points of Y. We say that p and q are a neutral pair of M, and write $p \sim_M q$, if

$$h^0(Y, M - p) = h^0(Y, M - q) = h^0(Y, M - p - q). \tag{5}$$

□

Remark 1.2. Notation as in Definition 1.1.

(A) The relation $p \sim_M q$ is an equivalence relation.

(B) If p and q lie in different connected components of Y, $p \sim_M q$ if and only if p and q are base points of M.

(C) $p \sim_{\mathcal{O}_Y} q$ if and only if p and q lie in the same connected component of Y.

(D) If M is very ample, then M has no neutral pair. □

Lemma 1.3. Let $Y = Z_1 \coprod Z_2/_{\{p_i = q_i,\ i=1,\dots,\beta\}}$, where Z_1 and Z_2 are two nodal curves, and $p_1, \dots, p_\beta$ (respectively, $q_1, \dots, q_\beta$) smooth points of Z_1 (resp. of Z_2). Let $M \in \operatorname{Pic} Y$ and let $p \in Z_1$ and $q \in Z_2$ be smooth points of Y. If $p \sim_M q$ then p is a base point of $M_{Z_1}(-\sum_{i=1}^{\beta} p_i)$ (and q is a base point of $M_{Z_2}(-\sum_{i=1}^{\beta} q_i)$). □

Proof. Suppose that p is not a base point of $M_{Z_1}(-\sum_{i=1}^{\beta} p_i)$. Then there exists $s_1 \in H^0(Z_1, M_{Z_1}(-\sum_{i=1}^{\beta} p_i))$ such that $s_1(p) \neq 0$. Since s_1 vanishes at p_i for $i \leq \beta$, s_1 can be glued to the zero section in $H^0(Z_2, M_{Z_2})$, to give a section $s \in H^0(Y, M)$. By construction, $s(p) \neq 0$ and $s(q) = 0$. Therefore $p \not\sim_M q$. ■

The next lemma follows trivially from [5, Lemmas 2.2.3 and 2.2.4].

Lemma 1.4. Let Y be a nodal curve, p and q be two nonsingular points of Y and $Y \to X = Y/_{\{p=q\}}$. Let $M \in \operatorname{Pic} Y$ be such that $h^0(Y, M) \neq 0$.

There exists $L \in F_M(X)$ such that $h^0(X, L) = h^0(Y, M)$ if and only if $p \sim_M q$. If Y is connected, such an L is unique (if it exists) if and only if p and q are not base points for M. □

Lemma 1.5. Let $Y = Z_1 \coprod Z_2 \to X = Y/_{\{p_i=q_i,\ i=1,\dots,\delta\}}$, where $p_1, \dots, p_\delta$ (respectively, $q_1, \dots, q_\delta$) are nonsingular points of Z_1 (resp. of Z_2). Let $M = (M_1, M_2) \in \operatorname{Pic} Z_1 \times \operatorname{Pic} Z_2 = \operatorname{Pic} Y$; assume $h^0(Y, M) \geq 2$, and $p_i \not\sim_M q_i$ $\forall i$. Then there exists $L \in F_M(X)$ such that $h^0(X, L) = h^0(Y, M) - 1$ if and only if

$$p_i \sim_{M_1} p_j \quad \text{and} \quad q_i \sim_{M_2} q_j \quad \forall i, j. \qquad \square$$

Proof. If $\delta = 1$, then we have $F_M(X) = \{L\}$ and our assumption $p_1 \not\sim_M q_1$ implies, by Lemma 1.4, that $h^0(X, L) = h^0(Y, M) - 1$. From now on we let $\delta \geq 2$. Assume first $\delta = 2$. Write $Y' = Y/_{\{p_1=q_1\}}$, and let $M' \in \operatorname{Pic} Y'$ be the (unique) line bundle corresponding to M. As we just said, Lemma 1.4 yields

$$h^0(Y', M') = h^0(Y, M) - 1.$$

Suppose $p_2 \not\sim_{M_1} p_1$. Then there is $s_1 \in H^0(Z_1, M_1)$ vanishing at p_1 but not at p_2. Hence p_2 is not a base point of $M_1(-p_1)$. By Lemma 1.3, we have $p_2 \not\sim_{M'} q_2$,, hence by Lemma 1.4, for every $L \in F_{M'}(X)$ we have $h^0(X, L) \leq h^0(Y', M') - 1 = h^0(Y, M) - 2$.

Conversely, assume $p_2 \sim_{M_1} p_1$ and $q_2 \sim_{M_2} q_1$. We claim that $p_2 \sim_{M'} q_2$. Indeed, pick $s \in H^0(Y', M')$ such that $s(p_2) = 0$. Let s_i be the restriction of s to Z_i. Then $s_1 \in H^0(Z_1, M_1)$, hence $s_1(p_1) = 0$ by hypothesis. Therefore, $s_2(q_1) = 0$. Finally, as $q_2 \sim_{M_2} q_1$, we get $s_2(q_2) = 0$, hence $s(q_2) = 0$. So $p_2 \sim_{M'} q_2$.

By Lemma 1.4, this implies that there exists $L \in F_{M'}(X)$ such that $h^0(X, L) = h^0(Y', M') = h^0(Y, M) - 1$, so we are done.

If $\delta \geq 3$, we just apply the previous argument by replacing p_2 and q_2 with p_i, q_i, $i \geq 3$, and use Remark 1.2(A). ■

Fact 1.6. Let X be connected, and assume $\underline{d} = \underline{0} = (0, \dots, 0)$. Then for every $L \in \operatorname{Pic}^{\underline{0}} X$ we have $h^0(X, L) \leq 1$ and equality holds if and only if $L = \mathcal{O}_X$ [5, Corollary 2.2.5]. □

The following easy observation will be applied several times.

Remark 1.7. *Let $X = V \cup Z$ and $L \in \operatorname{Pic}^{\underline{d}} X$; assume that $\underline{d}_Z = (0, \dots, 0)$. Then $h^0(X, L) \leq h^0(V, L_V)$.*

Indeed, let $Z = Z_1 \coprod \cdots \coprod Z_c$ be the connected component decomposition of Z. Then, by Fact 1.6, $h^0(Z_i, L_{Z_i}) \leq 1$ and equality holds if and only if $L_{Z_i} = \mathcal{O}_{Z_i}$, in which case L_{Z_i} has no base point. Set $X_1 = V \cup Z_1 \subset X$; if $h^0(Z_1, L_{Z_1}) = 0$ then, obviously, $h^0(X_1, L_{X_1}) \leq h^0(V, L_V)$. If instead $L_{Z_1} = \mathcal{O}_{Z_1}$, by Lemma 1.4 applied to X_1 we obtain $h^0(X_1, L_{X_1}) \leq h^0(V, L_V) + 1 - 1 = h^0(V, L_V)$. Iterating, we are done. □

Recall the notational conventions of Section 1.1.

Lemma 1.8. Let $X = C \cup Z$ with C irreducible, set $\delta_C = C \cdot Z$. Let $L \in \operatorname{Pic} X$ be such that $\deg L_C = 2g_C + e_C$ for some $e_C \geq 0$. Then

(i) $h^0(X, L) \leq h^0(C, L_C) + h^0(Z, L_Z) - \min\{\delta_C, e_C + 1\}$.

(ii) If $e_C \geq \delta_C - 1$, then $h^0(X, L) = h^0(C, L_C) + h^0(Z, L_Z) - \delta_C$.

(iii) If $e_C \leq \delta_C - 2$, then equality holds in (i) for at most one L. □

Proof. We simplify the notation setting $\delta = \delta_C$. Let $X_0 := C \coprod Z$ and $\nu_0 : X_0 \to X$ be the natural map (the normalization of X at $C \cap Z$). Write $M_0 = (L_C, L_Z) \in \operatorname{Pic} X_0 = \operatorname{Pic} C \times \operatorname{Pic} Z$. We can factor ν_0 by normalizing one node in $C \cap Z$ at the time, as follows. Write

$$\nu_0 : X_0 \xrightarrow{\nu_1^0} X_1 \xrightarrow{\nu_2^1} \cdots \longrightarrow X_{\delta-1} \xrightarrow{\nu_\delta^{\delta-1}} X_\delta = X,$$

so that

$$\nu_{i+1}^i : X_i \longrightarrow X_i/_{\{p_i = q_i\}} = X_{i+1}$$

is the normalization of exactly one node of X_{i+1}, the branches p_i and q_i of which satisfy $p_i \in C$ and $q_i \in Z$. For all $i < \delta$, denote $\nu_i : X_i \longrightarrow X$ the composition, and $M_i := \nu_i^* L$. We have, of course,

$$h^0(X, L) \leq h^0(X_i, M_i). \tag{6}$$

Note that $h^0(X_0, M_0) = h^0(C, L_C) + h^0(Z, L_Z)$.

We claim that, for every $e \leq \min\{\delta - 1, e_C\}$, we have

$$h^0(X_{e+1}, M_{e+1}) = h^0(C, L_C) + h^0(Z, L_Z) - e - 1. \tag{7}$$

By induction on e. If $e=0$, then $\deg L_C \geq 2g_C$, therefore L_C has no base points. By Lemma 1.4 we obtain

$$h^0(X_1, M_1) = h^0(X_0, M_0) - 1 = h^0(C, L_C) + h^0(Z, L_Z) - 1.$$

As induction hypothesis, assume $h^0(X_e, M_e) = h^0(C, L_C) + h^0(Z, L_Z) - e$. Now

$$\deg L_C\left(-\sum_{j=1}^{e} p_j\right) = \deg L_C - e \geq 2g_C,$$

therefore $L_C(-\sum_{j=1}^{e} p_j)$ does not have base points; in particular, p_{e+1} is not a base point. By Lemma 1.3 we have $p_{e+1} \not\sim_{M_e} q_{e+1}$. By Lemma 1.4, this implies

$$h^0(X_{e+1}, M_{e+1}) = h^0(X_e, M_e) - 1 = h^0(C, L_C) + h^0(Z, L_Z) - e - 1$$

proving (7), which, combined with (6), proves (i).

From (7) we also immediately derive (ii).

Finally, for (iii) it suffices to apply the uniqueness part of Lemma 1.4. ■

1.3 **Clifford index of a line bundle**

The Clifford index of a line bundle on a curve X is the number Cliff $L := \deg L - 2h^0(X, L) + 2$. If X is irreducible and $0 \leq \deg L \leq 2g$, then Cliff $L \geq 0$, by Clifford's theorem ([7]); in fact, the extension to irreducible nodal curves of the classical Clifford's theorem for smooth curves is well known, and easy to prove by induction on the genus. Note also that if Cliff $L = 0$, then L has no base points, and if Cliff $L = 1$, then L has at most one base point. Indeed, for nonsingular points this is a formal consequence of Clifford's theorem; for singular points it is easily proved by induction on the genus.

The next lemma relates Cliff L to the equivalence $\sim_L$ of Definition 1.1.

Lemma 1.9. Let C be an irreducible curve of genus g; fix $L \in \mathrm{Pic}^d C$ with $h^0(C, L) \geq 2$ and $d \leq 2g$. Let E be a set of nonsingular points of C such that $p \sim_L q$ for all $p, q \in E$. Then $\#E \leq$ Cliff $L + 2$. □

Proof. Let $p_1, \dots, p_e \in E$; for every $i = 1, \dots, e$ we have

$$1 \leq h^0(C, L - p_i) = h^0\left(C, L - \sum_{j=1}^{e} p_j\right) \leq \frac{d-e}{2} + 1$$

(by Clifford's theorem). On the other hand, $h^0(C, L) = d/2 + 1 - \text{Cliff}\, L/2$, hence

$$h^0(C, L - p_i) \geq \frac{d - \text{Cliff}\, L}{2}.$$

Therefore

$$\frac{\text{Cliff}\, L - d}{2} \geq \frac{e-d}{2} - 1 \Rightarrow \text{Cliff}\, L + 2 \geq e.$$ ■

Corollary 1.10. Let $X = (C_1 \coprod C_2)/_{\{p_i = q_i,\ i=1,\dots,\delta\}}$, with C_1 and C_2 irreducible, and $p_1, \dots, p_\delta$ (resp. $q_1, \dots, q_\delta$) nonsingular points of C_1 (resp. of C_2). Pick $L_1 \in \text{Pic}\, C_1$ globally generated, such that $h^0(C_1, L_1) \geq 2$ and $\text{Cliff}\, L_1 + 2 < \delta$. Then for any $L_2 \in \text{Pic}\, C_2$ and any $L \in F_{(L_1, L_2)}(X)$ we have $h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_2, L_2) - 2$. □

Proof. Since $\delta > \text{Cliff}\, L_1 + 2$, Lemma 1.9 yields that there exists at least a pair p_i and p_j such that $p_i \not\sim_{L_1} p_j$. As L_1 is globally generated, by Remark 1.2(B) we have $p_i \not\sim_L q_i$ for any L as above; hence Lemma 1.5 applies, giving the statement. ■

In what follows we shall frequently use, without mentioning it, the obvious fact that $\text{Cliff}\, L$ and $\deg L$ have the same parity.

Proposition 1.11. Let $X = C_1 \cup C_2$ with C_i irreducible of genus g_i. Assume $\delta := C_1 \cdot C_2 \geq 2$. Let $L \in \text{Pic}^{\underline{d}} X$, set $L_i = L_{C_i}$, $d_i = \deg_{C_i} L_i$ and assume $0 \leq d_i \leq 2g_i$ for $i = 1, 2$.

(i) If $\text{Cliff}\, L = 0$, then $\text{Cliff}\, L_1 = \text{Cliff}\, L_2 = 0$; moreover, if $\underline{d} \neq \underline{0}$ then $\delta = 2$.

(ii) If $\text{Cliff}\, L = 1$ we may assume d_1 odd and d_2 even. Then $\text{Cliff}\, L_1 = 1$ and $\text{Cliff}\, L_2 = 0$. Moreover, if $d_1 \geq 3$, then $\delta \leq 3$; if $d_2 \geq 2$, then $\delta = 2$.

(iii) If $0 \leq \text{Cliff}\, L \leq 1$, then

$$h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_2, L_2) - 1 \leq \frac{d}{2} + 1.$$ □

Proof. Write $l = h^0(X, L)$ and $l_i = h^0(C_i, L_i)$. Let $p_1, \ldots, p_\delta \in C_1$ and $q_1, \ldots, q_\delta \in C_2$ be the points corresponding to the nodes of X, so that

$$X = \left(C_1 \coprod C_2\right) \Big/_{\{p_i = q_i,\ i=1,\ldots,\delta\}}.$$

Now, as $l \le l_1 + l_2$ we always have

$$\text{Cliff } L = d - 2l + 2 \ge d - 2l_1 - 2l_2 + 2 = \text{Cliff } L_1 + \text{Cliff } L_2 - 2. \tag{8}$$

Moreover, if either L_1 does not have a base point at some p_i or L_2 does not have a base point at some q_i, we have $l \le l_1 + l_2 - 1$, by Lemma 1.4. Therefore

$$\text{Cliff } L = d - 2l + 2 \ge d - 2l_1 - 2l_2 + 2 + 2 = \text{Cliff } L_1 + \text{Cliff } L_2. \tag{9}$$

Recall that if Cliff $L_i \le 1$, then L_i has at most one base point. Therefore, as $\delta \ge 2$, (9) applies if either Cliff $L_1 \le 1$ or Cliff $L_2 \le 1$.

Assume Cliff $L = 0$. Then (8) yields Cliff $L_i \le 2$ for $i = 1, 2$ (as Cliff $L_i \ge 0$ by Clifford's theorem for irreducible curves). If Cliff $L_1 = 0$, then we can apply (9), obtaining Cliff $L_2 = 0$. Moreover, we have equality occurring in (9), hence $l = l_1 + l_2 - 1$. By Lemma 1.5 we obtain that $p_i \sim_{L_1} p_j$ and $q_i \sim_{L_2} q_j$ for all i, j. If $\underline{d} \neq \underline{0}$ and $\delta \ge 3$, then this is impossible by Lemma 1.9. We conclude that $\delta = 2$.

By switching roles between L_1 and L_2 this argument together with (8) shows that Cliff $L = 0$ implies Cliff $L_i \le 1$ for $i = 1, 2$. If Cliff $L_1 = 1$ applying (9) gives $0 \ge 1 +$ Cliff L_2, which is impossible. (i) is proved.

Now assume Cliff $L = 1$; (8) yields Cliff $L_1 +$ Cliff $L_2 \le 3$. If Cliff $L_1 = 1$, then (9) applies; we get $1 \ge 1 +$ Cliff L_2, hence Cliff $L_2 = 0$. Similarly, if Cliff $L_2 = 0$ by (9) we get Cliff $L_1 = 1$. We thus have that Cliff $L_1 = 1$ if and only if Cliff $L_2 = 0$. As d_1 is odd, the only remaining case is Cliff $L_1 = 3$; this would imply Cliff $L_2 = 0$ which implies Cliff $L_1 = 1$, a contradiction. Therefore, the case Cliff $L_1 = 3$ does not occur. In a similar way, we see that the case Cliff $L_2 = 2$ cannot occur (it would imply Cliff $L_1 = 1$ which implies Cliff $L_2 = 0$).

Finally, equality holds in (9), so that $l = l_1 + l_2 - 1$. Hence $p_i \sim_{L_1} p_j$ and $q_i \sim_{L_2} q_j$ for all i and j (by Lemma 1.5 as before). Now, if either $d_1 \ge 3$ and $\delta \ge 4$, or if $d_2 \ge 2$ and $\delta \ge 3$, this is impossible by Lemma 1.9. (ii) is proved.

Part (iii) follows from the previous ones, observing that in both cases L_2 has no base points. Therefore, by Lemma 1.4 we have $l \leq l_1 + l_2 - 1$. Finally, if Cliff $L = 0$ we have $l_1 + l_2 - 1 = d_1/2 + 1 + d_2/2 + 1 - 1 = d/2 + 1$. If Cliff $L = 1$ we have $l_1 + l_2 - 1 = (d_1 + 1)/2 + d_2/2 + 1 - 1 = d/2 + 1/2$; so we are done. ■

2 Riemann's Theorem for Semistable Curves

The well-known Riemann's theorem for a smooth curve C of genus g states that, if $d \geq 2g - 1$ and $L \in \mathrm{Pic}^d C$, then $h^0(C, L) = d - g + 1$. More generally, using the normalization and induction on the number of nodes, it is easy to prove the following:

Fact 2.1. Let X be a nodal irreducible curve (of genus g) and $L \in \mathrm{Pic}^d X$. Then

(1) If $d \geq 2g - 1$, then $h^0(X, L) = d - g + 1$.
(2) If $d \geq 2g$, then L is free from base points.

Part (1) follows from Riemann–Roch and Serre duality, (2) follows from (1). □

By contrast, if X is reducible, Riemann's theorem trivially fails. In fact, for every fixed $d \geq 2g - 1$ there exist infinitely many multidegrees $\underline{d}$, with $|\underline{d}| = d$, such that for any $L \in \mathrm{Pic}^{\underline{d}} X$ we have $h^0(X, L) > d - g + 1$ (see Example 2.6).

On the other hand, it is well known that, for every d, there exists a well-defined finite set of multidegrees, of total degree d, which appear as the multidegrees of all line bundles parameterized by the compactified Picard variety of a stable curve X. More precisely, for any stable curve X, we shall denote by $\overline{P_X^d}$ the compactified Picard scheme constructed (independently) in [4, 13, 15, 16] (known to be all isomorphic by [1, 15]). Recall that $\overline{P_X^d}$ is a reduced scheme of pure dimension g, which appears as the specialization of the degree-d Picard varieties of smooth curves specializing to X. There are several modular descriptions of $\overline{P_X^d}$; the one we shall use interprets its points as equivalence classes of balanced line bundles on curves stably equivalent to X.

The main result of this section, Theorem 2.3, states that if L is a line bundle on a semistable curve X, having degree at least $2g - 1$, and balanced multidegree, then, just as for smooth curves, we have $h^0(X, L) = d - g + 1$. Therefore, if X is stable, every line bundle parameterized by the compactified Picard scheme $\overline{P_X^d}$ satisfies Riemann's theorem.

2.1 Balanced line bundles

Let X be fixed. For every subcurve $Z \subset X$ with $\delta_Z := Z \cdot Z^c$, we set

$$w_Z := \deg_Z \omega_X = 2g_Z - 2 + \delta_Z \quad \text{and} \quad w := w_X = 2g - 2. \tag{10}$$

Recall that a (nodal connected) curve X of genus $g \geq 2$ is *stable* if for every subcurve $Z \subset X$ we have $0 < w_Z < w$. X is *semistable* if for every $Z \subset X$ we have

$$0 \leq w_Z \leq w, \tag{11}$$

and $w_Z = 0$ if and only if Z is a union of exceptional components of X (a component $E \subset X$ is called exceptional if $E \cong \mathbb{P}^1$ and if $\delta_E = 2$).

We say that a semistable curve X is *stably equivalent* to a stable curve $\overline{X}$ if $\overline{X}$ is the curve obtained from X by contracting all of its exceptional components. $\overline{X}$ is called the *stabilization* of X.

2.1.1

Let $\underline{d} \in \mathbb{Z}^\gamma$ with $d = |\underline{d}|$; also fix $g \geq 2$. Assume that X is stable. We say that $\underline{d}$ is *balanced* if for every (connected) subcurve $Z \subset X$ we have

$$d\frac{w_Z}{w} - \frac{\delta_Z}{2} \leq d_Z \leq d\frac{w_Z}{w} + \frac{\delta_Z}{2}. \tag{12}$$

More generally, if X is semistable, we say that $\underline{d}$ is balanced if (12) holds, and if for every exceptional component E of X we have $d_E = 1$ (note that if a semistable curve admits some balanced multidegree, then it is quasistable, that is, two exceptional components do not intersect). Set

$$B_d(X) := \{\underline{d} : |\underline{d}| = d,\ \underline{d} \text{ is balanced}\}. \tag{13}$$

A line bundle on a semistable curve is balanced if its multidegree is balanced.

Example 2.2. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$ and $1 \le g_1 \le g_2$. Pick $d = 2$.

$$B_2(X) = \begin{cases} \{(0,2)\} & \text{if } g_1 < \dfrac{g+1}{4}, \\ \{(0,2);\,(1,1)\} & \text{if } g_1 = \dfrac{g+1}{4}, \\ \{(1,1)\} & \text{if } g_1 > \dfrac{g+1}{4}. \end{cases}$$ □

The terminology "balanced" was introduced in [4] to indicate that balanced multidegrees are closely related to the topological characters of the curve. Indeed, the balanced multidegrees of total degree $d \in \mathbb{Z}$ are as close as they can be to the multidegree $d/(2g-2)\underline{\deg}\omega_X$. The word balanced is sometimes replaced by the word "semistable". As we mentioned, if X is stable its compactified Picard scheme parameterizes equivalence classes of balanced line bundles on semistable curves having X as stabilization. If X is semistable, then its compactified Picard scheme turns out to coincide with the compactified Picard scheme of its stabilization. Here we do not need to be more precise about this point; see [4] for details.

2.2 Positivity properties of balanced line bundles

We denote

$$X_{\text{sep}} := \{n \in X_{\text{sing}} : n \text{ is a separating node of } X\} \subset X. \tag{14}$$

Theorem 2.3 (Balanced Riemann). Let X be a semistable curve of genus $g \ge 2$, d an integer and $\underline{d} \in B_d(X)$. Let $L \in \text{Pic}^{\underline{d}} X$.

(i) If $d \ge 2g-1$, then $h^0(X, L) = d - g + 1$.
(ii) If $d \ge 2g$ and $X_{\text{sep}} = \emptyset$, then L has no base points.
(iii) If $d \ge 5(g-1)$, then L has no base points. □

Remark 2.4. Part (i) may fail if $\underline{d}$ is not balanced; see Example 2.6. Part (ii) may fail if $X_{\text{sep}} \neq \emptyset$; see Example 2.7. □

Proof. Let $Z \subsetneq X$ be a connected subcurve. We claim that, if $d \ge 2g-1$, then we have

$$d_Z \ge 2g_Z - 1 \tag{15}$$

and, if $d \geq 2g$ and $X_{\text{sep}} = \emptyset$, we have

$$d_Z \geq 2g_Z. \tag{16}$$

To prove this, set $d = 2g - 2 + a = w + a$ with $a > 0$. As $\underline{d}$ is balanced, we have

$$d_Z \geq d\frac{w_Z}{w} - \frac{\delta_Z}{2} = 2g_Z - 2 + \frac{\delta_Z}{2} + a\frac{w_Z}{w}.$$

Now, $\delta_Z \geq 1$ and $w_Z \geq 0$ (cf. (11)). Therefore, the above inequality yields $d_Z \geq 2g_Z - 1$, as claimed in (15).

To prove (16), assume $X_{\text{sep}} = \emptyset$. Then $\delta_Z \geq 2$, so the previous inequality yields $d_Z \geq 2g_Z$, unless $w_Z = 0$, that is, unless Z is a chain of exceptional components (recall that X is semistable). If that is the case, $d_Z = 1$ and $g_Z = 0$. So we have $d_Z = 2g_Z + 1 > 2g_Z$. Equation (16) is proved.

Now, part (i) of the theorem follows from Lemma 2.5.

We shall apply Lemma 2.5 also for part (ii). If $d_Z \geq 2g_Z$ for every Z, then for any nonsingular point $p \in X$ we obviously have $\deg_Z L(-p) \geq 2g_Z - 1$, hence Lemma 2.5 applies to $L(-p)$, yielding $h^0(X, L(-p)) = h^0(X, L) - 1$. Now let $n \in X_{\text{sing}}$. Let $\nu : Y \to X$ be the normalization of X at n, $M := \nu^* L$ and $\nu^{-1}(n) = \{q_1, q_2\}$. To prove that L has a section not vanishing at n, it suffices to prove that

$$h^0(Y, M(-q_1 - q_2)) = h^0(Y, M) - 2. \tag{17}$$

Let $Z' \subset Y$ be a connected subcurve and $Z := \nu(Z')$. Then

$$\deg_{Z'} M = \deg_Z L \geq 2g_Z,$$

also $g_Z \geq g_{Z'}$ and strict inequality holds if and only if both q_1 and q_2 lie on Z', in which case $g_Z = g_{Z'} + 1$. Therefore

$$\deg_{Z'} M(-q_1 - q_2) \geq \begin{cases} 2g_Z - 2 = 2g_{Z'} & \text{if } q_1, q_2 \in Z', \\ 2g_Z - 1 \geq 2g_{Z'} - 1 & \text{otherwise.} \end{cases}$$

We can thus apply Lemma 2.5, proving (17) as follows:

$$h^0(Y, M(-q_1 - q_2)) = \deg M - 2 - g_Y + 1 = h^0(Y, M) - 2.$$

By the same argument, to prove (iii) it suffices to show that $d_Z \geq 2g_Z$ for every $Z \subset X$. Now, $d \geq 5(g-1)$ implies $d \geq 2g$, so by the previous parts it suffices to consider subcurves Z having $\delta_Z = 1$. Let Z be such a subcurve of X; note that $g_Z \geq 1$ (X is semistable) hence $w_Z = 2g_Z - 2 + \delta_Z \geq 2 - 2 + 1 = 1$. As $\underline{d}$ is balanced, and $d \geq 2(g-1) + 3(g-1) = w + 3(g-1)$, we have

$$d_Z \geq \frac{dw_Z}{w} - \frac{1}{2} \geq w_Z + \frac{3(g-1)w_Z}{2(g-1)} - \frac{1}{2} = 2g_Z - \frac{3}{2} + \frac{3w_Z}{2} \geq 2g_Z.$$

Hence we are done. ■

Lemma 2.5. Let Y be a (possibly disconnected) curve of genus g and $L \in \mathrm{Pic}^d Y$. If $\deg_Z L \geq 2g_Z - 1$ for every connected subcurve $Z \subseteq Y$, then $h^0(Y, L) = d - g + 1$. □

Proof. Let $X_1, \ldots, X_c$ be the connected components of Y. Then $g = \sum_{i=1}^{c} g_{X_i} - c + 1$ and $h^0(Y, L) = \sum_{i=1}^{c} h^0(X_i, L_{X_i})$; therefore it suffices to prove the lemma for a connected curve X of genus g.

We shall use induction on the number of irreducible components of X. The base case, X irreducible, is known (cf. Fact 2.1). Assume X reducible. We begin by showing that there exists an irreducible component, C_1, of X such that

$$d_1 \geq 2g_1 + \delta_1 - 1. \tag{18}$$

By contradiction, assume the contrary. Then

$$d = \sum_{i=1}^{\gamma} d_i \leq \sum_{i=1}^{\gamma} (2g_i + \delta_i - 2) = 2\sum_{i=1}^{\gamma} g_i + \sum_{i=1}^{\gamma} \delta_i - 2\gamma.$$

Now, $\sum_{i=1}^{\gamma} \delta_i = 2\delta$ and $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$. Therefore

$$d \leq 2\left(\sum_{i=1}^{\gamma} g_i + \delta - \gamma\right) = 2(g-1),$$

contradicting the assumption $d \geq 2g - 1$. This proves (18).

Let us write $X = C_1 \cup Z$ with $Z = C_1^c$. Let $Z = Z_1 \coprod \cdots \coprod Z_c$, with Z_i connected. We use induction and get

$$h^0(Z_i, L_{Z_i}) = d_{Z_i} - g_{Z_i} + 1. \tag{19}$$

Now, by (18) we can apply Lemma 1.8(ii) and obtain

$$h^0(X, L) = h^0(C_1, L_1) + h^0(Z, L_Z) - \delta_1 = d - \left(g_1 + \sum_{i=1}^{c} g_{Z_i}\right) + c + 1 - \delta_1$$

(using $h^0(C_1, L_1) = d_1 - g_1 + 1$ and (19)). Now $g = g_1 + \sum_{i=1}^{c} g_{Z_i} + \delta_1 - c$, hence $h^0(X, L) = d - g + \delta_1 - c + c + 1 - \delta_1 = d - g + 1$. ■

Example 2.6. Fix X having $\gamma \geq 2$ components and genus g; let $d \geq 2g - 1$. The theorem of Riemann fails for all but finitely many $\underline{d}$ with $|\underline{d}| = d$. To prove that it will be enough to show the following. For every fixed $i \in \{1, \ldots, \gamma\}$ there exists m_i such that for every $\underline{d}$ such that $d_i \geq m_i$ and for every $L \in \mathrm{Pic}^{\underline{d}} X$ we have $h^0(X, L) > d - g + 1$.

So, pick $i = 1$, let $m_1 := d + g_1 + \delta_1 + 1$ ($\delta_1 = C_1 \cdot C_1^c$). If $d_1 \geq m_1$, then we have

$$d_1 \geq d + g_1 + \delta_1 + 1 \geq 2g - 1 + g_1 + \delta_1 + 1 \geq 2g_1 + g_1 + \delta_1 = 3g_1 + \delta_1 \geq 2g_1 + 1;$$

hence $h^0(C_1, L_1) = d_1 - g_1 + 1$. Now, for any $L \in \mathrm{Pic}^{\underline{d}} X$ such that $d_1 \geq m_1$ (we can adjust the remaining $d_2, \ldots, d_\gamma$ however we like so that $|\underline{d}| = d$)

$$h^0(X, L) \geq h^0(C_1, L_1) - \delta_1 = d_1 - g_1 + 1 - \delta_1 \geq d + g_1 + \delta_1 + 1 - g_1 + 1 - \delta_1$$

hence $h^0(X, L) \geq d + 2 > d - g + 1$ as wanted. □

Example 2.7. If X has a separating node, then part (ii) of Theorem 2.3 may fail. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$. Assume $g_1 = 1$ and $g_2 = g - 1$ and $d = 2g + b$ with $b \geq 0$. Let $\underline{d} = (1, d - 1) = (1, 2g + b - 1) = (1, 2g_2 + b + 1)$, if $g \geq b + 3$ one checks that $\underline{d}$ is balanced. Set $L = (\mathcal{O}_{C_1}(p), L_2)$ such that $p \neq C_1 \cap C_2$. Assume for simplicity that L_2 has no base point in $C_1 \cap C_2$. Then

$$h^0(X, L) = h^0(C_1, \mathcal{O}_{C_1}(p)) + h^0(C_2, L_2) - 1 = h^0(C_2, L_2).$$

Now, L has a base point in p, indeed

$$h^0(X, L(-p)) = h^0(C_1, \mathcal{O}_{C_1}) + h^0(C_2, L_2) - 1 = h^0(C_2, L_2). \qquad \square$$

3 Clifford's Theorem for All Degrees

In this section, we prove the following cases of Clifford's theorem: Theorem 3.3, for curves with two components and every balanced multidegree; Proposition 3.1 for all curves and all degrees, provided the hypothesis that the degree be at most twice the genus is "uniformly" satisfied on all irreducible components.

3.1 Uniform extension

Proposition 3.1 (Uniform Clifford). Let X be a connected curve of genus g. Let $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$ be such that $0 \leq d_i \leq 2g_i$ for every $i = 1, \ldots, \gamma$.

(i) Then $|\underline{d}| \leq 2g$ and for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have $h^0(X, L) \leq \deg L/2 + 1$.

(ii) If equality holds and $|\underline{d}| \leq 2g - 2$, then L has no nonsingular base points (i.e., if L admits a base point, then this point is a node of X). $\square$

Proof. As we said in Section 1.3, we may assume X reducible. Set $|\underline{d}| = d$.

Let us prove that $d \leq 2g$. We have $d = \sum_{i=1}^{\gamma} d_i \leq \sum_{i=1}^{\gamma} 2g_i$. Let δ be the number of nodes of X that lie in two different irreducible components. Then $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$. On the other hand, as X is connected, we have $\delta \geq \gamma - 1$. Therefore, $2g - d \geq 2g - 2\sum_{i=1}^{\gamma} g_i = 2(\delta - \gamma + 1) \geq 0$, as claimed.

We continue using induction on the number of irreducible components.

We decompose $X = Z_1 \cup Z_2$ so that the Z_i are connected. We set $l_i := h^0(Z_i, L_{Z_i})$; by the induction assumption, $l_i \leq d_{Z_i}/2 + 1$ and if equality holds, L_{Z_i} has no nonsingular base points. We distinguish three cases.

Case 1. $l_i < d_{Z_i}/2 + 1$ for both $i = 1, 2$.

If d_{Z_1} and d_{Z_2} are even, then $l_i \leq d_{Z_i}/2$. Hence $h^0(X, L) \leq l_1 + l_2 \leq d/2$.

If d_{Z_1} is even and d_{Z_2} is odd, then $l_1 \leq d_{Z_1}/2$ and $l_2 \leq (d_{Z_2} + 1)/2$. Hence $h^0(X, L) \leq l_1 + l_2 \leq (d + 1)/2 < d/2 + 1$.

Finally, assume d_{Z_1} and d_{Z_2} odd. Then $l_i \leq (d_{Z_i} + 1)/2$ hence

$$h^0(X, L) \leq l_1 + l_2 \leq \frac{d}{2} + 1.$$

If equality holds, then we get $l_i = (d_{Z_i} + 1)/2$ for $i = 1, 2$, and $h^0(X, L) = l_1 + l_2$. Therefore, L_{Z_1} and L_{Z_2} have a base point over every node in $Z_1 \cap Z_2$. This implies that $Z_1 \cdot Z_2 = 1$. Indeed, by induction, the Clifford inequality holds on Z_i, yielding that L_{Z_i} can have at most one base point (indeed, if L_{Z_i} had two base points, p and p', then $h^0(L_{Z_i}(-p-p')) = h^0(L_{Z_i}) = (d_{Z_i} + 1)/2 > (d_{Z_i} - 2)/2 + 1)$.

Let $q_i \in Z_i$ be the branch of the node $n = Z_1 \cap Z_2$. Let $p \in X$ be a point with $p \neq n$, say $p \in Z_1$. If p is a base point for L, then it is also a base point for L_{Z_1}, but this is not possible as we just proved that the only base point of L_{Z_1} is q_1.

The proof of (i) and (ii) in Case 1 is complete.

Case 2. $l_1 = d_{Z_1}/2 + 1$ and $l_2 < d_{Z_2}/2 + 1$.

By induction, L_{Z_1} has no nonsingular base point. Therefore, by Lemma 1.4

$$h^0(X, L) \leq l_1 + l_2 - 1 < \frac{d_{Z_1}}{2} + 1 + \frac{d_{Z_2}}{2} + 1 - 1 = \frac{d}{2} + 1.$$

So, in this case strict inequality always holds and we are done.

Case 3. $l_i = d_{Z_i}/2 + 1$ for both $i = 1, 2$.

By induction L_{Z_i} is free from nonsingular base points. We get, again by Lemma 1.4,

$$h^0(X, L) \leq l_1 + l_2 - 1 = \frac{d_{Z_1}}{2} + 1 + \frac{d_{Z_2}}{2} + 1 - 1 = \frac{d}{2} + 1.$$

Now equality holds if and only if $h^0(X, L) = l_1 + l_2 - 1$. Let $p \in X$ be a nonsingular point, say $p \in Z_1$. As p is not a base point of L_{Z_1}, we have

$$h^0(X, L(-p)) \leq h^0(Z_1, L_{Z_1}(-p)) + l_2 - 1 = l_1 - 1 + l_2 - 1 = h^0(X, L) - 1$$

hence p is not a base point of L, so we are done. ■

Corollary 3.2. Assumptions as in Proposition 3.1. Assume $0 < |\underline{d}| < 2g - 2$. If there exists $L \in \text{Pic}^{\underline{d}} X$ such that Cliff $L = 0$, then for every decomposition $X = Z_1 \cup Z_2$ with Z_1 connected and Z_2 irreducible, we have

(a) $Z_1 \cdot Z_2 \leq 2$.

(b) If d_{Z_1} and d_{Z_2} are even, then Cliff $L_{Z_i} = 0$ and $h^0(Z_i, L_{Z_i}(-Z_1 \cap Z_2)) = h^0(Z_i, L_{Z_i}) - 1$, for $i = 1, 2$.

(c) If d_{Z_1} and d_{Z_2} are odd, then $Z_1 \cdot Z_2 = 1$ and Cliff $L_{Z_i}(-Z_1 \cap Z_2) = 0$ for $i = 1, 2$. □

Proof. We use the proof of Proposition 3.1. In Case 1, Cliff $L=0$ exactly when the d_{Z_i} are both odd, Z_1 and Z_2 intersect in only one point, and

$$h^0(Z_i, L_{Z_i}) = h^0(Z_i, L_{Z_i}(-q_i)) = \frac{d_{Z_i}+1}{2} = \frac{d_{Z_i}-1}{2} + 1.$$

So Cliff $(L_{Z_i}(-q_i)) = 0$. Observe that we did not use the irreducibility of Z_2.

In Case 2, equality never holds.

In Case 3, we have Cliff $L=0$ exactly when the d_{Z_i} are even, Cliff $L_{Z_i}=0$ for $i=1,2$, and $h^0(X,L) = h^0(Z_1, L_{Z_1}) + h^0(Z_2, L_{Z_2}) - 1$. Note that by Lemma 1.5 this implies that for every pair of points $q, q' \in Z_1 \cap Z_2 \subset Z_2$ we have $q \sim_{L_{Z_2}} q'$ (and similarly for Z_1).

To complete the proof, we need to show that $Z_1 \cdot Z_2 \leq 2$. By contradiction, assume $Z_1 \cdot Z_2 \geq 3$; then a relation $q \sim_{L_{Z_2}} q' \sim_{L_{Z_2}} q''$ holds on Z_2. Observe also that L_{Z_2} has no nonsingular base points, as Cliff $L_{Z_2}=0$. Therefore

$$h^0(Z_2, L_{Z_2}(-q-q'-q'')) = h^0(Z_2, L_{Z_2}(-q)) = l_2 - 1 = \frac{d_{Z_2}}{2}.$$

But Z_2 is irreducible, hence Clifford applies to $L_{Z_2}(-q-q'-q'')$, and we get

$$h^0(Z_2, L_{Z_2}(-q-q'-q'')) \leq \frac{d_{Z_2}-3}{2} + 1 < \frac{d_{Z_2}}{2},$$

a contradiction. ■

3.2 Curves with two components

Clifford's inequality holds for curves with two irreducible components, by the following result.

Theorem 3.3. Let $X = C_1 \cup C_2$ be a semistable curve of genus $g \geq 2$. Let $0 \leq d \leq 2g$ and $\underline{d} \in B_d(X)$. Then for every $L \in \mathrm{Pic}^{\underline{d}} X$ we have

$$h^0(X, L) \leq \frac{d}{2} + 1. \tag{20}$$

□

Addendum 3.4. Let $\epsilon := 1 + \max\{d_1 - 2g_1, d_2 - 2g_2, 0\}$, and $\beta := \min\{C_1 \cdot C_2, \epsilon\}$. If $C_1 \cdot C_2 \geq 2$, then $h^0(X,L) \leq h^0(C_1, L_1) + h^0(C_2, L_2) - \beta \leq d/2 + 1$. □

Proof. Set $l := h^0(X, L)$, and for $i = 1, 2$, $L_i := L_{C_i}$, $l_i := h^0(C_i, L_i)$. As usual, set $\delta := C_1 \cdot C_2$. By Theorem 2.3 we can assume $d \leq 2g - 2$. We begin with

Case 0. If $d_1 < 0$ then (20) holds, with strict inequality if $d \leq 2g - 2$.

As $d_1 < 0$ we have $d_2 > 0$. Since $\underline{d}$ is balanced,

$$d_1 \geq \frac{dw_1}{w} - \frac{\delta}{2} \geq -\frac{\delta}{2} \tag{21}$$

($w_1 \geq 0$ as X is semistable). Of course $l_1 = 0$, therefore, denoting by $G_2 \in \operatorname{Div} C_2$ the degree δ divisor cut on C_2 by C_1, a section of L has to vanish on G_2, that is,

$$h^0(X, L) = h^0(C_2, L_2(-G_2)). \tag{22}$$

Note that $\deg L_2(-G_2) = d_2 - \delta$. If $d_2 - \delta < 0$, then we get $h^0(X, L) = 0$ and we are done. If $0 \leq d_2 - \delta \leq 2g_2$, then we can use Clifford on C_2 and obtain

$$h^0(C_2, L_2(-G_2)) \leq \frac{d_2 - \delta}{2} + 1 = \frac{d - d_1 - \delta}{2} + 1 \leq \frac{d + \delta/2 - \delta}{2} + 1$$

(using (21)). Combining the above with (22) yields

$$h^0(X, L) \leq \frac{d}{2} + 1 - \frac{\delta}{4} < \frac{d}{2} + 1$$

as stated. Finally, it remains to treat the case $d_2 - \delta \geq 2g_2$, that is,

$$l = h^0(C_2, L_2(-G_2)) = d_2 - \delta - g_2 + 1.$$

We argue by contradiction; assume $l \geq d/2 + 1$. That is to say

$$d_2 - \delta - g_2 + 1 \geq \frac{d}{2} + 1,$$

hence (using $d = d_1 + d_2$)

$$\frac{d_2 - d_1}{2} - \delta - g_2 \geq 0,$$

equivalently

$$d_2 - d_1 - 2\delta - 2g_2 \geq 0. \tag{23}$$

On the other hand, as $\underline{d}$ is balanced, we have

$$d_2 \leq \frac{dw_2}{w} + \frac{\delta}{2} \quad \text{and} \quad d_1 \geq \frac{dw_1}{w} - \frac{\delta}{2}.$$

Using these two inequalities we get

$$d_2 - d_1 - 2\delta - 2g_2 \leq \frac{dw_2}{w} + \frac{\delta}{2} - \frac{dw_1}{w} + \frac{\delta}{2} - 2\delta - 2g_2 = \frac{d}{w}(w_2 - w_1) - \delta - 2g_2.$$

Now, $w_2 - w_1 = 2g_2 - 2g_1$ and $d/w \leq 1$ (as $d \leq 2g - 2 = w$). We obtain

$$d_2 - d_1 - 2\delta - 2g_2 \leq \frac{d}{w}(2g_2 - 2g_1) - \delta - 2g_2 \leq -\frac{2dg_1}{w} - \delta < 0$$

contradicting (23). This finishes Case 0.

For the rest of the proof, we can restrict to $d_i \geq 0$ for $i = 1, 2$. By Propositions 3.1 and 1.11(iii), we can assume that $d_i \geq 2g_i + 1$ for at least one i, so let $d_1 \geq 2g_1 + 1$. Then $l_1 = d_1 - g_1 + 1$.

Case 1. If $d_1 \geq 2g_1 + \delta - 1$, then (20) holds, with strict inequality if $d \leq 2g - 1$.

By Lemma 1.8(ii),

$$l = l_1 + l_2 - \delta. \tag{24}$$

Subcase 1a. $d_2 \geq 2g_2$. Hence $l_2 = d_2 - g_2 + 1$. Combining with (24) we have

$$l = d_1 - g_1 + 1 + d_2 - g_2 + 1 - \delta = d - (g_1 + g_2 + \delta - 1) + 1 = d - g + 1.$$

Now $d \leq 2g$, hence

$$l = d - g + 1 \leq d - \frac{d}{2} + 1 = \frac{d}{2} + 1.$$

So we are done. Note that equality holds if and only if $d = 2g$.

Subcase 1b. $d_2 < 2g_2$. By Proposition 3.1, $l_2 \leq d_2/2 + 1$. Set

$$d_1 = 2g_1 + \delta - 1 + a$$

so that $a \geq 0$ and

$$g_1 = \frac{d_1 - \delta + 1 - a}{2}. \tag{25}$$

Using (24) and (25) we get

$$l \le d_1 - g_1 + 1 + \frac{d_2}{2} + 1 - \delta = d_1 - \frac{d_1 - \delta + 1 - a}{2} + 2 + \frac{d_2}{2} - \delta,$$

hence

$$l \le \frac{d}{2} + 1 + \frac{1 - \delta + a}{2}.$$

The subsequent Lemma 3.5 yields

$$a \le \begin{cases} \frac{\delta}{2} - 1 & \text{if } \delta \text{ is even,} \\ \frac{\delta - 1}{2} - 1 & \text{if } \delta \text{ is odd.} \end{cases}$$

Hence $1 + a \le \delta/2$, so that $1 + a - \delta \le -\delta/2 < 0$. We conclude $h^0(X, L) < d/2 + 1$ and we are done.

Case 2. Assume $2g_1 + 1 \le d_1 < 2g_1 + \delta - 1$.

Set $d_1 = 2g_1 + e_1$ where $1 \le e_1 \le \delta - 2$. Hence

$$g_1 = \frac{d_1 - e_1}{2}. \tag{26}$$

By Lemma 1.8 we have

$$l \le l_1 + l_2 - e_1 - 1. \tag{27}$$

If $d_2 \le 2g_2$, then $l_2 \le d_2/2 + 1$. Using (26) we have

$$l \le d_1 - g_1 + 1 + \frac{d_2}{2} + 1 - e_1 - 1 = d_1 - \frac{d_1 - e_1}{2} + \frac{d_2}{2} + 1 - e_1 = \frac{d}{2} + 1 - \frac{e_1}{2}.$$

Now $e_1 \ge 1$ hence $l < d/2 + 1$ and we are done. Also, strict inequality holds.

If $d_2 \ge 2g_2 + 1$, then set $d_2 = 2g_2 + e_2$ with $e_2 \ge 1$. We can also assume $e_2 \le \delta - 1$, otherwise we are done by Case 1 (interchanging C_1 with C_2).

Now the situation is symmetric between C_1 and C_2, so up to switching them we may assume $e_1 \ge e_2$. By Lemma 1.8 we have,

$$l \le l_1 + l_2 - e_1 - 1 = d_1 - g_1 + 1 + d_2 - g_2 + 1 - e_1 - 1.$$

Now, using (26) applied also to C_2

$$l \leq d_1 - \frac{d_1 - e_1}{2} + 1 + d_2 - \frac{d_2 - e_2}{2} + 1 - e_1 - 1 = \frac{d}{2} + 1 + \frac{e_2 - e_1}{2}.$$

As $e_1 \geq e_2$ we conclude $l \leq d/2 + 1$. Moreover, equality holds if $e_1 = e_2$ and $l = l_1 + l_2 - e_1 - 1$. ■

Lemma 3.5. Let X be a semistable curve of genus $g \geq 2$, $d \leq 2g - 2$, and $\underline{d} \in B_d(X)$. Let $Z \subset X$ be a subcurve, set $d_Z = 2g_Z + \delta_Z - 1 + a_Z$. Then

$$a_Z \leq \begin{cases} \dfrac{\delta_Z}{2} - 1 & \text{if } \delta_Z \text{ is even,} \\ \dfrac{\delta_Z - 1}{2} - 1 & \text{if } \delta_Z \text{ is odd.} \end{cases}$$ □

Proof. We just need to apply (12) and compute, using $d \leq 2g - 2 = w$:

$$d_Z \leq \frac{dw_Z}{w} + \frac{\delta_Z}{2} \leq w_Z + \frac{\delta_Z}{2} = 2g_Z - 2 + \delta_Z + \frac{\delta_Z}{2}.$$

Now the statement follows at once from

$$d_Z = 2g_Z + \delta_Z - 1 + a_Z \leq 2g_Z + \delta_Z - 2 + \frac{\delta_Z}{2}.$$ ■

4 Clifford's Theorem in Special Degrees

4.1 Line bundles of degree 0 and $2g - 2$

Let X be a fixed curve. For any $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$, we denote

$$Z_{\underline{d}}^- := \bigcup_{i: d_i < 0} C_i \subset X. \tag{28}$$

Remark 4.1. Let X be a curve, and let $\underline{d}$ be such that $|\underline{d}| < 0$ and $\underline{d} \leq 0$. Then, for every $L \in \mathrm{Pic}^{\underline{d}} X$ we have $h^0(X, L) = 0$.

Indeed $h^0(Z_{\underline{d}}^-, L_{Z_{\underline{d}}^-}) = 0$, of course. Now, for any connected component, Y, of $\overline{X \smallsetminus Z_{\underline{d}}^-}$, we have $\underline{d}_Y = (0, \ldots, 0)$, hence $h^0(Y, L_Y) \leq 1$ with equality if and only if $L_Y = \mathcal{O}_Y$, in which case L_Y has no base points. So the remark follows from Lemma 1.4. □

Theorem 4.2 (Clifford for $d=0$). Let X be a curve of genus $g\geq 2$. Let $\underline{d}$ be such that $|\underline{d}|=0$. Assume that one of the following conditions hold.

(1) $d_Z \leq \delta_Z - 1$ for every subcurve $Z \subsetneq X$.

(2) X is semistable and $\underline{d}$ is balanced.

(3) $\delta_i - 2 \leq d_i \leq 2g_i - 2 + \delta_i$ for every $i = 1, \dots, \gamma$.

Then $h^0(X, L) \leq 1$ for every $L \in \operatorname{Pic}^{\underline{d}} X$.

Moreover, let $L \in \operatorname{Pic}^{\underline{d}} X$ be such that $h^0(X, L) = 1$. If (1) or (2) holds, or if (3) holds with $X_{\text{sep}} = \emptyset$, then $L \cong \mathcal{O}_X$. □

Proof. If $\underline{d} = (0, \dots, 0)$ the entire statement follows from Fact 1.6; hence we can assume $\underline{d} \not\geq 0$.

Let us assume (1). We will show that $h^0(X, L) = 0$. By contradiction, suppose there exists a nonzero section $s \in H^0(X, L)$; we let Y_s be the subcurve of X where s does not vanish, and W_s its complementary curve:

$$Y_s := \bigcup_{i : s_{|C_i} \neq 0} C_i \subset X, \qquad W_s := Y_s^c. \tag{29}$$

With the notation introduced in (28) we have $Z^-_{\underline{d}} \subset W_s$; note also that $\underline{d}_{Y_s} \geq 0$. Therefore, as $Z^-_{\underline{d}}$ is nonempty, W_s is nonempty. Since s vanishes on $W_s \cap Y_s$ we have $d_{Y_s} \geq \delta_{Y_s}$. This is a contradiction, since by assumption we must have $d_{Y_s} < \delta_{Y_s}$.

Now, let us show that assumption (2) implies assumption (1). As $\underline{d}$ is balanced, for every subcurve $Z \subset X$ we have

$$d_Z \leq \frac{\delta_Z}{2};$$

hence $d_Z < \delta_Z$, as claimed. Therefore if (2) holds we are done.

Finally, let us assume (3). We must prove that $h^0(X, L) \leq 1$ and that strict inequality holds if $X_{\text{sep}} = \emptyset$. By Riemann–Roch and Serre duality, $h^0(X, L) \leq 1$ if and only if $h^0(X, \omega_X \otimes L^{-1}) \leq g$.

Now, for every $i = 1, \dots, \gamma$ assumption (3) implies

$$0 \leq \deg_{C_i} \omega_X \otimes L^{-1} = 2g_i - 2 + \delta_i - d_i \leq 2g_i.$$

We can hence apply Proposition 3.1 getting $h^0(X, \omega_X \otimes L^{-1}) \leq g$, as wanted.

Now, suppose $X_{\text{sep}} = \emptyset$. Then (3) implies $\underline{d} \geq 0$; by the observation at the beginning of the proof we are done. ■

Example 4.3. The hypothesis $X_{\text{sep}} = \emptyset$ is necessary in the last part of Theorem 4.2, as the present example shows. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$ and C_i smooth. Let $L = (\mathcal{O}_{C_1}(-p_1), \mathcal{O}_{C_2}(p_2))$, where $p_i = C_1 \cap C_2 \in C_i$. If $g_2 \geq 1$ then (3) is satisfied. It is clear that $h^0(X, L) = 1$. □

By Riemann–Roch and Serre duality, any statement about sections of line bundles of degree $2g - 2$ has a dual statement about sections of line bundles of degree 0. The following is the dual of Theorem 4.2 (it suffices to check the arithmetic).

Theorem 4.4. Let X be a connected curve of genus $g \geq 2$. Let $\underline{d}$ be a multidegree such that $|\underline{d}| = 2g - 2$. Assume that one of the following conditions hold.

(1) $d_Z \geq 2g_Z - 1$ for every subcurve $Z \subsetneq X$.
(2) X is semistable and $\underline{d}$ is balanced.
(3) $0 \leq d_i \leq 2g_i$, for every $i = 1, \dots, \gamma$.

Then $h^0(X, L) \leq g$ for every $L \in \operatorname{Pic}^{\underline{d}} X$.

Moreover, let $L \in \operatorname{Pic}^{\underline{d}} X$ be such that $h^0(X, L) = g$. If (1) or (2) holds, or if (3) holds with $X_{\text{sep}} = \emptyset$, then $L \cong \omega_X$. □

4.2 Clifford's theorem in degree at most 4

The main result of this section is Theorem 4.11, stating the Clifford inequality in degree at most 4 for balanced line bundles on semistable curves free from separating nodes. In Lemmas 4.6, 4.7, and Proposition 4.8, we study Clifford's inequality for $\underline{d} \geq 0$, without assuming that $\underline{d}$ is balanced. The proof of Theorem 4.11 is thus reduced to the case that $\underline{d}$ has some negative entry. Quite interestingly, if $d \geq 5$ Clifford's theorem fails even when X has no separating nodes. See Example 4.17.

4.2.1

Let $n \in X_{\text{sep}}$ be a separating node of X; then there exist two subcurves Z_1 and Z_2 of X such that $X = Z_1 \cup Z_2$ and $n = Z_1 \cap Z_2$. Such curves Z_1 and Z_2 are called the tails of X generated by n. So, a subcurve $Z \subset X$ is called a *tail* if $Z \cdot Z^c = 1$. As X is connected, its tails are connected.

Let $C \subset X$ be a subcurve. C is called a *separating line* if $C \cong \mathbb{P}^1$ and if C meets its complementary curve C^c only in separating nodes of X. Equivalently, a separating line $C \subset X$ is a smooth rational component such that C^c has a number of connected components equal to $C \cdot C^c$.

If $X \cong \mathbb{P}^1$, then X is a separating line of itself.

If Y is a disconnected curve and $C \subset Y$, then we say C is a separating line of Y if it is so for the connected component of Y containing C.

Observe that if C is a separating line, then we have

$$Z \cdot C \leq 1 \quad \text{for every connected } Z \subset C^c. \tag{30}$$

Remark 4.5. Assume $X_{\mathrm{sep}} = \emptyset$ (i.e., X has no tails). Let Z be a subcurve of X.

(A) If m is the number of connected components of Z, then $m \leq \delta_Z/2$.

(B) Let $X = D \cup Y$ with D connected. If $C \subset Y$ is a separating line of Y, then $\overline{X \smallsetminus C}$ is connected.

The only statement that is not obvious is (B). Let $Y_1, \ldots, Y_m$ be the connected components of Y and suppose $C \subset Y_1$. We can assume $C \neq Y_1$, otherwise we are done. Thus, every connected component of $\overline{Y_1 \smallsetminus C}$ is a tail of Y_1; as X has no tails D intersects every connected component of $\overline{Y_1 \smallsetminus C}$. On the other hand, D obviously intersects Y_i for all $i \geq 2$, therefore $\overline{X \smallsetminus C}$ is connected. □

Lemma 4.6. Let $L \in \mathrm{Pic}^{\underline{d}} X$ with $\underline{d} = (1, 0, \ldots, 0)$. Then either $h^0(X, L) \leq 1$, or C_1 is a separating line, $h^0(X, L) = 2$ and $L_{C_1^c} = \mathcal{O}_{C_1^c}$. □

Proof. Write $Y = C_1^c$ and let $Y = \coprod_{i=1}^m Y_i$ be the decomposition into connected components. Of course C_1 must intersect every Y_i.

If $g_1 \geq 1$ we have $h^0(C_1, L_{C_1}) \leq 1$ hence the lemma follows from Remark 1.7 (with $V = C_1$). So it suffices to assume $C_1 \cong \mathbb{P}^1$. If C_1 is not a separating line, then there exists at least one connected component of Y, Y_1 say, such that $C_1 \cdot Y_1 \geq 2$. Set $X_1 = C_1 \cup Y_1$, then by Remark 1.7 and Lemma 1.8 we conclude as follows:

$$h^0(X, L) \leq h^0(X_1, L_{X_1}) \leq h^0(C_1, L_1) + h^0(Y_1, L_{Y_1}) - 2 \leq 2 + 1 - 2 = 1.$$

If C_1 is a separating line and for some component of Y, Y_1 say, we have $L_{Y_1} \neq \mathcal{O}_{Y_1}$, then every section of L has to vanish on Y_1, hence not every section of $\mathcal{O}_{C_1}(1)$ extends to a section of L.

Conversely, if $L_{Y_i} = \mathcal{O}_{Y_i}$ for all i, then it is obvious that $h^0(X, L) = 2$. ■

Lemma 4.7. Let $L \in \mathrm{Pic}^{\underline{d}} X$. Assume that $|\underline{d}| = 2$ and $\underline{d} \geq 0$. Then either $h^0(X, L) \leq 2$, or $h^0(X, L) = 3$ and one of the following cases occurs.

(i) $\underline{d} = (2, 0, \dots, 0)$ with C_1 a separating line.

(ii) $\underline{d} = (1, 1, 0, \dots, 0)$, with C_1 and C_2 separating lines. □

Proof. Assume $h^0(L) \geq 3$. For every nonsingular point p of X we have

$$h^0(L(-p)) \geq h^0(L) - 1 \geq 2. \tag{31}$$

Of course, $\deg L(-p) = 1$ and, if p lies in a component C_1 such that $d_1 > 0$ we have $\underline{\deg} L(-p) \geq 0$. By Lemma 4.6 we get $h^0(L(-p)) \leq 1$, unless X has a separating line E with $\deg_E L(-p) = 1$. If X does not have such a separating line we got a contradiction to (31). Now, X admits such a separating line E if and only if either $d_1 = 2$ and $E = C_1$, or $d_1 = 1$, hence $d_2 = 1$, and C_2 is a separating line. By placing $p \in C_2$ we get that both C_1 and C_2 are separating lines. By Lemma 4.6 $h^0(L(-p)) = 2$, so $h^0(L) = 3$ by (31) and we are done. ■

Proposition 4.8. Let X be a stable curve free from separating nodes. Let $\underline{d}$ be such that $\underline{d} \geq 0$ and $|\underline{d}| = 3, 4$. Then $h^0(X, L) \leq |\underline{d}|/2 + 1$ for every $L \in \mathrm{Pic}^{\underline{d}} X$. □

Remark 4.9. The hypotheses X stable and $X_{\mathrm{sep}} = \emptyset$ are needed, as shown by Examples 4.15 and 4.16. □

Proof. We first treat the case $|\underline{d}| = 3$. Consider the irreducible component C_1 of X; we shall denote $C_1^c = Y_1 \coprod \dots \coprod Y_m$ the connected component decomposition. Observe that for every Y_i we have $Y_i \cdot C_1 \geq 2$. We set

$$X_1 := C_1 \cup Y_1 \subset X.$$

We shall repeatedly apply Lemma 1.8 and Remark 1.7.

Case 1. $\underline{d} = (3, 0, \dots, 0)$. We have $h^0(X, L) \leq h^0(X_1, L_{X_1})$ by Remark 1.7. Hence it suffices to assume that C_1 has genus $g_1 \leq 1$.

If $g_1 = 1$, by the initial observation and Lemma 1.8 we have $h^0(X_1, L_{X_1}) \leq 3 + 1 - 2 = 2$ and we are done.

If $C_1 \cong \mathbb{P}^1$ we have $h^0(C_1, L_1) = 4$ and $C_1 \cdot C_1^c \geq 3$. Suppose C_1^c has a connected component, Y_1, such that $C_1 \cdot Y_1 \geq 3$. Then by Lemma 1.8, as $h^0(Y_1, L_{Y_1}) \leq 1$, we get $h^0(X_1, L_{X_1}) \leq 4 + 1 - 3 = 2$, as wanted.

Let now $C_1 \cdot Y_i = 2$ for all $i = 1, \ldots, m$. Set $X_2 = Y_1 \cup Y_2 \cup C_1 \subset X$. Then $C_1 \cdot (Y_1 \cup Y_2) \geq 4 = d_1 + 1$, hence by Lemma 1.8,

$$h^0(X_2, L_{X_2}) \leq h^0(C_1, L_1) + h^0(Y_1, L_{Y_1}) + h^0(Y_2, L_{Y_2}) - 4 \leq 4 + 2 - 4 = 2.$$

By Remark 1.7 we are done.

Case 2. $\underline{d} = (1, 2, 0, \ldots, 0)$.

Write $l_i = h^0(C_i, L_i)$. Assume C_1^c connected; by Lemma 4.7, $h^0(C_1^c, L_{C_1^c}) \leq 3$ and equality holds if and only if C_2 is a separating line of C_1^c. If this is not the case, then by Lemma 1.8 and $\delta_1 \geq 2$, we get $h^0(X, L) \leq l_1 + 2 - 2 \leq 4 - 2 = 2$, as wanted.

If C_2 is a separating line of C_1^c, then $l_2 = 3$, and C_2^c is connected, by Remark 4.5(B); hence $h^0(C_2^c, L_{C_2^c}) \leq 2$. Since $\delta_2 \geq 3$ (as $d_2 = 2$) we obtain

$$h^0(X, L) \leq l_2 + h^0(C_2^c, L_{C_2^c}) - 3 \leq 5 - 3 = 2$$

and we are done. This last paragraph works regardless of C_1^c being connected.

Now let C_1^c have $m \geq 2$ connected components. We can assume that C_2 is not a separating line of C_1^c. Let $C_2 \subset Y_1$; we have $h^0(Y_1, L_{Y_1}) \leq 2$. By Lemma 1.8 we get $h^0(X_1, L_{X_1}) \leq h^0(C_1, L_1) + h^0(Y_1, L_{Y_1}) - 2 \leq 2$. By Remark 1.7 we are done.

Case 3. $\underline{d} = (1, 1, 1, 0, \ldots, 0)$. By Proposition 3.1 we may assume that $C_1 \cong \mathbb{P}^1$. Moreover, by Lemma 4.10, up to permuting the first three components, we can assume that C_2 and C_3 are not separating lines of C_1^c. If C_1^c is connected, by Lemma 4.7 we have $h^0(C_1^c, L_{C_1^c}) \leq 2$ (as C_2 and C_3 are not separating lines of C_1^c). By Lemma 1.8 we have $h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_1^c, L_{C_1^c}) - 2 \leq 2 + 2 - 2 \leq 2$ and we are done.

Now assume C_1^c has $m \geq 2$ connected components. If $C_2 \cup C_3$ lies in one connected component, Y_1, then $h^0(Y_1, L_{Y_1}) \leq 2$ (just as above). Therefore, $h^0(X, L) \leq h^0(X_1, L_{X_1}) \leq 2 + 2 - 2 = 2$. If instead C_2 lies in Y_1 and C_3 lies in Y_2, then for $i = 1, 2$ we have $h^0(Y_i, L_{Y_i}) \leq 1$ by Lemma 4.6 (C_2 and C_3 are not separating lines of, respectively, Y_1 and Y_2). We conclude $h^0(X_1, L_{X_1}) \leq 2 + 1 - 2 = 1$. Now, let $X_2 = X_1 \cup Y_2$, then

$$h^0(X, L) \leq h^0(X_2, L_{X_2}) \leq h^0(X_1, L_{X_1}) + h^0(Y_2, L_{Y_2}) \leq 2.$$

The proof for $d = 3$ is complete.

Now let $|\underline{d}| = 4$. By contradiction, suppose that $h^0(X, L) \geq 4$. As $\underline{d} \geq 0$, there exists a component, C_1 say, such that $d_1 \geq 1$. Let $p \in C_1$ be a nonsingular point of X, then $h^0(L(-p)) \geq h^0(L) - 1 \geq 3$. Now, $\deg L(-p) = 3$ and $\underline{\deg} L(-p) \geq 0$. By the previous part, $h^0(L(-p)) \leq 2$; impossible. ■

In the proof we used the following combinatorial lemma.

Lemma 4.10. Let X be stable, $X_{\text{sep}} = \emptyset$, and C_1 and C_2 be two irreducible components of X. Assume C_2 is a separating line of C_1^c, and C_1 is a separating line of C_2^c (i.e., (C_1, C_2) is a $\mathcal{B}$-pair, see Definition 5.7). Then for every other component D of X, C_1, and C_2 are not separating lines of D^c. □

Proof. Note that by Remark 4.5(B), C_1^c and C_2^c are connected. Let $T_1, \ldots, T_t$ be the tails of C_1^c generated by C_2. Thus $C_1^c = C_2 \cup T_1 \cup \cdots \cup T_t$, with $T_i \cap T_j = \emptyset$ and $T_i \cdot C_2 = 1$. As C_2^c is connected, C_1 must intersect every T_i. As C_1 is a separating line of C_2^c, we have

$$C_1 \cdot T_i = 1 \quad \forall i. \tag{32}$$

Let D be another component of X, assume $D \subset T_1$. Set $Z = C_2 \cup T_2 \cup \cdots \cup T_t$, so that $C_1^c = Z \cup T_1$, hence $\delta_{C_1} = Z \cdot C_1 + T_1 \cdot C_1 = Z \cdot C_1 + 1 \geq 3$, by (32) and the stability of X. We conclude $Z \cdot C_1 \geq 2$. This implies that C_1 cannot be a separating line of D^c, as Z is connected and $Z \subset D^c$ (cf. Section 4.2.1, (30)). The same argument with C_1 and C_2 switching roles yields that C_2 is not a separating line of D^c. ■

Theorem 4.11. Let X be a stable curve free from separating nodes. Let $\underline{d}$ be balanced with $0 < |\underline{d}| \leq 4$; let $L \in \operatorname{Pic}^{\underline{d}} X$. Then

(i) $h^0(X, L) \leq |\underline{d}|/2 + 1$.
(ii) If $|\underline{d}| = 1, 2$ and $h^0(X, L) = |\underline{d}|$, then $\underline{d} \geq 0$. □

If $|\underline{d}| = 1, 2$, then the hypotheses on X can be weakened as follows.

Addendum 4.12. If $|\underline{d}| = 1$ the same holds if X is semistable and has no separating lines. If $|\underline{d}| = 2$ the same holds if X is semistable and $X_{\text{sep}} = \emptyset$. □

Proof. If $\underline{d} \geq 0$ the statement follows from Lemmas 4.6, 4.7 and Proposition 4.8. So, assume $\underline{d} \not\geq 0$; set $d = |\underline{d}|$. We shall inductively define a useful subcurve $V \subseteq X$. Let

$V_0 := Z^-_{\underline{d}}$ (see (28)). Now define $V_1 \subset X$

$$V_1 := V_0 \cup \bigcup_{C_i \cdot V_0 > d_i = 0} C_i,$$

that is, V_1 is the union of V_0 with all components of degree 0 which intersect V_0. Next

$$V_2 := V_1 \cup \bigcup_{\substack{C_i \not\subset V_1, d_i \leq 1, \\ C_i \cdot V_1 > d_i}} C_i.$$

Iterating

$$V_{h+1} := V_h \cup \bigcup_{\substack{C_i \not\subset V_h, d_i \leq h, \\ C_i \cdot V_h > d_i}} C_i \subset X.$$

Of course, $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_h \subseteq V_{h+1} \subseteq \cdots \subseteq X$, therefore there exists an $m \geq 0$ minimum for which $V_n = V_m$ for every $n \geq m$. We set $V := V_m$.

We claim that every $s \in H^0(X, L)$ vanishes identically on V. It is clear that s vanishes on V_0; let us prove the claim inductively. Let $h \geq 0$ be such that V_{h+1} is not equal to V_h; by induction s vanishes identically on V_h. Let $C \subset V_{h+1}$ be such that C is not contained in V_h. Then s vanishes on $C \cap V_h$. Now, V_{h+1} is constructed so that $C \cdot V_h > \deg_C L > 0$, therefore s vanishes on C. The claim is proved.

If $V = X$, then we have $H^0(X, L) = 0$ and we are done. So assume that $Y := V^c$ is not empty. Denote by $G_Y \in \mathrm{Div}\, Y$ the divisor cut out by V, so that

$$\deg G_Y = \delta_Y. \tag{33}$$

Note that

$$H^0(X, L) \cong H^0(Y, L_Y(-G_Y)). \tag{34}$$

By construction we have

$$\underline{d}_Y - \underline{\deg} G_Y \geq 0. \tag{35}$$

We claim that

$$0 \leq d_Y - \delta_Y \leq d - 2. \tag{36}$$

Set $a = d_Y - \delta_Y$. That $0 \leq a$ follows from (33) and (35). Now, note that $w_Y < w$. Indeed, as $\underline{d}_V \not\geq 0$ by construction, $V = Y^c$ is not a union of exceptional components. Hence (cf. Section 2.1) $w_V > 0$ and $w_Y = w - w_V < w$. As $\underline{d}$ is balanced, we obtain

$$\delta_Y + a = d_Y \leq \frac{\delta_Y}{2} + \frac{dw_Y}{w} < \frac{\delta_Y}{2} + d. \tag{37}$$

Therefore $\delta_Y \leq 2d - 2a - 1$. As $X_{\text{sep}} = \emptyset$ we have $\delta_Y \geq 2$. We obtain

$$2d - 2a - 1 \geq 2$$

hence $a \leq d - \frac{3}{2}$, so that $a \leq d - 2$. Equation (36) is proved.

We continue the proof with a case-by-case analysis.

Case $d = 1$. The inequality (36) makes no sense, hence Y is empty, that is, $h^0(L) = 0$. We conclude that if $h^0(L) \neq 0$, then $\underline{d} \geq 0$, a case treated in Lemma 4.6. The assumptions X stable and $X_{\text{sep}} = \emptyset$ can clearly be weakened by, respectively, X semistable, and containing no separating line (needed for Lemma 4.6). If $d = 1$, then the theorem and the addendum are proved.

Case $d = 2$. By (36) we have $d_Y = \delta_Y$, hence $\deg L_Y(-G_Y) = 0$. Now, using (37) we get $\delta_Y = d_Y < \delta_Y/2 + 2$, hence $\delta_Y \leq 3$. This yields that Y is connected, by Remark 4.5(A). We can apply Fact 1.6 to $L_Y(-G_Y)$, obtaining, with (34),

$$h^0(X, L) = h^0(Y, L_Y(-G_Y)) \leq 1.$$

This concludes the proof if $d = 2$. We also showed that if $h^0(X, L) = 2$, then $\underline{d} \geq 0$. Observe that the argument works if X is semistable, so the theorem and the addendum are proved. The remaining cases will be treated similarly.

Case $d = 3$. By (36) we have two possibilities: either $\delta_Y = d_Y$ or $\delta_Y + 1 = d_Y$. If $\delta_Y = d_Y$ we have, using (37), $\delta_Y = d_Y < \delta_Y/2 + 3$, hence $\delta_Y \leq 5$. Therefore Y has at most two connected components (by Remark 4.5(A)). Let Y_i be a connected component of Y, then, by (35), $d_{Y_i} = \delta_{Y_i}$, and we can apply Fact 1.6 to $L_{Y_i}(-G_{Y_i})$ (with self-explanatory notation). Hence $h^0(Y_i, L_{Y_i}(-G_{Y_i}) \leq 1$; now Y has at most two connected components, hence by (34) we obtain $h^0(X, L) \leq 2$.

If $d_Y = \delta_Y + 1$, by (37) $\delta_Y + 1 = d_Y < \delta_Y/2 + 3$, hence $\delta_Y \leq 3$, so Y is connected. By (35) and (36) we can apply Lemma 4.6 to $L_Y(-G_Y)$; we get

$$h^0(X, L) = h^0(Y, L_Y(-G_Y)) \leq 2.$$

This finishes the proof in case $d = 3$.

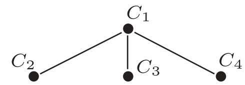

Fig. 1. Dual graph of the curve in Example 4.13.

Case $d=4$. By (36) we have three possibilities: $d_Y=\delta_Y$, $d_Y=\delta_Y+1$, or $d_Y=\delta_Y+2$.

If $d_Y=\delta_Y$, we get $\delta_Y=d_Y<\delta_Y/2+4$, hence $\delta_Y\leq 7$. Therefore, Y has at most three connected components (again by Remark 4.5(A)). Arguing as in the analogous case when $d=3$ ($d_Y=\delta_Y$) we see that $h^0(X,L)\leq 3$, so we are done.

If $d_Y=\delta_Y+1$, by (37) $\delta_Y+1=d_Y<\delta_Y/2+4$, hence $\delta_Y\leq 5$ and Y has at most two connected components. If Y is connected arguing as in the analogous case when $d=3$ we conclude $h^0(X,L)\leq 2$ and we are done. If Y has two connected components, Y_1 and Y_2, then we have $d_{Y_1}=\delta_{Y_1}$ and $d_{Y_2}=\delta_{Y_2}+1$. We can therefore apply Fact 1.6 to get $h^0(Y_1,L_{Y_1}(-G_{Y_1}))\leq 1$, and 4.6 to get $h^0(Y_2,L_{Y_2}(-G_{Y_2}))\leq 2$. Summing up we obtain

$$h^0(X,L)=h^0(Y_1,L_{Y_1}(-G_{Y_1}))+h^0(Y_2,L_{Y_2}(-G_{Y_2}))\leq 3$$

and we are done. Finally, if $d_Y=\delta_Y+2$, by the usual argument we get $\delta_Y\leq 3$ hence Y is connected. By Lemma 4.7 we have $3\geq h^0(Y,L_Y(-G_Y))=h^0(X,L)$ and we are done. ■

4.3 Counterexamples

Example 4.13. *Failure of Clifford's theorem:* $d=1$, $\underline{d}\geq 0$ *balanced* (X *contains a separating line*). Let $X=C_1\cup C_2\cup C_3\cup C_4$ with, for $i,j\geq 2$, $C_i\cap C_j=\emptyset$ and $C_1\cdot C_i=1$ (the dual graph of X is in Figure 1). Assume $C_1=\mathbb{P}^1$ (hence C_1 is a separating line) and $g_i=h\geq 1$ (hence X is stable). Thus, $g=3h$ and $w=6h-2$. Set $\underline{d}=(1,0,0,0)$, one checks that $\underline{d}\in B_1(X)$. Let

$$L:=(\mathcal{O}_{C_1}(1),\mathcal{O}_{C_2},\mathcal{O}_{C_3},\mathcal{O}_{C_4}).$$

Then, as all L_i are free from base points, we get $h^0(X,L)=\sum_1^4 h^0(C_i,L_i)-3=2$. □

Example 4.14. Cliff $L=0$ *with* $\underline{\deg}L\in B_1(X)$, $\underline{\deg}L\not\geq 0$ ($X_{\text{sep}}\neq\emptyset$). Let $X=C_1\cup C_2\cup C_3$ with, $C_1\cdot C_2=2$, $C_2\cdot C_3=1$, and $C_1\cap C_3=\emptyset$ (see Figure 2). Thus, $n=C_2\cap C_3$ is a separating node; for $i=2,3$, write $q_i\in C_i$ the point corresponding to this node. Assume $g_1=g_2=1$ and $g_3=4$, thus $g=7$. Set $\underline{d}=(1,-1,1)$; one checks that $\underline{d}\in B_1(X)$. Write

Fig. 2. Dual graph of the curve in Example 4.14.

$Z = C_1 \cup C_2 \subset X$ and let $L_{1,2} \in \mathrm{Pic}^{(1,-1)} Z$ be arbitrary. Note that $h^0(Z, L_{1,2}) = 0$. Set $L := (L_{1,2}, \mathcal{O}_{C_3}(q_3))$. Then, as $L_{1,2}$ and $\mathcal{O}_{C_3}(q_3)$ both have a base point in the respective branch (q_2 and q_3) of n, we get $h^0(X, L) = h^0(Z, L_{1,2}) + h^0(C_3, \mathcal{O}_{C_3}(q_3)) = 1$. □

Example 4.15. *Failure of Clifford's theorem: $d \geq 3$, $\underline{d}$ balanced, $X_{\mathrm{sep}} = \emptyset$ (X strictly semistable).* For $d \geq 3$ consider the curve $X = C_1 \cup \cdots \cup C_{2d}$ the dual graph of which is a $2d$-cycle, that is, a closed polygon with $2d$ vertices, $C_1, \ldots, C_{2d}$. We set $C_i \cdot C_{i+1} = C_{2d} \cdot C_1 = 1$ for all $i \geq 1$ and $C_i \cdot C_j = 0$ for all other intersections. So X has $2d$ nodes. Let $C_{2i-1} \cong \mathbb{P}^1$ for all i, so that the odd indexed components are exceptional; now let all the even indexed components be smooth of genus 1. Therefore, $g = d + 1$. Now choose the multidegree $\underline{d} = (1, 0, 1, \ldots, 1, 0)$ and set $L_{C_{2h}} \cong \mathcal{O}_{C_{2h}}$ for all h (of course $L_{C_{2h+1}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$). One easily checks that $\underline{d}$ is balanced. It is also clear that for any $L \in \mathrm{Pic}\, X$ the restrictions to the C_i of which are as above, we have $h^0(X, L) \geq 2d + d - 2d = d$. So Clifford's inequality fails. □

Example 4.16. *Failure of Clifford's theorem: $d \geq 3$, $\underline{d} \geq 0$, $X_{\mathrm{sep}} \neq \emptyset$.* Let $X = C_1 \cup C_2 \cup C_3$ with C_1 of genus 1 and $g_i \geq 1$. Let $C_1 \cdot C_2 = C_1 \cdot C_3 = 1$, and $C_2 \cdot C_3 = 0$ (the dual graph of X is obtained from the graph in Figure 1 by removing the vertex C_4 and the edge adjacent to it). Let $L = (L_1, \mathcal{O}_{C_2}, \mathcal{O}_{C_3}) \in \mathrm{Pic}^d X$ with $\deg L_1 = d$. Then $h^0(L) = d$. □

Example 4.17. *Failure of Clifford's theorem: $d = 5$, $\underline{d}$ balanced and $X_{\mathrm{sep}} = \emptyset$.* Let $X = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ with, for $i, j \geq 2$, $C_i \cap C_j = \emptyset$ and $C_1 \cdot C_i = 2$ for all $i \geq 2$. So every node of X lies on C_1, and $\delta = 8$ (the dual graph of X is in Figure 3). Now let h be any nonnegative integer. Let C_1 be of genus $g_1 = h$, and let C_i have genus $h + 3$ for every

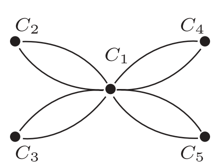

Fig. 3. Dual graph of the curve in Example 4.17.

$i \geq 2$. Hence $g = 5h + 16$. We now pick $d = 5$ and $\underline{d} = (-3, 2, 2, 2, 2)$. It is straightforward to check that $\underline{d}$ is balanced.

Now for $i \geq 2$, set $\{p_i, q_i\} = C_1 \cap C_i \subset C_i$. Let L be any line bundle the restrictions $(L_1, \ldots, L_5)$ of which are as follows. $L_1 \in \mathrm{Pic}^{-3} C_1$ is arbitrary, while $L_i = \mathcal{O}_{C_i}(p_i + q_i)$, for $i = 2, 3, 4, 5$.

Now every section s of L vanishes identically on C_1, hence s vanishes on p_i and q_i. Conversely, any quadruple of sections $s_i \in H^0(C_i, L_i(-p_i - q_i))$, for $i = 2, \ldots, 5$, glues to a section of L. We conclude $h^0(X, L) = \sum_{i=2}^{5} h^0(C_i, L_i(-p_i - q_i)) = 4$. So L violates Clifford inequality. Similar examples exist for higher degree d. □

5 Applications

If $g \geq 3$ we denote by $\overline{H}_g \subset \overline{M}_g$ the closure of the locus of hyperelliptic curves. Recall that $\overline{H}_g$ is an irreducible subscheme of dimension $2g - 1$. Following a common practice (see [11]), we say that a stable curve X is *hyperelliptic* if $[X] \in \overline{H}_g$.

Definition 5.1. We call a stable curve X *weakly hyperelliptic*, if there exists a balanced line bundle $L \in \mathrm{Pic}^2 X$ such that $h^0(X, L) \geq 2$. □

Lemma 5.2. If X is hyperelliptic, then X is weakly hyperelliptic. □

Remark 5.3. The converse is false, see Remark 5.6. □

Proof. As $[X] \in \overline{H}_g$ there exists a one parameter smoothing of X, $f : \mathcal{X} \to \mathrm{Spec} R$, the generic fiber of which is a smooth hyperelliptic curve. We can also assume that $\mathcal{X}$ is regular, and that there exists $\mathcal{L} \in \mathrm{Pic}\, \mathcal{X}$ such that the restriction of $\mathcal{L}$ to the generic fiber is the hyperelliptic bundle. Set $L = \mathcal{L}_{|X}$. Up to tensoring $\mathcal{L}$ with a divisor supported entirely on the closed fiber X we can assume that L is balanced. By uppersemicontinuity of h^0 we have $h^0(X, L) \geq 2$, so we are done. ■

5.1 Clifford index of two-components curves

Smooth hyperelliptic curves can be characterized using Clifford's inequality; the same holds for irreducible curves (see [5, Section 5]). We shall generalize this to stable curves having two components, for which we proved that Clifford's inequality holds.

The Clifford index of a line bundle has been introduced in Section 1.3. Now, if X is irreducible, its Clifford index is defined as Cliff $X = \min\{\text{Cliff } L\}$ where L varies in the set of line bundles on X such that $h^0(X, L) \geq 2$ and $h^1(X, L) \geq 2$. By Clifford's theorem, Cliff $X \geq 0$; moreover, Cliff $X = 0$ if and only if X is hyperelliptic. We extend the definition of the Clifford index to a semistable curve X as follows.

$$\text{Cliff } X = \min\{\text{Cliff } L \mid \underline{\deg} L \in B_d(X),\ h^0(X, L) \geq 2,\ h^1(X, L) \geq 2\}. \tag{38}$$

By Theorem 3.3, Cliff $X \geq 0$ if $X = C_1 \cup C_2$. We now ask: when is Cliff $X = 0$? To answer this question we use the following terminology. As in [6], a curve X (reduced, nodal, of genus g) is called a *binary curve* if it is the union of two copies of $\mathbb{P}^1$ meeting transversally in $g+1$ points.

Proposition 5.4. Let $X = C_1 \cup C_2$ be semistable.

(1) Cliff $X = 0$ if and only if X is weakly hyperelliptic.
(2) If X is weakly hyperelliptic, then $C_1 \cdot C_2 \leq 2$ unless X is a hyperelliptic binary curve. □

Proof. As we said, Theorem 3.3 yields Cliff $X \geq 0$. Therefore if X is weakly hyperelliptic, then Cliff $X = 0$.

Conversely, suppose Cliff $X = 0$; let $L \in \text{Pic}^{\underline{d}}(X)$ with $\underline{d} \in B_d(X)$, such that $h^0(L) = d/2 + 1$. If $d = 2$ there is nothing to prove, so assume $d > 2$. As usual, set $\delta = C_1 \cdot C_2$. We must prove that there exists a $J \in \text{Pic}^2 X$ such that $h^0(J) = 2$ and $\underline{\deg} J \in B_2(X)$.

• Assume first $d_i \leq 2g_i$ for $i = 1, 2$. By Corollary 3.2 we have $\delta \leq 2$.

Suppose $\delta = 2$; again by Corollary 3.2 we have Cliff $L_1 =$ Cliff $L_2 = 0$ and, if $d_i \geq 2$, then Cliff $L_i(-C_1 \cap C_2) = 0$.

If $d_1 = 0$ then $L_1 = \mathcal{O}_{C_1}$ and $L_2 = H_2^{d/2}$ for some $H_2 \in W_2^1(C_2)$ (see [5, Section 5.2]). By hypothesis $(0, d) \in B_d(X)$, which easily implies that $g_2 > g_1$, and hence that multidegree $(0, 2)$ is balanced. Consider the line bundle $M := (\mathcal{O}_{C_1}, H_2)$ on the normalization X^ν of X; as Cliff $H_2^{d/2}(-C_1 \cap C_2) = 0$ we have $h^0(C_2, H_2(-C_1 \cap C_2)) = 1$, hence by Lemma 1.4 there exists $J \in F_M(X)$ such that $h^0(X, J) = h^0(X^\nu, M) - 1 = 2$. Since $\underline{\deg} J = (0, 2)$ is balanced, we are done.

If $d_i > 0$ for $i = 1, 2$, then there exists $H_i \in W_2^1(C_i)$ such that $L_i = H_i^{d_i/2}$, for both i. Suppose $g_1 \leq g_2$; arguing as above we see that $(0, 2)$ is balanced and that there exists $J \in W^1_{(0,2)}(X)$ such that the pull-back of J to the normalization of X is $(\mathcal{O}_{C_1}, H_2)$. Up to switching C_1 and C_2, we are done.

Suppose $\delta = 1$. If $(1, 1)$ is balanced, then X is (trivially) weakly hyperelliptic (see Lemma 5.5). So assume $(1, 1)$ not balanced. By Example 2.2 we may assume $g_1 < g_2$ and $B_2(X) = \{(0, 2)\}$. By Corollary 3.2, Cliff $L_2 = 0$, therefore C_2 is hyperelliptic. Let H_{C_2} be its hyperelliptic bundle, and set $J = (\mathcal{O}_{C_1}, H_2)$; it is clear that $h^0(X, J) = 2$.

• Now assume that $d_1 = 2g_1 + e$ with $e \geq 1$. We will prove that X is a binary curve. In this case , the result is known: a binary curve is hyperelliptic if and only if it is weakly hyperelliptic [6, Section 3].

We are in the situation treated in the proof of Theorem 3.3, from which we now use the notation. We saw there that the Clifford inequality can be an equality only in Case 2, at the very end. More precisely, in order for Cliff $L = 0$ we must have $d_2 = 2g_2 + e$ (so that $d = 2g_1 + 2g_2 + 2e$) and

$$l = l_1 + l_2 - e - 1. \tag{39}$$

Now, as $d < 2g - 2$ and $g = g_1 + g_2 + \delta - 1$ we have $2(g_1 + g_2 + e) < 2(g_1 + g_2 + \delta - 2)$, hence

$$e \leq \delta - 3. \tag{40}$$

Now let $\beta := e + 1$, so that $\beta \leq \delta - 2$. Set

$$Y = \left(C_1 \coprod C_2\right) \Big/_{\{p_i = q_i,\ i = 1, \ldots, \beta\}} \stackrel{\nu}{\longrightarrow} X,$$

that is, ν is the normalization of X at $\delta - \beta$ nodes. Let $M = \nu^* L$; we have, by Lemma 1.8(ii),

$$h^0(Y, M) = l_1 + l_2 - e - 1 = l = h^0(X, L)$$

using (39). Therefore for all $i = \beta + 1, \ldots, \delta$, we have $p_i \sim_M q_i$, by Lemma 1.4. This implies that, for all $i \geq \beta + 1$, p_i is a base point of $L_1(-\sum_{j=1}^{\beta} p_j)$ and q_i is a base point of $L_2(-\sum_{j=1}^{\beta} q_j)$ (by Lemma 1.3). Now

$$\deg L_1\left(-\sum_{j=1}^{\beta} p_j\right) = 2g_1 + e - \beta = 2g_1 - 1, \quad \deg L_2\left(-\sum_{j=1}^{\beta} q_j\right) = 2g_2 + e - \beta = 2g_2 - 1.$$

If X is not a binary curve, we may assume $g_2 \geq 1$. Then, $L_2(-\sum_{j=1}^{\beta} q_j)$, having degree $2g_2 - 1$, can have at most one base point. Therefore $\delta - \beta \leq 1$, that is, $\delta - e \leq 2$, which is in contradiction with (40). We conclude that X is a binary curve. ■

5.1.1 *Curves of compact type*

For any integer h with $1 \le h \le g/2$, let Δ_h be the divisor in $\overline{M}_g$ the general point of which represents a curve $X = C_1 \cup C_2$ with C_i smooth, $C_1 \cdot C_2 = 1$ and $g_1 = h$. Fix such an X; for $i = 1, 2$ we shall denote by $q_i \in C_i$ the branches of the node of X. We computed $B_2(X)$ in Example 2.2.

Lemma 5.5. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$ and $1 \le g_1 \le g/2$.

Let $g_1 \ge (g+1)/4$. Then X is weakly hyperelliptic; more precisely, $(1,1)$ is balanced and $W^1_{(1,1)}(X) = \{(\mathcal{O}_{C_1}(q_1), \mathcal{O}_{C_2}(q_2)\}$.

Let $g_1 < (g+1)/4$. Then X is weakly hyperelliptic if and only if C_2 is hyperelliptic, if and only if $W^1_{(0,2)}(X) = \{(\mathcal{O}_{C_1}, H_{C_2})\}$. □

Proof. Set $L = (\mathcal{O}_{C_1}(q_1), \mathcal{O}_{C_2}(q_2)) \in \operatorname{Pic} X$. It is clear that $h^0(X, L) = 2$. If $g_1 \ge (g+1)/4$, then L is balanced. Conversely, let $L' \in W^1_{(1,1)}(X)$; by Corollary 3.2 we have $L' = (\mathcal{O}_{C_1}(q_1), \mathcal{O}_{C_2}(q_2))$, so the first part is proved.

Now suppose $g_1 < (g+1)/4$, then $(0, 2)$ is the unique balanced multidegree. If C_2 is hyperelliptic, the balanced line bundle $L = (\mathcal{O}_{C_1}, H_{C_2}) \in \operatorname{Pic} X$ has, of course, $h^0(X, L) = 2$. So, X is weakly hyperelliptic. Conversely, if there exists $L \in \operatorname{Pic}^{(0,2)} X$ such that $h^0(L) = 2$, we can apply Corollary 3.2 (we necessarily have $g_2 \ge 3$ by hypothesis) and conclude that $h^0(C_2, L_2) = 2$, so we are done. ■

Remark 5.6. The previous result shows that there exist (plenty of) weakly hyperelliptic curves that are not hyperelliptic. Indeed, it is well known that a curve of compact type $X = C_1 \cup C_2$ is hyperelliptic if and only if both C_1 and C_2 are hyperelliptic, and the two branches, q_1 and q_2, are Weierstrass points (cf. [8] for example). Also, there exist globally generated balanced line bundles $L \in W^1_2(X)$ which are not limits of hyperelliptic bundles of smooth curves (indeed $(\mathcal{O}_{C_1}, H_{C_2})$ is always globally generated). □

5.2 **Hyperelliptic and weakly hyperelliptic curves**

The next definition will be used only when $X_{\text{sep}} = \emptyset$.

Definition 5.7. A pair (C, D) of smooth, rational components of X is called a *binary-pair*, or a *$\mathcal{B}$-pair* for short, of X if C is a separating line of D^c and D is a separating line of C^c. Abusing terminology, the subcurve $C \cup D \subset X$ will be also called a $\mathcal{B}$-pair. □

Example 5.8. Let X be a binary curve (defined before Proposition 5.4); then its irreducible components form a $\mathcal{B}$-pair. Also, if $X' = C \cup D \cup E_1 \cup \cdots \cup E_s$ is a semistable curve the stabilization of which is a binary curve $X = C \cup D$, then (C, D) is a $\mathcal{B}$-pair of X'. □

Let (C, D) be a binary pair of X. Set $C \cap D = \{n_1, \dots, n_l\}$, with $l \geq 0$, and $q_C^i \in C$, $q_D^i \in D$ the two branches of n_i. If $C \cup D \neq X$, there is a decomposition $X = (C \cup D) \cup (Z_1 \coprod \dots \coprod Z_m)$ where Z_j are connected and $Z_j \cdot C = Z_j \cdot D = 1$ for all j. Write $p_C^j = C \cap Z_j$ and $p_D^j = D \cap Z_j$. Let $n = l + m$ ($m \geq 0$); now the ordered n-tuples

$$G_C := (q_C^1, \dots, q_C^l, p_C^1 \dots, p_C^m) \subset C, \quad G_D := (q_D^1, \dots, q_D^l, p_D^1 \dots, p_D^m) \subset D \tag{41}$$

give a structure of n-marked curve on C and D. We say that (C, D) is a *special $\mathcal{B}$-pair* if $(C; G_C)$ and $(D; G_D)$ are isomorphic as n-marked curves.

The next theorem is already known for irreducible curves; see [5, Proposition 5.2.1].

Theorem 5.9. Let X be semistable with $X_{\text{sep}} = \emptyset$; let $\underline{d}$ be such that $|\underline{d}| = 2$. Assume that $\underline{d}$ is balanced, or that X is stable and $\underline{d} \geq 0$. Suppose there exists $L \in \text{Pic}^{\underline{d}} X$ with $h^0(X, L) = 2$.

Then L is globally generated, and one of the two cases below occurs.

(1) $\underline{d} = (1, 1, 0, \dots, 0)$ and (C_1, C_2) is a special $\mathcal{B}$-pair of X. Also, the restriction of L to $\overline{X \smallsetminus (C_1 \cup C_2)}$ is trivial.

(2) $\underline{d} = (2, 0, \dots, 0)$ and, denoting $C_1^c = Z_1 \coprod \cdots \coprod Z_m$, with Z_i connected, $\forall i = 1, \dots, m$ we have

$$C_1 \cdot Z_i = 2, \quad L_{C_1} \cong \mathcal{O}_{C_1}(C_1 \cap Z_i), \quad L_{C_1^c} \cong \mathcal{O}_{C_1^c} \quad \text{and} \quad h^0(C_1, L_{C_1}) \geq 2.$$

Furthermore, if $C_1 \cong \mathbb{P}^1$, then we have $m \geq 2$ and, setting $\{p_i, q_i\} = Z_i \cap C_1 \subset C_1$, there exists a g_2^1, Λ, on C_1 such that $p_i + q_i$ is a divisor in Λ for every $i = 1, \dots, m$ (of course, $\Lambda \subset |\mathcal{O}_{C_1}(2)|$).

Conversely, if X and $\underline{d}$ satisfy either (1) or (2) above, there exists a unique line bundle $L \in \text{Pic}^{\underline{d}} X$ such that $W_{\underline{d}}^1(X) = \{L\}$. □

Proof. Assume that there exists $L \in W^1_{\underline{d}}(X)$; by Theorem 4.11(ii) and its addendum we obtain $\underline{d} \geq 0$, that is, $\underline{d}$ is as in (1) or (2). We will prove that L is globally generated as a consequence of (1) and (2). To ease the notation, we write $C = C_1$ and $D = C_2$.

Case 1. $\underline{d} = (1, 1, 0, \ldots, 0)$.

Suppose that C is a nondisconnecting component; set $Z = C^{\mathrm{c}}$. We first prove that (C, D) is a special $\mathcal{B}$-pair of X.

By contradiction, suppose D is not a separating line of Z; by Lemma 4.6 we have $h^0(Z, L_Z) \leq 1$. Let $C \ncong \mathbb{P}^1$, then $h^0(C, L_C) \leq 1$. So, in order to have $h^0(X, L) = 2$ we must have $h^0(C, L_C) = h^0(Z, L_Z) = 1$ and every point in $Z \cap C \subset C$ must be a base point for L_C (by Lemma 1.4). This is impossible, as $Z \cdot C \geq 2$ and $d_C = 1$. Now let $C \cong \mathbb{P}^1$, hence $h^0(C, L_C) = 2$. By Lemma 1.8 we have

$$h^0(X, L) \leq h^0(C, L_C) + h^0(Z, L_Z) - 2 \leq 2 + 1 - 2 = 1,$$

a contradiction.

Therefore, D is a separating line of Z and $h^0(Z, L_Z) = 2$. By Remark 4.5(B), D is a nondisconnecting component of X. Hence we can apply the previous argument replacing C by D; this yields that C is a separating line of D^{c}. In other words, (C, D) is a $\mathcal{B}$-pair of X.

We claim that the restriction of L to $(C \cup D)^{\mathrm{c}}$ is trivial. By contradiction, suppose $(C \cup D)^{\mathrm{c}}$ has a connected component, W, such that $h^0(W, L_W) = 0$. As (C, D) is a $\mathcal{B}$-pair we have $\#(W \cap C) = \#(W \cap D) = 1$; set $p_C = C \cap W$ and $p_D = D \cap W$; every section of L vanishes at p_C and p_D. On the other hand, L_C and L_D are free from base points, of course; hence, writing $X' = C \cup W \cup D$, we have $h^0(X', L_{X'}) \leq h^0(C, L_C) - 1 + h^0(D, L_D) - 1 = 2 = h^0(X, L)$. Therefore $h^0(X', L_{X'}) = 2$, which yields that $C \cap D = \emptyset$. Also, by Lemma 1.4 we easily get that $X = X'$; this is impossible, as $X_{\mathrm{sep}} = \emptyset$ whereas X' has separating nodes at $W \cap C$ and $W \cap D$. Our claim is proved.

Therefore, L determines a map $\psi : X \to \mathbb{P}^1$ such that $\psi(p^j_C) = \psi(p^j_D)$ for all j (notation as in (41)). Hence ψ induces an isomorphism of the n-marked curves C and D with the same n-marked $\mathbb{P}^1$. This shows that the $\mathcal{B}$-pair (C, D) is special.

Suppose now that both C and D disconnect X. We will prove that this case does not occur. Denote by $D^{\mathrm{c}} = Y_1 \coprod \cdots \coprod Y_m$ the connected components decomposition, so that $m \geq 2$. Let Y_1 be such that $C \subset Y_1$, so that $d_{Y_1} = 1$. Note that C is not a separating line of Y_1. Indeed, if C were a separating line of Y_1, then every connected component, V, of $\overline{Y_1 \smallsetminus C}$ must intersect D, for otherwise X has a separating node at $V \cap C$. But then C is a nondisconnecting component of X, contradicting our assumption. Therefore $h^0(Y_1, L_{Y_1}) \leq 1$, by Lemma 4.6.

Set $X_1 = D \cup Y_1 \subset X$; note that $D \cdot Y_1 \geq 2$.

If $D \cong \mathbb{P}^1$, by Lemma 1.8 we have $h^0(X_1, L_{X_1}) \leq 2 + 1 - 2 = 1 < h^0(X, L)$. By Remark 1.7, this is a contradiction.

If $D \ncong \mathbb{P}^1$, then $h^0(D, L_D) \leq 1$ and if equality holds L_D has at most one base point. It is clear that $h^0(X, L) = 2$ forces $h^0(D, L_D) = 1$ We, therefore, have $h^0(X_1, L_{X_1}) \leq h^0(D, L_D) + h^0(Y_1, L_{Y_1}) - 1 = 1 + 1 - 1 < h^0(X, L)$ (by Remark 1.2(B) and Lemma 1.4); a contradiction. Case 1 is complete.

Case 2. $\underline{d} = (2, 0, \ldots, 0)$.

Recall that $C \subset X$ is the component such that $d_C = 2$; set $Z = C^c$. Suppose first that Z is connected. Assume $C \ncong \mathbb{P}^1$. So $h^0(C, L_C) \leq 2$ with equality only if L_C has no base point; also, $h^0(Z, L_Z) \leq 1$ with equality if and only if $L_Z = \mathcal{O}_Z$ (by Fact 1.6). It is clear that, for $h^0(X, L) = 2$, we must have equality in both cases. Hence $h^0(C, L_C) = 2$ and $L_Z = \mathcal{O}_Z$. If $C \cdot Z \geq 3$, by Lemma 1.5 there exist three points $p, q, r \in C \cap Z \subset C$ such that $p \sim_{L_C} q \sim_{L_C} r$. Now L_C has no base points, hence we get

$$1 = h^0(C, L_C) - 1 = h^0(C, L_C(-p)) = h^0(C, L_C(-p - q - r))$$

which is impossible, as $\deg L_C(-p - q - r) = -1$. We thus proved that $C \cdot Z = 2$; set $C \cap Z = \{p, q\} \subset C$, arguing similarly we see that $h^0(C, L_C(-p - q)) = 1$, that is, $L_C = \mathcal{O}_C(p + q)$. Observe that Lemma 1.4 yields that L is unique.

Now let us prove that $C \ncong \mathbb{P}^1$. By contradiction, suppose $C \cong \mathbb{P}^1$. Note that X is a stable curve (an exceptional component must have degree 1), hence $\delta_C \geq 3$. By Lemma 1.8 we obtain (Z is connected)

$$h^0(X, L) \leq h^0(C, L_C) + h^0(Z, L_Z) - 3 \leq 3 + 1 - 3 = 1 \tag{42}$$

which is impossible.

Suppose now that $Z = Z_1 \coprod \cdots \coprod Z_m$ with Z_i connected and $m \geq 2$. For every $i = 1, \ldots, m$ set $X_i = C \cup Z_i$. If $C \ncong \mathbb{P}^1$, then we argue as in the previous part with X_i playing the role of X and Z_i (which is connected) playing the role of Z. This shows that L is unique and that for every i, C intersects Z_i in two points $p_i, q_i \in C$, that $L_C \cong \mathcal{O}_C(p_i + q_i)$, and that $L_{Z_i} \cong \mathcal{O}_{Z_i}$. If $C \cong \mathbb{P}^1$, then we have ($X_{\text{sep}} = \emptyset$) $C \cdot Z_i \geq 2$ for every i. We must prove that equality holds for every i. Indeed, if $C \cdot Z_i \geq 3$ we can argue as we did in (42), with Z replaced by Z_i and X replaced by X_i. We obtain $h^0(X_i, L_i) \leq 1$, which is impossible, by

Remark 1.7. We clearly have $L_C \cong \mathcal{O}_C(C \cap Z_i)$; now, the fact that as i varies from 1 to m, the divisors $p_i + q_i$ move in the same g^1_2, $\Lambda \subset |\mathcal{O}_C(2)|$, follows easily from Lemma 1.4 (or, from [5, Proposition 5.2.1] using the map σ described below). The remaining assertions of the theorem are clear, so Case 2 is proved.

It remains to show that L is globally generated. Let $X' \subset X$ be the maximal subcurve where $L_{X'} \cong \mathcal{O}_{X'}$. By what we proved above, it is clear that L has no base point at smooth points of X, or along X'. We need to show that L has no base point in $C \cap D$ in case (1), or in singular points of C in case 2. The latter case is clear, as C is irreducible and L_C is globally generated because $h^0(C, L_C) \geq 2$ (see the beginning of Section 1.3 for the case $C \ncong \mathbb{P}^1$). In case (1), if L has a base point $n \in C \cap D$ we get, denoting by $p_C \in C$ and $p_D \in D$ the branches of n,

$$h^0(C \cup D, L_{C \cup D}) \leq h^0(C, L_C(-p_C)) + h^0(D, L_D(-p_D)) = 2 = h^0(X, L),$$

which is easily ruled out using Lemmas 1.3 and 1.4.

Now the converse. In case 1 the statement holds if $X = C \cup D$ (i.e., X is a binary curve) by Lemma 1.4 (existence) and [6, Lemma 15] (uniqueness). In the general case, let $\sigma : X \to \overline{X}$ be the morphism contracting every connected component of $(C \cup D)^c$ to a node of $\overline{X}$ and mapping C and D isomorphically onto their image, so that $\overline{X}$ is a binary curve. The pull-back map $\sigma^* : \mathrm{Pic}\, \overline{X} \to \mathrm{Pic}\, X$ induces a bijection between line bundles on $\overline{X}$ and line bundles on X that are trivial on $(C \cup D)^c$. It is clear that this bijection preserves h^0. So the statement holds on X because it holds on $\overline{X}$. In case 2, existence follows from Lemma 1.4, and uniqueness has already been proved when $C \ncong \mathbb{P}^1$. If $C \cong \mathbb{P}^1$, then we proceed as before: let $\sigma : X \to \overline{X}$ be the map contracting every connected component of C^c to a node, and mapping C birationally onto its image, so that $\overline{X}$ is an irreducible nodal curve. Since for $\overline{X}$ the statement holds, it also holds for X. The proof is complete. ∎

Let X be a curve free from separating nodes. By Lemma 4.10, every irreducible component of X belongs to at most one $\mathcal{B}$-pair. Therefore we have the following.

Remark 5.10. Let X be a stable curve such that $X_{\text{sep}} = \emptyset$. Then X admits a decomposition, unique up to the order, $X = A_1 \cup \cdots \cup A_\alpha$ such that every A_i is either a $\mathcal{B}$-pair or an irreducible component of X not part of any $\mathcal{B}$-pair. □

We shall now apply the previous theorem to describe the combinatorics of hyperelliptic stable curves.

Proposition 5.11. Let X be a hyperelliptic stable curve such that $X_{\text{sep}} = \emptyset$. Consider the decomposition $X = A_1 \cup \cdots \cup A_\alpha$ defined in Remark 5.10. Then for every $i \neq j$ we have either $A_i \cap A_j = \emptyset$, or

$$A_i \cdot A_j = 2 \quad \text{and} \quad h^0(A_i, \mathcal{O}_{A_i}(A_i \cap A_j)) \geq 2. \qquad \square$$

Proof. We begin as in the proof of Lemma 5.2. Let $f : \mathcal{X} \to B$ be a one-parameter smoothing of X with $\mathcal{X}$ regular and hyperelliptic generic fiber. Let $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ be a balanced line bundle such that the restriction of $\mathcal{L}$ to the generic fiber is the hyperelliptic bundle, set $\mathcal{L}_{|X} = L$. By assumption $\underline{d} := \underline{\deg} L$ is balanced; moreover $h^0(X, L) \geq 2$ hence we may apply Theorem 5.9 to L. This enables us to write $X = A \cup (Z_1 \coprod \cdots \coprod Z_m)$, where either A is an irreducible component with $\underline{d}_A = 2$ or A is a special $\mathcal{B}$-pair with $\underline{d}_A = (1, 1)$. By Theorem 5.9 we have

$$\underline{d}_{Z_i} = \underline{0}, \quad Z_i \cdot A = 2, \quad h^0(A, \mathcal{O}_A(A \cap Z_i)) \geq 2 \quad \forall i = 1, \ldots, m;$$

moreover, if A is a $\mathcal{B}$-pair we have $\underline{\deg}_A Z_i = (1, 1)$. Comparing with the decomposition in Remark 5.10, we may set $A = A_1$.

Now, for any divisor $T \in \operatorname{Div} \mathcal{X}$ supported on X, we set $L_T := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(T) \otimes \mathcal{O}_X$. We have $\deg L_T = 2$ and, by uppersemicontinuity of h^0, $h^0(X, L_T) \geq 2$.

Consider L_T with $T = -Z_1$. We claim that $\underline{\deg} L_T \geq 0$. Indeed, let $C \subset X$ be an irreducible component (or a subcurve); by the previous discussion,

$$\deg_C L_T = \begin{cases} -\deg_C Z_1 = C \cdot A_1 \geq 0 & \text{if } C \subset Z_1, \\ 0 & \text{if } C \subset Z_1^c. \end{cases}$$

We can now apply Theorem 5.9 to L_T. Since $\underline{\deg}_{A_1} L_T = \underline{0}$ and $\underline{\deg}_{Z_i} L_T = \underline{0}$ for all $i \neq 1$, we derive that Z_1 contains one of the subcurves, A_2 say, of the decomposition in Remark 5.10. So, A_2 is either irreducible or a $\mathcal{B}$-pair, and $\deg_{A_2} L_T = 2$; therefore, by the above discussion, $A_1 \cdot A_2 = 2$. Hence $A_1 \cap Z_1 = A_1 \cap A_2$ and $h^0(A_1, \mathcal{O}_{A_1}(A_1 \cap A_2)) \geq 2$. Thus, the part of the statement concerning A_1 and A_2 is proved. If $A_2 = Z_1$, we pick Z_i with $i \geq 2$ and repeat the procedure with $T = -Z_i$. If $A_2 \subsetneq Z_1$, we iterate the procedure with A_2

playing the role of A_1 and with $T = -W$, where W is a connected component of $\overline{Z_1 \smallsetminus A_2}$. Obviously this iteration stops after finitely many steps, after which we are done. ■

5.3 Curves of genus 6 admitting a g^2_5

5.3.1

Throughout this section we shall consider curves $X = C_1 \cup C_2$, of genus 6, such that C_1 and C_2 are smooth, of respective genus g_1 and g_2; we set $\delta = C_1 \cdot C_2$. For any $L \in \operatorname{Pic} X$ we write $L_i = L_{|C_i}$ and $l_i = h^0(L_i) = h^0(C_i, L_i)$. We fix points $p_1, \dots, p_\delta \in C_1$ and $q_1, \dots, q_\delta \in C_2$ so that $X = (C_1 \coprod C_2)/_{(p_i = q_i,\ i=1,\dots,\delta)}$ and set

$$G_1 := \sum_{i=1}^{\delta} p_i, \quad G_2 := \sum_{i=1}^{\delta} q_i. \tag{43}$$

Finally, we set $\underline{g} := (g_1, g_2)$, and we always assume $g_1 \leq g_2$.

Theorem 5.12. With the above set-up, let $X = C_1 \cup C_2$ be semistable of genus 6, and let $\underline{d} \in B_5(X)$. Assume there exists a globally generated $L \in W^2_{\underline{d}}(X)$. Then

(I) If $\delta = 1$, C_2 is not hyperelliptic and one of the following cases occurs.

- (a) $\underline{g} = (1, 5)$, $\underline{d} = (0, 5)$, $L_1 = \mathcal{O}_{C_1}$, and $h^0(L_2) = 3$.
- (b) $\underline{g} = (2, 4)$ or $\underline{g} = (3, 3)$, $\underline{d} = (2, 3)$, and $h^0(L_1) = h^0(L_2) = 2$.

(II) If $\delta = 2$ one of the following cases occurs.

- (a) $\underline{g} = (0, 5)$, $\underline{d} = (1, 4)$, C_2 hyperelliptic, $L_2 = H_{C_2}^{\otimes 2}$.
- (b) $\underline{g} = (1, 4)$, $\underline{d} = (0, 5)$, $L_1 = \mathcal{O}_{C_1}$, C_2 not hyperelliptic, $h^0(L_2) = 3$.
- (c) $\underline{g} = (2, 3)$, $\underline{d} = (2, 3)$, $L_1 = H_{C_1} = \mathcal{O}_{C_1}(G_1)$, C_2 not hyperelliptic, $L_2 = \mathcal{O}_{C_2}(G_2 + q)$ and $h^0(L_2) = 2$.
- (d) $\underline{g} = (1, 4)$ or $\underline{g} = (2, 3)$, $\underline{d} = (2, 3)$, $L_1 = \mathcal{O}_{C_1}(G_1)$, C_2 not hyperelliptic, $L_2 = \mathcal{O}_{C_2}(G_2 + q)$ and $h^0(L_1) = h^0(L_2) = 2$.

(III) If $\delta = 3$ then $\underline{g} = (1, 3)$ and one of the following cases occurs.

- (a) $\underline{d} = (3, 2)$, $L_1 = \mathcal{O}_{C_1}(G_1)$, C_2 is hyperelliptic, $L_2 = H_{C_2}$.
- (b) $\underline{d} = (0, 5)$, $L_1 = \mathcal{O}_{C_1}$, and $h^0(L_2) = 3$.

(IV) If $\delta = 4$, then $\underline{g} = (0, 3)$, $\underline{d} = (1, 4)$ and $L_2 = K_{C_2} = \mathcal{O}_{C_2}(G_2)$.

(V) If $\delta = 6$, then $\underline{g} = (0, 1)$, $\underline{d} = (2, 3)$. □

Remark 5.13. The cases (I) and (II), that is, $\delta \leq 2$, are contained in Propositions 5.15 and 5.16, where a more precise statement is proved. □

Proof. Our curve X has a priori $\delta \leq 7$ nodes. The case that $\delta = 7$, that is, X is a binary curve, is ruled out as follows. Proposition 12 in [6] implies $\underline{\deg} L = (2, 3)$; by [6, Proposition 19 and Lemma 20] the curve X must be hyperelliptic. Therefore, the canonical morphism maps X two-to-one onto a rational normal quintic in $\mathbb{P}^5$. Now, we argue as for smooth curves (cf. [2, Ex. D-9, p. 41]): we have $h^0(X, \omega_X \otimes L^{-1}) = 3$, hence (as points on a rational normal curve are in general linear position) we easily get $L \cong H_X^{\otimes 2}(p)$ with $p \in X$ a base point of L. So L is not globally generated, and we are done.

From now on, by Remark 5.13, we assume $3 \leq \delta \leq 6$.

Pick $\underline{d}$ and $L \in W^2_{\underline{d}}(X)$ as in the statement. The fact that $\underline{d}$ is balanced means

$$g_i - 1 \leq d_i \leq g_i - 1 + \delta, \quad i = 1, 2, \tag{44}$$

and $d_i = 1$ if C_i is an exceptional component.

First of all, let us show that $\underline{d} \geq 0$. If $d_1 < 0$ we must have $\underline{d} = (-1, 6)$, and $g_1 = 0$. We have $h^0(X, L) = h^0(C_2, L_2(-\sum_{i=1}^{\delta} q_i)) \leq 2$, because $\deg L_2(-\sum_{i=1}^{\delta} q_i) = 6 - \delta$. This contradiction shows that $d_i \geq 0$ for $i = 1, 2$.

For $i = 1, 2$ we set $e_i := d_i - 2g_i$. Let

$$\epsilon := \max\{e_1, e_2, 0\} + 1 \quad \text{and} \quad \beta := \min\{\epsilon, \delta\}.$$

From Addendum 3.4 we have

$$h^0(X, L) \leq l_1 + l_2 - \beta \leq 3. \tag{45}$$

Step 1. We exclude all the cases for which $l_1 + l_2 - \beta \leq 2$. This only requires a trivial checking. To begin with, the following cases are all excluded:

$$\delta = 6, \quad \underline{g} = (0, 1), \quad \underline{d} \in \{(0, 5),\ (3, 2),\ (4, 1),\ (5, 0)\}. \tag{46}$$

Let us just show how to treat $\underline{d} = (0, 5)$. We have $l_1 = 1$, $l_2 = 5$, $\epsilon = e_2 + 1 = 4$, and $\beta = \min\{4, 6\} = 4$. Hence $h^0(X, L) \leq 2$. All other cases are treated in the same way. If $\delta = 6$, then we are left with $\underline{d} = (1, 4)$ and $\underline{d} = (2, 3)$ (of course $\underline{g} = (0, 1)$).

Let $\delta = 5$, by the same argument, we exclude

$$\delta = 5, \quad \underline{g} = (0, 2), \quad \underline{d} \in \{(2, 3)\ (3, 2),\ (4, 1),\ (5, 0)\} \tag{47}$$

and we exclude

$$\delta = 5, \quad \underline{g} = (1, 1), \quad \underline{d} \in \{(0, 5),\ (1, 4)\}. \tag{48}$$

Let $\delta = 4$. We exclude

$$\delta = 4, \quad \underline{g} = (0, 3), \quad \underline{d} \in \{(2, 3),\ (3, 2)\}. \tag{49}$$

and

$$\delta = 4, \quad \underline{g} = (1, 2), \quad \underline{d} = (4, 1). \tag{50}$$

Finally, this method applies to exclude

$$\delta = 3, \quad \underline{g} = (0, 4), \quad \underline{d} = (2, 3). \tag{51}$$

This finishes the list of cases for which $l_1 + l_2 - \beta \leq 2$.

From now on we always have $l_1 + l_2 - \beta = 3$ (by (45)).

Step 2. To exclude another group of cases we now use Lemma 1.3 and its consequence, Lemma 5.14. Let us begin with case $\delta = 6$, hence $\underline{g} = (0, 1)$ and $\underline{d} = (1, 4)$. In this case $\beta = 3$, so that we obviously have

$$3 = \beta < d_2 = 4 < \delta = 6. \tag{52}$$

Let $X' = (C_1 \coprod C_2)/_{\{p_i = q_i,\ i=1,\dots,3\}}$, let $\nu : X' \to X$ be the same map as in Lemma 5.14 and let $M = \nu^* L$. Then $h^0(X', M) = 3$ (by Lemma 1.8(ii), or by Clifford). By (52) Lemma 5.14 applies, yielding that $h^0(X, L) < 3$, a contradiction.

- By (46) if $\delta = 6$ the only remaining case is $\underline{d} = (2, 3)$. (V) is proved.

The previous argument can be repeated every time we have $\beta < d_i < \delta$ for some i, enabling us to exclude the following cases.

$\delta = 5$, $\underline{g} = (0, 2)$, and $\underline{d} = (1, 4)$. (Here $2 = \beta < d_2 = 4 < \delta = 5$.)

$\delta = 5$, $\underline{g} = (1, 1)$, and $\underline{d} = (2, 3)$. (Here $2 = \beta < d_2 = 3 < \delta = 5$.)

$\delta = 4$, $\underline{g} = (1, 2)$, and $\underline{d} \in \{(2, 3,)(3, 2)\}$ (if $\underline{d} = (2, 3)$ then $1 = \beta < d_2 = 3 < \delta = 4$; if $\underline{d} = (3, 2)$ then $2 = \beta < d_1 = 3 < \delta = 4$.)

We shall now exclude the two equal multidegree cases

$$\delta = 5, \quad \underline{g} = (0, 2), \quad \underline{d} = (0, 5) \quad \text{and} \quad \delta = 4, \quad \underline{g} = (1, 2), \quad \underline{d} = (0, 5),$$

with $l_1 + l_2 = 5$. Let $X' = (C_1 \coprod C_2)/(p_i = q_i, i = 1, 2)$ so that X' has two nodes. Let $L' \in \operatorname{Pic} X'$ be the pull back of L. Then $h^0(X', L') = 3$, so, for $h^0(X, L) = 3$ we must have $q_i \sim_{L'} p_i$ for $i \geq 3$. Now, by Lemma 1.3, this implies that $L_2(-q_1 - q_2)$ has at least two base points, which is clearly impossible.

• By Step 2, (47), and (48) there are no more cases with $\delta = 5$.

Step 3. Now we shall use Corollary 1.10 to exclude all the cases for which $l_1 + l_2 = 4$ and there is $i \in \{1, 2\}$ such that $l_i \geq 2$ and $\delta > \operatorname{Cliff} L_i + 2$. This amounts to the following list of cases.

$\delta = 4$, $\underline{g} = (0, 3)$, and $\underline{d} = (0, 5)$. $l_2 = 3$, and $\operatorname{Cliff} L_2 = 1$.

$\delta = 4$, $\underline{g} = (1, 2)$, and $\underline{d} = (1, 4)$. $l_2 = 3$, and $\operatorname{Cliff} L_2 = 0$.

By the previous step and (49) the only case left with $\delta = 4$ is $\underline{g} = (0, 3)$ and $\underline{d} = (1, 4)$. Now $\beta = 2$, therefore (as $l_1 + l_2 - 2 = 3$ by (45)) we have $l_2 = 3$, that is, L_2 is the canonical bundle of C_2. To prove that $L_2 = \mathcal{O}_{C_2}(\sum_1^4 q_i)$ it suffices to prove that $L_2(-q_1 - q_2)$ has q_3 and q_4 as base points (and note that we are free to permute the q_i). We argue as at the end of Step 2: let $X' = (C_1 \coprod C_2)/_{(p_i = q_i, i=1,2)}$ and let L' be the pull back of L to X'. Then $h^0(X', L') = 3 = h^0(X, L)$, so, $L_2(-q_1 - q_2)$ has q_3 and q_4 as base points.

• (IV) is proved.

$\delta = 3$, $\underline{g} = (1, 3)$. We exclude $\underline{d} = (1, 4)$ (as $l_2 = 3$ and $\operatorname{Cliff} L_2 = 0$), and $\underline{d} = (2, 3)$ (as $l_1 = 2$ and $\operatorname{Cliff} L_1 = 0$).

$\delta = 3$, $\underline{g} = (2, 2)$. We exclude $\underline{d} = (1, 4)$ (as $l_2 = 3$ and $\operatorname{Cliff} L_2 = 0$), and $\underline{d} = (2, 3)$ (as $l_1 = 2$ and $\operatorname{Cliff} L_1 = 0$).

Step 4. From now on we assume $\delta = 3$.

Let $\underline{g} = (2, 2)$ and $\underline{d} = (2, 3)$. Now $l_1 + l_2 = 4$ if and only if $L_2 = H_{C_2}(p)$. So L_2 has a base point, which is impossible by hypothesis. By Step 3, there are no more balanced multidegrees to treat when $\underline{g} = (2, 2)$.

Let $\underline{g} = (0, 4)$. By (49) there are two cases to rule out: $\underline{d} = (0, 5)$ and $\underline{d} = (1, 4)$.

Let $\underline{d} = (0, 5)$. As $l = 3$ we have $l_1 + l_2 = 1 + 3 = 4$. It is clear that Lemma 1.5 applies, giving $q_1 \sim_{L_2} q_2 \sim_{L_2} q_3$. Therefore, if $1 \leq i \neq j \leq 3$, we have

$$2 = h^0(C_2, L_2(-q_i)) = h^0(C_2, L_2(-q_i - q_j)) = h^0(C_2, L_2(-q_1 - q_2 - q_3)).$$

But then C_2 is hyperelliptic ($\deg L_2(-q_1-q_2-q_3)=2$), which implies that L_2 has a base point. A contradiction.

Let $\underline{d}=(1,4)$. As $\beta=2$ and $l=3$ we have $l_1+l_2=2+3$, so C_2 is hyperelliptic and $L_2=H_{C_2}^{\otimes 2}$. Consider $X'=(C_1\coprod C_2)/_{(p_i=q_i,\ i=1,2)}\stackrel{\nu}{\longrightarrow} X$ and let $M=\nu^*L$. Then $h^0(X',M)=3$, therefore $p_3\sim_M q_3$. By Lemma 1.3 we obtain that q_3 is a base point of $L_2(-q_1-q_2)$, hence (permuting the gluing points) $H_{C_2}\neq\mathcal{O}_{C_2}(q_i+q_j)$ for all $i\neq j$. So, $L_2(-q_1-q_2)=\mathcal{O}_{C_2}(q_1'+q_2')$ where q_1' is conjugate to q_1 under the hyperelliptic series, and the same for q_2' and q_2. But then, as q_3 is a base point of $L_2(-q_1-q_2)=\mathcal{O}_{C_2}(q_1'+q_2')$, we get that (say) $q_3=q_1'$, which is a contradiction.

• By Step 3, the remaining cases with $\delta=3$ have $\underline{g}=(1,3)$ and either $\underline{d}=(3,2)$ or $\underline{d}=(0,5)$. This is (III). ■

Lemma 5.14. Let δ and β be two positive integers with $\delta>\beta$. Consider the partial normalization of X defined as follows:

$$X'=\left(C_1\coprod C_2\right)\Big/_{\{p_i=q_i,\ i=1,\dots,\beta\}}\stackrel{\nu}{\longrightarrow} X=\left(C_1\coprod C_2\right)\Big/_{\{p_i=q_i,\ i=1,\dots,\delta\}}.$$

For $i=1,2$, pick $L_i\in\operatorname{Pic}C_i$ and $M\in\operatorname{Pic}(X')$ such that $M_{|C_i}=L_i$.

If $\beta<\deg L_i<\delta$ for some i, then $h^0(X,L)<h^0(X',M)$ for every $L\in F_M(X)$. □

Proof. We argue by contradiction, as follows. We prove that if $\beta<\deg L_1$, and if there exists $L\in F_M(X)$ such that $h^0(X,L)=h^0(X',M)$, then $\deg L_1\geq\delta$.

Let such an L be fixed. By Lemma 1.4 we have $p_i\sim_M q_i$ for all $i=\beta+1,\dots,\delta$. Now Lemma 1.3 yields that, for all $i\geq\beta+1$, p_i is a base point of $L_1(-\sum_{j=1}^{\beta}p_j)$.

As $\deg L_1>\beta$, $\deg L_1(-\sum_{j=1}^{\beta}p_j)\geq 1$. Now, a line bundle of positive degree can have at most as many base points as its degree. We just proved that $L_1(-\sum_{j=1}^{\beta}p_j)$ has $\delta-\beta$ base points, hence $\deg L_1-\beta\geq\delta-\beta$, that is, $\deg L_1\geq\delta$. We are done. ■

Proposition 5.15. With the set up of Section 5.3.1, let $X=C_1\cup C_2$ be semistable of genus 6, with $C_1\cdot C_2=1$, and let $\underline{d}\in B_5(X)$.

There exists a globally generated $L\in W^2_{\underline{d}}(X)$ if and only if C_2 is not hyperelliptic and one of the following cases occurs.

(1) $\underline{g}=(1,5)$, $\underline{d}=(0,5)$, and $L=(\mathcal{O}_{C_1},L_2)$ for some $L_2\in W^2_5(C_2)$.

(2) $\underline{g}=(2,4)$ or $\underline{g}=(3,3)$, $\underline{d}=(2,3)$, C_1 is hyperelliptic and $L=(H_{C_1},L_2)$ for some $L_2\in W^1_3(C_2)$. □

Proof. As X is semistable we have $g_1 \geq 1$. If L is globally generated, so are L_1 and L_2; hence if $h^0(X, L) = 3$ we have $3 = l_1 + l_2 - 1$ by Lemma 1.4. Therefore, $l_1 + l_2 = 4$.

Case $\underline{g} = (1, 5)$. The balanced multidegrees are $(0, 5)$ and $(1, 4)$. If $\underline{d} = (1, 4)$ and $l_1 = 1$ then L_1 has a base point, which is not possible. If $l_1 = 0$, then $h^0(X, L) \leq 2$. So $\underline{d} = (1, 4)$ is ruled out.

Assume $\underline{d} = (0, 5)$. By the initial observation, we must have $L_1 = \mathcal{O}_{C_1}$, $l_2 = 3$, and L_2 free from base points, hence C_2 is not hyperelliptic. Conversely, if $L_2 \in W^2_5(C_2)$ then L_2 is globally generated, because C_2 is not hyperelliptic; let $L = (\mathcal{O}_{C_1}, L_2)$, then obviously $h^0(X, L) = 3$.

Case $\underline{g} = (2, 4)$. The balanced multidegrees are $(1, 4)$ and $(2, 3)$. We rule out $\underline{d} = (1, 4)$ just as in the previous case. Assume $\underline{d} = (2, 3)$; as $l_i \leq 2$ we have $l_1 = l_2 = 2$ and C_2 cannot be hyperelliptic (for otherwise L_2 has a base point). The converse is easily proved as before.

Case $\underline{g} = (3, 3)$. This case is symmetric, so it suffices to consider the balanced multidegree $\underline{d} = (2, 3)$. We will show that C_1 is hyperelliptic and that C_2 is not. If C_1 is not hyperelliptic, then $l_1 \leq 1$; as $l_2 \leq 2$ to have $h^0(X, L) = 3$ both L_1 and L_2 must have a base point at the attaching point, which is not possible. So C_1 must be hyperelliptic. The rest of the argument is exactly as in the previous case. ■

Proposition 5.16. With the notations of Section 5.3.1, let $X = C_1 \cup C_2$ be of genus 6 with $C_1 \cdot C_2 = 2$, and let $\underline{d} \in B_5(X)$. There exists a globally generated $L \in W^2_{\underline{d}}(X)$ if and only if one of the following cases occurs.

(1) $\underline{g} = (0, 5)$, $\underline{d} = (1, 4)$, C_2 hyperelliptic and $L_2 = H^{\otimes 2}_{C_2}$.
(2) $\underline{g} = (1, 4)$, $\underline{d} = (0, 5)$, C_2 non-hyperelliptic, $L_1 = \mathcal{O}_{C_1}$, $h^0(L_2) = 3$ and $h^0(L_2(-G_2)) = 2$.
(3) $\underline{g} = (1, 4)$, $\underline{d} = (2, 3)$, $L_1 = \mathcal{O}_{C_1}(G_1)$, C_2 non-hyperelliptic, $L_2 = \mathcal{O}_{C_2}(G_2 + q)$, $h^0(L_2) = 2$.
(4) $\underline{g} = (2, 3)$, $\underline{d} = (2, 3)$, $H_{C_1} = \mathcal{O}_{C_1}(G_1) = L_1$, C_2 non-hyperelliptic and $L_2 = \mathcal{O}_{C_2}(G_2 + q)$, $h^0(L_2) = 2$. □

Proof. Note that, as L has no base points, L_1 and L_2 have no base points.

Let $\underline{g} = (0, 5)$ and $\underline{d} = (1, 4)$ (X is strictly semistable and C_1 its exceptional component). By Lemma 1.8 we have $h^0(X, L) \leq l_1 + l_2 - 2 \leq 2 + 3 - 2 = 3$, and equality holds if and only if $l_2 = 3$, if and only if C_2 is hyperelliptic and $L_2 = H^{\otimes 2}_{C_2}$, as stated. It is clear that every L pulling back to $(\mathcal{O}(1), H^{\otimes 2}_{C_2})$ on the normalization of X has $h^0(X, L) = 3$.

If $g_1 \geq 1$, one checks easily (by Proposition 1.11 and the fact that L_1 and L_2 have no base points) that $l_1 + l_2 = 4$. Hence by Lemma 1.5 we have

$$p_1 \sim_{L_1} p_2 \quad \text{and} \quad q_1 \sim_{L_2} q_2 \tag{53}$$

and L is uniquely determined by its pull-back to the normalization, by Lemma 1.4.

• Assume $\underline{g} = (1, 4)$. If $\underline{d} = (0, 5)$, by Proposition 1.11(ii) we obtain $L_1 = \mathcal{O}_{C_1}$ and Cliff $L_2 = 1$ so $h^0(L_2) = 3$. C_2 cannot be hyperelliptic, for otherwise L_2 will have a base point. Moreover, as $q_1 \sim_{L_2} q_2$, we have

$$h^0(L_2(-q_1 - q_2)) = h^0(L_2(-q_1)) = h^0(L_2(-q_2)) = 2$$

as claimed. The converse follows easily from Lemma 1.5. Suppose now $\underline{d} = (1, 4)$. As $p_1 \sim_{L_1} p_2$, we have $L_1 = \mathcal{O}_{C_1}(p)$ with $p \neq p_i$. So, L_1 has a base point in p, which is not possible. This case does not occur. Finally, let $\underline{d} = (2, 3)$. We must have $l_1 = l_2 = 2$ (as C_2 cannot be hyperelliptic, as before). By (53) we obtain $L_1 = \mathcal{O}_{C_1}(p_1 + p_2)$ and $L_2 = \mathcal{O}_{C_1}(q_1 + q_2 + q)$ for a (uniquely determined) $q \in C_2$. The converse follows from Lemma 1.5.

• Now assume $\underline{g} = (2, 3)$. If $\underline{d} = (2, 3)$, then we argue exactly as in the previous case ($\underline{g} = (1, 4)$, $\underline{d} = (2, 3)$). If $\underline{d} = (1, 4)$, then we have $l_1 = 1$ so that $L_1 = \mathcal{O}_{C_1}(p)$ with $p \neq p_i$ for $i = 1, 2$ (as $p_1 \sim_{L_1} p_2$). So L has a base point in p; this case is excluded. Finally, if $\underline{d} = (3, 2)$, arguing as before one obtains that L_1 has a base point in $p \in C_1$, which is impossible. This finishes all the possible cases, so we are done. ■

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