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NÉRON MODELS AND COMPACTIFIED PICARD SCHEMES OVER THE MODULI STACK OF STABLE CURVES

By Lucia Caporaso

Abstract. We construct modular Deligne-Mumford stacks $\mathcal{P}_{d,g}$ representable over $\overline{\mathcal{M}}_g$ parametrizing Néron models of Jacobians as follows. Let B be a smooth curve and K its function field, let $\mathcal{X}_K$ be a smooth genus-g curve over K admitting stable minimal model over B. The Néron model $\mathrm{N}(\mathrm{Pic}^d\mathcal{X}_K) \to B$ is then the base change of $\mathcal{P}_{d,g}$ via the moduli map $B \longrightarrow \overline{\mathcal{M}}_g$ of f, i.e.: $\mathrm{N}(\mathrm{Pic}^d\mathcal{X}_K) \cong \mathcal{P}_{d,g} \times_{\overline{\mathcal{M}}_g} B$. Moreover $\mathcal{P}_{d,g}$ is compactified by a Deligne-Mumford stack over $\overline{\mathcal{M}}_g$, giving a completion of Néron models naturally stratified in terms of Néron models.

0. Contents

1. Introduction.

1.1. Problems and results. The first goal of this paper is a parametrization result for Néron models of Jacobians of stable curves (Theorem 6.1). A technical part of the argument that yields results of independent interest is the strengthening of a construction of the compactified Picard variety over $\overline{\mathcal{M}}_g$. A further outcome is a geometrically meaningful compactification of such Néron models. We proceed to discuss all of that more precisely.

Let $K = k(B)$ be the field of rational functions of a nonsingular one-dimensional scheme B defined over an algebraically closed field k. Let $\mathcal{X}_K$ be a nonsingular connected projective curve of genus $g \geq 2$ over K whose regular minimal model over B is a family $f\colon\ \mathcal{X} \longrightarrow B$ of stable curves.

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For any integer d, denote by $\mathrm{Pic}^d_K := \mathrm{Pic}^d\, \mathcal{X}_K$ the degree-d Picard variety of $\mathcal{X}_K$ (parametrizing line bundles of degree d on $\mathcal{X}_K$), and let $\mathrm{N}(\mathrm{Pic}^d_K)$ be its Néron model over B. It is well known that (since the total space $\mathcal{X}$ is nonsingular) the fibers of $\mathrm{N}(\mathrm{Pic}^d_K) \longrightarrow B$ over the closed points of B depend only on the corresponding fibers of f.

It makes therefore sense to ask the following question: does there exist a space over $\overline{\mathcal{M}}_g$, such that, for every K and $\mathcal{X}_K$ as above, $\mathrm{N}(\mathrm{Pic}^d_K)$ is the base change of such a space via the moduli map $B \longrightarrow \overline{\mathcal{M}}_g$ associated to the family f?

In this paper we give a positive answer to this question for every $g \geq 3$ and for every d such that $(d-g+1, 2g-2) = 1$. Let us first state a result in scheme theoretic terms, postponing the stack-theoretic generalization for a moment (cf. Theorem 6.1). We construct a separated scheme P^d_g over the moduli scheme of stable curves $\overline{M}_g$, having the following property: for any family $f\colon\ \mathcal{X} \longrightarrow B$ of automorphism-free stable curves with $\mathcal{X}$ regular, there is a canonical isomorphism of B-schemes

$$\mathrm{N}(\mathrm{Pic}^d_K) \cong_B B \times_{\overline{M}_g} P^d_g$$

where B is viewed as an $\overline{M}_g$- scheme via the moduli map of the family f.

Working within the category of schemes, the restriction to automorphism-free curves is necessary: if X is a stable curve, $\mathrm{Aut}(X)$ injects into the automorphism group of its generalized Jacobian (Theorem 1.13 [DM69]), hence there cannot possibly exist a universal Picard scheme over the whole of $\overline{M}_g$ (for the same reason why there exists no universal curve).

The stack theoretic approach is thus necessary to answer the above question in general; the corresponding result is the following: there exists a smooth Deligne-Mumford stack $\mathcal{P}_{d,g}$, with a natural representable morphism to the stack $\overline{\mathcal{M}}_g$, such that for every family $f\colon\ \mathcal{X} \longrightarrow B$ of stable curves with $\mathcal{X}$ regular, the Néron model of Pic^d_K is the fiber product $\mathcal{P}_{d,g} \times_{\overline{\mathcal{M}}_g} B$.

The stack $\mathcal{P}_{d,g}$ has a geometric description, as it corresponds to the "balanced Picard functor", which is a separated partial completion of the degree-d component of the classical Picard functor on smooth curves (cf. 4.15 and 5.11). Similarly the scheme P^d_g is the fine moduli space for such a functor restricted to automorphism-free curves (5.3).

The requirement $(d-g+1, 2g-2) = 1$ is well known (by [MR85]) to be necessary and sufficient for the existence of a Poincaré line bundle for the universal Picard variety $\mathrm{Pic}^d_g \longrightarrow M^0_g$ (associated to the universal family of smooth curves); we extend such a result as follows. Our scheme P^d_g will be constructed as a dense open subset of the compactification $\overline{P}_{d,g}$ of Pic^d_g obtained in [C94]; we prove that the above Poincaré line bundle extends over $\overline{P}_{d,g}$. More precisely, we prove that such a numerical condition characterizes when the balanced Picard functor is representable (and separated), and when the corresponding groupoid is a Deligne-Mumford stack, representable over $\overline{\mathcal{M}}_g$ (cf. chapter 5). Thus the hypothesis that $(d-g+1, 2g-2) = 1$ plays a crucial role in various places of our argument; we are therefore led to conjecture that without it the parametrization result (6.1) would fail.

A consequence of the construction is a modular completion $\overline{\mathcal{P}}_{d,g}$ of $\mathcal{P}_{d,g}$ by a smooth Deligne-Mumford stack representable over $\overline{\mathcal{M}}_g$, which enables us to obtain a geometrically meaningful compactification of the Néron model for every family f as above.

We prove that our compactification of the Néron model is endowed with a canonical stratification described in terms of the Néron models of the connected partial normalizations of the closed fiber of f (Theorem 7.9). Moreover, in 8.5, we exhibit it as a "quotient" of the Néron model for a ramified degree 2-base change of f.

Notice that as d varies, the closed fibers of $P_g^d \longrightarrow \overline{M}_g$ do not, hence the question naturally arises as to how many isomorphism classes of these spaces there are; the exact number of them is computed in 6.9.

1.2. Context. The language and the techniques used in this paper are mostly those of [BLR] for the theory of Néron models, and of [GIT] for Geometric Invariant Theory and its applications to moduli problems.

As we said, we use the compactification $\overline{P}_{d,g} \longrightarrow \overline{M}_g$ of the universal Picard variety; however such a space existed only as a scheme, not as a stack. To answer our initial question about the parametrization of Néron models, we need to "stackify" such a construction and to build the standard universal elements for it (the universal curve and the Poincaré bundle). This occupies most of section 5. We are in a lucky position to apply the theory of stacks as was developed in recent years, in fact $\overline{P}_{d,g}$ and P_g^d are geometric GIT-quotients hence our stacks are "quotient stacks", which have been carefully studied by many authors. In particular, we use [AV01], [ACV01], [E00], [LM] and [Vi89], together with the seminal paper [DM69].

Why should $\mathcal{P}_{d,g}$ be a good candidate to glue Néron models together over $\overline{\mathcal{M}}_g$? The initial observation, already at the scheme level, is that if the condition $(d-g+1, 2g-2) = 1$ holds every closed fiber of $\overline{P}_{d,g}$ over $\overline{M}_g$ contains the fiber of the corresponding Néron model as a dense open subset.

Néron models provide the solution for a fundamental mapping problem (see the "Néron mapping property" in 2.5) and are uniquely determined by this. Their existence for abelian varieties was established by A. Néron in [N64]; the theory was developed by M. Raynaud (in [R70]) who, in particular, unraveled the connection with the Picard functor in a way that will be heavily used in this paper. Néron models have been widely applied in arithmetic and algebraic geometry; a remarkable example is the proof (valid in all characteristics) of the stable reduction theorem for curves given in [DM69]. Nevertheless they rarely appear in the present-day moduli theory of curves, where their potential impact looks promising (see Section 9).

Néron models are well known not to have good functorial properties: their formation does not commute with base change, unless it is an étale one. However there are advantages in having a geometric description for them (and for their completion), such as the possibility to interpret mappings in a geometric way (note that their universal property gives us the existence of many such mappings, some arising form remarkable geometric settings). This may be fruitfully used to study problems concerning limits of line bundles and linear series, as we briefly illustrate in 9.

We mention one further motivating issue; that is the problem of comparing various existing completions of the Picard functor and of some of its distinguished subfunctors (such as the spin-functors or the functor of torsion points in the Jacobian). It is fair to say that our understanding of the situation is insufficient, a clear picture of how the various compactifications mentioned above relate to each other is missing. An overview of various completions of the generalized Jacobian with some comparison results is in [Al04] (more details in 6.4); the interaction between compactified spin schemes and Picard schemes is studied in [F04] and [CCC04]; various basic questions remain open. Understanding the relation with Néron models can be used for such problems, thanks to the Néron mapping property (see 6.3).

1.3. Summary. The paper is organized as follows: Section 3 recalls some basic facts about our Néron models, Sections 4 and 5 are about the "balanced Picard functor" and the corresponding stack; in 6 the connection with Néron models is established, together with some comments and examples. The last two sections are devoted to the completion of the Néron model, which is described in 7 with focus on the stratification, and in 8 as a quotient of a Néron model of a certain base change. In the Appendix some comments about applications, together with some useful combinatorial facts, are collected.

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2. Notation and terminology.

2.1. All schemes are assumed locally of finite type over an an algebraically closed field k, unless otherwise specified. R denotes a discrete valuation ring (a DVR) with algebraically closed residue field k and quotient field K. For any scheme T over $\operatorname{Spec} R$ we denote T_K the generic fiber and T_k the closed fiber.

If $\phi\colon\ W \longrightarrow B$ is a morphism and $T \longrightarrow B$ is a B-scheme we shall denote $W_T := W \times_B T$ and $\phi_T\colon\ W_T \longrightarrow T$ the projection.

2.2. X will be a nodal connected curve projective over k; $C_1, \ldots, C_\gamma$ its irreducible components.

For any complete subcurve $Z \subset X$, g_Z is its arithmetic genus, $Z' := \overline{X \smallsetminus Z}$ its complementary curve and $k_Z := \#(Z \cap Z')$. Then

$$w_Z := \deg_Z \omega_X = 2g_Z - 2 + k_Z.$$

For a line bundle $L \in \operatorname{Pic} X$ its *multidegree* is $\underline{\deg}\, L := (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L)$.

We denote $\underline{d} = (d_1, \ldots, d_\gamma)$ elements of $\mathbb{Z}^\gamma$ (or in $\mathbb{Q}^\gamma$) and $|\underline{d}| := \sum_1^\gamma d_i$. We say that $\underline{d}$ is positive (similarly, non-negative, divisible by some integer, etc.) if all d_i are. If $\underline{d} \in \mathbb{Z}^\gamma$ (or in $\mathbb{Q}^\gamma$) we denote the "restriction of $\underline{d}$ to the subcurve Z" of X by $d_Z = \sum_{C_i \subset Z} d_i$.

We set $\operatorname{Pic}^{\underline{d}} X := \{L \in \operatorname{Pic} X\colon\ \underline{\deg}\, L = \underline{d}\}$. In particular, the *generalized Jacobian* of X is $\operatorname{Pic}^{\underline{0}} X := \{L \in \operatorname{Pic} X\colon\ \underline{\deg}\, L = (0, \ldots, 0)\}$. There are (non-canonical) isomorphisms $\operatorname{Pic}^{\underline{0}} X \cong \operatorname{Pic}^{\underline{d}} X$ for every $\underline{d} \in \mathbb{Z}^\gamma$. Finally we set $\operatorname{Pic}^d X := \{L \in \operatorname{Pic} X\colon\ \deg L = d\} = \coprod_{|\underline{d}|=d} \operatorname{Pic}^{\underline{d}} X$.

2.3. $f\colon \mathcal{X} \longrightarrow B$ will denote a family of nodal curves; that is, f is a proper flat morphism of schemes over k, such that every closed fiber of f is a connected nodal curve.

$\mathcal{P}ic_f$ denotes the Picard functor of such a family (often denoted $\mathcal{P}ic_{\mathcal{X}/B}$; see [GIT] chapter 0 part (d) and [BLR] chapter 8 for the general theory). $\mathcal{P}ic_f^d$ is the subfunctor of line bundles of (relative) degree d.

We shall often consider $B = \operatorname{Spec} R$; in that case the closed fiber of f will be denoted by X; let us assume that for the rest of the section. $\mathcal{P}ic_f$ (and similarly $\mathcal{P}ic_f^d$) is represented by a scheme Pic_f (due to D. Mumford, see [BLR] Theorem 2 in 8.2 and [M66]) which may very well fail to be separated: if all geometric fibers of f are irreducible, then Pic_f is separated (due to A. Grothendieck [SGA], see also [BLR] Theorem 1 in 8.2) and conversely (see 3.1).

The identity component of the Picard functor is well known to be represented by a separated scheme over B (the *generalized Jacobian*, see [R70] 8.2.1), which we shall denote $\operatorname{Pic}_f^{\underline{0}}$ (denoted by P^0 in [R70] and by $\operatorname{Pic}^0_{\mathcal{X}/B}$ in [BLR]).

For any $\underline{d} \in \mathbb{Z}^\gamma$ consider $\operatorname{Pic}_f^{\underline{d}} \subset \operatorname{Pic}_f^d$, parametrizing line bundles of degree d whose restriction to the closed fiber has multidegree $\underline{d}$. Just like $\operatorname{Pic}_f^{\underline{0}}$, these are fine moduli schemes; $\operatorname{Pic}_f^{\underline{d}}$ is a natural $\operatorname{Pic}_f^{\underline{0}}$-torsor.

The generic fiber of $\operatorname{Pic}_f^{\underline{0}}$ and of Pic_f^0 coincide and will be denoted by Pic_K^0; similarly Pic_K^d denotes the generic fiber of Pic_f^d (and of $\operatorname{Pic}_f^{\underline{d}}$).

2.4. A *stable* curve is (as usual) a nodal connected curve of genus $g \geq 2$ having ample dualizing sheaf. The moduli scheme (respectively stack) for stable curves of genus g is denoted by $\overline{M}_g$ (resp. $\overline{\mathcal{M}}_g$). If $g \geq 3$ the locus $\overline{M}_g^0 \subset \overline{M}_g$ of curves with trivial automorphism group is nonempty, open and nonsingular.

A *semistable* curve is a nodal connected curve of genus $g \geq 2$ whose dualizing sheaf has non-negative multidegree. A *quasistable* curve Y is a semistable curve such that any two of its *exceptional components* do not meet (an exceptional component of Y is a smooth rational component $E \cong \mathbb{P}^1$ such that $\#(E \cap \overline{Y \smallsetminus E}) = 2$).

If Y is a semistable curve, its *stable model* is the stable curve obtained by contracting all of the exceptional copmonents of Y. For a given stable curve X there exist finitely many quasistable curves having X as stable model; we shall call such curves thet *quasistable curves of X*.

2.5. Let B be a connected Dedekind scheme with function field K. If A_K is an abelian variety over K, or a torsor under a smooth group scheme, we denote by $\mathrm{N}(A_K)$ the Néron model of A_K, which is a smooth model of A_K over B uniquely determined by the following universal property (the *Néron mapping property*, cf. [BLR] definition 1): every K-morphism $u_K\colon Z_K \longrightarrow A_K$ defined on the generic fiber of some scheme Z smooth over B admits a unique extension to a B-morphism $u\colon Z \longrightarrow N(A_K)$.

Recall that $N(A_K)$ may fail to be proper over B, whereas it is obviously separated. Although $N(A_K)$ is endowed with a canonical torsor structure, induced by the one of A_K, we shall always consider it merely as a scheme.

3. The Néron model for the degree-d Picard scheme.

We begin by introducing Néron models of Picard varieties of curves, following Raynaud's approach ([R70]). Most of the material in this section is in chapter 9 of [BLR] in a far more general form (also in sections 2 and 3 of [E98], which is closer to our situation); we revisit it with a slightly different terminology, suitable to our goals.

3.1. Let $f\colon \mathcal{X} \longrightarrow B = \operatorname{Spec} R$ be a family of curves and denote $\mathcal{X}_K$ the generic fiber, assumed to be smooth. To construct the Néron model of the Picard variety $\operatorname{Pic}^d_K := \operatorname{Pic}^d \mathcal{X}_K$ of $\mathcal{X}_K$, it is natural to look at the Picard scheme $\operatorname{Pic}^d_f \longrightarrow B$ of the given family, which is smooth and has generic fiber equal to Pic^d_K. The problem is that Pic^d_f will fail to be separated over B as soon as the closed fiber X of f is reducible.

From now on we assume that X is a reduced curve having at most nodes as singularities. Its decomposition into irreducible components is denoted $X = \cup_1^\gamma C_i$. One begins by isolating line bundles on X that are specializations of the trivial bundle (so called *twisters*).

Definition 3.2. (i) Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R$ be a family of nodal curves. A line bundle $T \in \operatorname{Pic} X$ is called an *f-twister* (or simply a *twister*) if there exist integers $n_1, \ldots, n_\gamma$ such that $T \cong \mathcal{O}_{\mathcal{X}}(\sum_1^\gamma n_i C_i) \otimes \mathcal{O}_X$

(ii) The set of all f-twisters is a discrete subgroup of $\operatorname{Pic}^0 X$, denoted $\mathrm{Tw}_f X$.

(iii) Let $L, L' \in \operatorname{Pic} X$. We say that L and L' are *f-twist equivalent* (or just *twist equivalent*) if for some $T \in \mathrm{Tw}_f X$ we have $L^{-1} \otimes L' \cong T$

The point is: every separated completion of Pic $\mathcal{X}_K$ over B must identify twist equivalent line bundles.

Remark 3.3. Notice that the integers $n_1, \dots, n_\gamma$ are not uniquely determined, as X is a principal divisor (the base being Spec R) and we have for every integer n, $\mathcal{O}_{\mathcal{X}}(nX) \otimes \mathcal{O}_X \cong \mathcal{O}_X$

We need the following well known (6.1.11 in [R70] and [BLR], lemma 10, p. 272) list of facts (recall that $k_Z := \#Z \cap \overline{X \smallsetminus Z})$):

LEMMA 3.4. *Let* $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R$ *be a family of nodal curves with* $\mathcal{X}$ *regular.*
(i) $\underline{deg}\,(\mathcal{O}_{\mathcal{X}}(\sum_1^\gamma n_i C_i) \otimes \mathcal{O}_X) = \underline{0}$ *if and only if* $n_i = n_j$ *for all* $i, j = 1 \dots \gamma$.
(ii) *Let* T *be a non-zero twister. There exists a subcurve* $Z \subset X$ *such that*

$$\deg_Z T \geq k_Z$$

(iii) *There is a natural exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^\gamma \longrightarrow \mathrm{Tw}_f X \longrightarrow 0.$$

Proof. For (i), set $T = \mathcal{O}_{\mathcal{X}}(\sum_1^\gamma n_i C_i) \otimes \mathcal{O}_X$. One direction follows immediately from (3.3). Conversely assume that $\underline{deg}\, T = 0$. Define for $n \in \mathbb{Z}$ the subcurve D_n of X by

$$D_n = \cup_{n_i = n} C_i$$

(If $n = 0$ the curve D_0 is the union of all components having coefficient n_i equal to zero.) Now X is partitioned as $X = \cup_{n \in \mathbb{Z}} D_n$ and every irreducible component of X belongs to exactly one D_n. By construction

$$T = \mathcal{O}_{\mathcal{X}} \left(\sum_{n \in \mathbb{Z}} n D_n \right) \otimes \mathcal{O}_X$$

and our goal is to prove that there is only one nonempty D_n appearing above. Let m be the minimum integer such that D_m is not empty, thus $D_n = \emptyset$ for all $n < m$. We have

$$\begin{aligned} deg_{D_m} T &= -m k_{D_m} + \sum_{n > m} n (D_n \cdot D_m) \\ &\geq -m k_{D_m} + (m+1) \sum_{n > m} (D_n \cdot D_m) \\ &\geq \sum_{n > m} (D_n \cdot D_m) = k_{D_m} \geq 0 \end{aligned} \tag{1}$$

where in the last inequality we have equality if and only if all D_n are empty for $n > m$ (so that $X = D_m$). On the other hand the hypothesis was $\underline{deg}\, T = 0$ and hence equality must hold above, so we are done. This also proves (ii) by taking $D_m = Z$.

Now we prove (iii). The sequence is defined as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\sigma} & \mathbb{Z}^{\gamma} & \xrightarrow{\tau} & \mathrm{Tw}_f X & \longrightarrow & 0 \\
 & & 1 & \mapsto & (1,\dots,1) & & & & \\
 & & & & (n_1,\dots,n_{\gamma}) & \mapsto & \mathcal{O}_{\mathcal{X}}(\sum_1^{\gamma} n_i C_i) \otimes \mathcal{O}_X & &
\end{array}$$

The map τ defines a Cartier divisor because $\mathcal{X}$ is regular. The injectivity of σ and the surjectivity of τ are obvious. The fact that $\mathrm{Im}\,\sigma \subset \ker \tau$ was oserved before (in (3.3)). Finally, suppose that $(n_1,\dots,n_{\gamma})$ is such that the associated f-twister $T := \mathcal{O}_{\mathcal{X}}(\sum_1^{\gamma} n_i C_i) \otimes \mathcal{O}_X$ is zero. Then T must have multidegree equal to zero, therefore, by the first part, we obtain that $(n_1,\dots,n_{\gamma}) = (m,\dots,m)$ for some fixed m and hence $(n_1,\dots,n_{\gamma}) \in \mathrm{Im}\,\sigma$. $\square$

3.5. Twisters on a curve X depend on two types of data: (1) discrete data, i.e. the choice of the coefficients $n_1,\dots,n_{\gamma}$, (2) continuous data, namely the choice of $f\colon \mathcal{X} \longrightarrow B = \mathrm{Spec}\, R$. More precisely, while twisters may depend on $f\colon \mathcal{X} \longrightarrow B$, their multidegree only depends of the type of singularities of $\mathcal{X}$ (see 6.6). Let us assume that $\mathcal{X}$ is regular. For every component C_i of X denote, if $j \neq i$, $k_{i,j} := \#(C_i \cap C_j)$ and $k_{i,i} = -\#(C_i \cap \overline{C \setminus C_i})$ so that the matrix $M_X := (k_{i,j})$ is an integer valued symmetric matrix which can be viewed as an *intersection matrix* for X. It is clear that for every pair i,j and for every (regular) $\mathcal{X}$, $\deg_{C_j} \mathcal{O}_{\mathcal{X}}(C_i) = k_{i,j}$. We have that $\sum_{j=1}^{\gamma} k_{i,j} = 0$ for every fixed i. Now, for every $i = 1,\dots,\gamma$ set $\underline{c_i} := (k_{1,i},\dots,k_{\gamma,i}) \in \mathbb{Z}^{\gamma}$ and

$$\mathbf{Z} := \{\underline{d} \in \mathbb{Z}^{\gamma}\colon\ |\underline{d}| = 0\}$$

so that $\underline{c_i} \in \mathbf{Z}$ and we can consider the sublattice Λ_X of $\mathbf{Z}$ spanned by them

$$\Lambda_X := < \underline{c_1},\dots,\underline{c_{\gamma}} > .$$

Thus, Λ_X is the set of multidegrees of all twisters and has rank $\gamma - 1$ (by 3.4(iii)).

Definition 3.6. The *degree class group* of X is the (finite) group $\Delta_X := \mathbf{Z}/\Lambda_X$. Let $\underline{d}$ and $\underline{d}'$ be in $\mathbb{Z}^{\gamma}$; we say that they are equivalent, denoting $\underline{d} \equiv \underline{d}'$, iff their difference is the multidegree of a twister, that is if $\underline{d} - \underline{d}' \in \Lambda_X$

3.7. The degree class group is a natural invariant to consider in this setting, it was first (to our knowledge) defined and studied by Raynaud (in 8.1.2 of [R70], denoted $\ker \beta / \mathrm{Im}\, \alpha$). We here adopt the terminology and notation used in [C94] section 4.1, which is convenient for our goals.

Δ_X is the component-group of the Néron model of the Jacobian of a family of nodal curves $\mathcal{X} \longrightarrow \operatorname{Spec} R$ with $\mathcal{X}$ nonsingular (see thm.1 in 9.6 of [BLR] and also 3.11). The group of components of a Néron model, in more general situations than the one studied in this paper, has been the object of much research. In particular, bounds for its cardinality have been obtained by D. Lorenzini in [L90]; see also [L89], [L93] and[BL02] for further study and applications. It is quite clear that Δ_X is a purely combinatorial invariant of X, a description of it in terms of the dual graph (due to Oda and Seshadri [OS79]) is recalled in 9.10.

3.8. The group Δ_X parametrizes classes of multidegrees summing to zero. More generally, let us denote Δ_X^d the set of classes of multidegrees summing to d:

$$\Delta_X^d := \{\underline{d} \in \mathbb{Z}^\gamma\colon \ |\underline{d}| = d\}/_{\equiv}$$

(where "$\equiv$" is defined in 3.6). We shall denote the elements in Δ_X^d by lowercase greek letters δ and write $\underline{d} \in \delta$ meaning that the class $[\underline{d}]$ of $\underline{d}$ is δ. Of course, there are bijections $\Delta_X^d \leftrightarrow \Delta_X$.

3.9. Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R = B$ with $\mathcal{X}$ regular and, as usual, assume that the closed fiber has γ irreducible components. Let $\underline{d}$ and $\underline{d}'$ be equivalent multidegrees, then there is a canonical isomorphism (depending only on f)

$$\iota_f(\underline{d}, \underline{d}')\colon \ \operatorname{Pic}_f^{\underline{d}} \longrightarrow \operatorname{Pic}_f^{\underline{d}'}$$

which restricts to the identity on the generic fiber. To prove that, recall that by 3.4 part (i) there exists a unique $T \in \operatorname{Tw}_f X$ such that $\underline{deg}\, T = \underline{d}' - \underline{d}$ and that there is a unique line bundle $\mathcal{T} \in \operatorname{Pic} \mathcal{X}$ such that $\mathcal{T}$ is trivial on the generic fiber and $\mathcal{T} \otimes \mathcal{O}_X \cong T$; in fact $\mathcal{T}$ must be of the form $\mathcal{O}_{\mathcal{X}}(\sum n_i C_i)$ and the n_i are determined up to adding a multiple of the closed fiber (see 3.3), which does not change the equivalence class of $\mathcal{T}$, as X is a principal divisor on $\mathcal{X}$ ($\operatorname{Pic} B = 0$). The isomorphism $\iota = \iota_f(\underline{d}, \underline{d}')$ is thus given by tensor product with $\mathcal{T}$, so that if $L \in \operatorname{Pic}_k^{\underline{d}}$ we have $\iota(L) = L \otimes T$, whereas if $L \in \operatorname{Pic}_K^d$ then $\iota(L) = L$.

We shall therefore identify $\operatorname{Pic}_f^{\underline{d}}$ with $\operatorname{Pic}_f^{\underline{d}'}$ for all pairs of equivalent $\underline{d}$, $\underline{d}'$. Thus for every $\delta \in \Delta_X^d$ we define for every $\underline{d} \in \delta$

$$\operatorname{Pic}_f^\delta := \operatorname{Pic}_f^{\underline{d}}. \tag{2}$$

The schemes $\operatorname{Pic}_f^\delta$ for a fixed total degree d all have the same generic fiber, Pic_K^d; we can then glue them together identifying their generic fibers. We shall denote the so obtained scheme over B

$$\frac{\coprod_{\delta\in\Delta_X^d}\operatorname{Pic}_f^{\delta}}{\sim_K}$$

(where $\sim_K$ denotes the gluing along the generic fiber) so that its generic fiber is Pic_K^d. We have:

Lemma 3.10. *Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R$ be a family of nodal curves with $\mathcal{X}$ regular. Then we have a canonical B-isomorphism*

$$\operatorname{N}(\operatorname{Pic}_K^d) \cong \frac{\coprod_{\delta\in\Delta_X^d}\operatorname{Pic}_f^{\delta}}{\sim_K}.$$

Proof. We may replace B by its strict henselization, in fact all the objects involved in the statement are compatible with étale base changes (of course $\mathcal{X}$ remains regular under any such base change, and Δ_X does not change). Recall also that Néron models descend from the strict henselization of B to B itself ([BLR] 6.5/3).

Assume first that $d = 0$. The Néron model of Pic_K^0 is proved in [BLR] (Theorem 4 in 9.5) to be equal to the quotient Pic_f^0/E where E is the schematic closure of the unit section $\operatorname{Spec} K \longrightarrow \operatorname{Pic}_K^0$ (so that E is a scheme over B, see [BLR] p. 265).

We can explicitly describe the closed fiber of E: $E_k = \operatorname{Tw}_f X$. In fact if L belongs to the closed fiber of E, then L is a line bundle on X which is a specialization of the trivial line bundle on $\mathcal{X}_K$; thus there exists a line bundle $\mathcal{L}$ on the total space $\mathcal{X}$ which is trivial on the generic fiber of f and whose restriction to X is L. Therefore $\mathcal{L}$ is of the form $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(D)$ with D supported on X, hence $L \in \operatorname{Tw}_f X$. The converse, i.e. the fact that $\operatorname{Tw}_f X$ is in E_k, is obvious. Now we have

$$\operatorname{Pic}_f^0 = \frac{\coprod_{|\underline{d}|=0}\operatorname{Pic}_f^{\underline{d}}}{\sim_K},$$

where $\sim_K$ denotes (just as above) the gluing of the schemes $\operatorname{Pic}_f^{\underline{d}}$ along their generic fiber (which is the same for all of them: Pic_K^0).

We obtain that the quotient by E identifies $\operatorname{Pic}_f^{\underline{d}}$ with $\operatorname{Pic}_f^{\underline{d}'}$ for all pairs of equivalent $\underline{d}$ and $\underline{d}'$, and this identification is the same induced by $\iota_f(\underline{d},\underline{d}')$ which was used to define $\operatorname{Pic}_f^{\delta}$ in 3.9 formula (2). Hence we have canonical isomorphisms

$$\operatorname{Pic}_f^0/E \cong \frac{\coprod_{|\underline{d}|=0}\operatorname{Pic}_f^{\underline{d}}}{\sim_K} \cong \frac{\coprod_{\delta\in\Delta_X}\operatorname{Pic}_f^{\delta}}{\sim_K}.$$

For general d, we have that Pic^d_K is a trivial Pic^0_K-torsor (in the sense of [BLR] 6.4) and we can reason as we just did to obtain

$$\mathrm{N}(\operatorname{Pic}^d_K) = \operatorname{Pic}^d_f / E^d = \left(\coprod_{|\underline{d}|=d} \operatorname{Pic}^{\underline{d}}_f \right) / E^d \cong \left(\coprod_{\delta \in \Delta^d_X} \operatorname{Pic}^\delta_f \right) / \sim_K,$$

where E^d denotes the analog of E, that is the schematic closure of a fixed section $\operatorname{Spec} K \longrightarrow \operatorname{Pic}^d_K$ (which exists because, R being henselian, f has a section). ☐

Remark 3.11. The lemma clarifies 3.7: the degree class group Δ_X is the group of connected components of the closed fiber of $\mathrm{N}(\operatorname{Pic}^0_K)$. In fact (recalling 3.8) for the closed fiber we have

$$(\mathrm{N}(\operatorname{Pic}^d_K))_k \cong \operatorname{Pic}^d X / \operatorname{Tw}_f X \cong \coprod_{\delta \in \Delta^d_X} \operatorname{Pic}^\delta X.$$

4. The balanced Picard functor As stressed in 3.11, the scheme structure of the closed fiber of the Néron model does not depend on the family f (the hypothesis that $\mathcal{X}$ is a nonsingular surface is crucial, see 6.6). We shall now ask whether, for a fixed d, our Néron models "glue together" over $\overline{M}_g$. From the previous section, a good starting point would be to find a "natural" way of choosing representatives for multidegree classes.

Example 4.1. Let $d = 0$ and consider the identity in Δ_X; then $(0, \ldots, 0)$ is a natural representative for that. It is then reasonable to choose representatives for the other classes so that their entries have the smallest possible absolute value.

For example, let $X = C_1 \cup C_2$ with $C_1 \cap C_2 = k$ and k odd. Then $\Delta_X \cong \mathbb{Z}/k\mathbb{Z}$ and our choice is:

$$(0,0), (\pm 1, \mp 1), \ldots, \left(\pm \frac{k-1}{2}, \mp \frac{k-1}{2} \right).$$

Another natural case is $d = 2g - 2$; here the class $[\underline{deg}\, \omega_X]$, represented of course by $\underline{deg}\, \omega_X$, plays the role of the identity. Therefore, as before, the other representatives should be chosen as close to $\underline{deg}\, \omega_X$ as possible. For X as above the representatives would be (recalling that $w_{C_i} := \deg_{C_i} \omega_X$)

$$(w_{C_1}, w_{C_2}), (w_{C_1} \pm 1, w_{C_2} \mp 1), \ldots, \left(w_{C_1} \pm \frac{k-1}{2}, w_{C_2} \mp \frac{k-1}{2} \right).$$

In what follows we use the notation of 2.2.

Definition 4.2. Let X be a nodal curve of any genus.

(i) The *basic domain* of X is the bounded subset $\mathrm{B}_X \subset \mathbb{Z}^\gamma$ made of all $\underline{d} \in \mathbb{Z}^\gamma$ such that $|\underline{d}| = 0$ and such that for every subcurve $Z \subset X$ we have

$$-\frac{k_Z}{2} \leq d_Z \leq \frac{k_Z}{2}.$$

(ii) For any $\underline{b} \in \mathbb{Q}^\gamma$ such that $b := |\underline{b}| \in \mathbb{Z}$ denote $\mathrm{B}_X(\underline{b})$ the subset of $\mathbb{Z}^\gamma$ made of all $\underline{d} \in \mathbb{Z}^\gamma$ such that $|\underline{d}| = b$ and such that for every subcurve $Z \subset X$ we have

$$b_Z - \frac{k_Z}{2} \leq d_Z \leq \frac{k_Z}{2} + b_Z$$

Remark 4.3. Note that B_X (and similarly $\mathrm{B}_X(\underline{b})$) is the set of integral points contained in a polytope of $\mathbb{Q}^\gamma$, whose boundary is defined by the inequalities in 4.2. We shall refer to $\mathrm{B}_X(\underline{b})$ as a *translate* of B_X, although this is is slightly abusive.

In the definition one could replace "every subcurve Z of X" with "every connected subcurve Z of X" but not with "every irreducible component of X". In other words the basic polytope of X is not in general, a product of $\gamma - 1$ intervals (it is, of course, if X has only two components, in which case it is an interval).

To connect with the previous discussion, we have:

LEMMA 4.4. *Let X be a nodal (connected) curve of any genus. Fix any $\underline{b} \in \mathbb{Q}^\gamma$ with $b := |\underline{b}| \in \mathbb{Z}$. Then every $\delta \in \Delta_X^b$ has a representative contained in* $\mathrm{B}_X(\underline{b})$.

Proof. The proof of proposition 4.1 in [C94], apparently only a special case of this lemma (namely X quasistable (cf. 2.4) and $\underline{b} = \underline{b}_X^d$ as in (3) below), carries out word for word. □

4.5. We shall choose a special translate of B_X^d, according to the topological characters of X. Let $g \geq 2$, set

$$\underline{b}_X^d := \left(w_{C_1}\frac{d}{2g-2}, \ldots, w_{C_\gamma}\frac{d}{2g-2}\right) \quad \text{and} \quad \mathrm{B}_X^d := \mathrm{B}_X(\underline{b}_X^d). \tag{3}$$

Then:

Definition 4.6. Let X be a semistable curve of genus $g \geq 3$ and $L \in \operatorname{Pic}^d X$. Let $\underline{d}$ be the multidegree of L, We shall say that:

(i) $\underline{d}$ is *semibalanced* if for every subcurve Z of X the following ("Basic Inequality)" holds

$$m_Z(d) := d\frac{w_Z}{2g-2} - \frac{k_Z}{2} \leq \deg_Z L \leq d\frac{w_Z}{2g-2} + \frac{k_Z}{2} =: M_Z(d) \tag{4}$$

(equivalently, if $\underline{d} \in \mathrm{B}_X^d$) and if for every exceptional component E of X

$$0 \leq \deg_E L \leq 1 (= M_E(d)). \tag{5}$$

(ii) $\underline{d}$ is *balanced* if it is semibalanced and if for every exceptional component $E \subset X$

$$\deg_E L = 1. \tag{6}$$

(iii) $\underline{d}$ is *stably balanced* if it is balanced and if for every subcurve Z such that $d_Z = m_Z(d)$ we have that $\overline{X \smallsetminus Z}$ is a union of exceptional components.

If $\mathcal{X} \longrightarrow B$ is a family of semistable curves and $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ of relative degree d, we say that $\mathcal{L}$ is (respectively *stably*, *semi*) *balanced* if for every $b \in B$ the restriction of $\mathcal{L}$ to X_b has (stably, semi) balanced multidegree.

4.7. In particular if X is a stable curve the set B_X^d (cf. 4.5) equals the set of balanced multidegrees of total degree d.

The inequality (4) was discovered by D. Gieseker in the course of the construction of the moduli scheme $\overline{M}_g$. Proposition 1.0.11 in [Gie82] states that (4) is a necessary condition for the GIT-semistability of the Hilbert point of a (certain type of) projective curve; it was later proved in [C94] that it is also sufficient. We mention that there exist other interesting incarnations of that inequality, for example in [OS79] and [S94] ([Al04] connects one to the other). The terminology used in Definition 4.6 was introduced in [CCC04] (cf. Theorem 5.16 therein) to reflect the GIT-behaviour of Hilbert points.

Example 4.8. The representatives in 4.1 (for $d = 0$ and $d = 2g - 2$) are all stably balanced and they are all the balanced multidegrees for that X and those d's.

Remark 4.9. It is easy to check (combining (4) and (6) of 4.6) that balanced line bundles live on quasistable, rather than semistable curves, and hence on a "bounded" class of curves. In analogy with semistable curves, while semibalanced line bundles do not admit a nice moduli space (just like semistable curves) they do admit a "balanced line bundle model" (by contracting all of the exceptional components where the degree is 0, see 9.1).

Remark 4.10. Assume that d is very large with respect to g, then a balanced line bundle L on a quasistable curve X of genus g is necessarily very ample. In fact if $Z \subset X$, it suffices to show that the restriction of L to Z is very ample; if Z is exceptional then $\deg_Z L = 1$, otherwise we have $\deg_Z L \geq m_Z(d) = d\frac{w_Z}{2g-2} - \frac{k_Z}{2}$ and, since $w_Z \geq 1$ and $k_Z \leq g + 1$, the claim follows trivially.

Remark 4.11. Notation as in 4.6.

(a) Set $Z' := \overline{X \smallsetminus Z}$. Then $d = m_Z(d) + M_{Z'}(d)$, in particular $d_Z = m_Z(d)$ if and only if $d_{Z'} = M_{Z'}(d)$.

(b) Let X be stable; then $\underline{d}$ is stably balanced if and only if strict inequality holds in (4) for every $Z \subsetneq X$.

(c) Let X be quasistable. Then a balanced $\underline{d}$ is stably balanced if and only if the subcurves where strict inequality in (4) fails are all the Z' unions of exceptional components (in which case $d_{Z'} = M_{Z'}(d)$) and (by (a)) their complementary curves Z (in which case $d_Z = m_Z(d)$).

Proposition 4.12. *Fix $d \in \mathbb{Z}$ and $g \geq 2$.*

(i) *Let X be a quasistable curve of genus g and $\delta \in \Delta_X^d$. Then δ admits a semibalanced representative.*

(ii) *A balanced multidegree is unique in its equivalence class if and only if it is stably balanced.*

(iii) $(d - g + 1, 2g - 2) = 1$ *if and only if for every quasistable curve X of genus g and every $\delta \in \Delta_X^d$, δ has a unique semibalanced representative.*

Proof. (i) By 4.4 we know that every δ has a representative $\underline{d}$ in B_X^d; if X is stable this is enough. Assume that X has an exceptional component E, notice that $m_E(d) = -1$ thus we must prove that a representative for δ can be chosen so that its restriction to E is not -1. Assume first that E is the unique exceptional component. Observe that for any subcurve $Z \subset X$ and every decomposition $Z = A \cup B$ into two subcurves having no component in common and meeting in $k_{A,B}$ points, we have (omitting the dependence on d to simplify the notation)

$$M_Z = M_A + M_B - k_{A,B}. \tag{7}$$

Now let $\underline{d} \in \mathrm{B}_X^d$ and suppose that $d_E = -1 = m_E$, denote $Z = E'$ the complementary curve and note that by 4.11(a) we have that $d_Z = M_Z$. Let $\underline{e} \in \Lambda_X$ be the multidegree associated to E (notation of 3.5), we claim that $\underline{d}' := \underline{d} - \underline{e}$ is semibalanced. We have that $d'_E = (\underline{d} - \underline{e})_E = -1 + 2 = 1$ so we are OK on E, now it suffices check every connected subcurve $A \subset Z$ which meets E. Suppose first that $E \not\subset A$, then $d'_A = d_A - k_{A,E}$, where $k_{A,E} = \#(A \cap E) > 0$. By contradiction assume that $\underline{d}'$ violates (4) on A, then, as d_A satisfies (4) and $d'_A < d_A$ we must have that

$$d'_A = d_A - k_{A,E} < m_A = M_A - k_A.$$

Now let $Z = A \cup B$ as above, so that $k_{A,B} = k_A - k_{A,E}$, hence

$$d_A < M_A - k_{A,B}.$$

We conclude with the inequality

$$M_Z = d_Z = d_A + d_B < M_A - k_{A,B} + M_B$$

contradicting (7). Now let $E \subset A$; if $E \cap B = \emptyset$ then $d_A = d'_A$ and we are done. Otherwise E meets A in one point and one easily cheks that the basic inequality for A is exactly the same as for $\overline{A \smallsetminus E}$, so we are done by the previous argument.

Since X is quasistable, two of its exceptional components do not meet and hence this argument can be iterated; this proves (i).

For (ii), begin with a simple observation. For every subcurve Z of X, the interval allowed by the basic inequality contains at most $k_Z + 1$ integers and the maximum $k_Z + 1$ is attained if and only if its extremes $m_Z(d)$ and $M_Z(d)$ are integers.

Let now $\underline{d}$ be stably balanced and $\underline{t} \in \Lambda_X$ (that is, $\underline{t} = \underline{deg}\, T$ for some twister T); then, by 3.4 part (ii) there exists a subcurve $Z \subset X$ on which $(\underline{d}+\underline{t})_Z \geq d_Z + k_Z$. This implies that $\underline{d} + \underline{t}$ violates the Basic Inequality, in fact either d_Z lies in the interior of the allowed range and hence $d_Z + k_Z$ is out of the allowed range; or d_Z is extremal, and we use 4.11(c). Therefore a stably balanced representative is unique. Conversely, by what we said, two equivalent multidegrees that are both balanced must be at the extremes of the allowed range for some curve Z, so neither can be stably balanced (by 4.11).

Now part (iii). As explained above, it suffices to prove that $(d-g+1, 2g-2) = 1$ if and only if for every X quasistable of genus g and every subcurve $Z \subset X$ such that neither Z nor Z' is a union of exceptional components, $m_Z(d)$ is not integer. Suppose that $(d-g+1, 2g-2) = 1$ holds, then $(d, g-1) = 1$ (the converse holds only for odd g). By contradiction, let X be a quasistable curve having a subcurve Z as above for which $m_Z(d)$ is integer; thus

$$\frac{dw_Z}{2g-2} = \frac{n}{2} \quad \text{where} \quad n \in \mathbb{Z}\text{:}\ n \equiv k_Z \bmod (2), \tag{8}$$

hence $g - 1$ divides w_Z. Then (by 4.11(a)) $M_{Z'}$ and $m_{Z'}$ are also integer, therefore arguing as for Z, $g - 1$ divides $w_{Z'}$. Now notice that $2(g - 1) = w_Z + w_{Z'}$ and that w_Z and $w_{Z'}$ are not zero (because Z and Z' are not union of exceptional components). We conclude that

$$g - 1 = w_Z = w_{Z'} \quad \text{so that} \quad g = 2g_Z + k_Z - 1. \tag{9}$$

Thus by the (8)

$$\frac{dw_Z}{g-1} = d = n \quad \text{hence} \quad d \equiv k_Z \bmod (2).$$

On the other hand the second identity in (9) shows that

$$(g-1) \equiv k_Z \bmod (2), \quad \text{hence} \quad d \equiv (g-1) \bmod (2).$$

The latter implies that 2 divides $(d-g+1, 2g-2)$, a contradiction.

Conversely: suppose that for some X and $Z \subset X$ we have (see (8))

$$\frac{dw_Z}{g-1} = n \quad \text{with} \quad n \in \mathbb{Z}\colon\ n \equiv k_Z \bmod (2).$$

If $(d, g-1) \neq 1$ a fortiori $(d-g+1, 2g-2) \neq 1$. Suppose then that $g-1$ divides w_Z; we have just proved that this implies $g-1 = w_Z$, that $d = n$ and that

$$d \equiv (g-1) \bmod (2)$$

hence 2 divides $(d-g+1, 2g-2)$, and we are done. □

A weaker version of this result is proved in [C94] Section 4.2, where the assumption that d be very large is used. Despite the overlapping, we gave here the full general proof to stress the intrinsic nature of definition 4.6 and contrast the impression, which may arise from [Gie82] and [C94], that it be a technical condition deriving from Geometric Invariant Theory.

A consequence of 4.12 and its proof is the following useful:

COROLLARY - *Definition* 4.13. *Let d be an integer and X a stable curve, we shall say that X is d-*general *if the following equivalent conditions hold.*

(i) *A multidegree on X is balanced if and only if it is stably balanced.*

(ii) *The natural map sending a balanced multidegree to its class*

$$B_X^d \longrightarrow \Delta_X^d, \qquad \underline{d} \mapsto [\underline{d}]$$

is a bijection.

(iii) *For every quasistable curve Y of X, every element in Δ_Y^d has a unique semibalanced representative.*

Remark 4.14. The assumption $(d-g+1, 2g-2) = 1$ in part (iii) of 4.12 is a uniform condition ensuring that every stable curve of genus g is d-general. The terminology is justified by the fact that the locus in $\overline{M}_g$ of d-general curves is open (see 5.6).

At the opposite extreme is the case $d = (g-1)$ (and, more generally, $d = n(g-1)$ with n odd), which is uniformly degenerate in the sense that for evey $X \in \overline{M}_g$ there exists $\delta \in \Delta_X^d$ having more than one balanced representative.

We shall now define the moduli functor for balanced line bundles on stable curves.

Definition 4.15. Let $f\colon \mathcal{X} \longrightarrow B$ be a family of stable curves and d an integer. The *balanced Picard functor* $\mathcal{P}_f^d$ is the contravariant functor from the category of B-schemes to the category of sets which associates to a B-scheme T the set of equivalence classes of balanced line bundles $\mathcal{L} \in \operatorname{Pic} \mathcal{X}_T$ of relative degree d. We say that $\mathcal{L}$ and $\mathcal{L}'$ are equivalent if there exists $M \in \operatorname{Pic} T$ such that $\mathcal{L} \cong \mathcal{L}' \otimes f_T^* M$.

A B-morphism $\phi\colon T' \longrightarrow T$ is mapped by $\mathcal{P}_f^d$ to the usual pull-back morphism from $\mathcal{P}_f^d(T)$ to $\mathcal{P}_f^d(T')$.

It is clear that $\mathcal{P}_f^d$ is a subfunctor of $\mathcal{P}ic_f^d$. The point is that, in some "good" cases, $\mathcal{P}_f^d$ is representable by a separated scheme.

Example 4.16. Consider the "universal family of stable curves" of genus g

$$f_g\colon \mathcal{C}_g \longrightarrow \overline{M}_g^0 \subset \overline{M}_g.$$

(cf. 2.4). In this case we shall simplify the notation and set

$$\mathcal{P}_g^d := \mathcal{P}_{f_g}^d.$$

Observe that if $\mathcal{P}_g^d$ is representable by a separated scheme P_g^d, then for every family of automorphism-free stable curves $f\colon \mathcal{X} \longrightarrow B$, the functor $\mathcal{P}_f^d$ is representable by the scheme $\mu_f^* P_g^d = B \times_{\overline{M}_g} P_g^d$ where $\mu_f\colon B \longrightarrow \overline{M}_g$ is the moduli morphism of f.

5. Balanced Picard schemes and stacks. The purpose of this section is to build the "representable stack version" of the compactified universal Picard variety constructed in [C94] simply as a coarse moduli scheme. From now we fix integers d and $g \geq 3$ and set $r := d - g$.

5.1. We begin by recalling some facts about the restriction of the balanced Picard functor $\mathcal{P}_g^d$ (cf. 4.16) to nonsingular curves, which we denote $\mathcal{P}ic_g^d$. It is a well known fact that $\mathcal{P}ic_g^d$ is coarsely represented by the the so called "universal degree-d Picard variety", $\operatorname{Pic}_g^d \longrightarrow M_g$, over the moduli scheme of nonsingular curves (we use here the notation "Pic_g^d" in place of "$P_{d,g}$" used in [C94] and in [HM98]). The existence of the variety Pic_g^d follows from general results of A. Grothendieck ([SGA] and [M66], see [GIT] 0.5 (d) for an overview). As we already mentioned, for arbitrary values of d the functor $\mathcal{P}ic_g^d$ is only coarsely represented by Pic_g^d, in fact a Poincaré line bundle does not always exist. More precisely, by a result of N. Mestrano and S. Ramanan, a Poincaré line bundle exists if and only if $(d-g+1, 2g-2) = 1$ (in $\operatorname{char} k = 0$, see Cor. 2.9 of [MR85]).

5.2. To deal with singular stable curves we shall use the compactification $\overline{P}_{d,g} \longrightarrow \overline{M}_g$ of $\operatorname{Pic}_g^d \longrightarrow M_g$ constructed in [C94], from which we need to recall and improve some results. Assume that d is very large (which is irrelevant, see below); such a compactification is the GIT-quotient $\overline{P}_{d,g} = H_d/G$ of the action of the group $G = PGL(r+1)$ on the locus H_d of GIT-semistable points in the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^r}^{dt-g+1}$ (for technical reasons concerning linearizations, one actually carries out the GIT-construction using the group $SL(r+1)$, rather than $PGL(r+1)$; since the two groups have the same orbits this will not be a problem).

(1) Denote by $\mathcal{Z}_d$ the restriction to H_d of the universal family over the Hilbert scheme

$$\mathbb{P}^r \times H_d \supset \mathcal{Z}_d \longrightarrow H_d,$$

for $h \in H_d$ let Z_h be the fiber of $\mathcal{Z}_d$ over h and $L_h = \mathcal{O}_{Z_h}(1)$ the embedding line bundle. Z_h is a nondegenerate quasistable curve in $\mathbb{P}^r$ and L_h is balanced in the sense of 4.6 (by [Gie82]); conversely, every such a curve embedded in $\mathbb{P}^r$ by a balanced line bundle appears as a fiber over H_d (by [C94]). The point h is GIT-stable if and only if L_h is stably balanced.

(2) For $h \in H_d$ denote $X_h \in \overline{M}_g$ the stable model of the quasistable curve Z_h. If X_h is d-general (see 4.13) the point h is GIT-stable, which in turn implies that there is a natural injection (by [C94] Section 8.2)

$$\mathrm{Stab}_G(h) \hookrightarrow \mathrm{Aut}(X_h).$$

Conversely, if $X \in \overline{M}_g$ is not d-general, there exists a (strictly semistable) $h \in H_d$ lying over X having $\dim \mathrm{Stab}_G(h) > 0$.

(3) H_d is regular and irreducible (by Lemma 2.2 and Lemma 6.2 in [C94]).

(4) The GIT-quotient H_d/G is geometric (i.e. all semistable points are stable) if and only if d is such that $(d - g + 1, 2g - 2) = 1$ ([C94] Prop. 6.2).

(5) For every pair of integers d and d' such that $d \pm d' = n(2g-2)$, for $n \in \mathbb{Z}$, there are natural isomorphisms $\overline{P}_{d,g} \cong \overline{P}_{d',g}$ ([C94] Lemma 8.1). This allows us to define $\overline{P}_{d,g}$ for every $d \in \mathbb{Z}$, compatibly with the geometric description. That is, for $d \in \mathbb{Z}$, pick n such that $d' = d + n(2g - 2)$ is large enough, the above isomorphism $\overline{P}_{d,g} \cong \overline{P}_{d',g}$ is constructed by tensoring with the nth power of the relative dualizing sheaf. It is easy to verify that a line bundle L on a curve X is balanced if and only if $L \otimes \omega_X^{\otimes n}$ is balanced.

We begin with a scheme-theoretic result that will be generalized later on.

Proposition 5.3. *Let $g \geq 3$ and d be such that $(d - g + 1, 2g - 2) = 1$.*
(i) *The functor $\mathcal{P}_g^d$ is representable by a separated scheme P_g^d.*
(ii) *P_g^d is integral, regular and quasiprojective.*
(iii) *Let $[X] \in \overline{M}_g^0$ and denote P_X^d the fiber of P_g^d over it. Then P_X^d is regular of pure dimension g. In particular P_g^d is smooth over $\overline{M}_g^0$.*

Proof. Assume first that d is very large ($d \geq 20(g-1)$ will suffice). We use the notation and set up of 5.2 above. Denote by H_d^{st} the open subset of H_d

parametrizing points corresponding to stable curves, that is

$$H_d^{st} := \{h \in H_d \colon\ Z_h \text{ is a stable curve}\}.$$

By 5.2(1) there is a natural surjective map $\mu\colon H_d^{st} \longrightarrow \overline{M}_g$. Set $H := \mu^{-1}(\overline{M}_g^0)$ so that H parametrises points h such that Z_h is a projective stable curve free from automorphisms, L_h is a degree-d stably balanced line bundle on Z_h (by 5.2(4)) and $\operatorname{Stab}_G(h) \cong \operatorname{Aut}(Z_h) = \{1\}$ (by 5.2(2))

We have that H and H_d^{st} are G-invariant integral nonsingular schemes (by 5.2(3)). We shall denote $f_H\colon\ \mathcal{Z} \longrightarrow H$ the restriction to H of the universal family $\mathcal{Z}_d$ and define $P_g^d := H/G$, so that $H \longrightarrow P_g^d$ is the geometric quotient of a free action of G. Moreover, G acts naturally (and freely) also on $\mathcal{Z}$ so that the quotent $\mathcal{C}_{P_g^d} := \mathcal{Z}/G$ gives a universal family on P_g^d. Let us represent our parameter schemes and their families in a diagram

$$\begin{tikzcd}
\mathbb{P}^r & \mathbb{P}^r \times H \arrow[l, "\pi"'] \arrow[r, phantom, "\supset"] & \mathcal{Z} \arrow[r, "q"] \arrow[d, "f_H"] & \mathcal{C}_{P_g^d} \arrow[r, "p"] \arrow[d] & \mathcal{C}_g \arrow[d] & \\
& & H \arrow[r] & P_g^d = H/G \arrow[r, "\phi"] & \overline{M}_g^0 \arrow[r, phantom, "\subset"] & \overline{M}_g .
\end{tikzcd} \tag{10}$$

Notice that all squares are cartesian (i.e., fiber products) so that all vertical arrows are universal families.

Now let us consider the natural polarization $\mathcal{L} := \mathcal{O}_{\mathcal{Z}}(1) = \pi^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathcal{O}_{\mathcal{Z}}$. As we said in 5.2, $\mathcal{L}$ is stably balanced and, conversely, every pair (X, L), with X an automorphism free stable curve and $L \in \operatorname{Pic}^d X$ a stably balanced line bundle, is represented by a G-orbit in H. More generally, in Prop. 8.1(2) of [C94] it is proved that $\overline{P}_{d,g}$ is a coarse moduli scheme for the functor of stably balanced line bundles on quasistable curves.

In diagram (10) we have exhibited a universal family $\mathcal{C}_{P_g^d} \longrightarrow P_g^d$, to complete the statement we must show that there exists a universal or *Poincaré* line bundle $\overline{\mathcal{L}}$ over $\mathcal{C}_{P_g^d}$ (determined, of course, modulo pull-backs of line bundles on P_g^d). This follows from lemma 5.5, with $T = P_g^d$, $E = H$ and ψ the inclusion, so that $\mathcal{X} = \mathcal{C}_{P_g^d}$.

We have so far proved that, if d is large, the functor $\mathcal{P}_g^d$ is represented by the scheme P_g^d equipped with the universal pair $(\mathcal{C}_{P_g^d}, \overline{\mathcal{L}})$. The same result for all d is obtained easily using 5.2(5).

Now we prove (ii) and (iii). We constructed P_g^d as the quotient H/G obtained by restricting the quotient $\overline{P}_{d,g} = H_d/G$, that is, we have a diagram

$$\begin{tikzcd}
H \arrow[r, phantom, "\subset"] \arrow[d] & H_d \arrow[d] \\
P_g^d \arrow[r, phantom, "\subset"] & \overline{P}_{d,g} .
\end{tikzcd} \tag{11}$$

Thus P^d_g is quasiprojective because H is open and G-invariant. P^d_g is integral and regular because H is irreducible and regular (5.2(3)) and G acts freely on it. This concludes the second part of the statement.

The fact that P^d_X is smooth of pure dimension g follows immediately from Cor. 5.1 in [C94], which implies that P^d_X is a finite disjoint union of isomorphic copies of the generalized Jacobian of X.

Finally, P^d_g is flat over $\overline{M}^0_g$ (a consequence of the equidimensionality of the fibers) and, moreover, smooth because the fibers are all regular. □

5.4. Some notation before establishing the existence of Poincaré line bundles and thus complete the proof of 5.3. If $\psi\colon E \longrightarrow H_d$ is any map we denote by $f_E\colon \mathcal{Z}_E = \mathcal{Z}_d \times_{H_d} E \longrightarrow E$ and by $\mathcal{L}_E \in \operatorname{Pic} \mathcal{Z}_E$ the pull back of the polarization $\mathcal{O}_{\mathcal{Z}_d}(1)$ on $\mathcal{Z}_d$, so that $\mathcal{L}_E$ is a balanced line bundle of relative degree d. If, furthermore, $\pi\colon E \longrightarrow T$ is a principal G-bundle and the above map ψ is G-equivariant, we can form the quotient

(12)
$$\begin{tikzcd}
\mathcal{Z}_E \arrow[r] \arrow[d] & E \arrow[d] \\
\mathcal{X} = \mathcal{Z}_E/G \arrow[r, "f"] & E/G = T
\end{tikzcd}$$

so that f is a family of quasistable curves.

The proof of the next Lemma applies a well known method of M. Maruyama [M78]; we shall make the simplifying assumption that d be large, which will later be removed.

LEMMA 5.5. *Notation as in 5.4. Assume* $d \gg 0$ *and* $(d-g+1, 2g-2) = 1$. *Let* $\pi\colon E \longrightarrow T$ *be a principal* $PGL(r+1)$*-bundle and* $\psi\colon E \longrightarrow H_d$ *be an equivariant map. Then there exists a balanced line bundle* $\overline{\mathcal{L}} \in \operatorname{Pic} \mathcal{X}$ *of relative degree d such that for every* $e \in E$ *we have* $(\mathcal{L}_E)_{|Z_e} \cong \overline{\mathcal{L}}_{|X_{\pi(e)}}$.

Proof. The statement holds locally over T, since $E \longrightarrow T$ is a $PGL(r+1)$-torsor. Thus we can cover T by open subsets $T = \cup U_i$ such that, denoting the restriction of f to $\mathcal{X}_i := f^{-1}(U_i)$ by

$$f_i\colon \mathcal{X}_i \longrightarrow U_i,$$

there is an $\overline{\mathcal{L}_i} \in \operatorname{Pic} \mathcal{X}_i$ for which the thesis holds. We now prove that the $\overline{\mathcal{L}_i}$ can be glued together to a line bundle over the whole of $\mathcal{X}$, modulo tensoring each of them by the pull-back of a line bundle on U_i.

By hypothesis there exist integers a and b such that

$$a(d-g+1) + b(2g-2) = -1$$

which we re-write as

$$(a-b)(d-g+1)+b(d+2g-2-g+1)=-1. \tag{13}$$

Observe that, denoting by χ_{f_i} the relative Euler characteristic (with respect to the family f_i) we have that $\chi_{f_i}(\overline{\mathcal{L}}_i)=d-g+1$ and $\chi_{f_i}(\overline{\mathcal{L}}_i\otimes\omega_{f_i})=d+2g-2-g+1$. Note also that $\overline{\mathcal{L}}_i$ and $\overline{\mathcal{L}}_i\otimes\omega_{f_i}$ have no higher cohomology (d is very large) and hence their direct images via f_i are locally free of rank equal to their relative Euler characteristic. Define now for every i

$$\mathcal{N}_i:=f_i^*\Big(\det(f_{i*}\overline{\mathcal{L}}_i)^{\otimes a-b}\otimes\det(f_{i*}\overline{\mathcal{L}}_i\otimes\omega_{f_i})^{\otimes b}\Big).$$

Now look at the restrictions of the $\overline{\mathcal{L}}_i$'s to the intersections $\mathcal{X}_i\cap\mathcal{X}_j$, we oviously have isomorphisms $\epsilon_{i,j}\colon(\overline{\mathcal{L}}_i)_{|\mathcal{X}_i\cap\mathcal{X}_j}\xrightarrow{\cong}(\overline{\mathcal{L}}_j)_{|\mathcal{X}_i\cap\mathcal{X}_j}$ and hence for every triple of indeces i,j,k an automorphism

$$\alpha_{ijk}\colon(\overline{\mathcal{L}}_i)_{|\mathcal{X}_i\cap\mathcal{X}_j\cap\mathcal{X}_k}\xrightarrow{\cong}(\overline{\mathcal{L}}_i)_{|\mathcal{X}_i\cap\mathcal{X}_j\cap\mathcal{X}_k}$$

where $\alpha_{ijk}=\epsilon_{k,i}\epsilon_{j,k}\epsilon_{i,j}$; thus α_{ijk} is fiber multiplication by a nonzero constant $c\in\mathcal{O}^*_{\mathcal{X}}(\mathcal{X}_i\cap\mathcal{X}_j\cap\mathcal{X}_k)$.

The automorphism α_{ijk} naturally induces an automorphism β_{ijk} of the restriction of $\mathcal{N}_i$ to $\mathcal{X}_i\cap\mathcal{X}_j\cap\mathcal{X}_k$, where

$$\beta_{ijk}=f_i^*\Big(\det(f_{i*}\alpha_{ijk})^{\otimes a-b}\otimes\det(f_{i*}\alpha_{ijk}\otimes id_{\omega_{f_i}})^{\otimes b}\Big)$$

and one easily checks that, by (13), β_{ijk} is fiber multiplication by c^{-1}. We conclude that the line bundles $\overline{\mathcal{L}}_i\otimes\mathcal{N}_i\in\operatorname{Pic}\mathcal{X}_i$ can be glued together to a line bundle $\overline{\mathcal{L}}$ over $\mathcal{X}$. It is clear that $\overline{\mathcal{L}}$ satisfies the thesis (since the $\overline{\mathcal{L}}_i$'s do so). □

Remark 5.6. If the condition $(d-g+1,2g-2)=1$ is not satisfied the scheme P^d_g can still be constructed (as in the first part of the proof of 5.3). By 5.2(4) P^d_g is a geometric GIT-quotient if and only if $(d-g+1,2g-2)=1$; if such a condition does not hold, there exists an open subset $\overline{M}^d_g$ of $\overline{M}_g$ over which P^d_g (and $\overline{P}_{d,g}$) restricts to a geometric quotient. Such a nonempty open subset $\overline{M}^d_g$ is precisely the locus of d -general curves by 5.2(2).

5.7. An application of Lemma 5.5 gives the existence of the analog of a Poincaré line bundle for the compactified Picard variety of a family of automorphisim-free stable curves. More precisely, let $f\colon\mathcal{X}\longrightarrow B$ be such a family and let $\mu\colon B\longrightarrow\overline{M}^0_g$ be its moduli map; assume that $(d-g+1,2g-2)=1$.

Then we can form the compactified Picard scheme

$$\overline{P^d_f} := B \times_{\overline{M}^0_g} \overline{P}_{d,g} \longrightarrow B.$$

Now, on the open subset of $\overline{P}_{d,g}$ lying over $\overline{M}^0_g$ there is a tautological curve $\mathcal{D}$ which is constructed exactly as $\mathcal{C}_{P^d_g}$ over P^d_g (cf. proof of 5.3). Observe that $\mathcal{D}$ is a family of quasistable (not stable) curves. We can pull back $\mathcal{D}$ to $\overline{P^d_f}$ and obtain a tautological curve $\mathcal{D}_f := B \times_{\overline{M}^0_g} \mathcal{D} \longrightarrow \overline{P^d_f}$.

Lemma 5.5 yields the analog of the Poincaré line bundle on $\mathcal{D}$ and hence on $\mathcal{D}_f$; some care is needed as the boundary points of $\overline{P}_{d,g}$ correspond to equivalence classes of line bundles that disregard the gluing data over the exceptional component (see 7.2 and 7.3 for the precise statement).

The construction of Poincaré line bundles over compactified Jacobians is an interesting problem in its own right; a solution within the category of algebraic spaces was provided by E. Esteves in [E01] applying different techniques from ours.

As we indicated, our method allows us to construct Poincaré bundles for automorphism-free curves. Rather than providing the missing details, we "stackify" the construction of [C94] so that some of our results will generalize to all stable curves (with or without automorphisms).

5.8. Let us introduce the stacks defined by the group action used above:

$$\overline{\mathcal{P}}_{d,g} := [H_d/G] \quad \text{and} \quad \mathcal{P}_{d,g} := [H^{st}_d/G].$$

When are they Deligne-Mumford stacks (in the sense of [DM69] and [Vi89])? Do they have a modular description? We begin with the first question, adding to the picture the "forgetful" morphisms to $\overline{\mathcal{M}}_g$. To define it, pick a scheme T and a section of $\overline{\mathcal{P}}_{d,g}$ (or of $\mathcal{P}_{d,g}$) over T, that is a pair $(E \longrightarrow T, \psi)$ where E is a G-torsor and ψ: $E \longrightarrow H_d$ is a G-equivariant morphism. Then we apply 5.4 to obtain a family $\mathcal{X} \longrightarrow T$ of quasistable curves; the forgetful morphism maps $(E \longrightarrow T, \psi)$ to the stable model of $\mathcal{X} \longrightarrow T$ (the reason why we call it "forgetful" will be more clear from 5.11).

A map of stacks $\mathcal{P} \longrightarrow \mathcal{M}$ is called *representable* (respectively, *strongly representable*) if given any algebraic space (respectively, scheme) B with a map $B \longrightarrow \mathcal{M}$, the fiber product $B \times_{\mathcal{M}} \mathcal{P}$ is an algebraic space (respectively, a scheme).

THEOREM 5.9. *The stacks $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are Deligne-Mumford stacks if and only if* $(d - g + 1, 2g - 2) = 1$. *In that case the natural morphisms $\mathcal{P}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$ and $\overline{\mathcal{P}}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$ are strongly representable.*

Proof. As already said in 5.2 and in the proof of 5.3, H_d/G and H^{st}_d/G are geometric GIT-quotients (equivalently all stabilizers are finite and reduced) if and only if $(d-g+1, 2g-2)=1$. Hence the first sentence follows from the well known fact that a quotient stack like ours is a Deligne-Mumford stack if and only if all stabilizers are finite and reduced.

For the second sentence, we first apply a common criterion for representability (see for example [AV01] 4.4.3): our morphisms are representable if for every algebraically closed field k' and every section ξ of $\mathcal{P}_{d,g}$ (respectively of $\overline{\mathcal{P}}_{d,g}$) over $\operatorname{Spec} k'$ the automorphism group of ξ injects into the automorphism group of its image X in $\overline{\mathcal{M}}_g$. This follows from 5.2(2): ξ is a map onto a G-orbit in H_d and $\operatorname{Aut}(\xi)$ the stabilizer of such orbit (up to isomorphism, of course); the curve X is the stable model of the projective curve Z corresponding to such orbit, hence 5.2(2) gives us the desired injection.

We obtained that the two forgetful morphisms in the statements are representable, hence if B is any scheme and $B \longrightarrow \overline{\mathcal{M}}_g$ the map corresponding to a family of curves $f\colon \mathcal{X} \longrightarrow B$, the fiber product

$$\overline{P^d_f} := B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{P}}_{d,g}$$

is an algebraic space; it remains to show that $\overline{P^d_f}$ is a scheme (the fact that $B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g}$ is also a scheme follows in the same way, or observing that it is an open subspace of $\overline{P^d_f}$). To do that, fix $\mu_f\colon B \longrightarrow \overline{M}_g$ the moduli map of f and consider the scheme

$$Q_f := B \times_{\overline{M}_g} \overline{P}_{d,g},$$

which is projective over B (if the fibers of f are free from automorphisms then $Q_f = \overline{P^d_f}$). We shall prove that there is a (natural) finite projective morphism

$$\rho\colon \overline{P^d_f} \longrightarrow Q_f;$$

hence $\overline{P^d_f}$ is a scheme (cf. [Vie91] 9.4) projective over B.

To define ρ we use [Vi89] section 2 (in particular 2.1 and 2.11), which gives us that $\overline{M}_g$ and $\overline{P}_{d,g}$ are the coarse moduli schemes of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{P}}_{d,g}$ respectively and that we have a canonical commutative diagram where π and π' are proper

$$\begin{tikzcd} & \overline{\mathcal{P}}_{d,g} \arrow[r, "\pi"] \arrow[d] & \overline{P}_{d,g} \arrow[d] \\ B \arrow[r] & \overline{\mathcal{M}}_g \arrow[r, "\pi'"] & \overline{M}_g. \end{tikzcd} \tag{14}$$

The two above maps from B to $\overline{\mathcal{M}}_g$ and $\overline{M}_g$ are the same defining $\overline{P^d_f}$ and Q_f; we let ρ to be the base change over B of $\pi\colon \overline{\mathcal{P}}_{d,g} \longrightarrow \overline{P}_{d,g}$, so that ρ is proper.

Now let $\lambda \in Q_f$ be a closed point. Two different points in $\rho^{-1}(\lambda)$ correspond to two different maps $\psi, \psi'\colon G \longrightarrow H_d$ mapping onto the orbit determined by λ, hence (just as before) ψ and ψ' correspond to a nontrivial element in the stabilizer of a point in that orbit. Since stabilizers are finite ρ has finite fibers; as ρ is proper we are done. □

5.10. Geometric description of $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$. The modular description of $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ can be given by directly interpreting the quotient stacks that define them; what we are going to obtain is a rigidified "balanced Picard stack". The definition of the Picard scheme as a moduli scheme representing a certain functor, or a certain stack, is well known to require care, in fact a subtle "sheafification" procedure is needed to achieve representability. The crux of the matter is that line bundles always possess automorphisms that fix the scheme they live on, namely, fiber multiplication by nonzero constants; see for example [BLR] chapter 8 and [ACV01] section 5. We are here in a fortunate situation as the stacks already exist and have some good properties (by 5.9), all we have to do is to give them a geometric interpretation.

By 5.2(5) we are free to assume that d is very large.

Begin with an object in $\mathcal{P}_{d,g}$ (respectively in $\overline{\mathcal{P}}_{d,g}$), so let $E \longrightarrow T$ be a principal $PGL(r+1)$-bundle and $\psi\colon E \longrightarrow H_d^{st}$ (respectively $\psi\colon E \longrightarrow H_d$) an equivariant map. Pulling back to E the universal polarized family over the Hilbert scheme we obtain a polarized family of stable (respectively quasistable) curves over E, denoted as in 5.4 by $(f_E\colon \mathcal{Z}_E \longrightarrow E, \mathcal{L}_E)$. By construction $G = PGL(r+1)$ acts freely and we can form the quotient $f\colon \mathcal{X} = \mathcal{Z}_E/G \longrightarrow E/G = T$ which is a family of stable (respectively quasistable) curves. Applying lemma 5.5 we obtain a balanced line bundle $\overline{\mathcal{L}} \in \operatorname{Pic}\mathcal{X}$ of relative degree d. Notice that $\overline{\mathcal{L}}$ is determined up to tensor product by pull-backs of line bundles on T, note also that, using 4.10, we have a natural isomorphism $E \cong PGL(\mathbb{P}(f_*\overline{\mathcal{L}}))$.

Conversely let $(f\colon \mathcal{X} \longrightarrow T, \mathcal{L})$ be a pair consisting of a family f of stable (respectively quasistable) curves and a balanced line bundle of relative degree d on $\mathcal{X}$; we now invert the previous construction by producing a principal G-bundle $E \longrightarrow T$ and a G-equivariant map $E \longrightarrow H_d^{st}$ (resp. $E \longrightarrow H_d$). We argue similarly to [E00] 3.2. By 4.10 $\mathcal{L}$ is relatively very ample and $f_*\mathcal{L}$ is locally free of rank $r+1 = d-g+1$; let $E \longrightarrow T$ be the principal $PGL(r+1)$-bundle associated to the $\mathbb{P}^r$-bundle $\mathbb{P}(f_*\mathcal{L}) \longrightarrow T$. To obtain the equivariant map to the Hilbert scheme consider the pull-back family $f_E\colon \mathcal{X}_E = E \times_T \mathcal{X} \longrightarrow E$ polarized by the balanced, relatively very ample line bundle $\mathcal{L}_E$ (pull-back of $\mathcal{L}$). By construction $\mathbb{P}(f_{E*}\mathcal{L}_E) \cong \mathbb{P}^r \times E$ so that $\mathcal{X}_E$ is isomorphic over E to a family of projective curves in $\mathbb{P}^r \times E$ embedded by the balanced line bundle $\mathcal{L}_E$. By the universal property of the Hilbert scheme this family determines a map $\psi\colon E \longrightarrow \operatorname{Hilb}_{\mathbb{P}^r}^{dt-g+1}$ whose image is all contained in H_d^{st} (respectively in H_d). It is obvious that ψ is G-equivariant.

5.11. Let us summarize the construction of the previous paragraph, assume that $(d-g+1, 2g-2) = 1$, then:

(1) The stack $\mathcal{P}_{d,g}$ is the "rigidification" (in the sense of [ACV01] 5.1, see 5.12 below) of the category whose sections over a scheme T are pairs $(f\colon \mathcal{X} \longrightarrow T, \mathcal{L})$ where f is a family of stable curves of genus g and $\mathcal{L} \in \operatorname{Pic}\mathcal{X}$ is a balanced line bundle of relative degree d. The arrows between two such pairs are given by cartesian diagrams

$$\begin{tikzcd} \mathcal{X} \arrow[r, "h"] \arrow[d] & \mathcal{X}' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd} \tag{15}$$

and $\mathcal{L} \cong h^*\mathcal{L}' \otimes f^*M$ for $M \in \operatorname{Pic} T$.

(2) The stack $\overline{\mathcal{P}}_{d,g}$ is the rigidification of the category whose sections over a scheme T are pairs $(f\colon \mathcal{X} \longrightarrow T, \mathcal{L})$ where f is a family of quasistable curves of genus g and $\mathcal{L} \in \operatorname{Pic}\mathcal{X}$ is a balanced line bundle of relative degree d. Arrows are defined exactly as in (1).

Remark 5.12. The rigidification procedure removes those automorphisms of an $\mathcal{L}$ that fix $\mathcal{X}$; this is necessary for representability over $\overline{\mathcal{M}}_g$ (cf. 5.9 and [AV01] 4.4.3).

In [P96] section 10, the scheme $\overline{P}_{d,g}$ was given a geometric description in terms of rank-one torsion free sheaves rather than line bundles. This should enable one to obtain an alternative geometric description of the stacks $\mathcal{P}_{d,g}$, $\overline{\mathcal{P}}_{d,g}$ (and, obviously, of the scheme P^d_g).

5.13. Assume that $(d-g+1, 2g-2) = 1$ and let $f\colon \mathcal{X} \longrightarrow B$ be a family of stable curves of genus g; consider the schemes (cf. 5.9)

$$P^d_f = B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g} \quad \text{and} \quad \overline{P^d_f} = B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{P}}_{d,g}.$$

If $(d-g+1, 2g-2) \neq 1$ *the two schemes* P^d_f *and* $\overline{P^d_f}$ *can be defined in exactly the same way, provided that every singular fiber of* f *is* d*-general.*

In fact, by 5.2(2), the points in H_d lying over the open subset $\overline{M}^d_g$ of $\overline{M}_g$, parametrizing d-general curves, are all GIT-stable. Therefore the analogue of 5.9 holds, simply by restricting the quotient groupoids over $\overline{M}^d_g$ (the proof is the same).

In the special case $B = \operatorname{Spec} k$, so that the family f reduces to a fixed stable curve X, we naturally change the notation and denote by P^d_X (respectively by $\overline{P^d_X}$) the fiber of $\mathcal{P}_{d,g}$ (respectively of $\overline{\mathcal{P}}_{d,g}$) over X as above.

P^d_X is a finite disjoint union of isomorphic copies of the generalized Jacobian of X; the union is parametrized by the set of stably balanced multidegrees. Since X is d-general a multidegree is balanced if and only if it is stably balanced and every $\delta \in \Delta_X^d$ has a unique balanced representative (by 4.13). Therefore

$$P_X^d \cong \coprod_{\underline{d}\in \mathrm{B}_X^d} \mathrm{Pic}^{\underline{d}} X \cong \coprod_{\delta\in\Delta_X^d} \mathrm{Pic}^{\delta} X. \tag{16}$$

The next result generalizes 5.3.

COROLLARY 5.14. *Let $f\colon \mathcal{X} \longrightarrow B$ be a family of stable curves and d an integer. Assume that every singular fiber of f is d-general. Then the functor $\mathcal{P}_f^d$ is coarsely represented by the separated scheme P_f^d; if B is regular, P_f^d is smooth over B.*

Remark 5.15. Under the assumption that $(d-g+1, 2g-2) = 1$ the proof shows that P_f^d is a fine moduli scheme.

Proof. If we assume $(d-g+1, 2g-2) = 1$ the statement follows from 5.9 and 5.11 and we obtain (as stated in 5.15) that P_f^d is a fine moduli space. If, more generally, the singular fibers of f are d-general, we are still in the locus where the quotient defining $\mathcal{P}_{d,g}$ is geometric (cf. 5.13). Then the statement follows as before (the reason why we get only a coarse moduli space is that the Poincaré line bundle has been constructed only under the hypothesis $(d-g+1, 2g-2) = 1$). $P_f^d \longrightarrow B$ has equidimensional nonsingular fibers (cf. (16) above), hence P_f^d is smooth over B. □

6. Néron models and balanced Picard schemes.

With the notation introduced in 5.13, we are ready to prove our parametrization result.

THEOREM 6.1. *Let $f\colon \mathcal{X} \longrightarrow B$ be a family of stable curves of genus $g \geq 3$ such that $\mathcal{X}$ is regular and B is a one-dimensional regular connected scheme with function field K. Let d be such that every singular fiber of f is d-general (for example, assume that $(d-g+1, 2g-2) = 1$).*

(i) *Then P_f^d is the Néron model of* Pic_K^d *over B.*

(ii) *If f admits a section, P_f^d is isomorphic to the Néron model* $\mathrm{N}(\mathrm{Pic}_K^0)$ *of the Jacobian of the generic fiber of f.*

Proof. If f admits a section then $\mathrm{Pic}_K^d \cong \mathrm{Pic}_K^0$ hence $\mathrm{N}(\mathrm{Pic}_K^d) \cong \mathrm{N}(\mathrm{Pic}_K^0)$. Thus the second part of the statement is an immediate consequence of the first.

By 5.14 P_f^d is a smooth separated scheme of finite type over B; by [BLR] 1.2/Proposition 4 it suffices, for part (i), to prove that P_f^d is a *local* Néron model, that is, we can replace B by $\operatorname{Spec} R$ where R is the local ring of B at a closed point (hence a discrete valuation ring of K). Thus, we shall assume that $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R$ with $\mathcal{X}$ regular. By 3.10 we have

$$\mathrm{N}(\mathrm{Pic}_K^d) = \frac{\coprod_{\delta\in\Delta_X^d} \mathrm{Pic}_f^{\delta}}{\sim_K}$$

(where "$\sim_K$" denotes gluing along the generic fiber).

Since the closed fiber X is d-general, a multidegree $\underline{d}$ is balanced if and only if it is stably balanced and there is a natural bijection betweed the set of balanced multidegrees B_X^d and Δ_X^d (cf. 4.13). Therefore we have a canonical B-isomorphism

$$\mathrm{N}(\,\mathrm{Pic}^d_K) \cong \frac{\coprod_{\underline{d}\in \mathrm{B}_X^d} \mathrm{Pic}_f^{\underline{d}}}{\sim_K}.$$

We now claim that there is canonical B-isomorphism

$$P_f^d \cong \frac{\coprod_{\underline{d}\in \mathrm{B}_X^d} \mathrm{Pic}_f^{\underline{d}}}{\sim_K} \tag{17}$$

which, comparing the last two identities, concludes the proof.

To prove (17) it suffices to observe that the both schemes represent the balanced Picard functor for the given family f: for P_f^d this follows from 5.14, for the right-hand side this is clear. □

Remark 6.2. In 6.1 the hypothesis that $\mathcal{X}$ is regular is necessary; see 6.7 for an example illustrating why.

We can apply the previous result to compare at least birationally different completions of the generalized Jacobian.

COROLLARY 6.3. *Under the same hypotheses of 6.1(ii), let* $\overline{\mathrm{Pic}^0_K}$ *be any completion of* Pic^0_K *over B. Then there exists a regular map (canonical for any fixed group structure on P_f^d) from the smooth locus of* $\overline{\mathrm{Pic}^0_K} \longrightarrow B$ *to P_f^d, which restricts to an isomorphism on the generic fiber.*

Proof. Apply the Néron mapping property to P_f^d (which we can do by 6.1) and the unicity of the Néron model. □

Remark 6.4. It has been known for a long time that there is more than one good way of completing the generalized Jacobian of a family of nodal (reducible) curves. To our knowledge, the first to observe and study this phenomenon were T. Oda and C. S. Seshadri in [OS79]; their paper only dealt with a fixed curve and not with a family, nevertheless the insights contained there have deeply influenced the subsequent work of many authors.

Since then, diverse techniques have led to different models of compactified Jacobians. The problem remains as to which completions are more suitable for the miscellany of mathematical problems in which a compactified Picard variety is needed; the previous result may be viewed in this perspective, offering a way of comparing different constructions in different degrees.

A remarkable case is $d = g - 1$, which has been particularly studied (partly in relation with the problem of extending the theta-divisor). Some correlation results have been proved by V. Alexeev in [Al04] where there is also an overview of the various existing constructions. As mentioned in 4.14, the $d = g - 1$ case is "degenerate" from our point of view (arguing as 6.5, the compactified Picard variety is seen to have fewer components than the Néron model). For some aspects, however, it turns out to be easier to handle precisely because of certain degeneracy phenomena.

Example 6.5. The previous corollary applies to the compactified Jacobians given by the fibers of $\overline{P}_{d,g}$ over curves that that are not d -general. For any family $f\colon \mathcal{X} \longrightarrow B$ of (automorphism-free) stable curves of genus g denote, as usual, $\overline{P}^d_f := \overline{P}_{d,g} \times_{\overline{M}_g} B$ and note that $\overline{P}^d_f$ depends on d, in fact the fibers of $\overline{P}_{d,g}$ over $\overline{M}_g$ depend on d, as we are going to illustrate. If X is a singular fiber of f, the fiber of $\overline{P}_{d,g}$ over X is denoted $\overline{P^d_X}$.

The simplest case in which we find a "degenerate" compactification of the generalized Jacobian is $d = 0$ (this example works similarly if $d = g - 1$). Let $X = C_1 \cup C_2$ with $\#(C_1 \cap C_2) = k$ and assume, which is crucial, that k is even. Now, $\Delta_X = \mathbb{Z}/k\mathbb{Z}$ and the class

$$\delta := \left[\left(-\frac{k}{2}, \frac{k}{2}\right)\right] = \left[\left(\frac{k}{2}, -\frac{k}{2}\right)\right]$$

has two balanced representatives (the ones above). Correspondingly, in $\overline{P^0_X} \subset \overline{P_{0,g}}$, line bundles having such multidegrees are strictly GIT-semistable and get identified to points having a stabilizer of positive dimension (the so-called "ladders", curves obtained by blowing up all the nodes of X, see [C94] 7.3.3 for details). Therefore the corresponding component of the Néron model, $\operatorname{Pic}^{\delta} X$ (cf. 3.10), does not appear as an irreducible component of $\overline{P^0_X}$, where it collapses to a positive codimension boundary stratum.

In fact $\overline{P^0_X}$ has $k - 1$ irreducible components, each of which corresponds to one of the remaining classes in Δ_X. Thus 6.3 implies that if f and d are as in 6.3, with X as closed fiber, there is a diagram of birational maps

(18)

$$\begin{tikzcd}
f\colon & \overline{P^0_f} \arrow[r, dashed] & \overline{P^d_f} \\
& P^0_f \arrow[u] \arrow[r, hook] & P^d_f \arrow[u]
\end{tikzcd}$$

and the lower horizontal arrow is not an isomorphism.

6.6. Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R$ be a family of generically smooth curves with closed fiber X reduced, nodal and connected (not necessarily stable). Let $N(\operatorname{Pic}^0_K)$ be the Néron model of its Jacobian; then its special fiber $N(\operatorname{Pic}^0_K)_k$ only depends on the geometry of $\mathcal{X}$, or, which is the same, on the intersection form defined on the minimal desingularization of $\mathcal{X}$ (see [L90], [E98] and [BL02] for explicit details and computations). More precisely, the total space $\mathcal{X}$ can only have rational singularities of type A_n (i.e., formally equivalent to $xy = u^{n+1}$) at the nodes of X, and the singularities that will interfere with the structure of $\mathrm{N}(\operatorname{Pic}^0_K)_k$ are those occurring at the *external* nodes of X (i.e. nodes lying on two different components). Let δ be the number of external nodes of X and suppose that $\mathcal{X}$ has a singularity of type A_{n_i} at the ith external node. Then the structure of $\mathrm{N}(\operatorname{Pic}^0_K)_k$ only depends on $\underline{n} = (n_1, \ldots, n_\delta)$ so that we can denote $\mathrm{N}_X^{\underline{n}}$ the special fiber of a Néron model of this type.

We need the case where $\mathcal{X}$ is nonsingular, so that $\underline{n} = (0, \ldots, 0)$; then we denote the special fiber of the Néron model of the Jacobian of f by

$$\mathrm{N}_X := \mathrm{N}_X^{(0,\ldots,0)}$$

We have for any nodal (connected) curve X (see 3.10)

$$\mathrm{N}_X \cong \coprod_{\delta \in \Delta_X} \operatorname{Pic}^{\delta} X. \tag{19}$$

Example 6.7. We now exhibit an example showing that the assumption that $\mathcal{X}$ be regular in 6.1 cannot be weakened by assuming $\mathcal{X}$ normal. Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R$ having as closed fiber $X = C_1 \cup C_2$ with $k = \#(C_1 \cap C_2) \geq 2$. Assume that $\mathcal{X}$ has a singularity of type A_n at one of the nodes of X and it is smooth otherwise. Then the twister group $\operatorname{Tw}_f X$ of f is generated by $T_1 := \mathcal{O}_{\mathcal{X}}((n+1)C_1) \otimes \mathcal{O}_X$ which has multidegree $\underline{deg}\, T_1 = (-(nk + k - n), nk + k - n)$. Thus the group of multidegree classes for such an f will be (using a notation similar to the one introduced in 6.6)

$$\Delta_X^{(n,0,\ldots,0)} \cong \mathbb{Z}/(nk + k - n)\mathbb{Z},$$

which is bigger than Δ_X (if $n \geq 1$ of course). The closed fiber $\mathrm{N}_X^{(n,0,\ldots,0)}$ of the Néron model of the generalized jacobian of f has component group isomorphic to $\mathbb{Z}/(nk{+}k{-}n)\mathbb{Z}$, whereas the components of the closed fiber of P_f^d are parametrized by Δ_X (if X is d- general).

Finally, if X is not d- general so that we are in a degenerate case as described in 6.5, the number of components of the special fiber of P_f^d is smaller than $\#\Delta_X$ and hence also smaller than $\#\Delta_X^{(n,0,\ldots,0)}$.

6.8. A natural side question is: when are $\overline{P}_{d,g}$ and $\overline{P}_{d',g}$ isomorphic? Similar question for the stacks. This is easy to answer, we do it for the schemes but it is obvious that the same answer holds for the stacks. By 5.2(5) we have that $\overline{P}_{d,g} \cong \overline{P}_{d',g}$ if and only if $d \pm d' \equiv 0 \mod (2g-2)$ and these isomorphisms are canonical. Then we just need to count; denoting "Φ" the Euler ϕ-function on natural numbers we have:

LEMMA 6.9. *The number of nonisomorphic* $\overline{P}_{d,g}$ *for which* $(d-g+1, 2g-2) = 1$ *is equal to* $\Phi(g-1)$ *if* g *is odd and to* $\frac{\Phi(g-1)}{2}$ *if* g *is even.*

Proof. As we said, there are exactly g non isomorphic models for $\overline{P}_{d,g}$. We choose as representatives for each class of such models the values for d given by $d = 0, 1, \ldots, g-1$ so that we have

$$\overline{P}_{0,g} \cong \overline{P}_{2g-2,g}, \qquad \overline{P}_{1,g} \cong \overline{P}_{2g-3,g}, \ldots, \overline{P}_{g-2,g} \cong \overline{P}_{g,g}$$

and for any $d' \geq 2g-2$

$$\overline{P}_{d',g} \cong \overline{P}_{-d',g} \cong \overline{P}_{e,g},$$

where $0 \leq e < 2g-2$ and $d' = n(2g-2)+e$. Now $(d-g+1, 2g-2) = 1$ implies $(d, g-1) = 1$; if g is odd, one immediately sees that the converse holds, and we are done.

If g is even, the condition $(d-g+1, 2g-2) = 1$ is equivalent to d even and coprime with $g-1$. So the values of d that we are counting are the positive even integers d coprime with $g-1$ and smaller than $g-1$. This number equals $\frac{\Phi(g-1)}{2}$ (just notice that for any odd $m \in \mathbb{N}$, the Euler function $\phi(m)$ counts an equal number of odd and even integers; in fact if r is odd and coprime with m, the even number $m-r$ is also coprime with m; same thing starting with r even.) $\square$

7. Completing Néron models via Néron models.

7.1. From now on we shall assume that the stable curve X is d-general (4.13). For example, one may assume that $(d-g+1, 2g-2) = 1$.

Fix $f\colon \mathcal{X} \longrightarrow B = \operatorname{Spec} R$ a family of stable curves with smooth generic fiber and regular total space $\mathcal{X}$. In 5.13 we introduced the scheme $\overline{P^d_f}$, projective over B which, by 6.1, is a compactification of the Néron model of the Picard variety Pic^d_K (by 6.1); recall that $\overline{P^d_X}$ denotes its closed fiber. In the present section we shall exhibit a stratification of $\overline{P^d_X}$ in terms of Néron models associated to all the connected partial normalizations of X (Theorem 7.9). In section 8 we shall prove that $\overline{P^d_f}$ is dominated by the Néron model of a degree-2 base change of Pic^d_K. See [An99] for a different approach to the problem of compactifying Néron models of Jacobians.

7.2. With the notation introduced in 5.13, we shall refer to the points in $\overline{P^d_X} \smallsetminus P^d_X$ as the "boundary points of $\overline{P^d_X}$". To describe them precisely we need some simple preliminaries.

Let X be a stable curve, the quasistable curves of X (cf. 2.4) correspond bijectively to sets of its nodes: let S be a set of nodes of X, we shall denote

$\nu_S\colon X_S^\nu \longrightarrow X$ the normalization of X at the nodes in S and

$$Y_S := X_S^\nu \cup \left(\bigcup_1^{\#S} E_i \right)$$

the quasistable curve of X obtained by joining the two points of X_S^ν lying over the ith node in S with a smooth rational curve $E_i \cong \mathbb{P}^1$ (so that one may call Y_S the *blow up of X at S*).

7.3. A point of $\overline{P_X^d}$ corresponds to an equivalence class of pairs (Y_S, L) where $S \subset X_{sing}$ and $L \in \operatorname{Pic}^d Y_S$ is a balanced line bundle. Two pairs (Y_S, L) and (Y'_S, L') are equivalent if and only if $Y_S = Y'_S$ and $L_{|X_S^\nu} \cong L'_{|X_S^\nu}$.

The boundary points are those for which $S \neq \emptyset$.

Remark 7.4. Notice that a quasistable curve Y_S of X admits a (stably) balanced line bundle (of degree d) if and only if the subcurve X_S^ν (obtained by removing all of the exceptional components) is connected.

In fact if $X_S^\nu = Z_1 \cup Z_2$ with $Z_1 \cap Z_2 = \emptyset$ then a stably balanced $\underline{d}$ has to satisfy $d_{Z_1 \cup Z_2} = m_{Z_1 \cup Z_2}$, on the other hand $d_{Z_1 \cup Z_2} = d_{Z_1} + d_{Z_2}$ and hence $d_{Z_1} = m_{Z_1}$ (and $d_{Z_2} = m_{Z_2}$). This is impossible as the complementary curve of Z_1, containing Z_2, is not a union of exceptional components (cf 4.11).

7.5. Fix the quasistable curve Y_S and consider $\Delta_{Y_S}^d$; recall that a balanced multidegree must have degree 1 on all exceptional components of Y_S, so that not all elements in $\Delta_{Y_S}^d$ have a balanced representative. Denote

$$\Delta_{Y_S}^{d,1} := \{\delta \in \Delta_{Y_S}^d\colon \ \delta \text{ has a balanced representative}\}.$$

Thus for every $\delta \in \Delta_{Y_S}^{d,1}$ there exists a unique (by 7.1) balanced representative which we shall denote

$$(d_1^\delta, \dots, d_\gamma^\delta, 1, \dots, 1) \tag{20}$$

so that $[(d_1^\delta, \dots, d_\gamma^\delta, 1, \dots, 1)] = \delta$ and $\sum_1^\gamma d_i^\delta = d - s$, where $s := \#S$.

By 7.4 we have that $\Delta_{Y_S}^{d,1}$ is empty if and only if X_S^ν is not connected.

The next lemma will be used in the proof of Theorem 7.9.

Lemma 7.6. *Using the above notation, assume X_S^ν connected. Then the map*

$$\rho\colon \Delta_{Y_S}^{d,1} \longrightarrow \Delta_{X_S^\nu}^{d-s}, \qquad [(d_1^\delta, \dots, d_\gamma^\delta, 1, \dots, 1)] \mapsto [(d_1^\delta, \dots, d_\gamma^\delta)]$$

is bijective.

Proof. As we said ρ is well defined because of the assumption 7.1. We shall use the notation of 4.4 and 4.5, together with the following: let $Z \subset X_S^\nu \subset Y$, set $k_Z^S := \#(Z \cap \overline{X_S^\nu \smallsetminus Z})$ and denote by e_Z the number of points in which Z meets the the exceptional components of Y_S so that

$$k_Z = e_Z + k_Z^S. \tag{21}$$

The map ρ can be factored as follows:

$$\begin{array}{ccccccc} \rho\colon\ \Delta_{Y_S}^{d,1} & \longrightarrow & \mathrm{B}_{Y_S}(\underline{b}_{Y_S}^d) & \xrightarrow{\pi} & \mathrm{B}_{X_S^\nu}(\underline{b}) & \xrightarrow{\sigma} & \Delta_{X_S^\nu}^{d-s} \\ \delta & \mapsto & (d_1^\delta, \ldots, d_\gamma^\delta, 1, \ldots, 1) & \mapsto & (d_1^\delta, \ldots, d_\gamma^\delta) & \mapsto & [(d_1^\delta, \ldots, d_\gamma^\delta)] \end{array} \tag{22}$$

where $\underline{b} = (b_1, \ldots, b_\gamma)$ with

$$b_i := \frac{d}{2g-2} w_{C_i} - \frac{e_{C_i}}{2}$$

and $w_{C_i} = 2g_{C_i} - 2 + k_{C_i}$.

To prove that ρ is surjective, first of all observe that, by 4.4, σ is surjective. Now we claim that given $\underline{d} = (d_1, \ldots, d_\gamma, 1, \ldots, 1) \in \mathbb{Z}^{\gamma+s}$ such that $|\underline{d}| = d$, we have that $\underline{d}$ is balanced if and only if for every $Z \subset X_S^\nu$ we have

$$m_Z(d) \leq d_Z \leq M_Z(d) - e_Z, \tag{23}$$

where $M_Z(d) = \frac{d}{2g-2} w_Z + \frac{k_Z}{2}$ and $m_Z(d) = M_Z(d) - k_Z$ as usual. In fact for every exceptional component E of Y_S we have $w_Z = w_{E \cup Z}$ and hence the basic inequality on $Z \cup E$ gives

$$d_Z + 1 = d_{Z \cup E} \leq \begin{cases} M_Z(d) + 1 & \text{if } (E \cdot Z) = 0 \\ M_Z(d) & \text{if } (E \cdot Z) = 1 \\ M_Z(d) - 1 & \text{if } (E \cdot Z) = 2. \end{cases}$$

Iterating for all E we get the claim.

Therefore $\underline{d}$ is balanced if and only if (using (21))

$$\frac{d}{2g-2} w_Z - \frac{k_Z^S}{2} - \frac{e_Z}{2} \leq d_Z \leq \frac{d}{2g-2} w_Z + \frac{k_Z^S}{2} - \frac{e_Z}{2}$$

if and only if

$$(d_1, \ldots, d_\gamma) \in \mathrm{B}_{X_S^\nu}(\underline{b}).$$

This shows that ρ is surjective; to prove that it is injective it suffices to show that

σ is (the other two arrows of diagram (22) are obviously injective). If $\mathrm{B}_{X_S^\nu}(\underline{b})$ contains two equivalent multidegrees, then, using (23), we would get that there exists a subcurve $Z \subsetneq X_S^\nu$ for which $m_Z(d)$ is integer, which is impossible (as usual, by assumption 7.1). □

7.7. By 5.9 and 5.11, $\overline{P_f^d}$ is a coarse moduli scheme for the functor from B-schemes to sets which associates to a B-scheme T the set of equivalence classes of pairs $(h\colon \mathcal{Y} \longrightarrow T, \mathcal{L})$ where $h\colon \mathcal{Y} \longrightarrow T$ is a family of quasistable curves having $\mathcal{X}_T$ as stable model; and $\mathcal{L}$ is a balanced line bundle on $\mathcal{Y}$. The equivalence relation is the same as in 4.15.

7.8. The structure of the closed fiber $\overline{P_X^d}$ of $\overline{P_f^d}$ does not depend on d (by 7.1) and is a good compactification of N_X (see 6.6). Therefore we shall introduce the notation

$$\overline{\mathrm{N}_X} := \overline{P_X^d}.$$

Such a completion can be described by means of the Néron models of the Jacobians of all connected partial normalizations of X:

Theorem 7.9. *$\overline{\mathrm{N}_X}$ has a natural stratification as follows*

$$\overline{\mathrm{N}_X} \cong \coprod_{\substack{S\subset X_{sing}:\\ X_S^\nu \text{ connected}}} \mathrm{N}_{X_S^\nu}. \tag{24}$$

Denote $Q_S \subset \overline{\mathrm{N}_X}$ the stratum isomorphic to $\mathrm{N}_{X_S^\nu}$ under the decomposition (24); then

(i) *Q_S has pure codimension $\#S$.*
(ii) *$Q_S \subset \overline{Q_{S'}}$ if and only if $S' \subset S$.*
(iii) *The smooth locus of $\overline{\mathrm{N}_X}$ is N_X.*

Proof. As we explained in 7.2, the points of $\overline{P_X^d} = \overline{\mathrm{N}_X}$ parametrize pairs (Y_S, L) in such a way that for every $S \subset X_{sing}$ we have a well defined locus Q_S in $\overline{P_X^d}$, corresponding to balanced line bundles on Y_S. For example, P_X^d corresponds to the stratum $S = \emptyset$ (isomorphic to N_X).

In turn, Q_S is a disjoint union of irreducible components isomorphic to the generalized Jacobian of X_S^ν (cf. 7.3 and 5.13); there is one component for every (stably) balanced multidegree on Y_S. More precisely, for any balanced $\underline{d} = (d_1, \dots, d_\gamma, 1, \dots, 1)$ on Y_S let us denote $\underline{d}^S = (d_1, \dots, d_\gamma)$ its restriction to X_S^ν. Then the moduli morphism

$$\mathrm{Pic}^{\underline{d}}\, Y_S \longrightarrow \overline{P_X^d}$$

(associated to the universal line bundle on $\mathrm{Pic}^{\underline{d}}\, Y_S \times Y_S$, see 7.7) factors through a surjective morphism followed by a canonical embedding

$$\mathrm{Pic}^{\underline{d}}\, Y_S \twoheadrightarrow \mathrm{Pic}^{\underline{d}^S}\, X_S^\nu \hookrightarrow Q_S \subset \overline{P^d_X} \tag{25}$$

(see 7.3) whose image is open and closed in Q_S.

Set $\delta^S := [\underline{d}^S] \in \Delta^{d-s}_{X_S^\nu}$. We shall now see that the components of Q_S are in one-to-one correspondence with the elements of $\Delta^{d-s}_{X_S^\nu}$. The balanced multidegrees on Y_S are bijectively parametized by $\Delta^{d,1}_{Y_S}$ (cf. 7.5); by 7.6 the restriction to X_S^ν of a balanced multidegree induces the bijection

$$\rho\colon\ \Delta^{d,1}_{Y_S} \leftrightarrow \Delta^{d-s}_{X_S^\nu}$$

of 7.6, so we are done. In other words we obtain the stratification in the statement of our Theorem

$$Q_S \cong \coprod_{\delta^S \in \Delta^{d-s}_{X_S^\nu}} \mathrm{Pic}^{\delta^S}\, X_S^\nu \cong \mathrm{N}_{X_S^\nu}, \tag{26}$$

where the second isomorphism is (19).

Part (i) is a simple dimension count. We already know that each irreducible component of Q_S is isomorphic to the generalized Jacobian of X_S^ν; the genus of X_S^ν is equal to $g - s$ hence we are done.

By the previous results, part (ii) follows from Proposition 5.1 of [C94] (see below for more details).

Now (iii); quite generally, the Néron mapping property applied to étale points implies that any completion $\overline{N}$ of a Néron model N over B must be singular along $\overline{N} \smallsetminus N$ (If $\overline{N} \smallsetminus N$ contained regular points one would use 2.2/14 of [BLR] and find an étale point of N_K which does not come from an étale point of N). We include a direct proof to better illustrate the structure of $\overline{\mathrm{N}_X}$.

It suffices to prove that every component of every positive codimension stratum is contained in the closure of more than one irreducible component of $\mathrm{N}_X = P^d_X$. This also follows from Proposition 5.1 of [C94]. Let us treat the case $\#S = 1$; then Y_S has only one exceptional component E intersecting (say) C_1 and C_2 (viewed now as components of X_S^ν by a slight abuse of notation). If the point (Y_S, L) belongs to the component of Q_S corresponding to the multidegree $(d_1, d_2, \dots, d_\gamma, 1)$, we have that (Y_S, L) is contained in the closure of the two components of P^d_X that correspond to multidegrees $(d_1 + 1, d_2, \dots, d_\gamma)$ and $(d_1, d_2 + 1, \dots, d_\gamma)$. □

7.10. Let X be a stable curve; as we have seen, $\overline{\mathrm{N}_X}$ has a stratification (by equidimensional, possibly disconnected strata) parametrized by the sets of nodes of X which do not disconnect X, denote by $\mathbf{G}_X$ this set:

$$\mathbf{G}_X := \{S \subset X_{sing}\colon\ X_S^\nu \text{ is connected}\}.$$

For some more details on the stratification of Theorem 7.9, introduce the dual graph Γ_X of X, (cf. 9.5) and recall the genus formula $g = \sum_1^\gamma g_i + b_1(\Gamma_X)$ where g_i denotes the geometric genus of C_i and $b_1(\Gamma_X)$ is the first Betti number (see 9.6).

COROLLARY 7.11. *Let X be a stable curve and $S \in \mathbf{G}_X$; let $Q_S \subset \overline{\mathrm{N}_X}$ be a stratum as defined in Theorem 7.9. Then*

(i) $\dim Q_S \geq \sum_1^\gamma g_i$

(ii) $\dim Q_S = \sum_1^\gamma g_i \iff X_S^\nu$ *is of compact type* $\iff Q_S$ *is irreducible.*

(iii) *The number of minimal strata of* $\overline{\mathrm{N}_X}$ *(in the stratification of Theorem 7.9) is equal to* $\#\Delta_X$.

Proof. (i) is equivalent to $\dim Q_S \geq g - b_1(\Gamma_X)$, hence, by 7.9(i), it suffices to show that $\#S \leq b_1(\Gamma_X)$. Thus we must prove that the maximum number of nodes of X that can be normalized without disconnecting the curve is $b_1(\Gamma_X)$. Equivalently, that the maximum number of edges of Γ_X that can be removed without disconnecting Γ_X is $b_1(\Gamma_X)$. This follows from 9.6.

Now we prove (ii). $\dim Q_S = \sum_1^\gamma g_i$ if and only if Q_S is a minimal stratum of $\overline{\mathrm{N}_X}$ (by 7.9 and part (i)), if and only if all the nodes of X_S^ν are separating (i.e. any partial normalization of X_S^ν fails to be connected), if and only if X_S^ν is if compact type (by definition, cf. 9.8). This proves the first double arrow of part (ii).

X_S^ν is if compact type if and only if its dual graph is a tree, if and only if $\Delta_{X_S^\nu} = \{0\}$ (this can be easily shown directly or it follows from 9.10), if and only if Q_S has only one irreducible component (by 7.9 $Q_S \cong \mathrm{N}_{X_S^\nu}$ whose components correspond to elements in $\Delta_{X_S^\nu}$). This concludes (ii).

Now (iii). The strata of minimal dimension (equal to $\sum_1^\gamma g_i$) correspond bijectively to the connected partial normalizations of X that are of compact type which, in turn, correspond (naturally) to the spanning trees of the dual graph of X (cf. 9.7). Now, the number of spanning trees of Γ_X (the so called "complexity" of the graph) is shown to be equal to the cardinality of Δ_X in 9.10. So we are done. □

Example 7.12. Let $X = C_1 \cup C_2$ with C_i nonsingular and $\#(C_1 \cap C_2) = k$; then the set $\mathbf{G}_X$ is easy to describe: $\mathbf{G}_X = \{S \subset X_{sing}\colon\ S \neq X_{sing}\}$. Given $S \in \mathbf{G}_X$ let $\#S = s$ so that $X_S^\nu = C_1 \cup C_2$ with $\#(C_1 \cap C_2) = k - s$.

The connected components of N_X, each isomorphic to the generalized jacobian of X, are parametrized by $\mathbb{Z}/k\mathbb{Z}$.

The strata Q_S of codimension 1 of $\overline{\mathrm{N}_X}$ are parametrized by the nodes of X, denoted $n_1, \dots, n_k$. If $S = \{n_i\}$, Q_{n_i} is the special fiber $\mathrm{N}_{X_S^\nu}$ of the Néron model of the Jacobian of a family specializing to the normalization of X at n_i; hence it is made of $k-1$ connected components of dimension $g-1$.

And so on, going down in dimension till the minimal strata, which correspond to the k curves of compact type obtained from X by normalizing it at $k-1$ nodes. Each of these strata is isomorphic to the closed fiber of the Néron model of the Jacobian of a specialization to a curve of compact type having C_1 and C_2 as irreducible components; therefore it is an irreducible projective variety (isomorphic to $\mathrm{Pic}^{\underline{0}} C_1 \times \mathrm{Pic}^{\underline{0}} C_2$) of dimension $g-k+1$.

8. The compactification as a quotient. We begin with some informal remarks to motivate the content of this last section; consider a family of nodal curves $f\colon\ \mathcal{X} \longrightarrow B = \mathrm{Spec}\, R$ having regular $\mathcal{X}$ and singular closed fiber X. Let $p \in X$ be a nonsingular point, then p corresponds to a degree-1 line bundle of X which, up to an étale base change of f (ensuring the existence of a section through p) is the specialization of a degree 1 line bundle on the generic fiber. So p corresponds to a unique point in $\mathrm{N}(\mathrm{Pic}^1_K)$.

What if p is a singular point of X? Of course (intuitively) p can still be viewed as a limiting configuration of line bundles on X. On the other hand there will never be a section passing through p (not even after étale base change of f). What is needed to have such a section is a ramified base change, in fact a degree-2 base change will suffice (because X has ordinary double points). If $f_1\colon\ \mathcal{X}_1 \longrightarrow B_1$ is the base change of f under a degree-2 ramified covering $B_1 = \mathrm{Spec}\, R_1 \longrightarrow B$, then $\mathcal{X}_1$ has a singularity of type A_1 at each node of the closed fiber $X_1 \cong X$. If $p_1 \in X_1$ is the point corresponding to p, then f_1 (or some étale base change) does admit a section through p_1 , therefore p_1, and hence our original point p, corresponds to a unique point of $\mathrm{N}(\mathrm{Pic}^1_{K_1})$.

All of this suggests that to complete the Néron model of the Picard variety of $\mathcal{X}_K$ we could use the Néron model of the the Picard variety of a ramified base change of order 2. To better handle the Néron models $\mathrm{N}(\mathrm{Pic}^d_{K_1})$ we shall introduce and study the minimal desingularization of $\mathcal{X}_1$, whose closed fiber is the quasistable curve Y of X obtained by blowing up all the nodes of X.

8.1. Let X be a stable curve; consider the quasistable curve Y obtained by blowing up all the nodes of X so that, with the notation of 7.2, $Y := Y_{X_{sing}}$. Denote

$$\sigma\colon\ Y \longrightarrow X$$

the morphism contracting all of the exceptional components of Y.

Recall now that, by 7.9, $\overline{\mathrm{N}_X}$ has a stratification labeled by $\mathbf{G}_X$. We shall exhibit a decomposition of N_Y labeled by $\mathbf{G}_X$ and prove that it is naturally related to the stratification of $\overline{\mathrm{N}_X}$.

By 4.13 for any $\delta \in \Delta_Y^d$ there exists a unique semibalanced representative $\underline{d}^\delta$. Fix now a set S of nodes of X and define

$$\Delta_{Y,S}^d := \{\delta \in \Delta_Y^d\colon\ d_E^\delta = 1 \Leftrightarrow \sigma(E) \in S\}.$$

Let γ be the number of irreducible components of X and let $s = \#S$; order the exceptional components of Y so that the first s are those corresponding to S (i.e. mapped to S by σ). Connecting with 7.5 we can partition the component group Δ_Y of N_Y using $\mathbf{G}_X$:

LEMMA 8.2. *Let $Y = Y_{X_{sing}}$.*
(i) *For every S there is a natural bijection*

$$\Delta_{Y,S}^d \leftrightarrow \Delta_{Y_S}^{d,1},\quad [(d_1^\delta, \dots, d_\gamma^\delta, 1, \dots, 1, 0, \dots, 0)] \mapsto [(d_1^\delta, \dots, d_\gamma^\delta, 1, \dots, 1)]. \tag{27}$$

(ii) $\coprod_{S \in \mathbf{G}_X} \Delta_{Y,S}^d = \Delta_Y^d$.

Proof. Let $\underline{d} = (d_1, \dots, d_\gamma, 1, \dots, 1)$ a multidegree on Y_S and denote its "pull-back" to Y by $\underline{d}^* = (d_1, \dots, d_\gamma, 1, \dots, 1, 0, \dots, 0)$; to prove that (27) is a bijection it suffices to prove that $\underline{d}$ satisfies the basic inequality on Y_S if and only if $\underline{d}^*$ satisfies the basic inequality on Y. Denote $\sigma_S\colon\ Y \longrightarrow Y_S$ the contraction of all exceptional components of Y that do not correspond to S. Let $Z \subset Y$ be a subcurve and denote $Z_S = \sigma_S(Z) \subset Y_S$. Then it is easy to see that $w_Z = w_{Z_S}$ and that $k_Z = k_{Z_S} + 2t_Z$ where t_Z is the number of exceptional components E of Y that are not contained in Z and such that $\#(E \cap Z) = 2$. If we write the basic inequality for $Z_S \subset Y_S$ as usual (omitting the dependence on d which is fixed)

$$m_{Z_S} \leq d_{Z_S} \leq M_{Z_S} \tag{28}$$

the basic inequality for $Z \subset Y$ is

$$m_{Z_S} - t_Z \leq d_Z^* \leq M_{Z_S} + t_Z. \tag{29}$$

Under the correspondence (27) we have $d_{Z_S} = d_Z^*$, hence it is obvious that, if $\underline{d}$ satisfies (28), then $\underline{d}^*$ satisfies (29). Conversely, suppose that $\underline{d}^*$ satisfies the basic inequality and let $Z_S \subset Y_S$ be a subcurve. Denote by $Z = \sigma^{-1}(Z_S)$ so that $t_Z = 0$; thus the basic inequality for Z_S is the same as for Z_S and hence $\underline{d}$ satisfies it.

For the second part, recall that, because of 7.4, $\Delta_{Y_S}^{d,1}$ is empty if and only if $S \notin \mathbf{G}_X$ (see 7.5). Thus $\Delta_{Y,S}^d$ is empty if $S \notin \mathbf{G}_X$ and the second part of the lemma follows. □

Remark 8.3. As a consequence we get the $\mathbf{G}_X$-decomposition of N_Y mentioned in 8.1:

$$\mathrm{N}_Y = \coprod_{S\in\mathbf{G}_X}\left(\coprod_{\delta\in\Delta^d_{Y,S}} \mathrm{Pic}^{\delta} Y\right).$$

8.4. Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec} R = B$ with $\mathcal{X}$ regular and assume that f admits a section. The curve Y (defined in 8.1) is the closed fiber of the regular minimal model of the base change of $\mathcal{X}_K$ under a degree-2 ramified covering of $\operatorname{Spec} R$. More precisely, let t be a uniformizing parameter of R and let $K \hookrightarrow K_1$ be the degree-2 extension $K_1 = K(u)$ with $u^2 = t$. Denote R_1 the DVR of K_1 lying over R, so that $R \hookrightarrow R_1$ is a degree 2 ramified extension. Denote $B_1 = \operatorname{Spec} R_1$ and consider the covering $B_1 \longrightarrow B$. The corresponding base change of f is denoted

$$f_1\colon\ \mathcal{X}_1 := \mathcal{X} \times_B B_1 \longrightarrow B_1$$

and X_1 its closed fiber. At each of the nodes of X_1 the total space $\mathcal{X}_1$ has a singularity formally equivalent to $xy = u^2$, which can be resolved by blowing up once each of the nodes of X_1 (see [DM69] proof of 1.2). Denote $\mathcal{Y} \longrightarrow \mathcal{X}_1$ this blow-up and $h\colon \mathcal{Y} \longrightarrow B_1$ the composition; thus h is a family of quasistable curves having $\mathcal{X}_1$ as stable model and Y as closed fiber. We summarize with a diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "h"'] & \mathcal{X} \arrow[d, "f"] \\ B_1 \arrow[r] & B. \end{tikzcd} \tag{30}$$

Denote $\mathrm{Pic}^d_h \longrightarrow B_1$ the Picard variety for h and $\mathrm{Pic}^d_{K_1}$ its generic fiber.

Proposition 8.5. *In the set up of 8.4, let* $\mathrm{N}(\mathrm{Pic}^d_{K_1}) \longrightarrow B_1$ *be the Néron model of* $\mathrm{Pic}^d_{K_1}$*; then there is a canonical surjective B-morphism*

$$\pi\colon\ \mathrm{N}(\mathrm{Pic}^d_{K_1}) \longrightarrow \overline{P^d_f}.$$

The restriction of π to the closed fibers is compatible with their $\mathbf{G}_X$-stratifications in the following sense: for any $S \in \mathbf{G}_X$ the restriction π_S of π is a surjective morphism

$$\pi_S\colon\ \coprod_{\delta\in\Delta^d_{Y,S}} \mathrm{Pic}^{\delta} Y \longrightarrow Q_S \cong \mathrm{N}_{X^{\nu}_S}$$

(notation of 7.9) all of whose closed fibers are isomorphic to $(k^*)^s$ with $s = \#S$.

Remark 8.6. π is described as a quotient by a torus action in 8.7.

Proof. By 3.10 we have $\mathrm{N}(\operatorname{Pic}^d_{K_1}) \cong \dfrac{\coprod_{\delta\in\Delta^d_Y}\operatorname{Pic}^\delta_h}{\sim_K}$. The crux of the proof is to show that for every $\delta \in \Delta^d_Y$ there is a canonical morphism

$$\mu_\delta\colon\ \operatorname{Pic}^\delta_h \longrightarrow \overline{P^d_f}.$$

To do that, let S be the unique element in $\mathbf{G}_X$ such that $\delta \in \Delta^d_{Y,S}$ and consider the unique semibalanced representative $\underline{d}^\delta$ of δ (cf. 4.13). Denote by T and identify (by 3.9) $T := \operatorname{Pic}^\delta_h = \operatorname{Pic}^{\underline{d}^\delta}_h$. Set

$$h_T\colon\ \mathcal{Y}_T = \mathcal{Y} \times_{B_1} T \longrightarrow T$$

and let $\mathcal{P}$ be the Poincaré line bundle on $\mathcal{Y}_T$. Now we apply the construction of 8.8 to $h_T = p$ and $\mathcal{P} = \mathcal{N}$. Thereby we obtain a family, $\overline{\mathcal{Y}_T} \longrightarrow T$ (by contracting all the exceptional components of the fibers of h_T where $\mathcal{P}$ has degree equal to zero) and a line bundle $\mathcal{L}$ on $\overline{\mathcal{Y}_T}$ which pulls back to $\mathcal{P}$. The singular closed fibers of $\overline{\mathcal{Y}_T} \longrightarrow T$ are all isomorphic to Y_S and $\mathcal{L}$ has balanced multidegree $\underline{d} = (d^\delta_1, \dots, d^\delta_\gamma, 1, \dots, 1)$ (the fact that $\mathcal{L}$ is balanced follows from the proof of 8.2, whose notation we are here using). It may be useful to sum up the construction in a diagram where all squares are cartesian:

$$\begin{tikzcd}
\overline{\mathcal{Y}_T} \arrow[d] & \mathcal{Y}_T \arrow[l] \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] & \\
\mathcal{X}_T \arrow[d] \arrow[r, equal] & \mathcal{X}_T \arrow[r] \arrow[d] & \mathcal{X}_1 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
T \arrow[r, equal] & \operatorname{Pic}^\delta_h \arrow[r] & B_1 \arrow[r] & B.
\end{tikzcd} \tag{31}$$

Now the pair $(\overline{\mathcal{Y}_T} \longrightarrow T, \mathcal{L})$ is a family of quasistable curves with a balanced line bundle of degree d. The stable model of $\overline{\mathcal{Y}_T}$ is $\mathcal{X}_T$ therefore (by 7.7) we obtain a moduli morphism

$$\mu_\delta\colon\ T = \operatorname{Pic}^\delta_h \longrightarrow \overline{P^d_f}.$$

As δ varies, the morphisms μ_δ agree on the smooth fibers, that is, away from the closed point of B. Therefore (as in the proof of 6.1) they glue together to a B-morphism $\pi\colon\ \mathrm{N}(\operatorname{Pic}^d_{K_1}) \longrightarrow \overline{P^d_f}$ as stated.

To prove the rest of the statement it suffices to look at the closed fiber, as π_K is obviously a surjection, in fact

$$\mathrm{N}(\operatorname{Pic}^d_{K_1})_{K_1} = \operatorname{Pic}^d_K \times_B \operatorname{Spec} K_1 = (\overline{P^d_f})_K \times_B \operatorname{Spec} K_1 = (P^d_f)_K \times_B \operatorname{Spec} K_1$$

Now by 8.2 and 7.6 (and with the same notation) we have natural bijections

$$\begin{array}{ccccc} \Delta^{d}_{Y,S} & \leftrightarrow & \Delta^{d,1}_{Y_S} & \leftrightarrow & \Delta^{d-s}_{X^{\nu}_{S}} \\ \underline{d}^{\delta} & \mapsto & \underline{d} = [(d^{\delta}_1, \ldots, d^{\delta}_{\gamma}, 1, \ldots, 1)] & \mapsto & \underline{d}^S = [(d^{\delta}_1, \ldots, d^{\delta}_{\gamma})] \end{array} \tag{32}$$

As we said, the singular fibers of $\overline{\mathcal{Y}_T} \longrightarrow T$ are isomorphic to Y_S and we proved above that the restriction of μ_δ to the closed fibers factors

$$\operatorname{Pic}^{\underline{d}^{\delta}} Y \xrightarrow{\cong} \operatorname{Pic}^{\underline{d}} Y_S \twoheadrightarrow \operatorname{Pic}^{\underline{d}^S} X^{\nu}_{S} \hookrightarrow \overline{P^d_X}$$

where we used (25) for the last two arrows; the rest of the proof naturally continues as that of 7.9. □

8.7. Let $b = b_1(\Gamma_X)$. It is not difficult at this point to interpret π as a quotient by a natural action of $(k^*)^b$ on N_Y (extended to a trivial action on $\mathrm{N}(\operatorname{Pic}^{d}_{K_1})$). Observe that $\operatorname{Pic}^{\underline{d}} Y \cong \operatorname{Pic}^{\underline{d}} Y_S \cong \operatorname{Pic}^{\underline{d}^S} X$ (notation in the proof of 7.9) and that $b - s = b_1(\Gamma_{X^{\nu}_{S}})$; denote X^{ν} the normalization of X, we have a diagram of canonical exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & (k^*)^s \arrow[r, equal] \arrow[d] & (k^*)^s \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (k^*)^b \arrow[r] \arrow[d] & \operatorname{Pic}^{\underline{d}} Y_S \arrow[r, "\nu^*"] \arrow[d] & \operatorname{Pic}^{\underline{d}^S} X^{\nu} \times \prod_1^s \operatorname{Pic}^1 \mathbb{P}^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (k^*)^{b-s} \arrow[r] \arrow[d] & \operatorname{Pic}^{\underline{d}^S} X^{\nu}_{S} \arrow[r, "\nu^*"] \arrow[d] & \operatorname{Pic}^{\underline{d}^S} X^{\nu} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{33}$$

(where ν^* always denotes pull-back via the normalization map). The middle vertical sequence describes the restriction of π_S to any irreducible component, $\operatorname{Pic}^{\underline{d}} Y_S$, as the quotient of the action of $(k^*)^s$ on the gluing data over the exceptional components of Y_S.

We applied the following standard fact (included for completeness).

LEMMA 8.8. *Let p: $\mathcal{Z} \longrightarrow T$ be a family of semistable curves of genus at least 2 over a scheme T. Let $\mathcal{N} \in \operatorname{Pic} \mathcal{Z}$ having non-negative degree on each exceptional component of the fibers of p. Then there exist*

(a) *a factorization of p*

$$p\colon \mathcal{Z} \xrightarrow{\psi} \overline{\mathcal{Z}} \xrightarrow{\overline{p}} T$$

via a family of semistable curves $\overline{p}$ and a birational morphism ψ which contracts some exceptional components of the fibers of p;

(b) *a line bundle $\overline{\mathcal{N}} \in \operatorname{Pic} \overline{\mathcal{Z}}$ having positive degree on all exceptional components of the fibers of $\overline{p}$ and such that $\psi^*\overline{\mathcal{N}} \cong \mathcal{N} \otimes p^*M$, where $M \in \operatorname{Pic} T$.*

Proof. For n high enough (how high depends on $\mathcal{N}$) we have that $\omega_p^n \otimes \mathcal{N}$ is relatively base-point-free and $p_*(\omega_p^n \otimes \mathcal{N})$ is a vector bundle on T (trivial variation on Corollary to Theorem 1.2 in [DM69] p.78). Moreover $\omega_p^n \otimes \mathcal{N}$ defines a birational morphism $\psi\colon\ \mathcal{Z} \longrightarrow \overline{\mathcal{Z}} \subset \mathbb{P}(p_*(\omega_p^n \otimes \mathcal{N}))$ contracting the exceptional components of p where $\mathcal{N}$ has degree 0. The line bundle $\overline{\mathcal{N}}$ is given by $\overline{\mathcal{N}} = \mathcal{O}_{\overline{\mathcal{Z}}}(1) \otimes \omega_{\overline{p}}^{-n}$. □

Remark 8.9. It is clear that $\overline{\mathcal{Z}}$ is uniquely determined (just contract all the exceptional components of the fibers of p where $\mathcal{N}$ has degree 0) whereas $\overline{\mathcal{N}}$ is determined only up to pull-backs of line bundles on T. More precisely the lemma gives a map form $\operatorname{Pic} \mathcal{Z}/p^* \operatorname{Pic} T \longrightarrow \operatorname{Pic} \overline{\mathcal{Z}}/\overline{p}^* \operatorname{Pic} T$.

Remark 8.10. We conclude by observing that, as a consequence of 8.4, the completion $\overline{P_f^d}$ of the Néron model satisfies a mapping property for smooth schemes defined over quadratic, possibly ramified, coverings of B. This should be viewed as a strengthening of the mapping property of Néron models with respect to smooth schemes defined over étale coverings of B. It is in fact well known (see [A86] section 1) that Néron models are functorial with respect to étale base changes, but not in general.

To be more precise, let Z be a scheme smooth over $\operatorname{Spec} R_1$ (where $R \hookrightarrow R_1$ is a ramified quadratic extension as in 8.4), and let $v_K\colon\ Z_{K_1} \longrightarrow \operatorname{Pic}_K^d$ be a K-morphism. Then there exists a unique B-morphism $v\colon\ Z \longrightarrow \overline{P_f^d}$ extending v_K. Of course v is obtained by first extending the lifting of v_K to Z_{K_1}

$$u_{K_1}\colon\ Z_{K_1} \longrightarrow \mathrm{N}(\operatorname{Pic}_{K_1}^d)_{K_1} = \operatorname{Pic}_K^d \times_B \operatorname{Spec} K_1,$$

by the Néron mapping property u_{K_1} extends to $u\colon\ Z \longrightarrow \mathrm{N}(\operatorname{Pic}_{K_1}^d)$; thus v is the composition of u with π (defined in 8.5).

9. Appendix.

This appendix is made of two distinct parts. The first illustrates some applications of the results in the paper. The second part summarizes some well known combinatorial facts which have been used throughout.

Applications: towards Brill-Noether theory of stable curves.

9.1. Let $f\colon\ \mathcal{X} \longrightarrow B$ be a family of stable curves and T a scheme over B. Let $p\colon\ \mathcal{Z} \longrightarrow T$ be a family of semistable curves having $\mathcal{X}_T$ as stable model; if

$\mathcal{L} \in \operatorname{Pic}\mathcal{Z}$ is balanced of relative degree d, we can associate to $\mathcal{L}$ a unique map

$$\mu_{\mathcal{L}}\colon T \longrightarrow \overline{P^d_f}, \qquad t \mapsto [\mathcal{L}_{|p^{-1}(t)}]$$

which we call the *moduli map* of $\mathcal{L}$ (note that, $\mathcal{L}$ being balanced, $\mathcal{Z}$ is a family of quasistable curves).

More generally, suppose that $\mathcal{N} \in \operatorname{Pic}\mathcal{Z}$ is semibalanced (cf. 4.6). Apply the construction of 8.8 to obtain a pair $(\overline{\mathcal{Z}},\overline{\mathcal{N}})$ (so that $\overline{\mathcal{Z}} \longrightarrow T$ has $\mathcal{X}_T$ as stable model). Then $\overline{\mathcal{N}} \in \operatorname{Pic}\overline{\mathcal{Z}}$ is a balanced line bundle and its moduli map $\mu_{\overline{\mathcal{N}}}\colon T \longrightarrow \overline{P^d_f}$ can be viewed as induced by $\mathcal{N}$. In summary, to a semibalanced line bundle on $\mathcal{Z}$ we can associate a unique map $T \longrightarrow \overline{P^d_f}$.

9.2. Let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec}R = B$ be a family of curves with $\mathcal{X}$ regular and reducible closed fiber X (as usual), denote $f_d\colon \mathcal{X}^d_B \longrightarrow B$ its dth fibered power. Consider the degree-d Abel map of the generic fiber

$$\alpha^d_K\colon \mathcal{X}^d_K \longrightarrow \operatorname{Pic}^d \mathcal{X}_K, \qquad (p_1,\dots,p_d) \mapsto \left[\mathcal{O}_{\mathcal{X}_K}\left(\sum p_i\right)\right];$$

what is the limit of such a map as $\mathcal{X}_K$ specializes to X?

Not much is known about defining (and completing) Abel maps for reducible curves. A geometric construction for irreducible curves has been carried out in [EGK00] building upon previous well known work of A. Altman and S. Kleiman. Yet serious difficulties arise when reducible fibers occur (even when restricting, as we are, to nodal singularities).

As a first step towards understanding Abel maps of reducible curves, we consider the unique extension of α^d_K given by the Néron mapping property

$$\alpha^d_f\colon \dot{\mathcal{X}}^d_B \longrightarrow \mathrm{N}(\operatorname{Pic}^d \mathcal{X}_K)$$

where $\dot{\mathcal{X}}^d_B = \mathcal{X} \smallsetminus sing(f_d)$; We refer to α^d_f as the degree-d *Abel-Néron map* of f. The case $d = 1$ has been studied by B. Edixhoven in [E98], where there is also a characterization of when it is a closed immersion (in the example below it is).

The results of our paper enable us, on the one hand, to give a geometric description of the Abel-Néron map by identifying $\mathrm{N}(\operatorname{Pic}^d \mathcal{X}_K) \cong P^d_f$. On the other hand we have a natural ambient space where one can construct a completion for it, namely the compactification $\overline{P^d_f}$.

Example 9.3. Fix a stable curve $X = C_1 \cup C_2$ with C_1 and C_2 smooth of genus equal to $h \geq 1$ and $\#C_1 \cap C_2 = 2$ (thus $g = 2h + 1$); let $f\colon \mathcal{X} \longrightarrow \operatorname{Spec}R = B$ be a family of curves with $\mathcal{X}$ regular and X as closed fiber. Since our X is 1-general, we can identify $\mathrm{N}(\operatorname{Pic}^1 \mathcal{X}_K) = P^1_f$ by (6.1) so that the first Abel-Néron map becomes

$$\alpha_f\colon \dot{\mathcal{X}} \longrightarrow P^1_f.$$

We claim that:

(1) α_f can be completed to a map $\overline{\alpha_f}\colon \mathcal{X} \longrightarrow \overline{P^1_f}$;

(2) $\overline{\alpha_f}$ has a geometric description as the moduli map of a natural line bundle;

(3) the restriction to X of $\overline{\alpha_f}$ does not depend on f.

Consider $\mathcal{L} = \mathcal{O}_{\dot{\mathcal{X}}\times_B \mathcal{X}}(\Delta) \in \operatorname{Pic}(\dot{\mathcal{X}} \times_B \mathcal{X})$, where $\Delta \subset \dot{\mathcal{X}} \times_B \mathcal{X} = \mathcal{X}^2_B$ is the diagonal. Then, applying the set up of 9.1 with $T = \dot{\mathcal{X}}$, we claim that α_f is the moduli map of $\mathcal{L}$ (this is obviously true on the generic fiber $\mathcal{X}_K$ of $\dot{\mathcal{X}}$). For that it suffices to show that $\mathcal{L}$ is balanced, i.e. that for every nonsingular point $x \in X$ the line bundle $\mathcal{O}_X(x)$ (the restriction of $\mathcal{O}_{\dot{\mathcal{X}}\times_B \mathcal{X}}(\Delta)$ to the fiber over x) is balanced. This follows easily, by checking that for every subcurve Z of our X, we have $m_Z(1) < 0$ and $M_Z(1) > 1$ so that we have

$$\mathrm{N}(\operatorname{Pic}^1 \mathcal{X}_K) \cong P^1_f \cong \frac{\operatorname{Pic}^{(0,1)}_f \coprod \operatorname{Pic}^{(1,0)}_f}{\sim_K}.$$

Now let us denote $r\colon \mathcal{Z} \longrightarrow \mathcal{X}^2_B$ the resolution of singularities. A direct computation shows that $\mathcal{Z}$ is obtained by replacing each of the four singular points of $\mathcal{X}^2_B$ by a $\mathbb{P}^1$ so that $p\colon \mathcal{Z} \longrightarrow \mathcal{X} = T$ is a family of quasistable curves; moreover the proper transform $\tilde{\Delta} \subset \mathcal{Z}$ of Δ defines a line bundle $\mathcal{N} = \mathcal{O}_{\mathcal{Z}}(\tilde{\Delta})$ having non-negative degree on every exceptional component of the fibers of p. One checks that $\mathcal{N}$ is semibalanced hence, applying the construction of 9.1, we obtain a regular map

$$\mu_{\overline{\mathcal{N}}}\colon T = \mathcal{X} \longrightarrow \overline{P^1_f}$$

which defines the extension $\overline{\alpha_f} = \mu_{\overline{\mathcal{N}}}$ of α_f that we wanted.

To show that the restriction $\alpha_X := \overline{\alpha_f}_{|X}$ does not depend on f one simply observes that if $x \in X$ is a nonsingular point, then its image is just the class of $\mathcal{O}_X(x)$. If x is singular, denote by Y_x the quasistable curve obtained by blowing-up X at x and let $q \in Y_x$ be any nonsingular point of Y_x lying in the unique exceptional component. Then, as q varies, the line bundles $\mathcal{O}_{Y_x}(q)$ are all identified to the same point λ_x in $\overline{P^1_X}$ (by 7.3); then the image $\alpha_X(x)$ is exactly the point λ_x.

9.4. The method of the previous example can be applied to all stable curves, but nontrivial complications arise. First of all, it is not always true that the “diagonal” line bundle used above is balanced; a more delicate construction is needed to prove that the same properties (1)–(3) hold.

The global version of such a morphism (mapping the universal curve over $\overline{M}_g$ to $\overline{P_{1,g}}$) could also be carried out, as it is reasonable to expect, in view of the independence on f of the Abel-Néron map (property (3)).

Let us finish with a few words about the Abel-Néron maps for higher degree d. The problem can be approached similarly to what outlined for $d = 1$; however the situation is considerably more subtle. One important difference is that, as soon as $d \geq 2$, the dth Abel-Néron map will depend on f, for some combinatorially determined cases. In other words, the analogue of property (3) fails.

Another difficulty is the fact (observed by E. Esteves) that a completion of the Abel map will not be defined on $\mathcal{X}_B^d$, but only on some modification $\widetilde{\mathcal{X}_B^d} \longrightarrow \mathcal{X}_B^d$ of it.

These hurdles are to be expected, as the set up leads towards a construction of Brill-Noether varieties for singular curves. As a first step, we can define the Brill-Noether scheme $\overline{W_d^0}(X,f)$ (generalizing the Brill-Noether variety of effective line bundles of degree d on a smooth curve) as follows:

$$\overline{W_d^0}(X,f) := \overline{\mathrm{Im}\,(\alpha_f^d)_k} \subset \overline{P_X^d},$$

i.e., the closure of the image of the restriction $(\alpha_f^d)_k\colon\ \dot{X}^d \longrightarrow \overline{P_X^d}$, where $\dot{X}$ denotes the smooth locus of X. The closure symbol is used because such a scheme parametrizes "boundary points", that is, line bundles on quasistable curves $Y \neq X$ having X as stable model; we shall denote $W_d^0(X,f)$ its open subset parametrizing line bundles on X.

The presence of f in the notation is needed for $d \geq 2$; although we can prove that $\overline{W_1^0}(X,f)$ never depends on f, for $d \geq 2$ this turns out to fail. To be more precise, denote by $\mathcal{X}_{sep}^{\nu}$ the partial normalization of X at its separating nodes (so that $\mathcal{X}_{sep}^{\nu} = X$ if X has no separating node), then we conjecture the following. The restricted Abel-Néron map $(\alpha_f^d)_k$ is independent of f if and only if for every $k \leq d$ every connected component of $\mathcal{X}_{sep}^{\nu}$ admits no disconnecting subset of k nodes.

Combinatorics of stable curves.

9.5. Some features of stable curves are nicely expressed using graph theory. Chapter 1 of the article [OS79] contains a thorough study of the combinatorial aspects of the theory of compactified Jacobians. In the sequel we recall a few facts that can be found in that paper.

To a nodal curve X having γ irreducible components and δ nodes, one associates a graph Γ_X defined as the symplicial complex (of dimension at most 1) defined to have one *vertex* for every irreducible component of C, and one *edge* connecting two vertices for every node in which the two corresponding components intersect. Thus Γ_X has γ vertices, δ edges and among the edges there is a loop for every node lying on a single irreducible component of X.

9.6. The first Betti number $b_1(\Gamma_X)$ (sometimes called the *cyclomatic number*) is, for any orientation on Γ_X

$$b_1(\Gamma_X) := \dim_{\mathbb{Z}} H_1(\Gamma_X, \mathbb{Z}) = \delta - \gamma + 1.$$

Recall also that *the first Betti number of a connected graph is the maximal number of one-dimensional open symplices that can be removed from the graph without disconnecting it.*

Another important, somewhat less standard, invariant of a graph is its *complexity*.

Definition 9.7. Let Γ be a connected graph. A *spanning tree of* Γ is a subgraph $\Gamma' \subset \Gamma$ which is a connected tree and such that Γ and Γ' have the same vertices. The *complexity* of Γ, $\mu(\Gamma)$, is the total number of spanning trees of Γ.

Example 9.8. Let X be connected. (1) X is of compact type if and only if Γ_X is a tree, if and only if $b_1(\Gamma_X) = 0$, if and only if $\mu(\Gamma_X) = 1$.

(2) By the genus formula $g = \sum g_i + b_1(\Gamma_X)$ we get that $b_1(\Gamma_X) \leq g$. Moreover, $b_1(\Gamma_X) = g$ if and only if all irreducible components of X have geometric genus 0.

9.9. The complexity can be computed cohomologically. Fix an orientation on Γ and consider the standard homology operators

$$\partial\colon\ C_1(\Gamma,\mathbb{Z}) \longrightarrow C_0(\Gamma,\mathbb{Z}), \qquad e \mapsto v - w, \tag{34}$$

where e is an edge of Γ, starting in the vertex v and ending in the vertex w. And

$$\delta\colon\ C_0(\Gamma,\mathbb{Z}) \longrightarrow C_1(\Gamma,\mathbb{Z}), \qquad v \mapsto \sum e_v^+ - \sum e_v^-, \tag{35}$$

where e_v^+ are the edges starting at the vertex v and e_v^- are those ending in v. Then introduce the *complexity group* of the graph Γ

$$\frac{\partial C_1(\Gamma,\mathbb{Z})}{\partial\delta C_0(\Gamma,\mathbb{Z})}.$$

The name "complexity group" is due to the theorem of Kirchhoff-Trent ([OS79] p. 21) stating that *such a group is finite and its cardinality is equal to the complexity of* Γ.

The next lemma is Proposition 14.3 in [OS79] (see also [L89]).

LEMMA 9.10. *For a nodal connected curve X with dual graph Γ_X we have*

$$\Delta_X \cong \frac{\partial C_1(\Gamma_X,\mathbb{Z})}{\partial\delta C_0(\Gamma_X,\mathbb{Z})}.$$

In particular the cardinality of Δ_X is equal to the complexity of Γ_X.

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