Tropical methods in moduli theory and algebraic geometry
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Contents

1. Lecture 1: Basics on smooth curves

1.1. **Smooth curves and their Picard group.** Unless otherwise stated, by *curve* we mean a reduced, connected projective variety (not necessarily irreducible) of dimension one, defined over an algebraically closed field k.

Let C be a smooth curve.

The *Picard group* of C can be defined in various ways, and we shall use each time the most convenient version. First, as the set of divisors on C modulo linear equivalence, in symbols

$$\mathrm{Pic}(C) = \mathrm{Div}(C)/\sim$$

Next, via isomorphism classes of line bundles (equivalently, of invertible sheaves)

$$\mathrm{Pic}(C) = \{\text{Line bundles on } C\}/\cong$$

For any $D \in \mathrm{Div}(C)$ we denote by $\mathcal{O}(D)$ the corresponding line bundle.

$\mathrm{Pic}(C)$ is an abelian group, with the trivial bundle, $\mathcal{O}_C$, as neutral element. With divisors, for the operation on $\mathrm{Pic}(C)$ one uses the additive notation: $[D], [D'] \in \mathrm{Pic}(C)$ with $D, D' \in \mathrm{Div}(C)$ then $[D] + [D'] := [D + D']$ whereas for line bundles one uses the multiplicative notation: $L, L' \in \mathrm{Pic}(C)$ we write $LL' = L \otimes L'$. Usually, line bundles and their isomorphism classes are denoted in the same way.

We have a surjective homomorphism

$$\deg : \mathrm{Pic}(C) \longrightarrow \mathbb{Z}$$

such that if $D = \sum_{p \in C} n_p p$ then $\deg([D]) = \deg D = \sum_{p \in C} n_p$. Its kernel is a remarkable subgroup

$$\mathrm{Pic}^0(C) = \{L \in \mathrm{Pic}(C) : \deg L = 0\},$$

the Jacobian of C, also denoted as $\mathrm{Jac}(C)$.

The following is well known

Theorem 1.1.1.

$$\mathrm{Pic}(C) \cong \mathbb{Z} \Longleftrightarrow C \cong \mathbb{P}^1.$$

For any divisor $D \in \mathrm{Div}(C)$ the set of effective divisors linearly equivalent to D is written as follows

$$|D| := \{E \in \mathrm{Div}(C) : E \geq 0, E \sim D\}.$$

If $|D|$ is not empty, then it is identified with a projective space

$$|D| = \mathbb{P}^{r(D)} = \mathbb{P}(H^0(C, D))$$

where $H^0(C, D) = H^0(C, \mathcal{O}(D))$ is the vectors space of global sections of $\mathcal{O}(D)$. Its dimension is written $h^0(C, D)$ and, of course,

$$r(D) = h^0(C, D) - 1$$

so that

$$|D| = \emptyset \Longleftrightarrow \deg h^0(C, D) = 0$$

If $|D| \neq \emptyset$ we have a regular map

$$\phi_D : C \longrightarrow \mathbb{P}^{r(D)}$$

and if $r(D) \geq 1$ for any hyperplane $H \subset \mathbb{P}^{r(D)}$ (an effective divisor on $\mathbb{P}^{r(D)}$), the divisor on C given by the pull-back of H, satisfies

$$\phi_D^* H \in |D|;$$

conversely, every $E \in |D|$ is obtained in this way.

Definition 1.1.2. Let $D \in \mathrm{Div}(C)$.

(1) D is *very ample* if the map ϕ_D induces an isomorphism between C and $\phi_D(C)$;
(2) D is *ample* if there exists $n > 0$ such that nD is very ample.

Example 1.1.3. For the trivial line bundle, $\mathcal{O}_C$, we have $\deg \mathcal{O}_C = 0$ and $h^0(C, \mathcal{O}_C) = 1$. Moreover, these two conditions characterize $\mathcal{O}_C$.

Example 1.1.4. Let us introduce the most important line bundle on a curve C: the *canonical line bundle* denoted by K_C. This is the dual of the tangent bundle of C,

$$K_C = T_C^*$$

and the *genus* of C is defined as follows

$$g_C := h^0(C, K_C);$$

we have

$$\deg K_C = 2g_C - 2$$

and K_C is the unique line bundle on C satisfying these two conditions.

Remark 1.1.5. If $k = \mathbb{C}$ the genus defined above is equal to the topological genus of the surface, S_C, over $\mathbb{R}$ underlying C. Indeed, S_C is a compact, connected and orientable topological manifold of dimension 2: compactness and connectedness follow from the definition. Orientability follows from the fact that, in the analytic topology, C is covered by open subsets holomorphic to open subsets of $\mathbb{C}$, and holomorphic maps are conformal, hence they preserve the orientation. So the orientation of $\mathbb{C}$ induces an orientation on S_C.

Theorem 1.1.6. *(Riemann-Roch) For any $D \in \mathrm{Div}(C)$ we have*

$$h^0(C, D) - h^0(C, K_C - D) = \deg D - g_C + 1$$

equivalently, as by Serre's duality $h^0(C, K_C - D) = h^1(C, D)$,

$$h^0(C, D) - h^1(C, D) = \deg D - g_C + 1.$$

Corollary 1.1.7. *If $\deg D \geq 2g - 1$ then $h^0(C, D) = \deg D - g_C + 1$*

Proof. The hypothesis implies $\deg(K_C - D) < 0$ hence $h^0(C, K_C - D = 0)$ (for a divisor of negative degree $|D| = \emptyset$) hence by Riemann-Roch we are done. ♣

The following is a consequence of Riemann-Roch.

Fact 1.1.8. *Let $D \in \mathrm{Div}(C)$.*
(1) If $\deg D \geq 2g+1$ then D is very ample.
(2) D is ample if and only if $\deg D > 0$.

In particular

$$K_C \text{ is ample } \iff g_C \geq 2. \tag{1}$$

1.2. **Moduli spaces in low genus.** We will denote by M_g the *moduli spaces* of smooth curves of genus g, to be fully defined soon. As a first approximation, let us view M_g as the set of isomorphism classes of curves of genus g.

M_0 consists of one element, by the following.

Proposition 1.2.1. *If C has genus 0, then $C \cong \mathbb{P}^1$.*

Proof. If $g_C = 0$ then $\deg K_C = -2$ and $\deg T_C = 2$. Hence T_C is very ample and gives an embedding

$$\phi_{T_C} : C \hookrightarrow \mathbb{P}^2$$

whose image is a smooth conic, C', isomorphic to C. It is well known that any two smooth conics are isomorphic, and that every conic is isomorphic to $\mathbb{P}^1$ (the isomorphism is obtain by considering at the set of all lines passing through a point of the conic, which is a $\mathbb{P}^1$). ♣

Let $g = 1$, and assume *char*$k \neq 2$. The classical j-invariant gives is a bijection $\mathcal{M}_1 \leftrightarrow k$, hence one can endow M_1 with the structure of an algebraic variety, namely the affine line

$$M_1 = \mathbb{A}^1.$$

Let $g = 2$ and *char*$k \neq 2$. Now K_C has degree 2 and determines a morphism

$$\phi : C \longrightarrow |K_C| = \mathbb{P}^1$$

necessarily surjective of degree 2. Moreover, up to automorphisms of $\mathbb{P}^1$, the map ϕ is unique. We say that a point $p \in \mathbb{P}^1$ is a *branch point* if $|\phi^{-1}(p)| = 1$. Since ϕ has degree 2, the number of ramification points coincides with the degree of the ramification divisor of ϕ, which is given by the Riemann-Hurwitz formula

Theorem 1.2.2. *(Riemann-Hurwitz) Assume char$k \neq 2$. Let $\psi : C \longrightarrow D$ be a finite map of degree d between two smooth projective curves C and D of respective genus g_C and g_D. Let $R \in \mathrm{Div}(C)$ be the ramification divisor of ψ. Then*

$$\deg R = 2g_C - 2 - d(2g_D - 2).$$

By the Riemann-Hurwitz formula the ramification divisor of our ϕ has degree 6, hence ϕ has exactly 6 branch points.

Conversely, given 6 points in $\mathbb{P}^1$ there exists a unique curve C endowed with a degree 2 map to $\mathbb{P}^1$, given as follows. Let $a_1, \dots, a_6 \in k$ be the affine coordinates of the 6 points. Consider the quadratic extension of the function field, $k(x)$, of $\mathbb{P}^1$ given by adding $y := \sqrt{\Pi_{i=1}^{6}(x - a_i)}$. Then $k(x)(y)$ has transcendence degree 1 over k, and hence it is the function field of a unique curve C.

The degree two extension $k(x) \hookrightarrow k(C)$ is easily seen to correspond to a degree-2 map $C \to \mathbb{P}^1$ ramified exactly over the 6 given points.

On the other hand, any 6-tuple of points in $\mathbb{P}^1$ can be written, up to a unique automorphisms of $\mathbb{P}^1$ as

$$\{0, 1, \infty, b_1, b_2, b_3\} : \quad b_i \in k \smallsetminus \{0, 1\},\ i = 1, 2, 3.$$

Denote by $\Delta \subset (k \smallsetminus \{0, 1\})^3$ the union of all diagonals, then we have a surjection

$$(k \smallsetminus \{0, 1\})^3 \smallsetminus \Delta \longrightarrow M_2$$

which maps (b_1, b_2, b_3) to the curve C which has a degree-2 map to $\mathbb{P}^1$ ramified over $\{0, 1, \infty, b_1, b_2, b_3\}$. Let $U := (k \smallsetminus \{0, 1\})^3 \smallsetminus \Delta$, then one easily checks that U is an affine variety, hence M_2 is the quotient of an affine variety by a finite group, which is itself an affine variety.

2. Lecture 2

2.1. **The moduli scheme of smooth curves.** We have seen that the set of isomorphism classes of genus 1 and 2 is endowed with a natural structure of algebraic variety, dictated by the geometry of the objects it parametrizes. On the other hand, this structure tells us something about the parametrized curves. It tells us that there is a 1-dimensional (resp. 3-dimensional) family of curves of genus 1 (resp. 2). It also tells us that such curves do not form a complete space! This will be an important point in the sequel.

Let us list some properties that one would hope a moduli *scheme* M_g, for smooth curves of genus g satisfies.

(1) The (closed) points of M_g are in bijection with isomorphism classes of smooth curves of genus g. More precisely, $M_g(k)$ is in bijection with the set of isomorphism classes of smooth curves of genus g defined over k, where $M_g(k)$ denotes the set of closed points of M_g, i.e.

$$M_g(k) \leftrightarrow \mathrm{Hom}(\mathrm{Spec}\, k, M_g).$$

(2) For every *family* $f : \mathcal{C} \to B$ of smooth curves of genus g (i.e. for every flat proper morphism of schemes such that for every closed

point $b \in B$ the fiber $C_b = f^{-1}(b)$ is a smooth curve of genus g), the natural map

$$\mu_f : B \longrightarrow M_g; \qquad b \longmapsto \mu_f(b) = [C_b]$$

is a morphism of schemes. We call μ_f the *moduli map* of the family.

(3) Properties (1) and (2) determine M_g up to isomorphism.

(4) For any morphism of schemes $\phi : B \longrightarrow M_g$ there exists a family (as defined in (2)) of smooth curves $f : \mathcal{C} \to B$ such that $\phi = \mu_f$, and this family is unique up to B-isomorphisms, i.e. if $f' : \mathcal{C}' \to B$ is another family such that $\mu_{f'} = \phi$ then there is an isomorphism $\alpha : \mathcal{C} \to \mathcal{C}'$ such that $\alpha \circ f' = f$.

Remark 2.1.1. Property (4) is actually a generalization of (1). Indeed, the closed points of M_g are identified with the set of morphisms from $\operatorname{Spec} k$ to M_g. Hence property (1) is the case $B = \operatorname{Spec} k$ in (4).

The first three propeties are satisfied for all $g \geq 0$. The case $g = 0$ is trivial, so we omit it in the next statement

Theorem 2.1.2 (Mumford)**.** *For every $g \geq 1$ there exists an integral quasiprojective, non projective, scheme, M_g defined over* $\operatorname{Spec} \mathbb{Z}$ *which satisfies properties* (1), (2) *and* (3)*, but not* (4)*. Moreover*

If $g = 1$ then $\dim M_1 = 1$.

If $g \geq 2$ then $\dim M_g = 3g - 3$.

Remark 2.1.3. Property (4) cannot possibly be satisfied. In fact, both the existence part and the uniqueness part can fail, and the obstruction lies in the existence of curves having non trivial automorphism group. More precisely, there exists morphisms $\phi : B \to M_g$ for which there does not exist a family of smooth curves over B whose moduli map is ϕ. And, for every $g \geq 1$, there exist families of smooth curves over the same scheme B which are not isomorphic over B but have the same moduli map.

Exploring this phenomenon would take us too far from our main goal. Let us just mention that for a moduli space to satisfy property (4) one needs to weaken the requirement that it be a scheme, and require that it be a stack.

We shall limit ourselves to give an example showing the failure of uniqueness. The failure of uniqueness follows from the existence of isotrivial families that are not trivial, i.e. they are not of the form $B \times C$, for a fixed smooth curve, C, of genus g.

Example 2.1.4. (Isotrivial family 1) Consider the following family

$$by^2 = (x - a_1)(x - a_2) \cdot \ldots \cdot (x - a_6)$$

for $a_1, \ldots, a_6 \in k$ with $a_i \neq a_j$. This is a family of curves of genus 2 parametrized by $b \in B = \mathbb{A}^1 \smallsetminus \{0\}$. By what we already said, the fibers of this family are all isomorphic to the same curve of genus 2, hence the corresponding moduli map is a constant map to M_2. On the other hand, the total space of this family, $\mathcal{C}$, is birational to a suitable blow-up of $\mathbb{P}^2$, hence it is a rational surface. Therefore it cannot possibly be isomorphic to $C \times \mathbb{A}^1$, which is not rational as C is not rational.

On the other hand the trivial family $f' : C \times \mathbb{A}^1 \to \mathbb{A}^1$ has the constant map as moduli map, which coincides with μ_f away from 0.

What we will be mostly interested in is the fact that M_g is not complete. This fact says that there are families of smooth curves which degenerate to singular ones. So we will study the problem of completing M_g in a modular way, i.e. by constructing a projective scheme $\overline{M}_g$ which contains M_g as dense open subset, and which is itself a moduli space.

A remarkable solution to this problem, provided by Deligne and Mumford, consists in extending the set of smooth curves to the set of reduced (possibly reducible) curves having at most nodal as singularities, and having finitely many automorphisms.

Definition 2.1.5. A *stable curve* is a connected reduced curve X having at most nodes as singularities, and such that $\mathrm{Aut}(X)$ is finite.

Remark 2.1.6. Smooth curves of genus at least 2 are stable, since they have finitely many automorphisms.

We shall go back to this definition later. The next goal is to introduce tropical curves.

2.2. **Graphs and contractions.** We define graphs in purely combinatorial terms.

Definition 2.2.1. A *graph* G with n *legs* is the following set of data:

(1) A finite non-empty set $V(G)$, the set of *vertices*.
(2) A finite set $H(G)$, the set of *half-edges*.
(3) An involution

$$\iota : H(G) \longrightarrow H(G) \qquad h \mapsto \overline{h}$$

with n fixed points, called *legs*, whose set is denoted by $L(G)$.
(4) An *endpoint* map $\epsilon : H(G) \to V(G)$.

A pair $e = \{h, \overline{h}\}$ of distinct elements in $H(G)$ interchanged by the involution is called an *edge* of the graph; the set of edges is denoted by $E(G)$. If $\epsilon(h) = v$ we say that h, or e, is *adjacent* to v.

The *valence* or *degree* of a vertex v is the number $|\epsilon^{-1}(v)|$ of half-edges adjacent to v.

An edge adjacent to a vertex of valence 1 is called a *leaf edge*.

An edge whose endpoints coincide is called a *loop*.

Definition 2.2.2. A morphism α between combinatorial graphs G and G' is a map $\alpha : V(G) \cup H(G) \to V(G') \cup H(G')$ such that the two diagrams below are commutative.

$$\begin{array}{ccc} V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \\ \downarrow{\scriptstyle (id_V,\epsilon)} & & \downarrow{\scriptstyle (id_{V'},\epsilon')} \\ V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \end{array} \tag{2}$$

$$\begin{array}{ccc} V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \\ \downarrow{\scriptstyle (id_V,\iota)} & & \downarrow{\scriptstyle (id_{V'},\iota')} \\ V(G)\cup H(G) & \xrightarrow{\alpha} & V(G')\cup H(G') \end{array} \tag{3}$$

Remark 2.2.3. By the first diagram $\alpha(V(G)) \subset V(G')$. On the other hand the image of an edge $e \in E(G)$ is either an edge, or a vertex v' of G'; in the latter case the endpoints of e are also also mapped to v', and we say that e is contracted by α.

We will abuse notation and denote by $\alpha : G \longrightarrow G'$ a morphism as above. By definition, such a morphism induces the following maps on the set of vertices, edges and legs of G:

$$V(G) \xrightarrow{\alpha_V} V(G'), \quad E(G) \xrightarrow{\alpha_E} V(G') \cup E(G'), \quad \alpha_L : L(G) \xrightarrow{\alpha_L} V(G') \cup L(G').$$

We say that α is an *isomorphism* if α_V is a bijection, and if α_E and α_L induce bijections between the set of, respectively, edges and legs.

The genus $g(G)$ is its first Betti number

$$g(G) = b_1(G) := \mathrm{rk}_{\mathbb{Z}} H_1(G, \mathbb{Z}) = |E(G)| - |V(G)| + c$$

where c is the number of connected components of G.

We now describe a type of morphism between graphs, the *contraction of an edge* which will play an important role. Let G be graph and $e \in E(G)$. Let G/e be the graph obtained by contracting e to a vertex and leaving everything else unchanged. Then there is a natural map $G \to G/e$, called the *contraction* of e. If $v, w \in V(G)$ are the ends of e, then $V(G/E)$ is obtained from $V(G)$ by identifying v and w:

$$V(G/e) = V(G)/\{v = w\}$$

and

$$E(G/e) = E(G) \setminus \{e\},$$

and $L(G/e) = L(G)$.

Notice that if e is a loop, then

$$g(G/e) = g(G) - 1$$

hence the contraction does not preserve the genus.

3. Lecture 3

3.1. **Pure tropical curves.**

Definition 3.1.1. Let $n \geq 0$. A *n-marked (pure) tropical curve* of genus $g \geq 0$ is a pair $\Gamma = (G, \ell)$ where G is a graph of genus g with n legs and ℓ a *length* function

$$\ell : E(G) \cup L(G) \to \mathbb{R}_{>0} \cup \{\infty\}$$

such that $\ell(x) = \infty$ if and only if either $x \in E(G)$ and e is a leaf-edge, or $x \in L(G)$.

The underlying graph Γ is called the *combinatorial type* of Γ.

The legs of a tropical curve are sometimes called marked points, and they are usually labeled (i.e. ordered). This motivates the following definition of isomorphism between tropical curves.

Let $\Gamma = (G, \ell)$ and $\Gamma' = (G', \ell')$ be two n-marked tropical curves, with $G = (V, E, L = (x_1, \ldots, x_n))$ and $G' = (V', E', L' = (x'_1, \ldots, x'_n))$. An *isomorphism* $\alpha : \Gamma \to \Gamma'$ is an isomorphism of graphs $\alpha : G \to G'$ (defined earlier) such that $\alpha(x_i) = x'_i$ for $i = 1, \ldots, n$ and such that

$$\ell'(\alpha(e)) = \ell(e), \quad \forall e \in E.$$

Let us now define the notion of equivalence for tropical curves. This is weaker than isomorphism: equivalent curves need not be isomorphic.

Informally speaking, two tropical curves are *(tropically) equivalent* if, up to isomorphism, they can be obtained from one another by adding or removing vertices of valence 2, or vertices of valence 1 together with their adjacent leaf-edge. More precisely, if they become isomorphic after performing the following two moves a finite number of times.

(1) *Addition/removal of a vertex of valence* 1 *and of of its adjacent edge (a leaf).* The next picture illustrates the removal of the one-valent vertex u_0 and of the leaf e_0 adjacent to it. The opposite

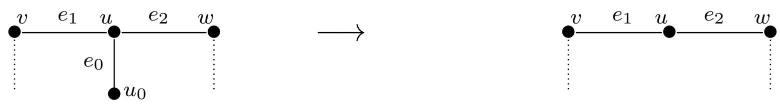


FIGURE 1. Removal of the 1-valent vertex u_0 and of its adjacent edge e_0.

move is the addition of a leaf, where the length of the added edge e_0 is set equal to ∞.

(2) *Addition/removal of a vertex of valence* 2. Pick an edge $e \in E(\mathbf{G})$ and denote by $v, w \in V(\mathbf{G})$ its endpoints. We can add a vertex u in the interior of e. This move replaces the edge e of length $\ell(e)$ by two edges e_v (with endpoints v, u) and e_w (with endpoints w, u), whose lengths satisfy $\ell(e) = \ell(e_v) + \ell(e_w)$. If one of the two endpoints of e, say v, has valence 1 we set the length of e_v equal to ∞, whereas the length of e_w can be arbitrary. The opposite procedure, which should be clear, is represented in the figure 2 below:

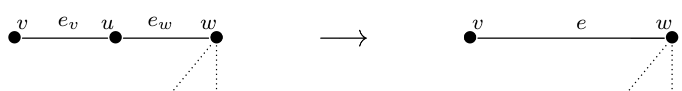


FIGURE 2. Removal of a vertex of valence 2.

Obviously, tropical equivalence preserves the number of marked points and the genus.

Example 3.1.2. Let Γ have one vertex and no edges or legs. Then Γ is an unpointed tropical curve of genus 0, equivalent to any tropical curve of genus 0. From the moduli point of view, its equivalence class is viewed as a trivial one, and will be excluded in future considerations. By a similar reasoning, if $g = 0$ we shall always assume $n \geq 3$.

Let now G be a graph with only one vertex, v, and one loop attached to it; so any $\Gamma = (G, \ell)$ is an unpointed tropical curve of genus 1. Now v is 2-valent, hence can be removed, leaving us with something which is not a tropical curve. For this reason, these curves are viewed as degenerate, and will also be excluded. Hence if $g = 1$ we shall always assume $n > 0$.

This motivates the future assumption $2g-2+n \geq 1$.

Proposition 3.1.3. *Let $2g-2+n>0$. Every equivalence class of n-pointed pure tropical curves of genus g has a unique (up to isomorphism) representative, called canonical or stable, having no vertex of degree less than* 3.

Proof. Easy. ♣

Example 3.1.4. Let $g=0$ and $n=4$. A graph with 4 legs and no vertex of valence ≤ 2 can have at most 1 edge. The graph with 0 edges is unique. On the other hand there are three non-isomorphic graphs with one edge and vertices of degree ≥ 3 according to how the 4 legs are distributed. They are drawn in the following picture. Each of these graphs supports a one dimensional family (as the length of their unique edge varies in $\mathbb{R}_+$) of isomorphism classes of stable tropical curves.

It is easy to check that these are all the combinatorial types of the stable representatives.

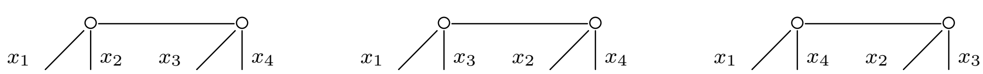


FIGURE 3. The three genus 0, stable graphs with 4 legs.

3.2. **(Weighted) tropical curves.** Let us now consider families of pure tropical curves. This can be done quite naturally by varying the lengths of the edges. Fix a graph G with no vertex of degree less than 3, and write $E(G)=\{e_1,\ldots,e_{|E(G)|}\}$; now consider the set of all tropical curves $\Gamma=(G,\ell)$ having G as combinatorial type. Now $\ell(e_i)$ is a positive real number for all i (G has no leaves), this space is identified with $\mathbb{R}_+^{E(G)}$ as follows: to a point

$$(l_1,\ldots,l_{|E(G)|}) \in \mathbb{R}_+^{E(G)}$$

there corresponds the tropical curve (G,ℓ) such that $\ell(e_i)=l_i$, $\forall i$. It is then natural to ask what happens when some of the lengths go to zero. Let l_1, say, tend to 0. How do we give the limit an interpretation? There is a simple candidate: as l_1 tends to zero, (G,ℓ) specializes to a tropical curve $(\overline{G},\overline{\ell})$ where $\overline{G}=G/e_1$ is obtained by contracting e_1, and $\overline{\ell}(e_i)=\ell(e_i)$, $\forall i\geq 2$. But there is a drawback with this limit: its genus may be smaller than that of G. Indeed we have

$$g(\overline{G}) = \begin{cases} g(G)-1 & \text{if } e_1 \text{ is a loop} \\ g(G) & \text{otherwise.} \end{cases}$$

From a geometric perspective this is quite unpleasant. We like the genus to remain constant under specialization. A solution to this problem is provided by S. Brannetti, M. Melo and F. Viviani. The idea is to extend the definition of a tropical curve by adding a weight function on the vertices.

Definition 3.2.1. A *weighted graph* (with n legs) is a pair (G, w) where G is a graph with n legs, and $w : V(G) \to \mathbb{Z}_+$ a *weight* function on the vertices.

A weighted graph is *stable* if every vertex of weight zero has degree at least 3 and every vertex of weight 1 has degree at least 1.

The genus $g(G, w)$ is defined as follows:

$$g(G, w) = b_1(G) + \sum_{v \in V(G)} w(v). \tag{4}$$

Definition 3.2.2. A *n-marked tropical curve* of genus $g \geq 0$ is a triple $\Gamma = (G, w, \ell)$ where (G, w) is a weighted graph of genus g with n legs and ℓ a *length* function

$$\ell : E(G) \cup L(G) \to \mathbb{R}_{>0} \cup \{\infty\}$$

such that $\ell(x) = \infty$ if and only if either $x \in L(G)$, or $x \in E(G)$ and x is a leaf-edge whose leaf-vertex v satisfies $w(v) = 0$.

The genus of Γ is $g(\mathbf{G}) := g(G, w)$.

The curve Γ is *stable* if so is (G, w).

The weighted graph (G, w) is called the *combinatorial type* of Γ.

To any tropical curve $\Gamma = (G, w, \ell)$ as above we can associate an underlying pure tropical curve $\Gamma^0 := (G, \ell)$ by disregarding the weight function w. Two tropical curves are *isomorphic* if there is an isomorphism of the underlying pure tropical curves which preserves the weights.

Remark 3.2.3. It is easy to check that n-marked stable curves of genus g exist if and only if g and n satisfy

$$2g - 2 + n \geq 1.$$

As for pure tropical curves, we have an equivalence relation which is the same as for tropical curves, with the requirement that it be performed only on vertices of weight zero. I.e. two tropical curves are equivalent if they become isomorphic after a finite sequence of the following operations.

(1) Addition/removal of a vertex of weight 0 and degree 1 and of of its adjacent edge.

(2) Addition/removal of a vertex of weight 0 and degree 2.

The analog of Proposition 3.1.3, with analogous proof, is the following.

Proposition 3.2.4. *Assume $2g-2+n \geq 1$. Then every equivalence class of n-pointed tropical curves of genus g has a unique stable representative.*

Lemma 3.2.5. *Let (G, w) be a genus g stable graph with n legs. Then $|E(G)| \leq 3g-3+n$ and the following are equivalent.*
(a) $|E(G)| = 3g-3+n$.
(b) Every vertex of G has weight 0 and degree 3.
(c) Every vertex of G has weight 0 and $|V(G)| = 2g-2+n$.

Proof. Suppose that (G, w) has a vertex, v such that $w(v) > 0$. Consider the graph (G', w') obtained from (G, w) by replacing v with a vertex, v' of weight zero and having $w(v)$ loops attached to it. Then it is easy to check that (G', w') is stable, has genus g and n legs. Moreover $|E(G')| = |E(G)| + w(v) > |E(G)|$. Hence a graph (G, w) with maximum number of edges (whose existence has yet to be proved) have all vertices weight zero. Then, by stability, $\deg v \geq 3$ for all $v \in V(G)$, hence

$$|E(G)| = 1/2\Big(\sum_{v \in V(G)} \deg v - n\Big) \geq 1/2(3|V(G)| - n),$$

i.e.

$$|V(G)| \leq 2|E(G)|/3 + n/3.$$

From $g = |E(G)| - |V(G)| + 1$ we get

$$|E(G)| = g - 1 + |V(G)| \leq g - 1 + 2|E(G)|/3 + n/3$$

so that

$$|E(G)| \leq 3g - 3 + n.$$

If equality holds, then we necessarily have $w \equiv 0$ and equality must hold everywhere above, in particular $\deg v = 3$ for all $v \in V(G)$ and $|V(G)| = 2g-2+n$. The implications (a)$\Longrightarrow$ (b) and (a)$\Longrightarrow$ (c) are proved.

Assume that $G = (E, V, L)$ has all vertices of degree 3 and weight 0. Then

$$|E| = 1/2\Big(\sum_{v \in V} \deg v - n\Big) = 1/2(3|V| - n)$$

and

$$g = |E| - |V| + 1.$$

Hence,

$$|E| = g - 1 + |V| = g - 1 + 2|E|/3 + n/3$$

hence $|E| = 3g - 3 + n$ and the implication (b)$\Longrightarrow$ (a) is proved.

If $G = (E, V, L)$ has all vertices weight 0 and $|V| = 2g - 2 + n$, then $\deg(v) \geq 3$, hence

$$|E| = 1/2(\sum_{v \in V} \deg v - n) \geq 1/2(3|V| - n) = 1/2(6g - 6 + 2n) = 3g - 3 + n.$$

We already proved that $|E| \leq 3g - 3 + n$, hence equality occur. This proves (c)$\Longrightarrow$ (a). ♣

Example 3.2.6. For $g \geq 3$ and $n = 0$ an example of a graph with $3g - 3$ edges is a polygon with $2g - 2$ vertices, and edges the $2g - 2$ sides and the $g - 1$ diagonals. We leave the case $g = 2$ for exercise (or, see Example 4.1.5).

Definition 3.2.7. A graph (G, w) such that $|E(G)| = 3g - 2 + n$ is called *3-regular*. A tropical curve whose combinatorial type is a 3-regular graph is called 3-regular.

4. Lecture 4

4.1. **Weighted contractions.** For any $g, n \in \mathbb{Z}_{\geq 0}$ we denote by $\mathcal{G}_{g,n}$ the set of all stable weighted graphs of genus g with n legs. Lemma 3.2.5 implies that $\mathcal{G}_{g,n}$ is a finite set, empty if $2g - 2 + n < 1$. We shall introduce a poset structure on $\mathcal{G}_{g,n}$.

Definition 4.1.1. Fix a set of edges, $S \subset E(G)$, of a weighted graph (G, w). We define the *weighted contraction* of S as the weighted graph $(G/S, w_{/S})$ such that

$$\sigma : G \longrightarrow G/S$$

is the contraction of all edges in S (defined earlier for a single edge), and the weight function $w_{/S}$ is defined, for every $\overline{v} \in V(G/S)$,

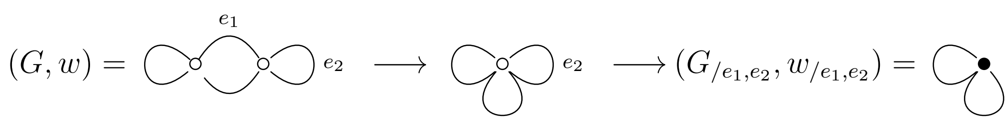
$$w_{/S}(\overline{v}) := b_1(\sigma^{-1}(\overline{v})) + \sum_{v \in \sigma_V^{-1}(\overline{v})} w(v) \tag{5}$$

where $\sigma^{-1}(\overline{v})$ is the subgraph of G spanned by all edges (and vertices) mapping to $\overline{v}$.

Remark 4.1.2. Set $T := E(G) \setminus S$. We have an obvious identification $E(G/S) = T$; moreover the map $\sigma_V : V(G) \to V(G/S)$ is surjective and $\sigma_V(v_1) = \sigma_V(v_2)$ if and only if v_1 and v_2 belong to the same connected component of the graph $G - T$.

Let $S = \{e\}$ where e is a loop based at a vertex v, let $\overline{v} \in V(G/e)$ be the image of v, and also the image of the contracted loop e. Then $w_{/e}(\overline{v}) = w(v) + 1$, whereas $\overline{u} \in V(G/e) \smallsetminus \{\overline{v}\}$ we have $w_{/e}(\overline{u}) = w(u)$.

Example 4.1.3. In the next picture we have two weighted contractions; the starting graph (G, w) has all vertices of weight zero, represented by a "$\circ$", so that G has genus 3. We first contract the non-loop edge e_1, so that the weighted contraction has again weight function equal to zero. Then we contract a loop edge, so that the weighted contraction has one vertex of weight 1, represented by a "$\bullet$".



Remark 4.1.4. It is not hard to check that

$$g(G/S, w_{/S}) = g(G, w).$$

and $(G/S, w_{/S})$ is stable if so is (G, w).

Let (G', w') be a weighted graph. We denote

$$(6) \quad (G, w) \geq (G', w') \text{ if } (G', w') \text{ is a weighted contraction of } (G, w)$$

and this is clearly a partial ordering on space of all graphs. This, of course, induces a poset structure on the set $\mathcal{G}_{g,n}$. By the previous remark $\mathcal{G}_{g,n}$ is closed (or down-closed) with respect to this partial order, i.e. if $(G, w) \in \mathcal{G}_{g,n}$ and $(G, w) \geq (G', w')$, then $(G', w') \in \mathcal{G}_{g,n}$

Example 4.1.5. The picture of the poset $\mathcal{G}_{2,0}$ is below.

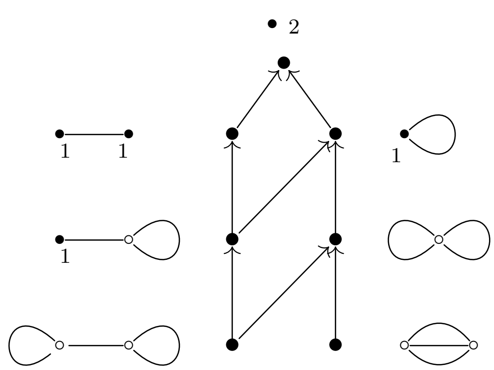


FIGURE 4. The poset $\mathcal{G}_{2,0}$

4.2. **Moduli of tropical curves: fixed combinatorial type.**

Notation 4.2.1. From now on we simplify the notation for weighted graphs as follows:

$$\mathbf{G} = (G, w)$$

where $G = (V, E, L)$ is a graph with legs and w a weight function on the vertices. Then a (weighted) n-marked tropical curve is denoted

$$\Gamma = (\mathbf{G}, \ell).$$

The set of equivalence classes of n-marked tropical curves of genus $g \geq 2$ is denoted by

$$M_{g,n}^{\mathrm{trop}}.$$

Since every equivalence class has a unique stable representative, $M_{g,n}^{\mathrm{trop}}$ is partitioned as follows

$$M_{g,n}^{\mathrm{trop}} = \bigsqcup_{\mathbf{G} \in \mathcal{G}_{g,n}} M_{\mathbf{G}}^{\mathrm{trop}} \tag{7}$$

where $M_{\mathbf{G}}^{\mathrm{trop}}$ denotes the set of all isomorphism classes of tropical curves whose combinatorial type is $\mathbf{G}$.

We shall give a topological structure to M_g^{trop} and thus make it a moduli space for tropical curves.

We begin by describing each stratum $M_{\mathbf{G}}^{\mathrm{trop}}$ appearing in (5.1.3). Write $G = (V, E, L)$, fix an ordering $E = \{e_1, \ldots, e_{|E|}\}$, and consider the open cone

$$\sigma_{\mathbf{G}}^{o} := \mathbb{R}_{+}^{|E|}$$

with its euclidean topology. To every point $(l_1, \ldots, l_{|E|}) \in \sigma_{\mathbf{G}}^{o}$ there corresponds a tropical curve $\Gamma = (G, w, \ell)$ such that $\ell(e_i) = l_i$.

Now consider the group of leg-fixing automorphisms, $\mathrm{Aut}(\mathbf{G})$, of $\mathbf{G}$. More precisely, $\mathrm{Aut}(\mathbf{G})$ is the set of automorphisms of $\alpha : G \to G$ (as defined earlier) such that $\alpha_L(x) = x$ for every leg $x \in L$. For example, $\mathrm{Aut}(\mathbf{G})$ is trivial for each of the three graphs in the picture of Example 3.1.4.

Clearly $\mathrm{Aut}(\mathbf{G})$ acts on E by permutations, so that we have a homomorphism from $\mathrm{Aut}(\mathbf{G})$ to the symmetric group on $|E|$ elements. Hence $\mathrm{Aut}(\mathbf{G})$ acts on $\sigma_{\mathbf{G}}^{o}$ by permuting the coordinates, and it is clear that the quotient by this action is in bijection with isomorphism classes of tropical curves having $\mathbf{G}$ as underlying graph:

$$M_{\mathbf{G}}^{\mathrm{trop}} = \sigma_{\mathbf{G}}^{o} / \mathrm{Aut}(\mathbf{G}).$$

We endow $M_{\mathbf{G}}^{\mathrm{trop}}$ with the quotient topology induced by the euclidean topology on the cone $\sigma_{\mathbf{G}}^{o}$.

Now we consider the boundary of the closed cone $\mathbb{R}^{|E|}_{\geq 0}$, i.e.

$$\mathbb{R}^{|E|}_{\geq 0} \setminus \mathbb{R}^{|E|}_{+}.$$

A point in this set has the form $\underline{l} = (l_1, \dots, l_{|E|}) \in \mathbb{R}^{|E|}_{\geq 0}$ such that at least one l_i is zero. We interpret this point as a *specialization* of tropical curves obtained as a weighted contraction of their underlying graphs. Indeed, denote by

$$S_{\underline{l}} = \{e_i : l_i = 0\} \subset E$$

then to the point $\underline{l}$ we associate the tropical curve

$$\Gamma_{\underline{l}} := (G/S_{\underline{l}}, w/_{S_{\underline{l}}}, \ell_{\underline{l}})$$

where $(G/S_{\underline{l}}, w/_{S_{\underline{l}}})$ is the weighted contraction of $S_{\underline{l}}$ and, identifying $E(G/S_{\underline{l}}) = E(G) \setminus S_{\underline{l}}$ we have

$$\ell_{\underline{l}}(e_i) = l_i$$

for all $e_i \in E(G/S_{\underline{l}})$. Therefore to every point in $\mathbb{R}^{|E|}_{\geq 0}$ there corresponds a tropical curve $\Gamma' = (G', w', \ell')$ such that $\mathbf{G}' = (G', w')$ is a (possibly trivial) contraction of $\mathbf{G}$. By Remark 4.1.4, the tropical curve $\mathbf{G}'$ is stable, with the same genus and number of legs as $\mathbf{G}$. We denote

$$\sigma_{\mathbf{G}} := \mathbb{R}^{|E|}_{\geq 0}$$

so that, by what we just said, we have a map

$$\sigma_{\mathbf{G}} \longrightarrow M^{\text{trop}}_{g,n}; \qquad \underline{l} \longmapsto [\mathbf{G}_{\underline{l}}].$$

5. Lecture 5

5.1. **Moduli spaces of tropical curves.** There are a few equivalent ways to construct M^{trop}_{g} as a topological space; we proceed starting from its "biggest" strata (i.e. strata of maximum dimension). By Lemma 3.2.5, the maximum number of edges of a stable graph of genus g is $3g - 3 + n$. Such graphs correspond to our biggest strata, and we proceed by considering the map

$$\bigsqcup_{\substack{\mathbf{G} \in \mathcal{G}_{g,n}: \\ |E| = 3g-3+n}} \sigma_{\mathbf{G}} \longrightarrow M^{\text{trop}}_{g,n}.$$

It turns out that the action of $\text{Aut}(\mathbf{G})$ on $\sigma^{o}_{\mathbf{G}}$ extends to an action on the closed cone $\sigma_{\mathbf{G}}$. Consider the quotient

$$q_{\mathbf{G}} : \sigma_{\mathbf{G}} \longrightarrow \widetilde{M^{\text{trop}}_{\mathbf{G}}} := \sigma_{\mathbf{G}}/\text{Aut}(\mathbf{G}),$$

if $q_{\mathbf{G}}(\underline{l}) = q_{\mathbf{G}}(\underline{l}')$ then the curve $\Gamma_{\underline{l}}$ is isomorphic to $\Gamma_{\underline{l}'}$ (but the converse may fail outside of $\sigma^o_{\mathbf{G}}$). In conclusion, $\widetilde{M^{\rm trop}_{\mathbf{G}}}$ maps to $M^{\rm trop}_g$ and we can factor

$$\bigsqcup_{\substack{\mathbf{G}\in\mathcal{G}_{g,n}:\\ |E|=3g-3+n}} \sigma_{\mathbf{G}} \longrightarrow \bigsqcup_{\substack{\mathbf{G}\in\mathcal{G}_{g,n}:\\ |E|=3g-3+n}} \widetilde{M^{\rm trop}_{\mathbf{G}}} \longrightarrow M^{\rm trop}_{g,n}$$

By the following Proposition 5.1.1, this map is surjective, i.e. every stable tropical curve can be obtained as a specialization of a stable tropical curve with $3g-3+n$ edges. Therefore we can endow $M^{\rm trop}_g$ with the quotient topology induced by the space on the left.

Proposition 5.1.1. *Let $\mathbf{G} \in \mathcal{G}_{g,n}$. Then there exists $\mathbf{G}' \in \mathcal{G}_{g,n}$ such that $E(G') = 3g-3+n$ and such that $\sigma^0_{\mathbf{G}} \subset \sigma_{\mathbf{G}'}$.*

Proof. It suffices to show that there exists $\mathbf{G}' \in \mathcal{G}_{g,n}$ with $E(G') = 3g-3+n$ admitting a weighted contraction $\mathbf{G}' \to \mathbf{G}$. We can, of course, assume $|E(G)| < 3g-3+n$. The proof is illustrated on an explicit case in Example 5.1.2.

Let $V_+ \subset V(G)$ be the set of vertices of positive weight; consider the graph G'' obtained from G by replacing every $v \in V_+$ by a weight-zero vertex with $w(v)$ new loops attached to it.

By construction, G'' has all vertices of weight zero, and specializes to $\mathbf{G}$ by contracting every one of the new loops ($\sum_{v\in V_+} w(v)$ of them). If every vertex of G'' has degree 3 we are done by Lemma 3.2.5.

So, suppose G'' has some vertex, v, of degree $d \geq 4$. We shall construct a graph G''' which contracts to G'' by contracting to v one edge whose ends have both degree less than d. Iterating this construction until there are no vertices of degree more than 3 we are done.

Let H_v be the set of d half-edges adjacent to v. We partition H_v, into two subsets, H^1_v and H^2_v, of respective cardinalities $d_1 = \lfloor d/2 \rfloor$ and $d_2 = \lceil d/2 \rceil$. As $d \geq 4$ we have

$$2 \leq d_i \leq d-2.$$

Consider the graph G''' obtained from G'' by replacing v by a non-loop edge e whose ends, u_1 and u_2, are attached to, respectively, H^1_v and H^2_v. As u_i has degree d_i+1, the graph G''' is stable and its vertices u_i have both degree less than d. It is clear that contracting e in G''' gives back our G''. So we are done. ♣

Example 5.1.2. The following picture illustrates the proof of Proposition 5.1.1 on the genus-2 graph G consisting of one vertex with weight 2 and no edges. On the right we see the two possible graphs G', corresponding (in the proof) to different distributions in G'' of the four half-edges adjacent to v. Of course, G'' is obtained from G' contracting the edge e, and G is obtained by contracting both loops of G''.

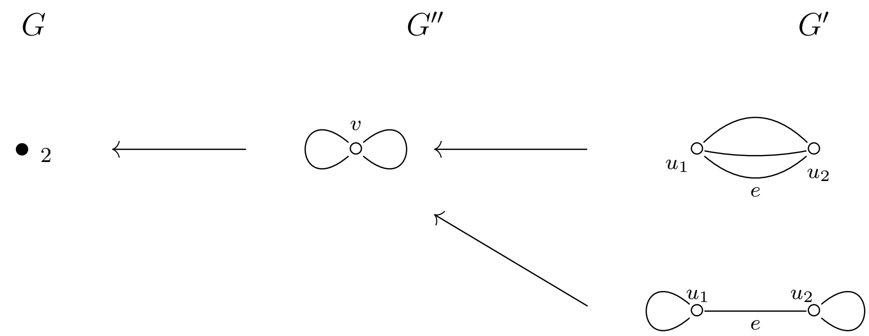


FIGURE 5. Proof of Proposition 5.1.1

For every $\mathbf{G} \in \mathcal{G}_{g,n}$ the space $M_{\mathbf{G}}^{\text{trop}}$ is the quotient of the topological manifold $\sigma^o_{\mathbf{G}}$ by the finite group $\text{Aut}(\mathbf{G})$. Its dimension is defined as the dimension of $\sigma^o_{\mathbf{G}}$, hence

$$\dim M_G^{\text{trop}} = |E(G)|. \tag{8}$$

Now, let X be a topological space containing a dense open subset which is an orbifold of dimension n (locally the quotient of an n-dimensional topological manifold by a finite group); then we say that X *has pure dimension* n. We apply this terminology below.

Recall that 3-regular curves $\Gamma = (\mathbf{G}, \ell)$ are those for which $|E(\mathbf{G})| = 3g - 3 + n$.

Theorem 5.1.3. *Let $2g - 2 + n \geq 1$; consider the stratified topological space*

$$M_{g,n}^{\text{trop}} = \bigsqcup_{\mathbf{G} \in \mathcal{G}_{g,n}} M_{\mathbf{G}}^{\text{trop}} \tag{9}$$

Then

(a) The closure of a stratum is a union of strata and

$$M_{\mathbf{G}'}^{\text{trop}} \subset \overline{M_{\mathbf{G}}^{\text{trop}}} \iff \mathbf{G} \geq \mathbf{G}'.$$

(b) Let $M_{g,n}^{\text{reg}} \subset M_{g,n}^{\text{trop}}$ be the subset parametrizing 3-regular curves, i.e.

$$M_{g,n}^{\text{reg}} := \bigsqcup_{\mathbf{G} \text{ 3-regular}} M_{\mathbf{G}}^{\text{trop}} \subset M_{g,n}^{\text{trop}},$$

then $M_{g,n}^{\text{reg}}$ is open and dense.

(c) Let $M_{g,n}^{\rm pure}$ be the subset parametrizing pure tropical curves. Then $M_{g,n}^{\rm pure}$ is open and dense.
(d) $M_{g,n}^{\rm trop}$ is a Hausdorff topological space of pure dimension $3g-3+n$.

5.2. **Nodal curves and their dual graph.** In definition 2.1.5 we said that a (connected reduced) curve is stable if it has at most nodes as singularities and if its automorphism group is finite.

We surveyed the geometry of smooth curves in the firs lecture. We now need to extend, whenever possible, that theory to singular nodal curves.

A point p of a curve X is a *node* if, locally at p, the curve X is formally analytically isomorphic to a neigborhood of the origin of the plane curve Y of equation $xy = 0$, i.e. if the complete local ring of X at p is isomorphic to the complete local ring of Y at the origin.

Let X be a connected nodal curve, and let $X_{\rm sing}$ be the set of its nodes.

The *Picard group* of X is the abelian group of Cartier divisors modulo linear equivalence, which will be identified with the set of isomorphism classes of line bundles

$$\text{Pic}(X) := \frac{\text{CaDiv}(X)}{\sim} = \frac{\{\text{Line bundles on } X\}}{\cong}$$

We use for divisors and linear systems on X the same notation we used in case X is a smooth curve. Now it is crucial to consider the *normalization* (i.e. the desingularization) of X, this is a canonical pair (X^ν, ν)

$$\nu : X^\nu \longrightarrow X \tag{10}$$

where X^ν is a smooth, possibly non connected curve and ν is a birational morphism which is an isomorphism away from $X_{\rm sing}$ and satisfies the following properties. Let

$$X = \cup_{i=1}^{\gamma} C_i$$

be the decomposition of X into its irreducible components. Then C_i is an irreducible nodal curve and X^ν is the disjoint union

$$X^\nu = \sqcup_{i=1}^{\gamma} C_i^\nu,$$

where C_i^ν is a smooth irreducible curve (the normalization of C_i) and the restriction of ν to C_i^ν, written ν_i, is a birational morphism $\nu_i : C_i^\nu \to C_i$. For every node $p \in X_{\rm sing}$ the preimage of p under ν consists of exactly two points, which are called the *branch points* of p, denoted

$\nu^{-1}(p) = \{p^+, p^-\}$. We write,

$$X = \frac{X^\nu}{p^+ \sim p^-, \ \forall p \in X_{\text{sing}}}$$

with the understanding that for every $p \in X_{\text{sing}}$ the curve X has a node at p (hence its complete local ring is uniquely determined).

Remark 5.2.1. The data of X^ν and the pairs of branch points $\{p^+, p^-\} \subset X^\nu$, determine the curve X uniquely.

Definition 5.2.2. The *dual graph* of a nodal curve X is the weighted graph (G_X, w_X) such that

(1) The set of vertices V_X is the set of all irreducible components of X i.e.
$$V_X = \{C_1, \dots, C_\gamma\}.$$
(2) The set of half edges H_X is the set of all branch points, i.e.
$$H_X = \{p^+, p^-, \ \forall p \in X_{\text{sing}}\}$$
(3) The involution $\iota_X : H_X \longrightarrow H_X$ is
$$\iota_X(p^+) = p^-, \qquad \forall p \in X_{\text{sing}}$$
(4) The endpoint map $\epsilon_X : H_X \longrightarrow V_X$ sends p^+ (respectively p^-), to the component C_i such that $p^+ \in C_i^\nu$ (resp. $p^- \in C_i^\nu$).
(5) The weight of a vertex/component is the genus of its normalization, i.e.
$$w_X(C_i) = g_{C_i^\nu}.$$

One easily checks that for the set of edges, E_X, we have an identification
$$E_X = X_{\text{sing}}.$$

Remark 5.2.3. Let $v \in V_X$ and let $C \subset X$ be the irreducible component corresponding to v. By definition, the half-edges adjacent to v correspond to points in C^ν; more precisely, if p is a node of X contained in the intersection of two different components one of which is C, then C^ν contains exactly one branch point of p, whereas if p is a singular point of C then both branches of p lie in C^ν. Hence
$$\deg(v) = |C \cap (\overline{X \smallsetminus C})| + 2|C_{\text{sing}}|$$
where C_{sing} denotes the set of nodes of C, each of which corresponds to a loop of G_X.

We shall now define the genus of a nodal curve and show it equals the genus of its dual graph. On a singular curve we cannot talk about a tangent or cotangent line bundle, since the tangent space of X at any of its nodes has dimension 2. Nonetheless there does exist a line bundle on X which plays the role of the canonical line bundle of a smooth curve. This line bundle is called the *dualizing line bundle*, or dualizing sheaf, and is denoted by

$$\omega_X;$$

it coincides with the canonical line bundle K_X if X is smooth. The name "dualizing bundle" follows from Serre duality, which holds for every nodal curve X in the following form: for every $L \in \text{Pic}(X)$ we have

$$h^0(X, \omega_X \otimes L^{-1}) = h^1(X, L).$$

The *(arithmetic) genus* of a connected nodal curve X is defined as follows

$$g_X := h^0(X, \omega_X) = h^1(X, \mathcal{O}_X)$$

(the second " $=$ " follows from Serre duality).

Lemma 5.2.4. *The genus of a connected nodal X is equal to the genus of its dual graph.*

Proof. The genus of X is defined as $g_X = h^1(X, \mathcal{O}_X)$. Write $G_X = (V, E)$, and consider the normalization

$$\nu : X^\nu = \bigsqcup_{v \in V} C_v^\nu \longrightarrow X.$$

The associated map of structure sheaves yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \nu_* \mathcal{O}_{X^\nu} \longrightarrow \mathcal{S} \longrightarrow 0 \tag{11}$$

where $\mathcal{S}$ is a skyscraper sheaf supported on the nodes of X; the associated exact sequence in cohomology is as follows (identifying the cohomology groups of $\nu_* \mathcal{O}_{X^\nu}$ with those of $\mathcal{O}_{X^\nu}$ as usual)

$$\begin{array}{rlll} 0 \longrightarrow & H^0(X, \mathcal{O}_X) \longrightarrow & H^0(X^\nu, \mathcal{O}_{X^\nu}) \stackrel{\tilde{\delta}}{\longrightarrow} k^{|E|} \longrightarrow \\ \longrightarrow & H^1(X, \mathcal{O}_X) \longrightarrow & H^1(X^\nu, \mathcal{O}_{X^\nu}) \longrightarrow 0. \end{array}$$

Hence

$$g_X = h^1(X^\nu, \mathcal{O}_{X^\nu}) + |E| - |V| + 1$$

Now,

$$h^1(X^\nu, \mathcal{O}_{X^\nu}) = \sum_{v \in V} h^1(C_v^\nu, \mathcal{O}_{C_v^\nu}) = \sum_{v \in V} g_{C_v^\nu} = \sum_{v \in V} w_X(v)$$

hence, as $|E| - |V| + 1 = b_1(G_X)$ we get

$$g_X = \sum_{v \in V} w_X(v) + b_1(G_X) = g(G_X, w_X).$$

♣

6. Lecture 6

6.1. **Geometry of stable curves.** By definition, a stable curve is a connected reduced nodal curve with finitely many automorphisms. Let us translate the stability condition into a combinatorial condition.

Lemma 6.1.1. *Let X be a connected nodal curve. Then X is stable if and only if $g_X \geq 2$ and the following equivalent conditions hold.*
(a) Every irreducible component $C \subset X$ such that $C^\nu \cong \mathbb{P}^1$ satisfies

$$|C \cap (\overline{X \smallsetminus C})| + 2|C_{\text{sing}}| \geq 3. \tag{12}$$

(b) The dual graph of X is stable.

Proof. Let us show that (a) and (b) are equivalent. In (G_X, w_X), vertices of weight zero correspond to components C such that $g_{C^\nu} = 0$, hence $C^\nu \cong \mathbb{P}^1$. By Remark 5.2.3, condition (12) is equivalent to the condition that the degree of such vertices be at least 3, i.e. that (G_X, w_X) is stable, and we are done.

Assume X is stable, i.e. that $\text{Aut}(X)$ is finite. If X has genus ≤ 1 then $\text{Aut}(X)$ is necessarily infinite, therefore a stable curve must have genus at least 2. Let us prove that (a) holds. Let $C \subset X$ be a component such that $C^\nu \cong \mathbb{P}^1$; by contradiction, let $|C \cap (\overline{X \smallsetminus C})| + 2|C_{\text{sing}}| \leq 2$, hence $|C_{\text{sing}}| \leq 1$. If $|C_{\text{sing}}| = 0$ then $C \cong \mathbb{P}^1$ and C intersects the remaining components of X in one or two points; the curve C has infinitely many automorphisms fixing the (at most two) intersection points, and these automorphisms correspond to automorphisms of X which restrict to the identity on $\overline{X \smallsetminus C}$. Hence $\text{Aut}(X)$ is infinite, a contradiction. If $|C_{\text{sing}}| = 1$ then $|C \cap (\overline{X \smallsetminus C})| = 0$, hence $X = C$, but this is impossible because X has genus at least two, while C has clearly genus 1.

Conversely, assume $g_X \geq 2$ and the dual graph (G_X, w_X) of X is stable; let V and E the set of vertices and edges of G_X. For every $v \in V_X$ let $C_v \subset X$ be the component corresponding to v. We must prove that $\text{Aut}(X)$ is finite. Consider the subgroup, $\text{Aut}^\#(X) \subset \text{Aut}(X)$ preserving the components of X, i.e.

$$\text{Aut}^\#(X) := \{\alpha \in \text{Aut}(X) : \alpha(C_v) = C_v, \ \forall v \in V).$$

Then we have an exact sequence of groups

$$0 \longrightarrow \text{Aut}^\#(X) \longrightarrow \text{Aut}(X) \longrightarrow \mathcal{S}_\gamma \longrightarrow 0$$

(where $\gamma = |V|$). Hence it suffices to prove that $\text{Aut}^\#(X)$ is finite. By definition, the half-edges adjacent to v correspond to the branch points lying in C_v^ν; denote by $B_v \subset C_v^\nu$ the set of such points, so that $|B_v| = \deg v$. Let $\alpha \in \mathrm{Aut}^{\#}(X)$; then α induces an automorphism of C_v^ν such that $\alpha(B_v) = B_v$, denote by $\mathrm{Aut}(C_v^\nu, B_v) \subset \mathrm{Aut}(C_v)$ the group of such automorphisms. If α is trivial on every C_v^ν then α is trivial, hence we have an injective map

$$\mathrm{Aut}^{\#}(X) \hookrightarrow \prod_{v \in V} \mathrm{Aut}(C_v^\nu, B_v)$$

(given by restricting α to each C_v and pulling it back to C_v^ν). We claim that the set on the right is finite. Indeed, let us show that $\mathrm{Aut}(C_v^\nu, B_v)$ is finite for every $v \in V_X$. This is obvious if C_v^ν has genus at least 2. If C_v^ν has genus 1 then $B_v \neq \emptyset$ (since X has genus at least 2), hence $\mathrm{Aut}(C_v^\nu, B_v)$ is finite. If C_v has genus zero, the hypothesis implies that $|B_v| \geq 3$, hence $\mathrm{Aut}(C_v^\nu, B_v)$ is finite. ♣

The stability condition on a curve can be seen directly from the behaviour of its smooth rational components. Indeed, smooth rational components correspond to vertices of the dual graph having no loop attached to it. The stability condition is equivalent to requiring that if $C \cong \mathbb{P}^1$ is any such component, then $|C \cap \overline{X \setminus C}| \geq 3$.

We say that the curve X is *semistable* if for every component $C \cong \mathbb{P}^1$ of X we have

$$|C \cap \overline{X \setminus C}| \geq 2.$$

Equivalently, X is semistable if its dual graph has no vertex of weight 0 and degree 1.

6.2. **Line bundles on nodal curves.** Let $X = \cup_{i=1}^{\gamma} C_i$ be a nodal curve with components $C_1, \ldots, C_\gamma$, and let

$$\nu : X^\nu = \sqcup_{i=1}^{\gamma} C_i^\nu \longrightarrow X$$

be its normalization. We said earlier that X has a dualizing bundle, ω_X, which coincides with the canonical bundle when X is smooth. We want to study the relation between ω_X and the canonical bundle of X^ν, and more generally, between line bundles on X and on X^ν. To do that, consider the pull-back homomorphism

$$\nu^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^\nu) = \prod_i \mathrm{Pic}(C_i^\nu).$$

This is surjective; an elementary way to verify this is to observe that given a line bundle, L^ν on X^ν we can construct a line bundle on X by prescribing an isomorphism between each pair of fibers, $L^\nu_{p^+}$ and $L^\nu_{p^-}$

for every $p \in X_{\text{sing}}$. A more complete proof is to recall that we have an isomorphism (Hartshorne Ex. III.4.5)

$$H^1(X, \mathcal{O}_X^*) \cong \operatorname{Pic}(X)$$

hence, similarly to the sequence (11), we have

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \nu_* \mathcal{O}_{X^\nu}^* \longrightarrow \mathcal{S}^* \longrightarrow 1 \tag{13}$$

from which we get, setting $\delta = |X_{\text{sing}}|$

$$1 \longrightarrow k^* \longrightarrow (k^*)^\gamma \longrightarrow (k^*)^\delta \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X^\nu, \mathcal{O}_{X^\nu}^*) \longrightarrow 0$$

hence, as $\delta - \gamma + 1 = b_1(G_X)$, we have the exact sequence

$$0 \longrightarrow (k^*)^{b_1(G_X)} \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X^\nu) \longrightarrow 0. \tag{14}$$

For every $L \in \operatorname{Pic}(X)$ its degree on the component C_i of X is defined as the degree of $\nu^* L$ on C_i^ν, in symbols

$$\deg_{C_i} L := \deg(\nu^* L)_{|C_i^\nu}.$$

Now the *multidegree* of L is defined as

$$\underline{\deg} L = (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L) \in \mathbb{Z}^\gamma$$

and the *degree* of L is

$$\deg L = \sum_{\imath=1}^{\gamma} \deg_{C_i} L.$$

For any $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$ we denote by $\operatorname{Pic}^{\underline{d}}(X) \subset \operatorname{Pic}(X)$ the set of line bundles of multidegree $\underline{d}$, hence we have a surjection

$$\operatorname{Pic}^{\underline{d}}(X) \longrightarrow \operatorname{Pic}^{\underline{d}}(X^\nu).$$

We write $|\underline{d}| = \sum d_i$ and for any $d \in \mathbb{Z}$ we write $\operatorname{Pic}^d(X) \subset \operatorname{Pic}(X)$ for the set of line bundles of degree d. Of course, if X is reducible the degree is not as fine an invariant as the multidegree. Indeed, for any fixed $d \in \mathbb{Z}$ we have

$$\operatorname{Pic}^d(X) = \coprod_{\underline{d} \in \mathbb{Z}: |\underline{d}| = d} \operatorname{Pic}^{\underline{d}}(X).$$

The Jacobian variety of X is defined as the set of line bundles on X having degree 0 on every irreducible component, i.e.

$$\operatorname{Jac}(X) = \operatorname{Pic}^{(0,\ldots,0)}(X).$$

As for the smooth case, $\operatorname{Jac}(X)$ is an abelian group and from (14) we have an exact sequence of groups

$$0 \longrightarrow (k^*)^{b_1(G_X)} \longrightarrow \operatorname{Jac}(X) \longrightarrow \operatorname{Jac}(X^\nu) = \prod_{i=1}^{\gamma} \operatorname{Jac}(C_i^\nu) \longrightarrow 0.$$

From the above sequence, we get the following

Definition/Lemma 6.2.1. *Let X be a nodal curve of genus g. Then* $\dim \mathrm{Jac}(X) = g$ *and the following conditions are equivalent*
(a) $\mathrm{Jac}(X)$ *is projective.*
(b) $\mathrm{Jac}(X) \cong \mathrm{Jac}(X^\nu)$
(c) $b_1(G_X) = 0$ *i.e.* G_X *is a tree.*
If X satisfies such conditions, we say that X is of compact type.

Now, the relation between ω_X and the canonical bundle of the normalization of X is quite explicit. As X^ν is smooth, its dualizing/canonical bundle $\omega_{X^\nu} = K_{X^\nu}$ is defined as the canonical bundle of every connected component. Now we have the following fact

$$\nu^*\omega_X = K_{X^\nu}\Big(\sum_{p\in X_{\mathrm{sing}}}(p^+ + p^-)\Big).$$

7. Lecture 7

Lemma 7.0.1. *Let X be a nodal curve of genus g. Then*
(a) $\deg \omega_X = 2g-2$.
(b) X is stable if and only if for every irreducible component $C \subset X$ we have

$$\deg_C \omega_X > 0. \tag{15}$$

Proof. Set $\delta = |X_{\mathrm{sing}}|$; we have $g = \sum_{i=1}^{\gamma} g_{C_i^\nu} + \delta - \gamma + 1$. Now

$$\deg \omega_X = \deg K_{X^\nu}\big(\sum_{p\in X_{\mathrm{sing}}}(p^+ + p^-)\big) = \sum_{i=1}^{\gamma}\deg_{C_i^\nu} K_{X^\nu} + 2\delta$$

and

$$\deg_{C_i^\nu} K_{X^\nu} = \deg K_{C_i^\nu} = 2g_{C_i^\nu} - 2.$$

Hence

$$\deg \omega_X = \sum_{i=1}^{\gamma}(2g_{C_i^\nu} - 2) + 2\delta = 2(\sum_{i=1}^{\gamma} g_{C_i^\nu} - \gamma + \delta) = 2g-2$$

as claimed. By the above discussion we have

$$\deg_C \omega_X = \deg K_{C^\nu}\big(\sum_{p\in C_{\mathrm{sing}}}(p^+ + p^-)\big) + |C \cap (\overline{X \smallsetminus C})| = 2g_{C^\nu} - 2 + 2|C_{\mathrm{sing}}| + |C \cap (\overline{X \smallsetminus C})|.$$

In the dual graph (G_X, w_X) of X, let v be the vertex corresponding to C, then

$$\deg_C \omega_X = 2g_{C^\nu} - 2 + \deg v = 2w_X(v) - 2 + \deg v$$

(see Remark 5.2.3).

Suppose X is stable. If $g_{C^\nu} \geq 2$ then obviously $\deg_C \omega_X > 0$. If $g_{C^\nu} = 1$ then $\deg v > 1$ (as X has genus ≥ 2), hence $\deg_C \omega_X > 0$. If $g_{C^\nu} = 0$ then, by stability, $\deg v \geq 3$ hence again $\deg_C \omega_X > 0$.

Conversely, assume (15) holds and let us show that the dual graph of X is stable. If $C \subset X$ is a component such that $g_{C^\nu} = 0$, then (15) gives

$$0 < \deg_C \omega_X = -2 + \deg v$$

hence $\deg v \geq 3$, hence X is stable. ♣

The Riemann-Roch theorem holds for nodal curves:

Theorem 7.0.2. *(Riemann-Roch) Let X be a nodal curve of (arithmetic) genus g. For any $L \in \mathrm{Pic}(X)$ we have*

$$h^0(X, L) - h^0(X, \omega_X L^{-1}) = \deg L - g + 1$$

equivalently, by Serre's duality

$$h^0(X, L) - h^1(X, L) = \deg L - g + 1.$$

The following is a consequence of Riemann-Roch and Lemma 7.0.1.

Fact 7.0.3. *Let $X = \cup_{i=1}^{\gamma} C_i$ be a nodal curve and $L \in \mathrm{Pic}(X)$.*

(1) If $\deg_{C_i} L \geq 2g_{C_i} + 1$ for every $i = 1, \ldots, \gamma$, then L is very ample.

(2) L is ample if and only if $\deg_{C_i} L > 0$ for every $i = 1, \ldots, \gamma$.

(3) The curve X is stable if and only if ω_X is ample.

7.1. Stabilization and Stable Reduction.

Stabilization. If X is a non-stable nodal curve of genus $g \geq 2$, its dualizing line bundle ω_X is not ample. What happens in this case? Let $C \subset X$ be a destabilizing component of X, i.e. an irreducible component such that $C^\nu \cong \mathbb{P}^1$ and such that the vertex, v_C, corresponding to C has degree ≤ 2. Then one easily check that $C^\nu \cong \mathbb{P}^1$ and $\deg_C \omega_X \leq 0$. More precisely, if v_C is a leaf vertex, then $\deg_C \omega_X = -1$ while if $\deg(v_C) = 2$ then $(\omega_X)_{|C} = \mathcal{O}_C$. On the other hand, X has genus at least 2, hence $\deg \omega_X > 0$, hence X contains some component over which the restriction of ω_X has positive degree, i.e. it is ample. Hence there exists some $n > 0$ such that the restriction of ω_X^n to every non-destabilizing component is very ample. Therefore ω_X^n induces a morphism in projective space $\mathbb{P}^r$ with $r = n(2g-2)-g$, which contracts every destabilizing component

$$X \longrightarrow \mathrm{st}(X) \subset \mathbb{P}^r.$$

Such a morphism can be also constructed abstractly, without using the dualizing sheaf, by contracting all the destabilizing components to a point. The point to which the component C is contracted is either a smooth point of $\mathrm{st}(X)$, if $\deg v_C = 1$, or a node if $\deg v_C = 2$.

One easily checks that $\mathrm{st}(X)$ is stable of genus g. Its dual graph is obtained by removing every vertex of weight 0 and degree 1 and its adjacent edge, and every vertex of weight 0 and degree 2, exactly as the two removal operations described to define equivalence of tropical curves, before Proposition 3.2.4

We say that two nodal curves X and X' are *stably equivalent* if $\mathrm{st}(X) \cong \mathrm{st}(X')$.

Stable reduction. Why are stable curves so important among all singular curves? Since every nodal curve has a unique stabilization (up to isomorphism), the above question is asking why nodal curves are so important. A simple natural answer is: because nodes are the simplest type of singularities a curve can have. But there is a deeper fact which provides a stronger motivation. This is the so-called stable reduction theorem, which states the following.

Theorem 7.1.1 (Stable Reduction Theorem)**.** *Let B be a smooth, connected, one-dimensional variety and $b_0 \in B$ a fixed point. Let $f : \mathcal{X} \to B$ be a family of curves such that for every $b \in B \smallsetminus b_0$ the fiber, X_b, over b is smooth. Then there exists a finite cover $\phi : B' \to B$ and a map $h : \mathcal{Y} \to B'$ all of whose fibers are stable curves and such that on $\phi^{-1}(B \smallsetminus b_0) \subset B'$ the restriction of h is the base change of f. In particular, for every $b' \in B' \smallsetminus \phi^{-1}(b_0)$ the fiber $Y_{b'}$ is isomorphic to $X_{\phi(b')}$.*

The family $h : \mathcal{Y} \to B'$ is uniquely determined by ϕ.

We will not prove this theorem. We observe that the existence part still holds if we replace the word stable by the word semistable, or nodal. One can achieve uniqueness for families of semistable curves by requiring that the total space $\mathcal{Y}$ be a nonsingular <u>minimal</u> surface. Then uniqueness follows from the uniqueness of minimal models for surfaces. In such a case the singular fibers will necessarily be semistable (a rational tail in a fiber would be contractible).

Remark 7.1.2. It is not hard to show that the stable reduction theorem extends to the case where the fibers of f are stable curves (rather than smooth ones) over $B \setminus \{b_0\}$.

The stable reduction theorem, applied to $B = \operatorname{Spec} K$ with K is a discrete-valued field implies that the moduli space of stable curves (if it exists) satisfies the valuative criterion for properness.

7.2. **The moduli space of stable curves.** The original construction of the (coarse) moduli space $\overline{M}_g$ as a projective variety, is due to D. Gieseker, and can be can be summarized as follows. Fix $g \geq 2$.

(1) The dualizing bundle ω_X of a stable curve X of genus g is ample, hence X can be embedded in a projective space using a high enough power, ω_X^n, such that ω_X^n is very ample. We call the image of this embedding an *n-canonical* model of X. The power n can be chosen to be the same for all stable curves of fixed genus g, hence all n-canonical models of all stable curves are in the same projective space $\mathbb{P}^r$ and have degree $2n(g-1)$. From now on n and $r = 2n(g-1) - g$ are fixed with $n \geq 10$.
(2) The dualizing bundle is preserved by isomorphisms, hence so are its powers, hence two n-canonical models are abstractly isomorphic if and only if they are projectively equivalent, i.e. conjugate by an element of $G := PGL(r+1)$.
(3) Let us denote by K_g the set of all n-canonical models of stable curves of genus g. Then the group G acts on K_g and there is a bijection between the quotient set K_g/G and the set of isomorphism classes of genus g stable curves.
(4) The set K_g has a natural an algebraic structure via theory of Hilbert schemes. The Hilbert scheme, $H = \mathrm{Hilb}^r_{d,g}$, we need here is the fine moduli space for projective curves in $\mathbb{P}^r$ of degree $d = n(g-1)$ and genus g, and $K_g \subset H$ is a quasiprojective variety; moreover, K_g is smooth and irreducible.
(5) To give the quotient K_g/G the structure of an algebraic variety, one applies Geometric Invariant Theory. The group G acts on the Hilbert scheme H by leaving K_g invariant. It turns out that K_g is made of GIT-stable points for the action of G on H, and that K_g is closed in set of GIT-semistable points of H. Therefore the GIT-quotient, denoted by $K_g /\!\!/ G$, is a projective irreducible variety, and its points coincide with the orbits of G, in other words there is a canonical bijection $K_g /\!\!/ G = K_g/G$ We set

$$\overline{M}_g := K_g /\!\!/ G.$$

Now let us illustrate some consequences of the construction.

(a) Stable and smooth curves are treated at the same time, and the construction yields together with the moduli space of stable curves, the moduli space of smooth curves as an open subset $M_g \subset \overline{M}_g$.
(b) This construction is purely algebraic and works in any characteristic. Hence one obtains that M_g is irreducible (because so is $\overline{M}_g$) a fact that was not known in positive characteristic.
(c) The variety $\overline{M}_g$ is normal because so is K_g (which is even smooth), and the GIT quotient of a normal variety is normal. Moreover $\overline{M}_g$ has mild singularties, and can be explicitly described, locally at every point. The stabilizers of the action of G are the automorphism groups of our curves, which are finite. Hence $\overline{M}_g$ has at most finite quotient singularties, and is smooth at curves having trivial automorphism group (but the smooth locus of $\overline{M}_g$ is bigger than the locus of curves with trivial automorphism group).
(d) We know that $\overline{M}_g$ is not a fine moduli space, on the other hand the Hilbert scheme H is, hence so is K_g. Therefore K_g is the base of a family of stable curves, $f : \mathcal{C} \longrightarrow K_g$ and the moduli map of this family, $\mu_f : K_g \to \overline{M}_g$, coincides with the quotient map $K_g \to \overline{M}_g$.

A somewhat different construction of $\overline{M}_g$ has been given later, by considering the related algebraic algebraic stack, $\overline{\mathcal{M}_g}$, and then showing that this stack admits a projective moduli scheme $\overline{M}_g$. The stack $\overline{\mathcal{M}_g}$, albeit more sophisticated, is preferable from some points of view, as it retains more information.

8. Lecture 8

8.1. **Stable n-pointed curves.**

Definition 8.1.1. Let $g, n \geq 0$. A *nodal n-pointed curve* of genus g, written $(X; p_1, \ldots, p_n) = (X; \underline{p})$, is a (connected) nodal curve X of arithmetic genus g, together with n distinct nonsingular points $p_i \in X \setminus X_{\text{sing}}$. A nodal n-pointed curve is *stable* if the set of automorphisms of X mapping p_i to p_i for all $i = 1, \ldots, n$ is finite.

The dual graph, $(G_{(X;\underline{p})}, w_X)$, of the nodal n-pointed curve $(X; \underline{p})$ is the weighted graph with n legs obtained by adding to the dual graph (G_X, w_X) of X (in 5.2.2) a leg l_i at the vertex corresponding to the component containing p_i, for every $i = 1, \ldots, n$.

Notice that $(G_{(X;\underline{p})}, w_X)$ has the same vertices, edges and weights as (G_X, w_X), whereas the set, $H_{(X;\underline{p})}$, of half-edges is as follows

$$H_{(X;\underline{p})} = \{p^+, p^-, l_i \quad \forall p \in X_{\text{sing}}, \forall i = 1, \ldots, n\},$$

now $l_1, \dots, l_n$ are the fixed points of the involution on $H_{(X;\underline{p})}$, which otherwise swaps p^+ with p^-. The endpoint map sends l_i to the vertex corresponding to the component containing p_i, and equals the endpoint map of (G_X, w_X) otherwise.

Remark 8.1.2. It is easy to check that if $g = 0$ and $n \leq 2$, or if $g = 1$ and $n = 0$, there are no stable curves, and that stable curves exist if and only if $2g - 2 + n > 0$.

On a pointed curve $(X; \underline{p})$ the line bundle

$$\omega_X(\sum_1^n p_i)$$

plays a fundamental role. For every component C of X we have

$$\begin{aligned}\deg_C \omega_X(\sum_1^n p_i) &= 2g_C - 2 + |C \cap \{p_1, \dots, p_n\}| + |C \cap \overline{X \smallsetminus C}| \\ &= 2g_{C^\nu} - 2 + 2|C_{\text{sing}}| + |C \cap \{p_1, \dots, p_n\}| + |C \cap \overline{X \smallsetminus C}|\end{aligned}$$

Proposition 8.1.3. *Assume* $2g-2+n > 0$ *and let* $(X; \underline{p})$ *be a* n*-pointed nodal curve of genus* g*. The following are equivalent*

(1) $(X; \underline{p})$ *is stable.*
(2) The dual graph of $(X; \underline{p})$ *is stable.*
(3) The line bundle $\omega_X(p_1 + \dots + p_n)$ *is ample* *(equivalently,* $\deg_C \omega_X(p_1 + \dots + p_n) > 0$ *for every component* $C \subset X$*).*
(4) For every component $C \subset X$ *such that* $g_C \leq 1$ *we have*

$$|C \cap \{p_1, \dots, p_n\}| + |C \cap \overline{X \smallsetminus C}| \geq \begin{cases} 3 & \textit{if } g_C = 0 \\ 1 & \textit{if } g_C = 1. \end{cases}$$

Proof. Easy generalization of the proofs of the analogous statements for $n = 0$. ♣

There are many reasons for extending our field of interest from curves to pointed curves. Here is one

Remark 8.1.4. *Stability of pointed curves is preserved under normalization.*

More precisely, let $(X; p_1, \dots, p_n)$ be an n-pointed stable curve of genus g. Pick a node $q \in X_{\text{sing}}$, let $\nu_q : X_q^\nu \to X$ be the normalization at q only; let $q^+, q^- \in X_q^\nu$ be the two branches of q. Now abuse notation by setting $p_i = \nu_q^{-1}(p_i)$ for $i = 1, \dots, n$. Then one easily checks that the $(n+2)$-pointed curve $(X_q^\nu; p_1, \dots, p_n, q^+, q^-)$ is either stable of genus

$g-1$ if q is not a separating node of X, or the disjoint union of two pointed stable curves of genera summing to g if q is a separating node.

8.2. The moduli spaces $\overline{M}_{g,n}$.

Theorem 8.2.1. *Let $2g-2+n \geq 1$. There exists a projective, irreducible, normal, variety of dimension $3g-3+n$, denoted by $\overline{M}_{g,n}$, which is the coarse moduli space of n-pointed stable curves of genus g. The moduli space of nonsingular n-pointed curves of genus g is an open subset $M_{g,n} \subset \overline{M}_{g,n}$.*

Remark 8.2.2. The fact that $\overline{M}_{g,n}$ is not a fine moduli space follows from the existence of curves with nontrivial automorphisms (in fact if $g=0$ then $M_{0,n}$ is a fine moduli space for $n \geq 3$). On the other hand there do exist finite covers of $\overline{M}_{g,n}$ which are fine moduli spaces of stable curves with some extra structure. In particular, such coverings are endowed with universal families of stable pointed curves whose moduli map to $\overline{M}_{g,n}$ coincides with the covering map.

Recall that we denote by $\mathcal{G}_{g,n}$ the set of all stable graphs with n legs of genus g. For every $\mathbf{G} \in \mathcal{G}_{g,n}$ we denote by $M^{\text{alg}}_{\mathbf{G}} \subset \overline{M}_{g,n}$ the locus of isomorphism classes of n-pointed curves with $\mathbf{G}$ as dual graph:

$$M^{\text{alg}}_{\mathbf{G}} := \{(X;\underline{p}) \in \overline{M}_{g,n} : (G_{(X,\underline{p})}, w_X) = \mathbf{G}\}$$

hence we have the following combinatorial partition of $\overline{M}_{g,n}$

$$\overline{M}_{g,n} = \bigsqcup_{\mathbf{G} \in \mathcal{G}_{g,n}} M^{\text{alg}}_{\mathbf{G}}. \tag{16}$$

In the sequel, we shall identify $E(\mathbf{G})$ with X_{sing} when no confusion is likely. Similarly $V(\mathbf{G})$, respectively $L(\mathbf{G})$, will be identified with the set of components, resp. marked points, of X.

Lemma 8.2.3. *Assume $2g-2+n>0$ and fix $\mathbf{G} \in \mathcal{G}_{g,n}$. Then $M^{\text{alg}}_{\mathbf{G}}$ is an irreducible quasiprojective variety and its codimension in $\overline{M}_{g,n}$ is equal to $|E(\mathbf{G})|$.*

Proof. Set $\mathbf{G} = (G,w)$ and $\delta := |E(G)|$. If $\delta = 0$ then $M^{\text{alg}}_{\mathbf{G}} = M_{g,n}$ which is irreducible of codimension 0 by Theorem 8.2.1. Let $\delta > 0$. Pick a curve $(X;\underline{p}) \in M^{\text{alg}}_{\mathbf{G}}$, denote by $C_1, \ldots, C_\gamma$ its irreducible components, by $\nu : \sqcup_1^\gamma C_i^\nu \to X$ its normalization, and set $g_i = g_{C_1^\nu} = w(v_i)$ where $v_i \in V(G)$ is the vertex corresponding to C_i.

Let n_i be the number of marked points contained in C_i, so that $\sum_1^\gamma n_i = n$; denote by $p_1^{(i)}, \ldots, p_{n_i}^{(i)} \in C_i^\nu$ the preimages via ν of such points. Set $\delta_i := |\nu^{-1}(C_i \cap X_{\text{sing}}) \cap C_i^\nu|$ so that we have

$$\deg(v_i) = n_i + \delta_i$$

For every $i = 1, \dots, \gamma$ we have a nonsingular pointed curve of genus g_i with $n_i + 2\delta_i$ marked points in it, $(C_i^\nu; \underline{p^{(i)}})$, where

$$\{p_j^{(i)},\ j = 1, \dots, \deg(v_i)\} := (p_1^{(i)}, \dots, p_{n_i}^{(i)}) \cup \nu^{-1}(C_i \cap X_{\text{sing}}) \cap C_i^\nu$$

where we fixed an ordering on the δ_i points in $\nu^{-1}(C_i \cap X_{\text{sing}})$. Moreover, as we observed in Remark 8.1.4, the pointed (smooth) curve $(C_i^\nu; \underline{p^{(i)}})$ is stable and

$$(C_i^\nu; \underline{p^{(i)}}) \in M_{g_i, \deg(v_i)}.$$

Once we have such γ smooth pointed curves, the gluing data of $\nu^{-1}(X_{\text{sing}})$ are uniquely determined by the graph $\mathbf{G}$ and by the orderings we have chosen on $\nu^{-1}((C_i)_{\text{sing}})$. Therefore we have a surjective morphism

$$\prod_{v \in V(\mathbf{G}}^{\gamma} M_{w(v), \deg(v)} = \prod_{i=1}^{\gamma} M_{g_i, n_i + \delta_i} \longrightarrow M_{\mathbf{G}}^{\text{alg}} = \prod_{i=1}^{\gamma} M_{g_i, \deg(v_i)} / \text{Aut}(\mathbf{G}).$$

Theorem 8.2.1 implies that $M_{g_i, n_i + \delta_i}$ is irreducible of dimension $3g_i - 3 + n_i + \delta_i$ for all $i = 1, \dots, \gamma$. The above morphism is the quotient by the action of the finite group $\text{Aut}(\mathbf{G})$, hence $M_{\mathbf{G}}^{\text{alg}}$ is quasiprojective and irreducible. Since the above surjection has finite fibers we get

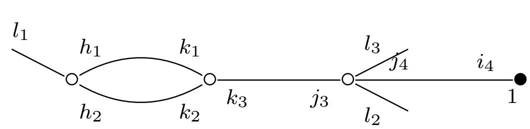
$$\dim M_{\mathbf{G}}^{\text{alg}} = \sum_{i=1}^{\gamma} (3g_i - 3 + n_i + 2\delta_i) = 3 \sum_{i=1}^{\gamma} g_i - 3\gamma + n + 2\delta$$

(since $\sum_{i=1}^{\gamma} \delta_i = \delta$). Now $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$ hence

$$\dim M_{\mathbf{G}}^{\text{alg}} = 3g - 3\delta + 3\gamma - 3 - 3\gamma + n + 2\delta = 3g - 3 + n - \delta.$$

♣

Example 8.2.4. Let $\mathbf{G}$ be the stable graph of genus 2 with $n = 3$ below. We have



$$M_{\mathbf{G}} \cong M_{0,3} \times M_{0,3} \times M_{0,4} \times M_{1,1}$$

since the action of $\text{Aut}(\mathbf{G})$ on the product is trivial. Indeed, the only non-trivial automorphism of $\mathbf{G}$, interchanging the two edges on the left, acts trivially on $M_{0,3}$ which is a point. The stable curve of genus 2 associated to

$$\Big((C_0; p_1, n_1, n_2),\ (D_0; q_1, q_2, q_3),\ (E_0; p_2, p_3, s_3, s_4),\ (C_1; t_4)\Big)$$

with $C_0 \cong D_0 \cong E_0 \cong \mathbb{P}^1$ and C_1 of genus 1, is given by identifying

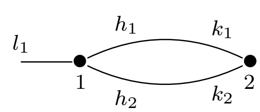
$$n_1 = q_1, \qquad n_2 = q_2, \qquad q_3 = s_3, \qquad s_4 = t_4$$

so that the legs correspond to p_1, p_2, p_3.

Example 8.2.5. Consider the graph, $\mathbf{G}$, in the picture below, stable of genus 4 with one leg. We marked the identifications on the edges. Now $\mathrm{Aut}(\mathbf{G})$ is generated by the involution, α, which interchanges the two edges, i.e. such that

$$\alpha(h_1) = (h_2), \qquad \alpha(k_1) = (k_2)$$

and fixes everything else.



We have

$$M_{\mathbf{G}} \cong \frac{M_{1,3} \times M_{2,2}}{\mathrm{Aut}(\mathbf{G})}$$

where the action of α on $M_{1,3} \times M_{2,2}$ swaps the two marked points of $M_{2,2}$ and the last two marked points of $M_{1,3}$. Indeed, the following two elements in $M_{1,3} \times M_{2,2}$

$$\big((C_1; p_1, p, p'),\ (C_2; q, q')\big), \qquad \big((C_1; p_1, p', p),\ (C_2; q', q)\big)$$

are conjugated by $\mathrm{Aut}(\mathbf{G})$ and give to the same point of $M_{\mathbf{G}}$. In other words, they are different presentations for the same curve.

9. Lecture 9

9.1. **Partition analogies.** We have two similar partitions:

$$M^{\mathrm{trop}}_{g,n} = \bigsqcup_{\mathbf{G} \in \mathcal{G}_{g,n}} M^{\mathrm{trop}}_{\mathbf{G}}$$

(see Theorem 5.1.3) and (16)

$$\overline{M}_{g,n} = \bigsqcup_{\mathbf{G} \in \mathcal{G}_{g,n}} M^{\mathrm{alg}}_{\mathbf{G}}.$$

In the sequel, by $\dim M^{\text{alg}}_{\mathbf{G}}$, resp. $\dim M^{\text{trop}}_{\mathbf{G}}$, we mean the dimension as an algebraic variety, resp. as an orbifold, while by $\operatorname{codim} M^{\text{alg}}_{\mathbf{G}}$, resp. $\operatorname{codim} M^{\text{trop}}_{\mathbf{G}}$, we mean the codimension in $\overline{M}_g$, resp. in $M^{\text{trop}}_{g,n}$.

Theorem 9.1.1. *Assume $2g-2+n \geq 1$. Consider the above partitions and the following bijection between their parts*

$$M^{\text{alg}}_{\mathbf{G}} \mapsto M^{\text{trop}}_{\mathbf{G}},$$

where $\mathbf{G}$ varies among all stable graphs of genus g with n legs. Then the following hold.

(a) $\dim M^{\text{alg}}_{\mathbf{G}} = \operatorname{codim} M^{\text{trop}}_{\mathbf{G}} = 3g-3+n-|E(\mathbf{G})|$.

(b) With the notation (6),

$$M^{\text{alg}}_{\mathbf{G}} \subset \overline{M^{\text{alg}}_{\mathbf{G}'}} \iff M^{\text{trop}}_{\mathbf{G}'} \subset \overline{M^{\text{trop}}_{\mathbf{G}}} \iff \mathbf{G} \geq \mathbf{G}'.$$

Proof. Set $\mathbf{G} = (G, w)$. For (a), we know from Lemma 8.2.3 that $\dim M^{\text{alg}}_{\mathbf{G}} = 3g-3+n-|E(\mathbf{G})|$. By (8), we have $\dim M^{\text{trop}}_{\mathbf{G}} = |E(\mathbf{G})|$, hence $\operatorname{codim} M^{\text{trop}}_{\mathbf{G}} = \dim M^{\text{trop}}_{g,n} - |E(\mathbf{G})| = 3g-3+n-|E(\mathbf{G})|$.

Now in (b), the second double implication follows from Theorem 5.1.3. We give the proof of the remaining part only in case $n = 0$, leaving the generalization to the reader.

Fix a stable curve $X \in M^{\text{alg}}_{\mathbf{G}}$. By the moduli properties of $\overline{M}_{g,n}$ (see subsection 7.2), if X lies in the closure of $M^{\text{alg}}_{\mathbf{G}'}$, there exist families of curves with dual graph $\mathbf{G}'$ specializing to X. We pick one such family, $f : \mathcal{X} \to B$ with a one dimensional base B with a marked point b_0, so that the fiber of f over every $b \neq b_0$ is a stable curve $X_b \in M^{\text{alg}}_{\mathbf{G}'}$, while the fiber over b_0 is isomorphic to X.

In such a family, every node of X_b specializes to a node of X, and distinct nodes specialize to distinct nodes. This singles out a set $T \subset E(\mathbf{G})$ of nodes of X, namely T is the set of nodes that are specializations of nodes of X_b. Let $S = E(\mathbf{G}) \smallsetminus T$, so S parametrizes the nodes of X that do not come from nodes of X_b. Consider the graph $(G/S, w_{/S})$; we claim that $(G/S, w_{/S}) = \mathbf{G}'$.

By construction we have a bijection

$$E(\mathbf{G}') \longleftrightarrow E(G/S) = T$$

mapping an edge of $\mathbf{G}'$, i.e. a node of X', to the node in X to which it specializes.

The total space $\mathcal{X}$ of our family of curves is singular along the nodes of the fibers X_b, for $b \neq b_0$. Let us desingularize $\mathcal{X}$ at such $|T|$ loci, we thus obtain a new family $\mathcal{Y} \to B$ whose fiber over $b \neq b_0$ is the normalization of X_b. The fiber over b_0 is the partial normalization of

X at T, which we denote by Y; so its dual graph satisfies

$$G_Y = G - T.$$

Notice that $\mathcal{Y} \to B$ is a union of families parametrized by the irreducible components of X_b, i.e. by the vertices of $\mathbf{G}'$. Let us denote these families by $\mathcal{Y}(v') \to B$; so if $b \neq b_0$ the fiber of $\mathcal{Y}(v')$ over b is the smooth irreducible component corresponding to $v' \in V(\mathbf{G}')$. The fiber over b_0 of $\mathcal{Y}(v') \to B$ is a connected component of Y, which we denote $Y(v')$, so that the special fiber Y is decomposed as follows

$$Y = \bigsqcup_{v' \in V(\mathbf{G}')} Y(v')$$

Now, $Y(v')$ determines a set of vertices of $\mathbf{G}$ (those corresponding to its irreducible components). Notice that two different vertices of $\mathbf{G}'$ determine in this way disjoint sets of vertices of $\mathbf{G}$. Therefore we have a surjection $\phi : V(\mathbf{G}) \to V(\mathbf{G}')$ mapping each vertex v to the vertex v' such that the component corresponding to v lies in $Y(v')$. It is clear that $\phi(v_1) = \phi(v_2)$ if and only if v_1 and v_2 belong to the same connected component of $G - T$. Therefore ϕ is the same map as the map $\sigma_V : V(G) \to V(G/S)$ (cf. Remark 4.1.2). This shows that $G_Y \cong G/S$. Finally, since the arithmetic genus of a family of algebraic curves is constant, for any $v' \in V(\mathbf{G}')$ we have that the genus of the component of Y corresponding to v', i.e. the weight $w'(v')$, is equal to the (arithmetic) genus of the limit curve $Y(v')$. Therefore

$$w'(v') = b_1(G_{Y(v')}) + \sum_{v \in \sigma_V^{-1}(v')} w(v) = b_1(\sigma^{-1}(v')) + \sum_{v \in \sigma_V^{-1}(v')} w(v).$$

By (5) the weight function w' coincides with $w/_S$; so we are done.

Conversely, assume $\mathbf{G} \geq \mathbf{G}'$ with $\mathbf{G}' = (G/S, w/_S)$ for some $S \subset E(\mathbf{G})$; let $T := E(\mathbf{G}) \smallsetminus S$. Let $X \in M^{\text{alg}}_{\mathbf{G}}$. We shall reverse the procedure we just described to prove that there exists a family of curves whose dual graph is $\mathbf{G}'$ specializing to X. Let $Y \to X$ be the normalization of X at T, so that $G_Y = G_X - T$. Notice that Y has $|S|$ nodes and is endowed with $|T|$ pairs of smooth points, namely the branches over the nodes in T, and it is a disjoint union of stable pointed curves, by Remark 8.1.4. Therefore, by Theorem 8.2.1, our curve Y is the limit of smooth, possibly disconnected, curves with $2|T|$ marked points; more precisely, there exists a family $\mathcal{Y} \to B$ over some smooth curve B, whose fiber over a fixed point $b_0 \in B$ is Y, and whose fibers over $b \in B \smallsetminus \{b_0\}$ is a smooth, possibly disconnected, curve. Furthermore, this family is endowed with $|T|$ pairs of disjoint sections in such a way that every connected component of $\mathcal{Y} \to B$ is a family of stable curves, smooth away from b_0.

We let $\mathcal{X}$ be the surface obtained by gluing together, transversally, the $|T|$ pairs of sections. Then, by construction, $\mathcal{X}$ is a family of nodal curves over B, whose fiber over b_0 is X and whose fiber over $b \neq b_0$ lies in $M_{\mathbf{G'}}^{\mathrm{alg}}$. ♣