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Enumerating rational curves: The rational fibration method

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Abstract. A new method to approach enumerative questions about rational curves on algebraic varieties is described. The idea is to reduce the counting problems to computations on the Néron–Severi group of a ruled surface. Applications include a short proof of Kontsevich's formula for plane curves and the solution of the analogous problem for the Hirzebruch surface $\mathbb{F}_3$.

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1. Introduction

1.1. Statement of the problem

Let S be a smooth projective rational surface and let D be an effective devisor on S. Let $V(D) \subset |D|$ be the closure of the locus of irreducible rational curves. For general results about the geometry of $V(D)$, we refer to $[H]$ and to $[CH]$.

If D has nonnegative self-intersection and $V(D)$ is nonempty the dimension of $V(D)$ is known (cf. [K]):

$$r_0(D) := \dim V(D) = -(K_S \cdot D) - 1.$$

The problem that will study here is to compute the degrees

$$N(D) := \deg V(D)$$

of these varieties as subvarieties of $|D| \cong \mathbb{P}^r$. Alternatively, $N(D)$ is the number of irreducible rational curves in $|D|$ that pass through $r_0(D)$ general points of S. If S is the projective plane $\mathbb{P}^2$ and $d = \deg D$, then one also uses the notation $N(d)$ to denote the number of irreducible, rational curves of degree d passing through $3d - 1$ general points.

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1.2. TERMINOLOGY AND NOTATION

We will work over the complex numbers. Throughout, the words 'surface' and 'curve' will refer to projective varieties.

If D and D' are effective divisors (or divisor classes) on a surface, we will say that $D > D'$ if $D - D'$ is effective and nonzero.

We will denote by $\mathbb{F}_n$ the *Hirzebruch surface* $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. On each $\mathbb{F}_n$ with $n \geqslant 1$ there exists a unique curve of negative self intersection, which we will denote by E and refer to as the *exceptional curve* on $\mathbb{F}_n$. We will denote by F a fiber of the projection $\mathbb{F}_n \to \mathbb{P}^1$; the classes of E and F generate the Picard group of $\mathbb{F}_n$, with intersection pairing given by

$$E^2 = -n; \qquad (E \cdot F) = 1 \quad \text{and} \quad F^2 = 0.$$

Another useful divisor class is the class of a complementary section, that is, a section C of the $\mathbb{P}^1$-bundle $\mathbb{F}_n \to \mathbb{P}^1$ disjoint from E. Since $(C \cdot E) = 0$ and $(C \cdot F) = 1$, we see that $C \equiv E + nF$; so the classes C and F also generate the Picard group, with intersection numbers

$$C^2 = n; \qquad (C \cdot F) = 1 \quad \text{and} \quad F^2 = 0.$$

For any positive integer m, we will denote by $V_m(D) \subset V(D) \subset |D|$ the closure of the locus of irreducible rational curves, X having contact of order at least m with E at a smooth point of X. These varieties will also be referred to a Severi varieties. We set $N_m(D) := \deg V_m(D)$.

1.3. METHODS AND RESULTS

Until very recently, the basic enumerative problem of determining the degrees of Severi varieties was unsolved even in the case of $\mathbb{P}^2$. In 1989 Ziv Ran [R] described a recursive procedure for calculating the degrees of the Severi varieties parametrizing plane curves of any degree and genus (see also [R2]). Recently, M. Kontsevich discovered a beautiful and simple recursive formula in the case of rational curves on $\mathbb{P}^2$ (see [KM] and [RT] for proofs). Kontsevich's method was based on his description of a compactified moduli space for maps of $\mathbb{P}^1$ into the surface $S = \mathbb{P}^2$; others (e.g., [DI], [KP] and [CM]) were able to use the same method to derive similar formulas in the case of other surfaces S for which a Kontsevich-style moduli space existed, such as $S = \mathbb{P}^1 \times \mathbb{P}^1$, the ruled surface $S = \mathbb{F}_1$ and del Pezzo surfaces.

It was our feeling that the reliance of Kontsevich's method on the existence of a well-behaved moduli space was not essential. We were especially interested in whether a similar formula might be derived for the Hirzebruch surfaces $\mathbb{F}_n$.

In [CH], we succeeded in recasting the Kontsevich method so as to remove the apparent dependence on the existence of a moduli space: as we set it up, it was necessary only to understand the degenerations of the rational curves in the one-parameter families corresponding to general one-dimensional linear sections of $V(D)$. The resulting 'cross- ratio method' allowed us to derive a complete recursion for all divisor classes on the ruled surace $S = \mathbb{F}_2$ – that is, a formula expressing $N(D)$ in terms of $N(D')$ for $D' < D$ - and a closed-form formula for certain divisor classes on the ruled surfaces $\mathbb{F}_n$ for any n. (In fact, compactifications of the moduli space of maps $\mathbb{P}^1 \to S$ do exist for these surfaces, but they contain in general many components, only one of which parametrizes generically irreducible rational curves and the others of which may have strictly larger dimension. Kontsevich's method can be carried out in these cases, as was done by Kleiman and Piene [KP]; but at present we do not see how to use the resulting formulas to enumerate irreducible rational curves.)

However, we were unable to go significantly beyond this point: a similarly derived formula in [CH] for the degrees $N(D)$ of Severi Varieties $V(D)$ on $\mathbb{F}_n$ espresses $N(D)$ not solely in terms of $N(D')$ for $D' < D$, but also in terms of the degrees $N_k(D'')$ of the Severi varieties $V_k(D'')$ parametrizing curves with a point of k-fold tangency with a fixed curve $E \subset S$. For example, if $S = \mathbb{F}_3$, then $N(D)$ is expresses as a function of $N(D')$ and of $N_2(D'')$, where $N_2(D'')$ is the number of irreducible rational curves in $|D''|$ that are simply tangent to E and pass through the appropriate number (that is, $r_0(D'') - 1$) of general points of $\mathbb{F}_3$. A complete recursion in this case would have required a similar analysis of linear sections of the Severi varieties $V_2(D)$, which in turn would have necessitated an analysis of Severi varieties parametrizing curves with more complicated tangency conditions.

In the end, it seems that one way or another we need to deal with the degrees of these 'tangetial' loci as well. This difficulty led us to the discovery of a computational technique different from and simpler than the cross-ratio method, which we will describe in the present paper. It involves an analysis of the same basic object as the cross-ratio method – that is, the one-parameter family $\chi \to \Gamma$ of rational curves through $r_0(D) - 1$ general points of S and their limits – but extracts more information from it. It is based on a description of the Néron–Severi group of a minimal desingularization of χ (we will therefore refer to it as the 'rational fibration method'). The main advantage of this technique for our present purposes is that we are in fact able to compute the degrees of the tangential loci involved; at least in all cases that we studied. It also yields other related formulas, such as the number of irreducible rational curves having a node at a given general point $p \in S$ and passing through $r_0(D) - 2$ other general points.

1.4. Contents of this paper

In the following section we will describe the rational fibration method in general setting. In the succeeding sections we will apply it in the cases $S = \mathbb{P}^2, S = \mathbb{F}_2$ and $S = \mathbb{F}_3$. In the first of these cases, we obtain another (simpler) proof of Kontsevich's formula, as well as some related formulas derived by Pandharipande

[P]. In the second, we will recover the general recursion formula found originally in [CH] for degrees of Severi varieties on $\mathbb{F}_2$. Finally, in the last section we derive a complete set of recursions for $\mathbb{F}_3$, the first case for which the cross-ratio method does not give a complete answer.

We have tried to keep this paper relatively self-contained; in particular it should be intelligible to a reader unfamiliar with [CH]. We will, however, have to appeal to Chapter 2 of [CH] for proofs of some of the basic assertions about the local geometry of Severi varieties and the families of curves they parametrize.

2. The rational fibration method in general

2.1. Objects and morphisms

As we indicated in the introduction, the rational fibration method, like the cross-ratio method, involves studying a suitable general one-parameter family of rational curves. To set it up, first let S be a smooth rational surface and D an effective divisor on S. We will assume that D has nonnegative self-intersection and that the Severi variety $V(D) \neq \emptyset$, so that in particular we have $\dim V(D) = r_0(D) = -(K_S \cdot D) - 1$. Now, choose $r_0(D) - 1$ general points $p_1, \ldots, p_{r_0(D)-1} \in S$ and let $\Gamma \subset V(D)$ be the closure of the locus of points $[X] \in V(D)$ corresponding to irreducible rational curves X passing through these points. Equivalently, if for any point $p \in S$ we let $H_p \subset \mathbb{P}^r$ by the hyperplane of points corresponding to curves passing through p, Γ will be the one-dimensional linear section of $V(D)$

$$\Gamma = V(D) \cap \left(\bigcap_{i=1}^{r_0(D)-1} H_{p_i} \right).$$

Now, let $\chi \subset \Gamma \times S \to \Gamma$ be the family of curves corresponding to $\Gamma \subset |D|$. Consider the normalization Γ^ν of $\mathcal{X} \times_\Gamma \Gamma^\nu$ to arrive at a family

$$\chi^\nu \to \Gamma^\nu$$

over a smooth curve Γ^ν, whose general fiber is isomorphic to $\mathbb{P}^1$.

Next, we apply semi-stable reduction (which we should rather call 'nodal reduction', since our curves have genus zero): after making a base change $B \to \Gamma^\nu$ and blowing up the total space of the pullback family $\chi^\nu \times_{\Gamma^\nu} B$, we arrive at a family $f\colon \mathcal{Y} \to B$ whose total space is smooth, whose general fiber is a smooth rational curve and whose special fibers are all nodal curves. In fact, a base change will turn out to be unnecessary in each of the three cases considered below – the minimal desingularization $\mathcal{Y}$ of the total space χ^ν already has this property – but this is not relevant, since even a superfluous base change will not affect the subsequent calculations. We will denote by $\pi : \mathcal{Y} \to S$ the composite map

$$\pi\colon \mathcal{Y} \to \chi^\nu \times_{\Gamma^\nu} B \to \chi^\nu \to \chi \subset \Gamma \times S \to S.$$

Notice that π is a generically finite map, whose degree is equal to the product of the degree of the map $B \to \Gamma^{\nu}$ times the number of irreducible rational curves in the linear series $|D|$ passing through the points $p_1, \ldots, p_{r_0(D)-1}$ and p (that is, the degree $N(D)$ of the Severi variety $V(D)$).

Here is a diagram of the basic objects and morphisms we have introduced:

$$\begin{tikzcd}
 & & & & \mathcal{Y} \arrow[dl] \arrow[ddr, "f"] \arrow[dddllll, bend right, "\pi"'] & \\
 & & & \mathcal{X}^{\nu} \arrow[ddl] \arrow[dddr] & & \\
 & & & & & B \arrow[ddl] \\
S & \Gamma \times S \arrow[l] \arrow[r, phantom, "\supset"] & \mathcal{X} \arrow[ddr] & & & \\
 & & & & \Gamma^{\nu} \arrow[dl] & \\
 & & & \Gamma & &
\end{tikzcd}$$

Figure 1.

2.2. Outline of the Method

As we indicated, our method involves calculating in the Néron–Severi group of the total space $\mathcal{Y}$ of our family. This is motivated by a simple observation: given any two line bundles L and M on S, we have

$$(\pi^* L \cdot \pi^* M) = \deg\{B \to \Gamma^{\nu}\} \cdot N(D) \cdot (L \cdot M).$$

Thus, in order to derive a formula for $N(D)$, we want to compute intersection numbers in the Néron–Severi group of $\mathcal{Y}$. For example, if S is the projective plane $\mathbb{P}^2$ we can take $L = M = \mathcal{O}_{\mathbb{P}^2}(1)$. Then $(\pi^* L)^2 = N(D)$; so if we can compute $(\pi^* L)^2$ we get a formula for $N(D)$.

What makes is possible to perform such calculations is the fact that f expresses $\mathcal{Y}$ as the total space of a one-parameter family of generically smooth rational curves, so that to determine the class of a given divisor it is enough to know its degree on each component of each fiber of f. More precisely, the Picard group of $\mathcal{Y}$ will be freely generated by the class of a fiber, the class of any section A of f, and the classes of all the irreducible curves contained in fibers of f and disjoint from A. Moreover, in terms of these generators the intersection pairing on Pic$(\mathcal{Y})$ is (except possibly for the self-intersection of A) easy to describe. This means two things: first, we can express a given divisor class as a linear combination of these generators once we know its degree on each component of each reducible fiber and on A; and second, having expressed two divisor classes as linear combinations of these generators, we can readily compute their intersection number.

The method we will apply in each case thus consists of five steps:

- First, we need to describe the reducible fibers of $\mathcal{Y} \to B$; that is (given that B will be in practice just the normalization Γ^{ν} of the base of our original family $\chi \to \Gamma$), the set of reducible curves in the linear series $|D|$ through the points $p_1, \ldots, p_{r_0(D)-1}$ that are limits of irreducible rational curves through these points, and the branches of Γ at each one. The characterization of such curves is straightforward in the case of $S = \mathbb{P}^2$ by simple dimension-counting. In the case of $S = \mathbb{F}_n$ with $n \geqslant 2$ it is less obvious, since in contrast with the case of $\mathbb{P}^2$ most reducible curves through the points $p_1, \ldots, p_{r_0(D)-1}$ whose components are all rational are not limits of irreducible rational such curves; the answer is worked out in [CH]. In either case the number of such fibers will be known inductively.
- Second, we need to describe the local structure of the family $\chi^{\nu} \to \Gamma^{\nu}$ near each reducible fiber; specifically, we need to know whether χ^{ν} is smooth or if we have to blow up. This likewise is straightforward in the case of the plane, where in fact χ^{ν} is smooth. It is more interesting in the case of the Hirzebruch surfaces $\mathbb{F}_n$, where for $n \geqslant 3$ we see that χ^{ν} will indeed have singularities; again, this is worked out in [CH] and we will refer there for the relevant results.
- Third, we choose a basis for the Néron–Severi group of $\mathcal{Y}$, and calculate the intersection pairing on these classes.
- Fourth, since we know the images in S of the components of reducible fibers of $f\colon \mathcal{Y} \to B$, we can calculate the degrees on all such components of the pullback $\pi^* L$ of any line bundle L on S; and
- Fifth, we are able therefore to express the intersection numbers $(\pi^* L \cdot \pi^* M)$ for pairs $L, M \in \operatorname{Pic} S$ of line bundles on S.

Evidently, the particulars of this process will depend on S and D; for the moment we shall just fix some notation and make some preliminary observations. First, for $b \in B$ we use the common notation $Y_b := f^{-1}(b)$ to denote the fiber of f over b. The class in $\mathrm{NS}(\mathcal{Y})$ of such a fiber is denoted by Y.

Secondly, recall that our family parametrizes curves through certain base points. We pick two of them, q and q', and we denote by A and A' the corresponding sections of f. The following relations are clear:

$$Y^2 = A \cdot A' = 0 \quad \text{and} \quad A \cdot Y = A' \cdot Y = 1.$$

Notice also that by symmetry $A^2 = \frac{1}{2}(A - A')^2$ which will be useful to compute the left-hand side. In fact $A - A'$ is supported on exactly those fibers where q and q' lie on different components, the number of which we will be able to count.

One further note: the description above of the Néron–Severi group of $\mathcal{Y}$ as generated by the classes of A, Y and components of reducible fibers assumes that the base B of the family is connected, which we will not always know in practice. This assumption is not essential, however: in case B has irreducible components

$B_1, \ldots, B_k$ we simply have to replace every multiple of Y in the formulas below by a suitable linear combination of fibers Y_i lying over points of B_i. As the reader may verify, this does not alter the outcome of the subsequent calculations.

3. Plane curves

Here we study the cases $S = \mathbb{P}^2$. If D and D_i are divisors on $\mathbb{P}^2$, we denote their degrees respectively by d and d_i. Since a divisor class in the plane is determined by its degree, we will introduce the notation $N(d) := N(D)$.

We have that $r_0(D) = 3d - 1$, so we choose general points $p_1, \ldots, p_{3d-2} \in \mathbb{P}^2$, let $\Gamma \subset V(D)$ be the locus of curves in $V(D)$ containing the points p_i, and proceed as described in the preceding section. To describe the resulting family of curves, let Δ be the locus of $V(D)$ parametrizing degenerate curves (that is, curves that are reducible or have singularities other than nodes). Since our curve $\Gamma \subset V(D)$ will intersect Δ only at general points of components of Δ, we may apply the results of [DH] and [H] to conclude the following

A. Any fiber X_γ of $\chi \to \Gamma$ is either
   1. an irreducible curve with exactly $\delta = (d-1)(d-2)/2$ nodes;
   2. an irreducible curve with exactly $\delta - 1$ nodes and a cusp;
   3. an irreducible curve with exactly $\delta - 2$ nodes and a tacnode;
   4. an irreducible curve with exactly $\delta - 3$ nodes and an ordinary triple point; or
   5. a curve having exactly two irreducible components X_1, X_2, of degrees d_1 and d_2, with exactly $(d_1 - 1)(d_1 - 2)/2$ and $(d_2 - 1)(d_2 - 2)/2$ nodes respectively, and intersecting transversally in $d_1 d_2$ points.

B. In cases 1, 3 and 4, the curve Γ is smooth at γ and the family $\chi^\nu \to \Gamma^\nu$ is smooth at the unique point of Γ^ν lying over γ. In case 2, Γ has a cusp at γ but the family $\chi^\nu \to \Gamma^\nu$ is still smooth at the unique point of Γ^ν lying over γ.

C. In case 5 the curve Γ has $d_1 d_2$ smooth branches at γ (corresponding to deformations of X_γ smoothing any one of the $d_1 d_2$ nodes of X_γ coming from a point $p \in X_1 \cap X_2$ of intersection of X_1 and X_2). At each point of Γ^ν lying over γ the fiber of the family $\chi^\nu \to \Gamma^\nu$ has two smooth rational components meeting transversally at one point (more precisely, it is the normalization of X_γ at the remaining $(d_1-1)(d_1-2)/2 + (d_2-1)(d_2-2)/2 + d_1 d_2 - 1 = \delta$ nodes of X_γ), and smooth total space.

D. Finally, if $X \subset \mathbb{P}^2$ is any curve of type 1–5 passing through the points $p_1, \ldots, p_{3d-2}$, then conversely $[X] \in \Gamma$; that is, X is a limit of irreducible rational curves X_γ through $p_1, \ldots, p_{3d-2}$.

We see in particular that the total space χ^ν is smooth and that the fibers of $\chi^\nu \to \Gamma^\nu$ are all nodal, so that no further base changes or blow-ups are necessary; that is, we may take $B = \Gamma^\nu$ (as we stated earlier) and $\mathcal{Y} = \chi^\nu$. Note also that every reducible fiber of $\mathcal{Y}$ has precisely two irreducible components, meeting transversally at one point.

We shall call a reducible fiber of f a *fiber of type* $\mathbf{J}$ and we shall denote by B_J the subset of points of B such that the corresponding fiber is of type $\mathbf{J}$, that is, reducible. (This new piece of terminology probably seems pointless, but it will be useful in the sequel.) For any $b \in B_J$, we then denote the two irreducible components of the fiber over b by $J_{1,b}$ and $J_{2,b}$. We shall always denote by $J_{1,b}$ the component containing the point $q = p_1$. The picture of $\mathcal{Y}$ is thus:

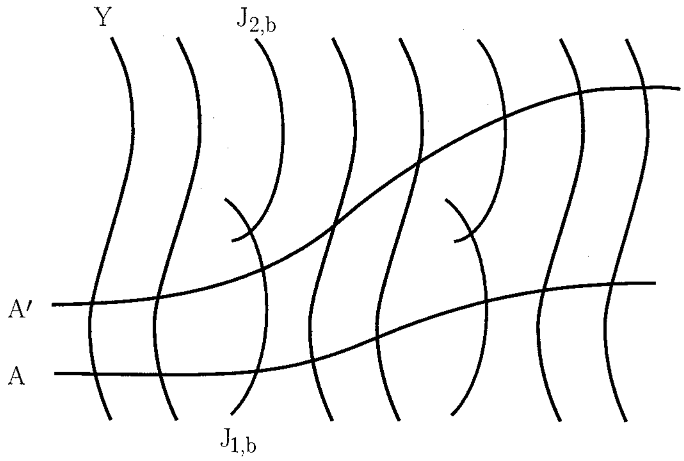

Figure 2.

Let D_i be the class of $\pi(J_{i,b})$. We denote by $j(D_1, D_2)$ the number of all such fibers, for any given decomposition $D = D_1 + D_2$. To determine $j(D_1, D_2)$, note first that if $X = X_1 \cup X_2 \in \Gamma$ is any reducible curve, D_i the class of X_i, then X_i can contain at most $r_0(D_i)$ of the $r_0(D) - 1 = 3d - 2$ points $p_1, \ldots, p_{3d-2}$. Since

$$r_0(D_1) + r_0(D_2) = r_0(D) - 1,$$

it follows that each component X_i must contain exactly $r_0(D_i)$ of the points $p_1, \ldots, p_{3d-2}$. Thus, to specify such a curve, we have first to choose a decomposition of the set $\Phi = \{p_1, \ldots, p_{3d-2}\}$ into disjoint subsets Φ_1, Φ_2 of cardinality $r_0(D_1)$ and $r_0(D_2)$ respectively, with the point $q = p_1 \in \Phi_1$; and then to choose, for each i, one of the $N(D_i)$ curves $x_i \in V(D_i)$ containing Φ_i. The number of such curves X is thus

$$\binom{r_0(D) - 2}{r_0(D_1) - 1} \cdot N(D_1) \cdot N(D_2)$$

and since we have seen there are $(D_1 \cdot D_2) = d_1 d_2$ points of B lying over each point $[X] \in \Gamma$ corresponding to a curve of this type, we have

$$\begin{aligned} j(D_1, D_2) &= N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r_0(D)-2}{r_0(D_1)-1} \\ &= N(d_1)N(d_2)d_1d_2\binom{3d-3}{3d_1-2}. \end{aligned}$$

Note that by a simple dimension count, any of the curves $X_i \in V(D_i)$ passing through $r_0(D_i)$ of the points $p_1, \ldots, p_{3d-2}$ will be irreducible and nodal. By a standard further argument as in Lemma 2.1 of [CH], we see that any pair X_1, X_2 of such curves will intersect transversally, so that the union $X = X_1 \cup X_2$ will indeed be a curve as described in (5) above.

This completes the first two steps in the general method. Next, we give a basis for the Néron–Severi group NS($\mathcal{Y}$). We now choose as a system of generators for NS($\mathcal{Y}$) the class A of the section of $f\colon \mathcal{Y} \to B$ coming from the base point $q = p_1$ of our family; the class Y of a fiber of the map $f\colon \mathcal{Y} \to B$, and the classes $\{J_{2,b}\}_{b\in B_J}$. Most of the pairwise intersection numbers of these classes are readily given: we clearly have

$$\begin{aligned} (A \cdot Y) &= 1 \\ Y^2 &= 0 \\ (A \cdot J_{2,b}) &= 0 \quad \forall\, b \\ (Y \cdot J_{2,b}) &= 0 \quad \forall\, b \\ (J_{2,b} \cdot J_{2,b'}) &= 0 \quad \forall\, b \neq b'; \text{ and} \\ (J_{2,b} \cdot J_{2,b}) &= -1 \quad \forall\, b. \end{aligned}$$

In fact, there is only one intersection number that is not evident: A^2. To compute it we choose a base point $q' \neq q$, so that q' determines a second section A' of $\mathcal{Y} \to B$ disjoint from A. Since the base points $p_1, \ldots, p_{3d-2}$ of our family are general points in the plane, by symmetry we have $A'^2 = A^2$; hence we can write

$$2A^2 = (A - A')^2.$$

To compute the right-hand side, let

$$S_J = \{b \in B_J \text{ such that } q' \in \pi(J_{2,b})\}$$

be the collection of points in B over which the sections A and A' meet different components of the fiber; let $n_J = |S_J|$ be the cardinality of S_J. For every $b \notin S_J$, A and A' have the same intersection number with each component of the fiber Y_b. For $b \in S_J$, on the other hand, we have $(A \cdot J_{1,b}) = 1$ and $(A \cdot J_{2,b}) = 0$, while $(A' \cdot J_{1,b}) = 0$ and $(A' \cdot J_{2,b}) = 1$. It follows that the classes

$$A \quad \text{and} \quad A' - \sum_{b\in S_J} J_{2,b}$$

have the same intersection number with every component of every fiber of $\mathcal{Y} \to B$, and so must differ by a multiple of the class Y of a fiber: that is ,

$$A - A' = -\sum_{b \in S_J} J_{2,b} + nY$$

for some integer n. In fact, n must be equal to $n_J/2$ by symmetry, but that is irrelevant in any case: squaring both sides, we find that

$$(A - A')^2 = \sum_{b \in S_J} J_{2,b}^2 = -n_J$$

and hence

$$A^2 = -\frac{n_J}{2}.$$

Thus, it remains only to determine the number n_J of reducible fibers of $\mathcal{Y} \to B$ lying over curves in our original family in which the points q and q' lie in different components. We can do this in exactly the same way as we determined the total number of reducible fibers: the only difference is that now we want to count only decompositions $\Phi = \Phi_1 \cup \Phi_2$ in which $q = p_1 \in \Phi_1$ and $q' = p_2 \in \Phi_2$. We thus replace the binomial coefficient $\binom{r_0(D)-2}{r_0(D_1)-1}$ in the formula for $j(D_1, D_2)$ above with $\binom{r_0(D)-3}{r_0(D_1)-1}$ and sum over all pairs D_1, D_2 with $D_1 + D_2 = D$ to obtain

$$n_J = \sum_{D_1 + D_2 = D} N(D_1) N(D_2)(D_1 \cdot D_2) \binom{r_0(D)-3}{r_0(D_1)-1}.$$

This completes the third step of the process.

Now let $L \in \mathrm{Pic}(\mathbb{P}^2)$ be any line bundle on the plane, and write the class of its pullback to $\mathcal{Y}$ as a general linear combination of our chosen generators

$$\pi^* L \equiv c_a A + c_y Y + \sum_{b \in B_J} c_b J_{2,b}.$$

We will denote by J^L the third term on the right, that is, we set

$$J^L := \sum_{b \in B_J} c_b J_{2,b};$$

this is not immediately useful, but will become so in the succeeding calculations.

We now intersect both sides of the above equivalence with each of our chosen generators of $\mathrm{NS}(\mathcal{Y})$ to determine the coefficients c_a, c_y and c_b. First, by intersecting both sides with Y we find that

$$c_a = (\pi^* L \cdot Y) = (L \cdot \pi_* Y) = (L \cdot D).$$

Next we intersect with A: we have

$$(\pi^* L \cdot A) = (L \cdot \pi_* A) = 0$$

sinceπ is constant on the curve A; and hence

$$c_y = -A^2(L \cdot D) = \frac{nJ}{2}(L \cdot D).$$

Finally, to determine c_b we naturally intersect both sides with the class of $J_{2,b}$; we find that

$$c_b = -(\pi^* L \cdot J_{2,b}) = -(L \cdot \pi_* J_{2,b}) = -(L \cdot D).$$

Thus, in sum,

$$\pi^* L \equiv (L \cdot D)A + \frac{n_J}{2}(L \cdot D)Y - \sum_{b \in B_J} (L \cdot D_2) J_{2,b}.$$

For the final step in the process, we evaluate the self- intersection of $\pi^* L$: we find

$$\begin{aligned}
(\pi^* L)^2 &= \frac{n_J}{2}(L \cdot D)^2 - \sum_{b \in B_J} (L \cdot D_2)^2 \\
&= \sum_{D_1 + D_2 = D} \left[\tfrac{1}{2} N(D_1) N(D_2) (D_1 \cdot D_2) \binom{r_0(D) - 3}{r_0(D_1) - 1} (L \cdot D)^2 \right. \\
&\qquad \left. - N(D_1) N(D_2) (D_1 \cdot D_2) \binom{r_0(D) - 2}{r_0(D_1) - 1} (L \cdot D_2)^2 \right].
\end{aligned}$$

Applying this in case $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and recalling that $(\pi^* \mathcal{O}_{\mathbb{P}^2}(1))^2 = N(d)$, we have

$$N(d) = \sum_{d_1 + d_2 = d} N(d_1) N(d_2) \left[\frac{d^2 d_1 d_2}{2} \binom{3d - 4}{3d_1 - 2} - d_1^2 d_2^2 \binom{3d - 3}{3d_1 - 2} \right];$$

and expanding out $d^2 = (d_1 + d_2)^2$ and using the symmetry with respect to d_1 and d_2 we get the well known recursive formula of Kontsevich

$$N(d) = \sum_{d_1+d_2=d} N(d_1)N(d_2)\left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1 d_2^3 \binom{3d-4}{3d_1-3}\right].$$

Remark. In parts (A) and (B) of the statement of results quoted from [H] and [DH], we describe completely all curves X_γ in the family $\chi \to \Gamma$ having other than δ nodes, and the local geometry of $\chi^\nu \to \Gamma^\nu$ along each. This is in fact necessary to describe the Néron–Severi group of $\mathcal{Y}$, since even in those cases where a fiber is irreducible it is a priori possible that χ^ν will be singular along such a fiber, giving rise to a reducible fiber of $\mathcal{Y} \to B$. Looking back over the preceding calculation, though, we see that even if this did happen, it would not affect the outcome of the calculation, as long as the non-nodal singularities did not occur at base points of the family: while the resolution of the singularities of χ^ν would create additional curves on $\mathcal{Y}$ independent in $\mathrm{NS}(\mathcal{Y})$, the sections A and A' and any line bundle pulled back via π from $\mathbb{P}^2$, would all have degree 0 on these curves, and so the relations of linear equivalence above would still hold.

Thus it was only necessary to observe that every curve singular at a base point is an irreducible curve with δ nodes. Since this statement will also hold for the families of curves on $\mathbb{F}_n$ that we will be considering in the following two sections, we will in the sequel omit the description of the fibers $\chi \to \Gamma$ other than reducible ones.

As another application, we give a formula for the number $N_2(d)$ of plane, irreducible, rational curves $X \subset \mathbb{P}^2$ of degree d passing through $3d - 2$ given general points and tangent to a given general line ℓ in the plane. Equivalently, this is the degree of the subvariety $V_2(D)$ of $V(D)$ defined as the closure of the locus of irreducible rational curves that are tangent to ℓ in $\mathbb{P}^2$ at a smooth point of [X] (notice that $V_2(D)$ has codimension 1 in $V(D)$). To calculate this number, let $\tilde{L} = \pi^{-1}(\ell) \subset \mathcal{Y}$ be the preimage of ℓ under π. Then $\tilde{L}$ is an irreducible smooth curve, and the morphism $f: \mathcal{Y} \to B$ restricts to a finite morphism $\tilde{f} : \tilde{L} \to B$ of degree d on $\tilde{L}$. Moreover, the set of fibers $Y_b \subset \mathcal{Y}$ of f tangent to $\tilde{L}$ – that is, such that the intersection $Y_b \cap \tilde{L}$ has cardinality strictly less than d – corresponds to the set of curves X_b in our original family $\chi \to \Gamma$ tangent to ℓ. Thus, $N_2(D)$ is equal to the degree of the ramification divisor of the morphism $\tilde{f}$.

Now, using the adjunction formula, this degree is given by

$$N_2(d) = (\pi^*\mathcal{O}_{\mathbb{P}^2}(1))^2 + (\omega_{\mathcal{Y}/B} \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1)),$$

where $\omega_{\mathcal{Y}/B}$ is the relative dualizing sheaf of the family. Since we have already calculated the class of $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ above, it remains only to determine the class of $\omega_{\mathcal{Y}/B}$ in similar terms, and then we will be able to evaluate this expression. We do this as for $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$: we first express $\omega_{\mathcal{Y}/B}$ as a linear combination of the generators:

$$\omega_{\mathcal{Y}/B} \equiv e_a A + e_y Y + \sum_{b\in B_J} e_b J_{2,b}$$

and then intersect both sides with the generators of $\mathrm{NS}(\mathcal{Y})$ to determine the coefficients. First, intersecting with Y, we find that

$$e_a = (\omega_{\mathcal{Y}/B}\cdot Y) = -2$$

and then intersecting with A and using the fact that $(\omega_{\mathcal{Y}/B}\cdot A) = -A^2$ we find that

$$e_y = A^2.$$

Finally, we have $(\omega_{\mathcal{Y}/B}\cdot J_{2,b}) = -1, \forall b \in B_J$, and it follows that the coefficients e_b are all 1. Thus, in sum,

$$\omega_{\mathcal{Y}/B} = -2A + A^2 Y + \sum_{b\in B_J} J_{2,b}.$$

We finally obtain a formula first found by Pandharipande [P]:

$$\begin{aligned} N_2(d) \;&=\; N(d) + \sum_{d_1+d_2=d} N(d_1)N(d_2)d_1 d_2^2 \binom{3d-4}{3d_1-3} \\ &= \sum_{d_1+d_2=d} N(d_1)N(d_2)d_1 d_2 \Bigg[d_1 d_2 \binom{3d-4}{3d_1-2} \\ &\qquad\qquad - (d_2^2 - d_2)\binom{3d-4}{3d_1-3}\Bigg]. \end{aligned}$$

This technique can also be used to recover another formula of Pandharipande, for the degree of the closure of the locus of irreducible rational curves of degree d having a cusp. To obtain this, we simply apply Porteous' formula to the differential

$$d(f\times\pi) : T_{\mathcal{Y}} \to (f\times\pi)^* T_{B\times\mathbb{P}^2}$$

of the map $f\times\pi : \mathcal{Y} \to B\times\mathbb{P}^2$; the classes on $\mathcal{Y}$ involved have already been calculated. It should also be possible to determine is similar fashion the degrees on Γ of all the divisor classes introduced in [DH], and in particular obtain formulas for the number of irreducible rational curves through $3d-2$ points and having a tacnode, or the number of irreducible rational curves through $3d-2$ points and having a triple point, etc. At this point, however, we conclude our study of the plane and turn to the Hirzebruch surfaces.

4. The general recursion for $\mathbb{F}_2$

Let now $S = \mathbb{F}_2$. Let C, E and F be the curves on $\mathbb{F}_2$ described in Section 1. We apply our method exactly as before: for any effective divisor class D on S with $V(D) \neq \emptyset$, we choose $r_0(D) - 1$ general points $p_1, \ldots, p_{r_0(D)-1}$ of S and consider the family $\chi \to \Gamma$ of curves $X \in V(D)$ passing through the points p_i; we let $\chi^\nu \to \Gamma^\nu$ and $\mathcal{Y} \to B$ be derived from this family as in the general set-up.

We first describe the various types of reducible fibers that our family $\mathcal{Y} \to B$ has. The following analysis is based on Propositions 2.1, 2.5, 2.6 and 2.7 of [CH]. In particular, the various types of degenerations can be classified as an application of Proposition 2.5, and the singularities of Γ at the points $\gamma \in \Gamma$ corresponding to each as an application of Proposition 2.6. Moreover, Proposition 2.7 assures us that, just as in the case of $\mathbb{P}^2$, the normalization χ^ν is smooth and so the total space $\mathcal{Y}$ coincides with χ^ν. In particular, we see that no irreducible component of any fiber of $\mathcal{Y}$ is mapped to a point by π.

With that said, we have the following calssification of irreducible fibers of $\mathcal{Y} \to B$:

Type **J**. Fibers having two smooth irreducible components J_1 and J_2, meeting transversally at one point, such that $\pi(J_i) = D_i$ with $D_i > 0$ and not equal to E. We will always assume that $q \in J_1$. For any decomposition $D = D_1 + D_2$ we have that the number $j(D_1, D_2)$ of fibers of type **J** such that $\pi(J_i) = D_i$ is

$$j(D_1, D_2) = N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r_0(D) - 2}{r_0(D_1) - 1}.$$

The factor $(D_1 \cdot D_2)$ appears because, just as in the case of $\mathbb{P}^2$, if $[X] \in \Gamma$ corresponds to a curve of type **J**, in the normalization map $\nu \colon B \to \Gamma$ the fiber over $[X]$ contains exactly $(D_1 \cdot D_2)$ points (here we are using Proposition 2.6 of [CH]).

We let B_J be the subset of points b in B whose fiber X_b is a curve of type **J**.

Type **G**. Fibers having two smooth irreducible components G_1 and G_E, meeting transversally at one point, such that $\pi(G_E) = E$ and $\pi(G_1)$ is simply tangent to E. Clearly $q \in G_1$. The total number of such fibers will not matter in the subsequent calculation.

We let B_G be the subset of points b in B whose fiber X_b is a curve of type **G**.

Type **H**. Fibers having three irreducible components H_1, H_2, H_E, such that $\pi(H_E) = E$ and $\pi(H_i) = D_i$, with $D_i > 0$ and $D_1 + D_2 = D - E$ (again, we will choose the labelling so that $q \in H_1$ always). By Proposition 2.6 of [CH], if $[X] \in \Gamma$ is a point corresponding to this type of curve, then the fiber of B over Γ contains exactly $(D_1 \cdot E)(D_2 \cdot E)$ points. Hence the total number of fibers of type **H** that correspond to a given decomposition $D = D_1 + D_2 + E$ is

$$h(D_1, D_2) = \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot E)(D_2 \cdot E) \binom{r_0(D) - 2}{r_0(D_1) - 1}.$$

And just like for the other types, we define B_H to be the subset of points of B parametrizing curves of type **H**.

The picture of $\mathcal{Y} \to B$ thus looks like this:

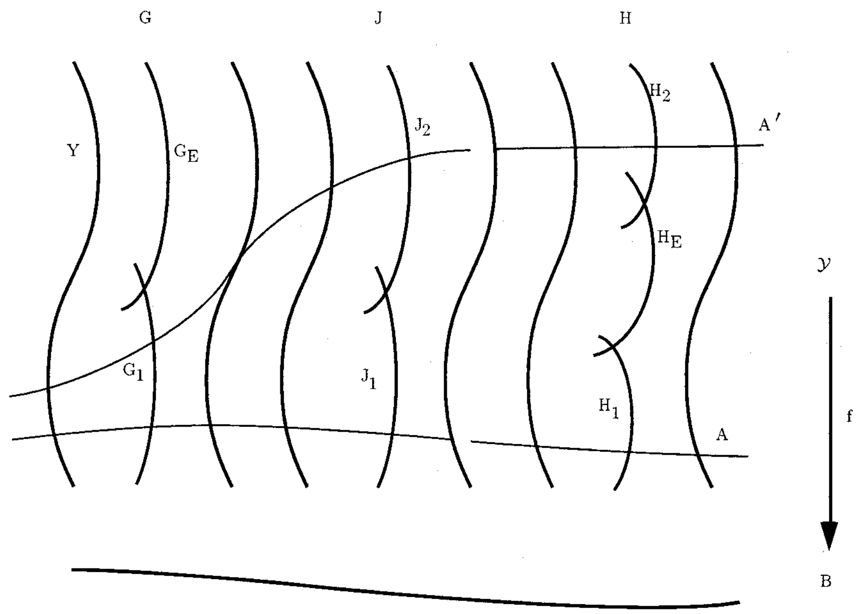

Figure 3.

Now we choose the following set of generators for the Néron–Severi group of $\mathcal{Y}$

$$\{A, Y\} \cup \{J_{2,b}\}_{b \in B_J} \cup \{G_{E,b}\}_{b \in B_G} \cup \{H_{2,b}, H_{E,b}\}_{b \in B_H}.$$

The following relations are obvious

$$G_{E,b}^2 = J_{2,b}^2 = H_{2,b}^2 = -1; \qquad H_{E,b}^2 = -2$$

and the intersection number of A with any generator other than A and Y is zero.

We compute A^2 by the same argument as used in the preceding section. We define

$$S_H = \{b \in B_H \text{ such that } q' \in H_{2,b}\}$$

and we let $n_H = |S_J|$. Then we have

$$n_H = \sum_{D_1+D_2=D-E} N(D_1)N(D_2)(D_1 \cdot E)(D_2 \cdot E) \binom{r_0(D)-3}{r_0(D_1)-1}$$

Similarly, we let S_J and n_J to be defined exactly as in the preceding section, and notice that the value of n_J is expressed by the same formula that we had in the plane. We obtain

$$A - A' = -\sum_{b\in S_J} J_{2,b} - \sum_{b\in S_H} (H_{E,b} + 2H_{2,b}) + nY$$

for some integer n. Hence

$$A^2 = -\frac{n_J + 2n_H}{2}.$$

Let now $L \in \mathrm{Pic}(\mathbb{F}_2)$. We want to compute the coefficients of π^*L as a linear combination of the chosen generators of $\mathrm{NS}(\mathcal{Y})$. The number of generators being quite large, it is now convenient to use the following notation: if W is any of the chosen generators, we shall denote by $\{\pi^*L\}_W$ the coefficient of π^*L with respect to W. We shall then write

$$\pi^*L = \{\pi^*L\}_A A + \{\pi^*L\}_Y Y + G^L + J^L + H^L$$

where J^L is defined just as in the preceeding section, and similarly

$$G^L = \sum \{\pi^*L\}_{G_{E,b}} G_{E,b} \quad \text{and}$$

$$H^L = \sum (\{\pi^*L\}_{H_{2,b}} H_{2,b} + \{\pi^*L\}_{H_{E,b}} H_{E,b}).$$

We could now easily compute all the missing numbers in term of intersection numbers on $\mathbb{F}_2$; only we don't really need it. All we need is the expression for π^*C; in fact we shall obtain a formula for $N(D)$ by using the fact that $(\pi^*C)^2 = 2N(D)$. The following numbers are obtained in a straightforward way, just as in the case of $\mathbb{P}^2$.

$$\{\pi^*C\}_A = (C \cdot D) = a.$$

$$\{\pi^*C\}_Y = -(C \cdot D)A^2 = \frac{a(n_J + 2n_H)}{2}.$$

$$\{\pi^*C\}_{G_E} = 0 \text{ for any generator of type } \mathbf{G}_E.$$

$$\{\pi^*C\}_{J_{2,b}} = -(C \cdot \pi(J_{2,b})).$$

And for any curve $H = H_1 + H_2 + H_E$ of type **H** such that $\pi(H_i) = D_i$ we have

$$\{\pi^*C\}_{H_E} = -(C \cdot D_2) \text{ and } \{\pi^*C\}_{H_2} = -2(C \cdot D_2).$$

In conclusion, we get the same recursive formula that we obtained in [CH]:

THEOREM. *For any effective divisor $D \neq E$ on $\mathbb{F}_2$ with $V(D) \neq \emptyset$,*

$$\begin{aligned} N(D) = {} & \frac{1}{2} \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \\ & \left[\binom{r_0(D)-3}{r_0(D_1)-1} (D_1 \cdot C)(D_2 \cdot C) - \binom{r_0(D)-3}{r_0(D_1)-2} (D_2 \cdot C)^2 \right] \\ & + \sum_{D_1+D_2=D-E} N(D_1)N(D_2)(D_1 \cdot E)(D_2 \cdot E) \\ & \left[\binom{r_0(D)-3}{r_0(D_1)-1} (D_1 \cdot C)(D_2 \cdot C) - \binom{r_0(D)-3}{r_0(D_1)-2} (D_2 \cdot C)^2 \right]. \end{aligned}$$

5. The general recursion for $\mathbb{F}_3$

Let now $S = \mathbb{F}_3$ and let C, E and F be as in 1.2. Let D be an effective divisor class D on S with $V(D) \neq \emptyset$. We also introduce in this case two additional subvarieties of the linear series $|D|$: the subvariety $V_2(D)$ of $V(D)$ defined to be the closure of the locus of irreducible rational curves X tangent to E at a smooth point X; and the closure $F(D)$ of the subvariety of $V(D)$ parametrizing irreducible curves having a smooth point of intersection multiplicity 3 with E. Their degrees will be denoted by $N_2(D)$ and $N_3(D)$ respectively.

Now we proceed as before: we choose $r_0(D)-1$ general points $p_1, \ldots, p_{r_0(D)-1}$ of S and consider the family $\chi \to \Gamma$ of curves $X \in V(D)$ passing through the points p_i; we let $\chi^\nu \to \Gamma^\nu$ and $\mathcal{Y} \to B$ be derived from this family as in the general set-up. Our method will again provide us with a recursive formula for the degree of $V(D)$, but there will be now an important difference: the recursion will involve as well the degrees $N_2(D)$ and $N_3(D)$ of the varieties $V_2(D)$ and $F(D)$. More precisely, we are going to obtain three formulas:

(a) A formula expressing $N(D)$ in terms of $N(D')$ and $N_2(D'')$, where $D' < D$ and $D'' < D - E$;
(b) A formula expressing $N_2(D)$ in terms of $N(D)$, $N(D')$, $N_2(D'')$ and $N_3(D - E)$, where $D' < D$ and $D'' < D - E$; and
(c) A formula expressing $N_3(D - E)$ in terms of $N(D')$ and $N_2(D'')$, where $D' < D$ and $D'' < D - E$.

We now describe the various reducible fibers of $\mathcal{Y} \to B$. Again we use the results of [CH], in particular, Propositions 2.5 and 2.7 for the geometry of the normalization map $B \to \Gamma$ and of the total space χ^ν. By 2.7 we have that χ^ν is smooth at points lying on fibers corresponding to types **J**, **G** and **H** below; in other words, no irreducible component of a fiber belonging to one of these types is mapped to a point of $\mathbf{F}_3$.

Type **J**. (This is the exact analog of the type **J** for $\mathbb{F}_2$.) Fibers having two smooth irreducible components J_1 and J_2, meeting transversally at one point, such that $\pi(J_i) = D_i$ with $D_i > 0$ and not equal to E. We will always assume that $q \in J_1$. For any decomposition $D = D_1 + D_2$ we have that the number $j(D_1, D_2)$ of fibers of type **J** such that $\pi(J_i) = D_i$ is

$$j(D_1, D_2) = N(D_1)N(D_2)(D_1 \cdot D_2)\begin{pmatrix} r_0(D) - 2 \\ r_0(D_1) - 1 \end{pmatrix}.$$

We have the coefficient $(D_1 \cdot D_2)$ because, just as in the case of $\mathbb{P}^2$, if $[X] \in \Gamma$ corresponds to a curve of type **J**, in the normalization map $\nu : B \to \Gamma$ the fiber over $[X]$ contains exactly $(D_1 \cdot D_2)$ points.

We let B_J be the subset of points b in B whose fiber X_b is a curve of type **J**.

Type **G**. Fibers having two smooth irreducible components G_1 and G_E, meeting transversally at one point, such that $\pi(G_E) = E$ and $\pi(G_1)$ has a smooth point of contract of order 3 with E. Clearly $q \in G_1$. The total number of such fibers is $N_3(D)$.

We let B_G be the subset of points b in B whose fiber X_b is a curve of type G.

Type **K**. Fibers having four irreducible components K_1, K_E, K_0 and K_2 forming a chain in the given order, that is $K_1 \cap K_E = K_0 \cap K_E = K_0 \cap K_2 = 1$ so that $K_1^2 = K_2^2 = -1$ and $K_E^2 = K_0^2 = -2$. As usual, we have that $q \in \pi(K_1)$. Moreover, $\pi(K_E) = E$ and $\pi(K_1)$ is tangent to E; $\pi(K_0)$ is a point of E (namely, the point $E \cap \pi(K_2)$), in fact the exceptional curve K_0 arises from the fact that the surface χ^ν is singular at the point corresponding to $E \cap \pi(K_2)$ (cf. Proposition 2.7 in [CH]). Let B_K be the subset of B whose corresponding fiber is a curve of type **K**. Finally, for any given decomposition $D = E + D_1 + D_2$ we have that the number $k(D_1, D_2)$ of corresponding fibers of type **K** is given by

$$k(D_1, D_2) = N_2(D_1)N(D_2)(E \cdot D_2)\begin{pmatrix} r_0(D) - 2 \\ r_0(D_1) - 2 \end{pmatrix}.$$

$\mathbf{K}'$. These are just like the fibers of type $\mathbf{K}$ with the only difference that the point q belongs to the curve that is not tangent to E, that is, we have now $\pi(K_2)$ tangent to E. We denote the irreducible components of such a fiber K_1', K_0', K_E' and K_2', forming a chain in the given order, so that

$$(K_1' \cdot K_0') = (K_0' \cdot K_E') = (K_E' \cdot K_2) = 1$$

and

$$(K_1')^2 = (K_2')^2 = -1 \quad \text{and} \quad (K_E')^2 = (K_0')^2 = -2.$$

Moreover, $\pi(K_E') = E$ and $\pi(K_0')$ is a point of E (namely, the point $E \cap \pi(K_1')$). We define as usual $B_{K'}$ to be the subset of B whose corresponding fiber is a curve of type $\mathbf{K}'$. Finally we see that the number $k'(D_1, D_2)$ of such fibers is

$$k'(D_1, D_2) = N(D_1)N_2(D_2)(D_1 \cdot E)\binom{r_0(D) - 2}{r_0(D_1) - 1}.$$

Type **H**. Fibers having four irreducible components H_1, H_2, H_3 and H_E, such that $\pi(H_E) = E$ and $\pi(H_i) = D_i$, with $D_i > 0$ and $D_1 + D_2 + D_3 = D - E$. In Proposition 2.6 of [CH] we proved that if $[X] \in \Gamma$ is a point corresponding to this type of curve, then the fiber of B over Γ contains exactly $(D_1 \cdot E)(D_2 \cdot E)(D_3 \cdot E)$ points. Hence the total number of fibers of type **H** that correspond to a given decomposition $D = D_1 + D_2 + D_3 + E$ is given by

$$\begin{aligned} &h(D_1, D_2, D_3) \\ &\quad = N(D_1)N(D_2)N(D_3)(D_1 \cdot E)(D_2 \cdot E)(D_3 \cdot E) \\ &\quad\quad \binom{r_0(D) - 2}{r_0(D_1) - 2, r_0(D_2)}. \end{aligned}$$

And, as usual, we define B_H to be the subset of points of B corresponding to curves of type **H**.

Here is a picture displaying the various types of reducible fibers in our family:

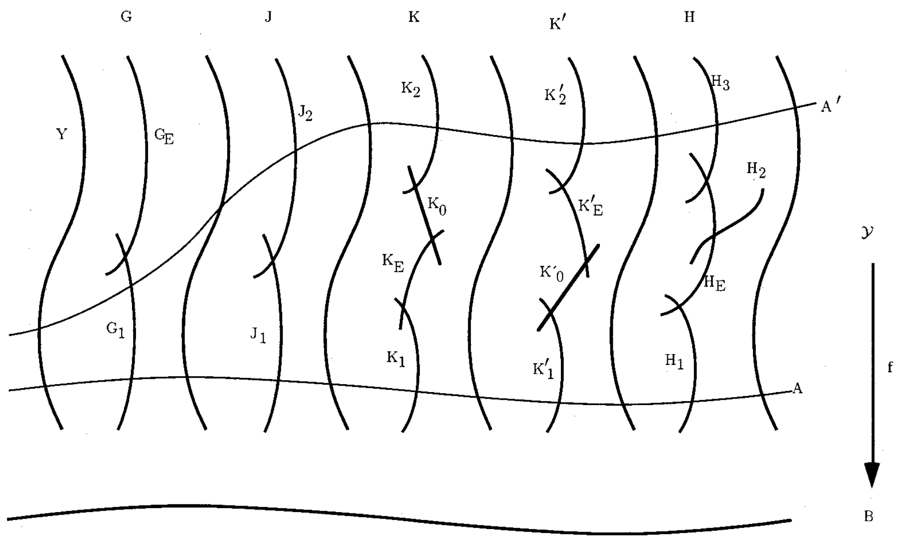

Figure 4.

Now we choose the following set of generators for the Néron–Severi group of $\mathcal{Y}$:

$$\{A, Y\} \cup \{J_{2,b}\}_{b\in B_J} \cup \{G_{E,b}\}_{b\in B_G} \cup \{K_{2,b}, K_{E,b}, K_{0,b}\}_{b\in B_K} \cup$$

$$\cup \{K'_{2,b}, K'_{E,b}, K'_{0,b}\}_{b\in B_{K'}} \cup \{H_{3,b}, H_{2,b}, H_{E,b}\}_{b\in B_H}.$$

The following relations are obvious:

$$G^2_{E,b} = J^2_{2,b} = K^2_{2,b} = (K'_{2,b})^2 = K^2_{1,b} = H^2_{2,b} = -1;$$

$$K^2_{E,b} = K^2_{0,b} = (K'_{E,b})^2 = (K'_{0,b})^2 = -2;$$

$$H^2_{E,b} = -3$$

and the intersection number of A with any generator other than A and Y is zero.

It will also be convenient to have a symbol denoting the class in $\mathrm{NS}(\mathcal{Y})$ of all generators of the same type. Therefore we introduce the classes

$$\mathcal{G}_E := \sum_{b\in B_G} G_{E,b}, \qquad \mathcal{J}_2 := \sum_{b\in B_J} J_{2,b},$$

$$\mathcal{K}_E := \sum_{b\in B_K} K_{E,b}, \qquad \mathcal{K}_0 := \sum_{b\in B_K} K_{0,b}, \qquad \mathcal{K}_2 := \sum_{b\in B_K} K_{2,b},$$

$$\mathcal{K}'_E := \sum_{b\in B'_K} K'_{E,b}, \qquad \mathcal{K}'_0 := \sum_{b\in B'_K} K'_{0,b}, \qquad \mathcal{K}'_2 := \sum_{b\in B'_K} K'_{2,b},$$

and

$$\mathcal{H}_E := \sum_{b \in B_H} H_{E,b}, \qquad \mathcal{H}_3 := \sum_{b \in B_H} H_{3,b}, \qquad \mathcal{H}_2 := \sum_{b \in B_H} H_{2,b}.$$

We now use this notation immediately to write the class of the relative dualizing sheaf $\omega_{\mathcal{Y}/B}$ of the family $f\colon \mathcal{Y} \to B$. We have

$$\begin{aligned}\omega_{\mathcal{Y}/B} &= -2A + A^2 Y + \mathcal{G}_E + \mathcal{J}_2 + \mathcal{K}_E + 2\mathcal{K}_0 + 3\mathcal{K}_2 \\ &\quad + \mathcal{K}'_0 + 2\mathcal{K}'_E + 3\mathcal{K}'_2 + \mathcal{H}_e + 2h_3 + 2h_2.\end{aligned}$$

Now we compute A^2. Let q' be a basic point different from q, then q' determines a section A' such that $(A \cdot A') = 0$. As before, we get $2A^2 = (A - A')^2$ and we can compute the right-hand side by expressing the difference $A - A'$ as a linear combination of components of fibers. So, let n_J be the number of fibers of type **J** such that q' lies on a different component than q. We have

$$n_J = \sum_{D_1 + D_2 = D} N(D_1)N(D_2)(D_1 \cdot D_2) \binom{r_0(D) - 3}{r_0(D_1) - 1}.$$

Now let n_K be the number of fibers of type **K** such that q' lies on a different component than q; we have

$$n_K = \sum_{D_1 + D_2 = D - E} N_2(D_1)N(D_2)(E \cdot D_2) \binom{r_0(D) - 3}{r_0(D_1) - 2}.$$

We define $n_{K'}$ analogously, and obtain

$$n_{K'} = \sum_{D_1 + D_2 = D - E} N(D_1)N_2(D_2)(D_1 \cdot E) \binom{r_0(D) - 3}{r_0(D_2) - 2};$$

and similarly n_H, for which we have

$$\begin{aligned}n_H &= \sum_{D_1 + D_2 + D_3 = D - E} N(D_1)N(D_2)N(D_3)(D_1 \cdot E)(D_2 \cdot E)(D_3 \cdot E) \\ &\quad \binom{r_0(D) - 3}{r_0(D_1) - 1, r_0(D_2) - 1}\end{aligned}$$

(where we will denote by $\binom{n}{a, b}$ the multinomial $n!/a!b!(n - a - b)!$).

Let S_J be the subset of B consisting of those points b such that Y_b is a fiber of type **J** for which q and q' lie on different components. Obviously S_J contains

n_J points. If $b \in S_J$ we write $Y_b = J_{1,b} + J_{2,b}$ (hence $q \in J_{1,b}$ and $q' \in J_{2,b}$). In a completely analogous fashion we define S_K, $S_{K'}$ and S_H. If $b \in S_K$ then we write $Y_b = K_{1,b} + K_{E,b} + K_{0,b} + K_{2,b}$ and similarly if b is in $S_{K'}$ or S_H. We therefore have that if $b \in S_K$ (respectively, $b \in S_{K'}$ and $b \in S_H$), then q' lies on $K_{2,b}$ (respectively on $K'_{2,b}$ and $H_{2,b}$). Now we have

$$A' - A = \sum_{b \in S_J} J_{1,b} + \sum_{b \in S_K} (3K_{1,b} + 2K_{E,b} + K_{0,b})$$
$$+ \sum_{b \in S_{K'}} (3K'_{1,b} + 2K'_{0,b} + K'_{E,b}) + \sum_{b \in S_H} (H_{1,b} - H_{2,b}) + nY.$$

where n is some integer that is irrelevant for our computation. Finally we obtain

$$A^2 = \frac{-n_J - 2n_H - 6n_K}{2}.$$

Now, for any $L \in \mathrm{Pic}(\mathbb{F}_3)$, we have

$$\{\pi^* L\}_A = (L \cdot D),$$
$$\{\pi^* L\}_Y = -(L \cdot D)A^2$$

and

$$\{\pi^* L\}_{G_{E,b}} = -(L \cdot E)$$

for any b in B_G. These are obtained, in the given order, from the products $(\pi^* L \cdot Y) = (L \cdot D), (\pi^* L \cdot A) = 0$ and $(\pi^* L \cdot G_{E,b}) = (L \cdot E)$

Let us fix a fiber of type **J** which we write as $J_{1,b} + J_{2,b}$ as usual; let D_2 be the class in $\mathbb{F}_3$ of $\pi(J_{2,b})$. From the product $(\pi^* L \cdot J_{2,b}) = (L \cdot D_2)$ we see that

$$\{\pi^* L\}_{J_{2,b}} = -(L \cdot D_2).$$

Fix now a fiber of type **K**, which we shall write as $K_{1,b} + K_{E,b} + K_{0,b} + K_{2,b}$, such that the image in $\mathbb{F}_3$ has corresponding divisor classes D_1 for $K_{1,b}$ and D_2 for $K_{2,b}$. The relation $(\pi^* L \cdot K_{1,b}) = (L \cdot D_1)$ implies

$$\{\pi^* L\}_{K_{E,b}} = (L \cdot D_1) - (L \cdot D);$$

the above formula together with $(\pi^* L \cdot K_{E,b}) = (L \cdot E)$ gives

$$\{\pi^* L\}_{K_{0,b}} = 2(L \cdot D_1) - 2(L \cdot D) + (L \cdot E);$$

and the two previous formulas combined with $(\pi^* L \cdot K_0) = 0$ gives

$$\{\pi^* L\}_{K_{2,b}} = 3(L \cdot D_1) - 3(L \cdot D) + 2(L \cdot E).$$

With completely analogous notation and procedure, for a fixed fiber of type $\mathbf{K}'$ we have

$$\{\pi^* L\}_{K'_{0,b}} = (L \cdot D_1) - (L \cdot D),$$

$$\{\pi^* L\}_{K'_{E,b}} = 2(L \cdot D_1) - 2(L \cdot D),$$

$$\{\pi^* L\}_{K'_{2,b}} = 3(L \cdot D_1) - 3(L \cdot D) + (L \cdot E).$$

Finally, fix a fiber of type $\mathbf{H}$ such that the class in $\mathbf{F}_3$ corresponding to $H_{i,b}$ is D_i; again the same procedure yields

$$\{\pi^* L\}_{H_{E,b}} = (L \cdot D_1) - (L \cdot D),$$

$$\{\pi^* L\}_{H_{3,b}} = (L \cdot D_1) - (L \cdot D) - (L \cdot D_3)$$

and

$$\{\pi^* L\}_{H_{2,b}} = (L \cdot D_1) - (L \cdot D) - (L \cdot D_2).$$

We shall also use the following short notation:

$$\begin{aligned} \pi^* L \;=\; & (L \cdot D)A + (L \cdot D)\left(\frac{-n_J - n_H - 6n_K}{2}\right)Y \\ & -(L \cdot E)\mathcal{G}_E + J^L + K^L + K'^L + H^L \end{aligned} \tag{$*$}$$

Now we want to compute the intersection product on $\mathcal{Y}$ of the pull-back of two line bundles L and M on $\mathbb{F}_3$. We easily have

$$\begin{aligned} (\pi^* L \cdot \pi^* M) \;=\; & -(L \cdot D)(M \cdot D)A^2 - (L \cdot E)(M \cdot E)f(D - E) \\ & +(J^L \cdot J^M) + (K^L \cdot K^M) + (K'^L \cdot K'^M) + (H^L \cdot H^M). \end{aligned}$$

And now a completely straightforward computation yields

$$(J^L \cdot J^M) = -\sum_{D_1 + D_2 = D} j(D_1, D_2)(L \cdot D_2)(M \cdot D_2),$$

$$\begin{aligned} (K^L \cdot K^M) \;=\; & -\sum_{D_1 + D_2 = D - E} k(D_1, D_2)((L \cdot E)((M \cdot D_1) - (M \cdot D)) + \\ & -(L \cdot D_2)(3(M \cdot D_2) + (M \cdot E))), \end{aligned}$$

$$
\begin{aligned}
(K'^L \cdot K'^M) \;=\; & -\sum_{D_1+D_2=D-E} k'(D_1,D_2)((L\cdot E)(2(M\cdot D_1)-2(M\cdot D))+ \\
& -(L\cdot D_2)(3(M\cdot D_2)+2(M\cdot E)))
\end{aligned}
$$

and

$$
\begin{aligned}
(H^L \cdot H^M) \;=\; & -\frac{1}{2}\sum_{D_1+D_2+D_3=D-E} h(D_1,D_2,D_3) \\
& \times(-(L\cdot D)(M\cdot D)+((L\cdot D)(M\cdot D_1) \\
& +(L\cdot D_1)(M\cdot D)-(L\cdot D_1)(M\cdot D_1) \\
& -(L\cdot D_2)(M\cdot D_2)-(L\cdot D_3)(M\cdot D_3)).
\end{aligned}
$$

In the last formula, we divide by 2 because D_2 and D_3 are not distinguished from one another.

Now we are ready to write down the three formulas that we mentioned at the beginning of this chapter. Before we carry out the computation, we can explain briefly the procedure. We have to look at the relation $(*)$ and keep track of the Severi degrees on which the characteristic numbers depend.

(a) The first relation we shall use is

$$(\pi^*C\cdot\pi^*C)=3N(D).$$

This will give a formula expressing

$N(D)$ in terms of $N(D')$ and $N_2(D'')$ with

$D'<D$ and $D''<D-E$.

This is clear; since $(C\cdot E)=0$ if we apply (LM) to $L=M=\mathcal{O}_S(C)$ the Severi degree $N_3(D-E)$ disappears.

(b) Now we need a formula for $N_2(D)$. We will imitate what we did to compute the degree of the variety of rational curves tangent to a fixed line in the plane. We define $\tilde{E}$ to be the class of the irreducible component of $\pi^{-1}(E)$ that dominates B. Then we have

$$\tilde{E}=\pi^*E-3\mathcal{G}_E-2\mathcal{K}_E-\mathcal{K}_0-2\mathcal{K}'_E-\mathcal{K}'_0-\mathcal{H}_E.$$

This is obtained as follows: for the coefficient of $\mathcal{G}_E$ we notice that for any $b\in B_G$ we have $(\tilde{E}\cdot G_{E,b})=0$, while on the other hand $(\pi^*E\cdot G_{E,b})=-3$. The same procedure yields the remaining terms.

$$N_2(D)=\tilde{E}^2+(\tilde{E}\cdot\omega_{\mathcal{Y}/B}).$$

This will give a recursion expressing

$$N_2(D) \quad \text{in terms of} \quad N(D), N(D'), N_3(D-E), \quad \text{and} \quad N_2(D'') \quad \text{with}$$
$$D' < D \quad \text{and} \quad D'' < D-E.$$

(c) The third and last step will be to find a formula for $N_3(D-E)$. This will be done by using

$$(\pi^* F \cdot \pi^* F) = 0$$

which, as one can imply using $(*)$, will give

$$N_3(D-E) \quad \text{in terms of} \quad N(D') \quad \text{and} \quad N_2(D'') \quad \text{with}$$
$$D' < D \quad \text{and} \quad D'' < D-E.$$

Example. If $D = 2C$ we have on one hand that $(\pi^* C \cdot \pi^* C) = 3N(2C)$, and on the other hand our formulas give

$$\begin{aligned}(\pi^* C \cdot \pi^* C) \;=\; & 3[-12A^2 - 3j(C,C) - k(C+2F,F) - 25k'(F,C+2F)\\ & -14h(F,F,C+F) - 2h(C+F,F,F)].\end{aligned}$$

Here are the relevant numbers for the case $D = 2C$.

$$\begin{aligned}&j(C,C) = 105, \qquad k(C+2F,F) = 14, \qquad k'(F,C+2F) = 2,\\ &h(F,F,C+F) = 7, \qquad h(C+F,F,F) = 21, \qquad n_J = 60,\\ &n_K = n_{K'} = 2, \qquad n_H = 13,\end{aligned}$$

so that

$$A^2 = 49$$

and we can conclude that $N(2C) = 69$.

We will now state our main result for $\mathbb{F}_3$:

THEOREM. *Let $D \in \mathrm{Pic}(\mathbb{F}_3)$. Let $N(D)$ be the number of irreducible rational curves in $|D|$ that pass through $r_0(D)$ general points. Then*

$$\begin{aligned}N(D) \;=\; & \frac{1}{3} \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2)\\ & \cdot \left[\binom{r_0(D)-3}{r_0(D_1)-1} (D_1 \cdot C)(D_2 \cdot C) - \binom{r_0(D)-3}{r_0(D_1)-2} (D_2 \cdot C)^2 \right]\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{D_1+D_2=D-E} N_2(D_1)N(D_2)(E\cdot D_2)\\
&\cdot\left[\binom{r_0(D)-3}{r_0(D_1)-2}(D_1\cdot C)(D_2\cdot C) - \binom{r_0(D)-3}{r_0(D_1)-3}(D_2\cdot C)^2\right]\\
&+ \sum_{D_1+D_2=D-E} N(D_1)N_2(D_2)(E\cdot D_1)\\
&\cdot\left[\binom{r_0(D)-3}{r_0(D_1)-1}(D_1\cdot C)(D_2\cdot C) - \binom{r_0(D)-3}{r_0(D_1)-2}(D_2\cdot C)^2\right]\\
&+\frac{1}{3}\sum_{D_1+D_2+D_3=D-E} N(D_1)N(D_2)N(D_3)(E\cdot D_1)(E\cdot D_2)(E\cdot D_3)\cdot\\
&\cdot\Bigg[\binom{r_0(D)-3}{r_0(D_1)-1, r_0(D_2)-1}\\
&\quad\times[2(C\cdot D_1)(C\cdot D_2)+(C\cdot D_1)(C\cdot D_3)\\
&\quad+(C\cdot D_2)(C\cdot D_3)-(C\cdot D_3)^2]\\
&\quad-\binom{r_0(D)-3}{r_0(D_1)-2, r_0(D_2)}\\
&\quad\times[(C\cdot D_2)^2+(C\cdot D_3)^2+(C\cdot D_2)(C\cdot D_3)]\Bigg].
\end{aligned}
$$

Proof. We just have to compute. Applying $(*)$ to π^*C gives

$$
\begin{aligned}
(\pi^*C)^2 = &-(C\cdot D)^2A^2 - \sum_{D_1+D_2=D} j(D_1,D_2)(C\cdot D_2)^2\\
&- \sum_{D_1+D_2=D-E} 3k(D_1,D_2)(C\cdot D_2)^2\\
&- \sum_{D_1+D_2=D-E} 3k'(D_1,D_2)(C\cdot D_2)^2\\
&- \sum_{D_1+D_2+D_3=D-E} h(D_1,D_2,D_3)[(C\cdot D_2)^2\\
&+(C\cdot D_3)^2+(C\cdot D_2)(C\cdot D_3)].
\end{aligned}
$$

This gives

$$
\begin{aligned}
N(D) = & \frac{1}{3} \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \\
& \cdot \left[\binom{r_0(D)-3}{r_0(D_1)-1} (D_1 \cdot C)(D_2 \cdot C) - \binom{r_0(D)-3}{r_0(D_1)-2} (D_2 \cdot C)^2 \right] \\
& + \sum_{D_1+D_2=D-E} N_2(D_1)N(D_2)(E \cdot D_2) \\
& \cdot \left[2\binom{r_0(D)-3}{r_0(D_1)-2} (D_1 \cdot C)(D_2 \cdot C) \right. \\
& \left. - \binom{r_0(D)-3}{r_0(D_1)-1} (D_1 \cdot C)^2 - \binom{r_0(D)-3}{r_0(D_1)-3} (D_2 \cdot C)^2 \right] \\
& + \frac{1}{3} \sum_{D_1+D_2+D_3=D-E} N(D_1)N(D_2)N(D_3)(E \cdot D_1)(E \cdot D_2)(E \cdot D_3) \\
& \cdot \left[(C \cdot D)^2 \binom{r_0(D)-3}{r_0(D_1)-1, \quad r_0(D_2)-1} \right. \\
& - ((C \cdot D_2)^2 + (C \cdot D_3)^2 + (C \cdot D_2)(C \cdot D_3)) \\
& \left. \cdot \binom{r_0(D)-2}{r_0(D_1)-1, \quad r_0(D_2)} \right].
\end{aligned}
$$

And this concludes the proof.

We will write as well the formulas for the degrees of the other loci that we need. The first formula is obtained by

$$N_2(D) = \tilde{E}^2 + (\tilde{E} \cdot \omega_{\mathcal{Y}/B}),$$

which gives

$$
\begin{aligned}
N_2(D) = & -3N(D) + 9N_3(D-E) + (E \cdot D)A^2 \\
& + \sum_{D_1+D_2=D} j(D_1, D_2)(E \cdot D_2) \\
& + \sum_{D_1+D_2=D-E} 6(k(D_1, D_2) + k'(D_1, D_2)) \\
& + \sum_{D_1+D_2+D_3=D-E} h(D_1, D_2, D_3)[2(E \cdot D_2) + 2(E \cdot D_3) - 1].
\end{aligned}
$$

Finally, the degree of the Severi variety parametrizing rational curves having a point of contact of order at least 3 with E is obtained by $\pi^* F^2 = 0$, which translates into

$$
\begin{aligned}
N_3(D-E) \;=\; & -(F\cdot D)^2 A^2 \\
& - \sum_{D_1+D_2=D} j(D_1,D_2)(F\cdot D_2)^2 \\
& - \sum_{D_1+D_2=D-E} k(D_1,D_2)[1+2(F\cdot D_2)+3(F\cdot D_2)^2] \\
& - \sum_{D_1+D_2=D-E} k'(D_1,D_2)[2+4(F\cdot D_2)+3(F\cdot D_2)^2] \\
& - \sum_{D_1+D_2+D_3=D-E} h(D_1,D_2,D_3) \\
& \times[(F\cdot D)^2 - 2(F\cdot D)(F\cdot D_1) + (F\cdot D_1)^2 \\
& +(F\cdot D_2)^2 + (F\cdot D_3)^2].
\end{aligned}
$$

References

[CH] Caporaso, L. and Harris, J.: *Parameter spaces for curves on surfaces and enumeration of rational curves* 1995, Preprint.

[CM] Crauder, B. and Miranda, R.: Quantum cohomology of rational surfaces. The Moduli Space of Curves, *Progress in Mathematics* 129, Birkhauser (1995) 33–80.

[DH] Diaz, S. and Harris, J.: Geometry of the Severi variety. *Trans. A.M.S.* 309 (1988), 1–34.

[DI] Di Francesco, P. and Itzykson, C.: Quantum intersection rings. The Moduli Space of Curves, *Progress in Mathematics* 129, Birkhauser (1995) 81–148.

[H] Harris, J.: On the Severi problem. *Invent. Math.* 84 (1986) 445–461.

[K] Kollár, J.: *Rational Curves on Algebraic Varieties*, Springer 1996.

[KP] S. Kleiman and Piene, R.: Private communication.

[KM] Kontsevich, M. and Manin, Y.: *Gromov–Witten Classes, Quantum Cohomology and Enumerative Geometry.*

[P] Pandharipande, R.: *Intersections of $\mathbb{Q}$-divisors on Kontsevich's Moduli Space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and Enumerative Geometry.* 1995, Preprint.

[R] Ran, Z.: Enumerative geometry of singular plane curves. *Invent. Math.* 97 (1989) 447–465.

[R2] Ran, Z.: *On the Quantum Cohomology of the Plane Old and New.* (1995) Preprint, alg-geom 9508011.

[RT] Ruan, Y. and Tian, G.: A mathematical theory of quantum cohomology, *J. Diff. Geom.* 42(2) (1995) 259–367.