

Compactified Jacobians, Abel maps and Theta divisors

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1. INTRODUCTION

Outline of the paper. This paper is an expository account about compactified Picard schemes of nodal curves and some related topics. After some preliminaries, Néron models are used to classify different compactified Picard schemes, Abel maps are studied accordingly, and finally some recent results on the Theta divisor are reviewed.

Several examples are included, as an attempt to elucidate some important parts that require a consistent amount of technical work to be rigorously settled. To fill out the unavoidable gaps, various references are given throughout the paper.

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Conventions. We fix an algebraically closed field k and work with schemes locally of finite type over k , unless differently specified.

X will always be a connected, reduced, projective curve over k , having at most nodes as singularities. We denote by g the arithmetic genus of X , by δ the number of its nodes and by γ the number of its irreducible components. The dual graph of X (having as vertices the γ irreducible components of X and as edges the δ nodes of X) is denoted Γ_X .

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X is called of *compact type* if Γ_X is a tree; more generally, X is called *tree-like* if Γ_X becomes a tree after all loops are removed (in particular every irreducible curve is tree-like).

By $f : \mathcal{X} \rightarrow B$ we denote a *one-parameter smoothing* of X . That is, $B = \text{Spec } R$ is the spectrum of a discrete valuation ring R having residue field k and quotient field K ; the closed fiber of f is X and the generic fiber, denoted \mathcal{X}_K , is a smooth projective curve over K . Everything we shall say holds (mutatis mutandis) if one replaces B by any Dedekind scheme.

The total space \mathcal{X} is a normal surface with singularities of type A_n at the nodes of $X \subset \mathcal{X}$. If \mathcal{X} is nonsingular, we will say that f is a *regular smoothing*.

2. COMPACTIFIED PICARD SCHEMES

2.1. Set up.

2.1.1. Jacobians, their torsors and their models. We consider the Picard scheme of \mathcal{X}_K

$$\text{Pic } \mathcal{X}_K = \coprod_{d \in \mathbb{Z}} \text{Pic}^d \mathcal{X}_K$$

where $\text{Pic}^d \mathcal{X}_K$ is the variety parametrizing isomorphism classes of line bundles of degree d on \mathcal{X}_K . In particular $\text{Pic}^0 \mathcal{X}_K$ is an abelian variety over K and $\text{Pic}^d \mathcal{X}_K$ a torsor under it.

What about models of $\text{Pic}^d \mathcal{X}_K$ over B ? The analysis of such models (by which we mean integral schemes over B whose generic fiber is $\text{Pic}^d \mathcal{X}_K$) is a key to understand the properties of any *compactified Jacobian* or *compactified Picard scheme* of X , which is one of the main themes of this paper.

We are going to introduce three different types of models for $\text{Pic}^d \mathcal{X}_K$ over B . The first two (the Picard scheme and the Néron model) are uniquely determined by their defining properties, the third consists of a miscellany of different models (compactified Picard schemes, see 2.3.1). We shall see how they relate to one another; in particular, they are all isomorphic if and only if X is nonsingular.

2.1.2. The (relative) Picard scheme. For every $f : \mathcal{X} \rightarrow B$ and for every d there exists the (relative, degree d) Picard scheme

$$\lambda_d : \text{Pic}_f^d \rightarrow B$$

(often denoted $\text{Pic}_{\mathcal{X}/B}^d \rightarrow B$). The existence and basic theory are due to A. Grothendieck, P. Deligne and D. Mumford (see [D79], [SGA], [M66]); we refer to [BLR] for a unified account. Over every point in B , the fiber of λ_d is the variety of isomorphism classes of line bundles of degree d on the corresponding fiber of f . So, the generic fiber of λ_d is $\text{Pic}^d \mathcal{X}_K$ and the closed fiber is $\text{Pic}^d X$ (see (2) for an explicit description of $\text{Pic}^d X$).

The moduli property of the Picard scheme is expressed in Proposition 4, section 8.1. p. 204 of [BLR]; loosely speaking it amounts to the following. For every B -scheme $T \rightarrow B$, denote $f_T : \mathcal{X}_T := \mathcal{X} \times_B T \rightarrow T$ the base change of f . For every line bundle \mathcal{L} of relative degree d on \mathcal{X}_T there exists a unique “moduli morphism” $\mu_{\mathcal{L}} : T \rightarrow \text{Pic}_f^d$ mapping $t \in T$ to the isomorphism class of the restriction of \mathcal{L} to $f_T^{-1}(t)$; for any $M \in \text{Pic } T$, we have $\mu_{\mathcal{L}} = \mu_{\mathcal{L} \otimes f_T^* M}$. Conversely, given a morphism $\mu : T \rightarrow \text{Pic}_f^d$, the obstruction to the existence of a line bundle \mathcal{L} on \mathcal{X}_T having

μ as moduli map lies in the Brauer group of T . If f has a section, there is no obstruction.

The union $\text{Pic}_f := \coprod_{d \in \mathbb{Z}} \text{Pic}_f^d$ is a group scheme over B with respect to tensor product.

An important fact is that $\text{Pic}_f^d \rightarrow B$ is separated if and only if X is irreducible (see 2.2.4).

2.1.3. The generalized jacobian. The generalized jacobian of X , denoted $J(X)$, parametrizes isomorphism classes of line bundles having degree 0 on every irreducible component of X ; thus $J(X)$ is a commutative algebraic group with respect to tensor product. It is well known that $J(X)$ is a semiabelian variety, i.e. there exists an exact sequence

$$(1) \quad 0 \longrightarrow (k^*)^b \longrightarrow J(X) \xrightarrow{\nu^*} J(X^\nu) \longrightarrow 0$$

where $\nu : X^\nu \rightarrow X$ is the normalization of X (hence $J(X^\nu)$ is an abelian variety) and

$$b = \delta - \gamma + 1 = b_1(\Gamma_X)$$

is the first betti number of the dual graph of X , Γ_X . As a consequence we see that $J(X)$ is projective if and only if X is a curve of compact type.

To relate the generalized jacobian to the Picard scheme, note that for every $d \in \mathbb{Z}$ we have

$$(2) \quad \text{Pic}^d X = \coprod_{\underline{d} \in \mathbb{Z}^\gamma} \text{Pic}^{\underline{d}} X$$

where $\text{Pic}^{\underline{d}} X$ parametrizes line bundles of multidegree \underline{d} on X . For example $\text{Pic}^{(0, \dots, 0)} X = J(X)$. Of course $\text{Pic}^{\underline{d}} X$ is, non-canonically, isomorphic to $J(X)$ for every \underline{d} .

For every $f : \mathcal{X} \rightarrow B$ there exists the relative jacobian, $J_f \rightarrow B$ (often denoted $J_{\mathcal{X}/B} \rightarrow B$) which is a group scheme over B having fibers the generalized jacobians of the fibers of f .

The connected component of the identity, $(\text{Pic}_f)^0 \rightarrow B$, of the group scheme $\text{Pic}_f \rightarrow B$ is canonically identified with $J_f \rightarrow B$.

2.1.4. The Néron model We here describe some well known results of A. Néron and M. Raynaud ([N64], [R70]) on the existence of Néron models and on Néron models of Picard varieties. We refer to [BLR] for all details, and to [A86] for a synthesis of the basic theory. The Néron model of $\text{Pic}^d \mathcal{X}_K$ will here be denoted

$$\sigma_d : N_f^d \longrightarrow B$$

(a common notation is $N(\text{Pic}^d \mathcal{X}_K) \rightarrow B$); σ_d is a smooth, separated morphism of finite type. If $d = 0$ then $N_f^0 \rightarrow B$ is a group scheme whose identity component is $J_f \rightarrow B$; for a general d , $N_f^d \rightarrow B$ is a torsor under $N_f^0 \rightarrow B$ (the Néron model of $\text{Pic}^0 \mathcal{X}_K$). The Néron model is uniquely determined by the *Néron mapping property* ([BLR], Def. 1, p. 12), which is the following. For every scheme Z smooth over B , every map u_K from its generic fiber to $\text{Pic}^d \mathcal{X}_K$ extends uniquely to a regular B -morphism $N(u_K) : Z \rightarrow N_f^d$.

The closed fiber of N_f^d , viewed simply as a scheme (forgetting its torsor structure), depends only on the type of singularities of the surface \mathcal{X} , in particular it

does not depend on d . So, if $f : \mathcal{X} \rightarrow B$ is a regular smoothing of X , we denote the closed fiber of $N_f^d \rightarrow B$ by N_X^d , or by N_X when no confusion is possible.

The scheme N_X can be described in various ways; we begin using combinatorics, as in [OS79] sections 4 and 14, to which we refer for more details. Consider the dual graph Γ_X of X , let $c(X)$ be the *complexity* of Γ_X ; recall that the complexity of a graph is the number of trees contained in it and passing through all of its vertices (i.e. the number of so-called “spanning trees”). Now, N_X is the disjoint union of $c(X)$ copies of the generalized jacobian of X . Therefore N_X is irreducible (and isomorphic to $J(X)$) if and only if X is a tree-like curve. An alternative description of N_X will be given in the sequel.

Example 2.1.5. *The curve X_δ .* Denote by

$$X_\delta = C_1 \cup C_2, \quad \delta = \#(C_1 \cap C_2) = \#X_{\text{sing}}$$

the union of two smooth curves meeting transversally at δ points. We call g_i the genus of C_i so that the genus g of X_δ is $g = g_1 + g_2 + \delta - 1$. X_δ is sometimes called a *vine* curve; it will be our leading example throughout the paper.

Γ_{X_δ} consists of two vertices joined by δ edges; therefore every edge is a spanning tree and $c(X) = \delta$. We obtain that N_X is the disjoint union of δ copies of $J(X)$.

2.2. Relation between Picard scheme and Néron model.

2.2.1. *A canonical quotient.* Fix $f : \mathcal{X} \rightarrow B$ a regular (for simplicity) smoothing of X and consider the two models of $\text{Pic}^d \mathcal{X}_K$ that we have introduced so far: the Picard scheme $\lambda_d : \text{Pic}_f^d \rightarrow B$ and the Néron model $\sigma_d : N_f^d \rightarrow B$. The Néron mapping property yields a canonical regular B -map

$$(3) \quad q_f : \text{Pic}_f^d \rightarrow N_f^d$$

extending the identity on the generic fibers (if $d = 0$ our q_f is the map “Ner” in diagram 1.21 of [A86]); we omit d for simplicity. q_f is a surjection, indeed if $d = 0$ it is a quotient of group schemes. So N_f^d is sometimes called the “largest separated quotient of the degree- d Picard scheme”. The restriction of q_f to the closed fibers depends on f ; we shall now concentrate on it.

2.2.2. *Twisters.* To a regular smoothing f of X we can associate a discrete subgroup $\text{Tw}_f X$ of $\text{Pic}^0 X$; $\text{Tw}_f X$ is the set of all line bundles of the form $\mathcal{O}_{\mathcal{X}}(D)|_X$ where D is a divisor on \mathcal{X} supported on the closed fiber X . Elements of $\text{Tw}_f X$ are called *twisters* (or f -twisters). The multidegree map

$$\underline{\text{deg}} : \text{Tw}_f X \longrightarrow \mathbb{Z}^\gamma$$

has image a group called Λ_X :

$$\Lambda_X = \underline{\text{deg}}(\text{Tw}_f X) \subset \{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = 0\} \subset \mathbb{Z}^\gamma.$$

Remark 2.2.3. Observe that while $\text{Tw}_f X$ depends on f (unless X is tree-like) Λ_X does not. For example, if $X = C_1 \cup C_2$ with $\#C_1 \cap C_2 = \delta \geq 2$, for every $n \neq 0$ we have that there exist regular smoothings $f : \mathcal{X} \rightarrow B$ and $f' : \mathcal{X}' \rightarrow B$ of X such that $\mathcal{O}_{\mathcal{X}}(nC_1)|_X \not\cong \mathcal{O}_{\mathcal{X}'}(nC_1)|_X$. On the other hand, for any f we have $\underline{\text{deg}} \mathcal{O}_{\mathcal{X}}(nC_1)|_X = (-n\delta, n\delta)$.

2.2.4. *Towards separatedness.* Consider for a moment the case $d = 0$; recall that N_f^0 is a separated model of $\text{Pic}^0 \mathcal{X}_K$ endowed with a universal mapping property. Consider the line bundles $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{L} := \mathcal{O}_{\mathcal{X}}(D)$ for any divisor D on \mathcal{X} such that

Supp $D \subset X$. Suppose that D is not a multiple of X , which is equivalent to ask that $\deg \mathcal{O}_{\mathcal{X}}(D)|_X \neq (0, \dots, 0)$. Thus \mathcal{L} and $\mathcal{O}_{\mathcal{X}}$ determine two different sections, $\mu_{\mathcal{O}_{\mathcal{X}}} : B \rightarrow \text{Pic}_f^0$ and $\mu_{\mathcal{L}} : B \rightarrow \text{Pic}_f^0$, of the Picard scheme $\text{Pic}_f^0 \rightarrow B$ (cf. 2.1.2). These two sections of course coincide on the generic point $\text{Spec } K$.

The generic fiber of N_f^0 is the same as that of Pic_f^0 , so we may interpret $(\mu_{\mathcal{O}_{\mathcal{X}}})|_{\text{Spec } K} = (\mu_{\mathcal{L}})|_{\text{Spec } K}$ as a map from $\text{Spec } K$ to the generic fiber of N_f^0 . By the Néron mapping property, there exists a unique morphism $\bar{\mu} : B \rightarrow N_f^0$ extending $(\mu_{\mathcal{O}_{\mathcal{X}}})|_{\text{Spec } K}$ and $(\mu_{\mathcal{L}})|_{\text{Spec } K}$ and such that

$$\bar{\mu} = q_f \circ \mu_{\mathcal{O}_{\mathcal{X}}} = q_f \circ \mu_{\mathcal{L}}.$$

This implies that in the closed fiber of N_f^0 there is a unique point corresponding to both $(\mathcal{O}_{\mathcal{X}})|_X = \mathcal{O}_X$ and $\mathcal{O}_{\mathcal{X}}(D)|_X$ for any D ; in other words q_f maps $\text{Tw}_f X$ to one point. More generally, we have

Lemma 2.2.5. *Let $L, L' \in \text{Pic}^d X$ and assume that there exists a smoothing $f : \mathcal{X} \rightarrow B$ of X and a $T \in \text{Tw}_f X$ such that $L' \cong L \otimes T$. Then $q_f(L) = q_f(L')$.*

Proof. (The lemma and its proof hold for every smoothing f of X , of its being regular). As $T \in \text{Tw}_f X$ there exists a line bundle $\mathcal{T} = \mathcal{O}_{\mathcal{X}}(D)$ on \mathcal{X} which restricts to T on the closed fiber X . Suppose that there exists $\mathcal{L} \in \text{Pic } \mathcal{X}$ restricting to L on X ; set $\mathcal{L}' := \mathcal{L} \otimes \mathcal{T}$ so that $\mathcal{L}'|_X = L'$ and $\mathcal{L}'|_{\mathcal{X}_K} = \mathcal{L}'|_{\mathcal{X}_K}$. We can argue as we did in 2.2.4 (with respect to the pair \mathcal{L} and $\mathcal{O}_{\mathcal{X}}$) with respect to the pair \mathcal{L} and \mathcal{L}' . So we are done.

Now, up to replacing f with an étale base change, we can assume that such a line bundle \mathcal{L} exists. Since the formation of Néron models commutes with étale base change, we are done. \blacksquare

2.2.6. Multidegree classes. The previous discussion motivates the following definition (from [C94]). Let \underline{d} and \underline{d}' be two multidegrees (so that $\underline{d}, \underline{d}' \in \mathbb{Z}^\gamma$); we define them to be equivalent if their difference is the multidegree of a twister, i.e.: $\underline{d} \equiv \underline{d}' \Leftrightarrow \underline{d} - \underline{d}' \in \Lambda_X$. Now the quotient of the set of multidegrees with fixed total degree by this equivalence relation is a finite set Δ_X^d :

$$\Delta_X^d := \frac{\{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = d\}}{\equiv}.$$

It is well known that the cardinality of Δ_X^d is independent of d (so that we shall sometimes simply write Δ_X) and it is equal to $c(X)$ (defined in 2.1.4). Δ_X^d naturally labels the connected/irreducible components of N_X^d ; indeed we have

$$(4) \quad N_X^d = \coprod_{\mu \in \Delta_X^d} N_{X,\mu}$$

with non canonical isomorphisms

$$N_{X,\mu} \cong J(X), \quad \forall \mu \in \Delta_X^d.$$

Finally, the restriction of q_f to the closed fiber $\text{Pic}^d X$ of λ_d is surjective and induces an isomorphism of each connected component $\text{Pic}^d X$ of $\text{Pic}^d X$ with the connected component of N_X^d corresponding to the class μ of \underline{d} in Δ_X^d :

$$(q_f)|_{\text{Pic}^d X} : \text{Pic}^d X \xrightarrow{\cong} N_{X,\mu}, \quad \text{where } \mu = [\underline{d}] \in \Delta_X^d$$

(more details in chapter 9 of [BLR] or in [A86]).

Example 2.2.7. *Structure of $\Delta_{X_\delta}^d$.* In the situation of Example 2.1.5, one has that the group of multidegrees of twistors is

$$\Lambda_{X_\delta} = \{(n\delta, -n\delta), \forall n \in \mathbb{Z}\} \subset \mathbb{Z}^2$$

(see remark 2.2.3). Hence $\Delta_{X_\delta}^d$ has cardinality δ (indeed $\Delta_{X_\delta}^0 \cong \mathbb{Z}/\delta\mathbb{Z}$).

2.3. Types of Compactified Picard schemes.

2.3.1. *A general notion of compactified Picard scheme.* In this paper, a (degree- d) compactified Picard scheme for X is a projective, reduced scheme \overline{P}_X^d such that for every $f : \mathcal{X} \rightarrow B$ there exists a unique integral scheme \overline{P}_f^d with a projective morphism

$$(5) \quad \pi_d : \overline{P}_f^d \rightarrow B$$

whose generic fiber is $\text{Pic}^d \mathcal{X}_K$ and whose closed fiber is \overline{P}_X^d . We call \overline{P}_f^d the (relative, degree- d) compactified Picard scheme associated to f . We denote by $P_f^d \subset \overline{P}_f^d$ the smooth locus of π_d .

A compactified Picard scheme is also endowed with some geometric meaning. More precisely, \overline{P}_X^d and \overline{P}_f^d will be (coarse or fine) moduli schemes for certain functors strictly related to the Picard functor.

2.3.2. *References.* In the literature, there exist several constructions of compactified Picard schemes (see for example [I78], [Ds79], [OS79], [AK80], [S94] [C94], [P96], [E01]) which differ from one another in various aspects, such as the functorial interpretation. A survey may be found in [Al04].

Our goal here is to classify them according to their relation with the Néron model; see 2.3.5. In doing so we shall not go through the details concerning various constructions, but we shall focus on their formal properties.

We first describe an apparently simple, yet challenging case.

Example 2.3.3. The following is a special case of Example 2.1.5, whose notation we continue to use. Let $X = X_\delta = C_1 \cup C_2$ with $\delta = 1$. So X is a curve of compact type whose generalized jacobian is projective (cf. 2.1.3):

$$J(X) = \text{Pic}^{(0,0)} X \cong \text{Pic}^0 C_1 \times \text{Pic}^0 C_2 = J(C_1) \times J(C_2).$$

If $f : \mathcal{X} \rightarrow B$ is a regular smoothing of X , the Picard scheme $\text{Pic}_f^d \rightarrow B$ is not separated (cf. 2.1.2). Let us look for a separated, even projective, model for $\text{Pic}^d \mathcal{X}_K$; in other words, let us look for a compactified Picard scheme. This does not seem too hard: every connected component of $\text{Pic}^d X$ is projective. Recall also (see example 2.2.7) that Δ_X consists of only one element, so we expect a separated model of $\text{Pic}^d \mathcal{X}_K$ to have only one irreducible component.

If $d = 0$ there is no problem: it suffices to take the generalized jacobian $J_f \rightarrow B$ which is certainly projective. In doing so we have made a choice of multidegree for the closed fiber, namely the multidegree $(0, 0)$.

If $d \neq 0$ there does not seem to be a canonical choice of multidegree. Indeed, consider the case $d = g - 1$ (which will turn out to be quite interesting in the sequel). Since $g = g_1 + g_2$, there exists no natural choice of a multidegree of total degree $g - 1$. The lesson we should draw from this is that a compactified degree- d Picard scheme for X may fail to contain a subset corresponding to line bundles of degree d on X itself (this does not happen if X is irreducible; see 2.3.6).

In the present example, we shall see that there exists a canonical compactified Picard scheme $\overline{P_X^{g-1}}$ endowed with a canonical isomorphism with $\text{Pic}^{(g_1-1, g_2-1, 1)} \widehat{X}$, where $\widehat{X} = C_1 \cup C_2 \cup C_3$ is the curve obtained by “blowing-up” X at the node and calling the “exceptional” component C_3 .

An equivalent description of $\overline{P_X^{g-1}}$ is as the moduli space of Euler characteristic 0, rank-1, torsion-free sheaves on X that are not locally free at the node.

In either case, $\overline{P_X^{g-1}}$ does not parametrize any set of line bundles of degree $g-1$ on X . The above description yields an isomorphism

$$\overline{P_X^{g-1}} \cong \text{Pic}^{(g_1-1, g_2-1)} X,$$

so, an interpretation in terms of line bundles of degree $g-2$ rather than $g-1$ (see 2.4.6 for more details).

2.3.4. Néron models and compactified Picard schemes. Let X be any curve and $f : \mathcal{X} \rightarrow B$ a regular smoothing for X . Let us now consider a fixed compactified Picard scheme $\pi_d : \overline{P_f^d} \rightarrow B$ and its closed fiber $\overline{P_X^d}$ (cf. 2.3.1). Recall that we denote $P_f^d \rightarrow B$ the smooth locus of π_d . We apply the Néron mapping property to obtain a canonical B -morphism

$$(6) \quad n_f : P_f^d \rightarrow N_f^d$$

extending the identity map from the generic fiber of π_d to the generic fiber of $N_f^d \rightarrow B$.

Definition 2.3.5. We say that $\overline{P_X^d}$, or $\overline{P_f^d}$, is of *N-type* (or of *Néron type*) if the map $n_f : P_f^d \rightarrow N_f^d$ is an isomorphism. We say that $\overline{P_X^d}$, or $\overline{P_f^d}$, is of *D-type* (or of *Degeneration type*) otherwise.

2.3.6. Irreducible curves. If X is irreducible, up to isomorphisms there exists in the literature a unique compactified Picard scheme $\overline{P_X^d}$, which is of N-type and does not depend on d . $\overline{P_X^d}$ is irreducible and reduced; it is singular unless X is smooth. It is also true that $\text{Pic}^d X$ is naturally isomorphic to an open subset of $\overline{P_X^d}$ (the first such constructions are [I78], [MM64], [Ds79] and [AK80]).

As we said, if X is reducible and not tree-like, then the structure and the type of $\overline{P_X^d}$ varies.

Example 2.3.7. Let us consider a curve $X_\delta = C_1 \cup C_2$ as in Example 2.1.5, and assume $\delta \geq 2$.

The fact that there exist different, non isomorphic, compactified jacobians for X_δ was first discovered by Oda and Seshadri in their fundamental paper [OS79], the first paper providing a construction of a compactified jacobian for any reducible, nodal, fixed (i.e. not moving in a family) curve. They proved that, depending on a certain choice of polarization, there exists a compactified jacobian of N-type, hence having δ irreducible components, or a compactified jacobian of D-type, having $\delta-1$ irreducible components (see [OS79] Chapter II, section 13).

The same was later found in [C94] in a different framework. There is no polarization involved in this construction, and the structure of $\overline{P_X^d}$ depends solely on the degree d . For every curve X_δ both types of compactification appear, and they are isomorphic to the ones of [OS79]. More precisely, if $d = g-1$ then for every $\delta \geq 2$,

we have that $\overline{P_{X_\delta}^d}$ has $\delta - 1$ components (hence it is of D-type). If $d = 0$ then $\overline{P_{X_\delta}^d}$ is of N-type (resp. of D-type with $\delta - 1$ components) if δ is odd (resp if δ is even). See Theorem 2.3.9 for values of d for which $\overline{P_{X_\delta}^d}$ is of N-type.

2.3.8. Reducible curves. Once we choose the specific construction (so that we choose a $\overline{P_X^d}$ for every X and d) the following problem arises naturally: for a fixed d classify all curves for which $\overline{P_X^d}$ is of Néron type. A satisfactory answer to this question is known if $d = g - 1$ (and the answer is: only the obvious ones; see Theorem 2.4.1 below), and in a few other cases. For instance, this issue is rather well understood for the construction of [C94], or of [P96] (the two are naturally isomorphic). Here is what is known.

Theorem 2.3.9. *For every d there exists an open substack $\overline{\mathcal{M}}_g^d$ of $\overline{\mathcal{M}}_g$, whose moduli scheme \overline{M}_g^d we call the locus of d -general stable curves, such that the following holds.*

- (i) *There exists a Deligne-Mumford stack $\overline{\mathcal{P}}_{d,g}^{Ner}$ with a strongly representable morphism onto $\overline{\mathcal{M}}_g^d$, such that for every $X \in \overline{\mathcal{M}}_g^d$ and every regular smoothing $f : \mathcal{X} \rightarrow B$ of X , the base change $\overline{\mathcal{P}}_{d,g}^{Ner} \times_{\overline{\mathcal{M}}_g^d} B \rightarrow B$ is a compactified Picard scheme of Néron-type.*
- (ii) *\overline{M}_g^d is open in \overline{M}_g and contains the locus of irreducible curves.*
- (iii) *$\overline{\mathcal{M}}_g^d = \overline{\mathcal{M}}_g$ if and only if $(d - g + 1, 2g - 2) = 1$.*
- (iv) *If $d = g - 1$ then \overline{M}_g^d equals the locus of irreducible curves.*

See [C05] for the case described in part (iii) and [M07] for the remaining cases; in the latter paper there is also a complete description of \overline{M}_g^d , based on the combinatorics of the curves.

Although Theorem 2.3.9 is proved using the [C94] construction, by results of [Al04] section 1, it applies also for the compactifications of [OS79] and [S94] with respect to the canonical polarization.

Finally, the construction of [E01] (which concerns curves with singularities more general than nodes) has been studied in [B07]. It is there shown that the compactified Picard schemes called $J_{\mathcal{E}}^g$ are of N-type for all regular smoothings (endowed with a section σ and a polarization \mathcal{E} , needed for the construction).

2.4. The case $d = g - 1$.

As we mentioned, the case $d = g - 1$ has been studied closely by various authors and it is thus much better understood. The following is a summary of known results:

Theorem 2.4.1. *For every curve X of genus $g \geq 2$ the following facts hold.*

- (i) *The compactified degree- $(g - 1)$ Picard schemes constructed in [OS79], [S94] and [C94] are all isomorphic to a projective, reduced scheme, denoted P_X^{g-1} from now on.*
- (ii) *P_X^{g-1} possesses a theta divisor $\Theta(X)$ which is Cartier and ample. If X is smooth, $\Theta(X)$ coincides with the classical theta divisor i.e. $\Theta(X) = W_{g-1}(X)$ in the standard notation (see [ACGH]).*

- (iii) The pair $(\overline{P_X^{g-1}}, \Theta(X))$ is a semiabelic stable pair in the sense of [Al02].
 (iv) If X is not tree-like, $\overline{P_X^{g-1}}$ has less than $c(X)$ components (hence it is of D -type).

Parts (i) and (iii) are due to V. Alexeev, [Al04]. Part (ii) is due to A. Soucaris [So94] and E. Esteves [E97] in case X is irreducible, and to Alexeev if X is reducible, in which case some results of A. Beauville in [B77] play a key role; see [Al04] for details. For part (iv) see [C05] Section 4.

2.4.2. The canonical compactified jacobian in degree $g - 1$. As we said, the above results, especially part (iii), lead us to regard such compactification $\overline{P_X^{g-1}}$ as canonical. Throughout this paper, the compactified Picard schemes $\overline{P_X^{g-1}}$ and $\overline{P_f^{g-1}}$ will be the ones of Theorem 2.4.1. We shall give it a more explicit description in the sequel.

Having Theorem 2.3.9 in mind we now ask: how do these Picard schemes $\overline{P_X^{g-1}}$ glue together over $\overline{\mathcal{M}}_g$, as X varies among all stable curves of genus g ? Of course Theorem 2.3.9 does not tell us much. The following result, answering this question, is due to M. Melo; see [M07].

Proposition 2.4.3. *There exists an Artin stack $\overline{\mathcal{P}}_{g-1,g}$ with a non representable morphism to $\overline{\mathcal{M}}_g$, such that for every $X \in \overline{\mathcal{M}}_g$ and every smoothing $f : \mathcal{X} \rightarrow B$ of it, the stack $\overline{\mathcal{P}}_{g-1,g} \times_{\overline{\mathcal{M}}_g} B$ admits a canonical proper B -morphism onto $\overline{P_f^{g-1}}$.*

Remark 2.4.4. In particular, the stack $\overline{\mathcal{P}}_{g-1,g}$ is not Deligne-Mumford. We like to interpret this phenomenon as a reflection of the fact that $\overline{P_X^{g-1}}$, being of D -type, does not have the best moduli properties that one may hope for. Recall in fact that $\overline{P_X^{g-1}}$ has fewer components than the Néron model (by 2.4.1 (iv)). This tells us that some multidegree classes are not “finely” represented by points in $\overline{P_X^{g-1}}$ (see Example 2.4.5).

However, if we restrict our attention to the moduli scheme of automorphism-free stable curves, $\overline{\mathcal{M}}_g^0$ (so that there is a universal family $\overline{\mathcal{C}}_g \rightarrow \overline{\mathcal{M}}_g^0$), there does exist a scheme

$$(7) \quad \overline{P}_{g-1,g} \longrightarrow \overline{\mathcal{M}}_g^0$$

whose fiber over every curve X is the (canonical) $\overline{P_X^{g-1}}$.

Example 2.4.5. Consider the curve $X_\delta = C_1 \cup C_2$ and assume that $\delta \geq 2$ (notation in Example 2.1.5). Then $\overline{P_X^{g-1}}$ has $\delta - 1$ irreducible components each of which contains a copy of $J(X)$ as a dense open subset. So, we seem to have lost a multidegree class (cf. example 2.2.7)!

What actually happens is that there is one multidegree class, call it $\mu_0 \in \Delta_X^{g-1}$, such that line bundles having multidegree of class μ_0 are represented by points in the boundary of $\overline{P_X^{g-1}}$. Furthermore, different such line bundles get identified.

The simplest case when that happens is $\delta = 2$. $\overline{P_X^{g-1}}$ is thus irreducible, and it turns out to contain a dense open subset (equal to its smooth locus) naturally identified to $\text{Pic}^{(g_1, g_2)} X$ (see Proposition 2.4.9 and Definition 2.4.7). We shall henceforth identify the smooth locus of $\overline{P_X^{g-1}}$ with $\text{Pic}^{(g_1, g_2)} X$.

The class μ_0 (defined above) is thus $\mu_0 = [(g_1 - 1, g_2 + 1)] = [(g_1 - 1 - 2n, g_2 + 1 + 2n)]$, $n \in \mathbb{Z}$. Now the boundary of $\overline{P_X^{g-1}}$ is an irreducible $(g-1)$ -dimensional closed subscheme which is isomorphic to the jacobian of the normalization of X . More precisely, we have that the boundary has a canonical isomorphism

$$(8) \quad \overline{P_X^{g-1}} \setminus \text{Pic}^{(g_1, g_2)} X \cong \text{Pic}^{(g_1-1, g_2-1)} X^\nu = \text{Pic}^{g_1-1} C_1 \times \text{Pic}^{g_2-1} C_2$$

where $X^\nu = C_1 \amalg C_2$ is the normalization of X .

Let $L \in \text{Pic}^{(g_1-1, g_2+1)} X$, pick a regular smoothing $f : \mathcal{X} \rightarrow B$ of X such that there exists a line bundle \mathcal{L} on \mathcal{X} restricting to L on the closed fiber. Then there is a map $\phi : B \rightarrow \overline{P_f^{g-1}}$ such that $\phi(\text{Spec } K) = [\mathcal{L}_{\mathcal{X}_K}]$ (of course, ϕ is regular as B is a smooth curve and $\overline{P_f^{g-1}}$ is projective). By what we claimed before, ϕ must map the closed point of B to a boundary point of $\overline{P_X^{g-1}}$; which point?

The answer is, using (8), the point of the line bundle $(L_1, L_2(-p_2 - q_2))$ on X^ν , where L_i denotes the restriction of L to C_i , and p_2, q_2 are the branches of the nodes of X lying in C_2 (see also 2.4.11).

We conclude by noticing that this gives a map from $\text{Pic}^{(g_1-1, g_2+1)} X$ to the boundary of $\overline{P_X^{g-1}}$, which is surjective and has one-dimensional fibers. More details will be in 2.4.11.

2.4.6. *A stratification of $\overline{P_X^{g-1}}$.* We need to recall what a *semistable*, equivalently a *balanced*, multidegree is. The two notions, “semistable” and “balanced”, coincide; as the first is more common when the total degree is $g-1$, we shall use that one.

Definition 2.4.7. Let $\underline{d} \in \mathbb{Z}^\gamma$ be such that $|\underline{d}| = g-1$. Then \underline{d} is *semistable* (resp. *stable*) on X if for every connected subcurve $Z \subsetneq X$ we have $d_Z \geq p_a(Z) - 1$ (resp. $d_Z > p_a(Z) - 1$), where $d_Z = \sum_{C_i \subset Z} d_i$ denotes the total degree of the restriction of \underline{d} to Z and $p_a(Z)$ the arithmetic genus of Z .

If Y is a nodal, disconnected curve, we say that a multidegree is semistable or stable if it is so on every connected component of Y .

We denote by $\Sigma^{ss}(X)$ (resp. by $\Sigma(X)$) the set of semistable (resp. stable) multidegrees on X .

If X is irreducible, then of course $\Sigma^{ss}(X) = \Sigma(X) = \{g-1\}$.

It is easy to check that $\Sigma^{ss}(X)$ is finite and not empty, whereas $\Sigma(X)$ may be empty (see example 2.4.8).

It is known (cf. [C05] Prop. 4.12) that every class in Δ_X^{g-1} has some semistable representative.

Example 2.4.8. If X_δ is the vine curve of example 2.1.5, then

$$\#\Sigma^{ss}(X_\delta) = \delta + 1 \quad \text{and} \quad \#\Sigma(X_\delta) = \delta - 1.$$

More precisely

$$\Sigma^{ss}(X_\delta) = \{(g_1 - 1, g_2 - 1 + \delta), \dots, (g_1 - 1 + \delta, g_2 - 1)\}$$

and $\Sigma(X_\delta) = \{(g_1, g_2 - 2 + \delta), \dots, (g_1 - 2 + \delta, g_2)\}$. In particular, $\Sigma(X_\delta)$ is empty if and only if $\delta = 1$.

If $X^\nu = C_1 \amalg C_2$ is the normalization of X (so that X^ν is disconnected), one easily checks that

$$(9) \quad \Sigma^{ss}(C_1 \amalg C_2) = \Sigma(C_1 \amalg C_2) = \{(g_1 - 1, g_2 - 1)\}.$$

Here is a concise way of describing $\overline{P_X^{g-1}}$ (for the proof, one may choose any of the constructions [OS79], [C94], [S94] and use their equivalence established in [Al04] (cf. Theorem 2.4.1).

Proposition 2.4.9. *The points of $\overline{P_X^{g-1}}$ bijectively parametrize all line bundles having stable multidegree on all partial normalizations of X (including X itself).*

More precisely, for every subset $S \subset X_{\text{sing}}$ consider the partial normalization $X_S^\nu \rightarrow X$ of X at exactly S . Consider now $\Sigma(X_S^\nu)$; if it is nonempty let \underline{d} be a multidegree in it (note that $|\underline{d}| = p_a(X_S^\nu) - 1$) and consider $\text{Pic}^{\underline{d}} X_S^\nu$. Then there exists a canonical injective morphism $\epsilon_S^{\underline{d}} : \text{Pic}^{\underline{d}} X_S^\nu \hookrightarrow \overline{P_X^{g-1}}$ whose image we denote

$$P_S^{\underline{d}} := \epsilon_S^{\underline{d}}(\text{Pic}^{\underline{d}} X_S^\nu) \subset \overline{P_X^{g-1}}.$$

The sets $P_S^{\underline{d}}$ form a stratification of $\overline{P_X^{g-1}}$ in disjoint strata, i.e. we have $\overline{P_X^{g-1}} = \coprod_{S \subset X_{\text{sing}}} \coprod_{\underline{d} \in \Sigma(X_S^\nu)} P_S^{\underline{d}}$. More details can be found in [C07].

The explicit description of $\epsilon_S^{\underline{d}}$ depends on how we choose to describe $\overline{P_X^{g-1}}$ functorially (whether we use rank-1 torsion free sheaves, line bundles on semistable curves, cell decompositions...). In the next example we will use line bundles on semistable curves.

Example 2.4.10. Let $X = X_\delta = C_1 \cup C_2$. If we interpret $\overline{P_X^{g-1}}$ as parametrizing line bundles on blow-ups of X , then the map $\epsilon_S^{\underline{d}}$ is described as follows. If $S = \emptyset$ and \underline{d} is a stable multidegree on S (in particular, $\delta \geq 2$, by example 2.4.8), then $\epsilon_S^{\underline{d}}$ is the identity map.

If $S = \{n\}$ is one node, let X_S^ν be the normalization of X at n and $\widehat{X}_S = X_S^\nu \cup C_3$ be the curve obtained by joining the two branches over n by a smooth rational curve C_3 . Let $\underline{d} \in \Sigma(X_S^\nu)$, so that $|\underline{d}| = p_a(X_S^\nu) - 1 = g - 2$. Now $\epsilon_S^{\underline{d}}$ maps a line bundle $L \in \text{Pic}^{\underline{d}} X_S^\nu$ to the (unique) point in $P_S^{\underline{d}} \subset \overline{P_X^{g-1}}$ corresponding to line bundles on \widehat{X}_S whose restriction to X_S^ν is L and whose restriction to $C_3 \cong \mathbb{P}^1$ is $\mathcal{O}(1)$.

Finally, if $X = X_{\text{sing}}$ then $X_S^\nu = X^\nu$ is the normalization of X . Using Example 2.4.8 we see that $\epsilon_S^{\underline{d}} = \epsilon_{X_{\text{sing}}}^{\underline{d}} = \epsilon_{X_{\text{sing}}}^{(g_1-1, g_2-1)}$. With a procedure analogous to the previous case, we get that the smallest stratum of $\overline{P_X^{g-1}}$ is isomorphic to $\text{Pic}^{g_1-1} C_1 \times \text{Pic}^{g_2-1} C_2$. Call \widehat{X} the (connected, nodal) curve obtained by blowing up every node of X so that \widehat{X} is the union of X^ν with δ copies of \mathbb{P}^1 , one for each node. Now every point ℓ of this stratum corresponds to the set of line bundles on \widehat{X} whose restriction to X^ν is a fixed line bundle L of multidegree $(g_1 - 1, g_2 - 1)$, and whose restriction to each of the remaining components is $\mathcal{O}(1)$. Thus $\ell = \epsilon_S^{\underline{d}}(L)$.

2.4.11. Functorial properties of $\overline{P_f^{g-1}}$. Fix the curve X and a regular smoothing f for it. We are going to illustrate some moduli properties of $\overline{P_f^{g-1}}$, using the same notation as for the moduli property of Pic_f (cf. 2.1.2). For every B -scheme $T \rightarrow B$ and every line bundle \mathcal{L} on \mathcal{X}_T , such that for every $t \in T$ the restriction $\mathcal{L}_{f_T^{-1}(t)}$ has semistable multidegree, there exists a canonical morphism

$$\bar{\mu}_{\mathcal{L}} : T \longrightarrow \overline{P_f^{g-1}}$$

having the following properties. The restriction of $\bar{\mu}_{\mathcal{L}}$ over the generic point is the usual moduli map to the Picard variety $\text{Pic}^{g-1} \mathcal{X}_K$. If t lies over the closed point of

B , then obviously $f_T^{-1}(t) = X$ and we are assuming that $\underline{\deg} \mathcal{L}|_{f_T^{-1}(t)} \in \Sigma^{ss}(X)$. We denote $\bar{\mu}_{\mathcal{L}}(t) = [\mathcal{L}|_{f_T^{-1}(t)}] \in \overline{P_X^{g-1}}$. What is this class, in terms on the description given in 2.4.6?

Call $L = \mathcal{L}|_{f_T^{-1}(t)}$ and \underline{d} its multidegree. If \underline{d} is stable, then there is no ambiguity: by Proposition 2.4.9 there is a point $[L]$ in $\overline{P_X^{g-1}}$ corresponding to L .

If \underline{d} is strictly semistable some cumbersome notation is needed for an arbitrary curve. Therefore, to give a more efficient explanation, we shall precisely describe only the case of a curve with two components.

So, let $X = X_{\delta} = C_1 \cup C_2$, recall (cf. example 2.4.8) that there are exactly two strictly semistable multidegrees, and they are equivalent; namely

$$(g_1 - 1, g_2 + \delta - 1) \equiv (g_1 + \delta - 1, g_2 - 1).$$

Call L_1 and L_2 the restrictions of L to C_1 and C_2 . Denote $\{p_1, \dots, p_{\delta}\} \subset C_1$ (resp. $\{q_1, \dots, q_{\delta}\} \subset C_2$) the δ points of C_1 (resp. of C_2) lying over the nodes of X .

If $\underline{\deg} L = (g_1 - 1, g_2 + \delta - 1)$ then $\bar{\mu}_{\mathcal{L}}(t)$ is the point in the stratum $P_{X_{\text{sing}}}^{(g_1-1, g_2-1)} \cong \text{Pic}^{(g_1-1, g_2-1)} X^{\nu}$ given by

$$[(L_1, L_2(-\sum_{i=1}^{\delta} q_i))]$$

(see 9 and example 2.4.8). If instead $\underline{\deg} L = (g_1 + \delta - 1, g_2 - 1)$, then

$$\bar{\mu}_{\mathcal{L}}(t) = [(L_1(-\sum_{i=1}^{\delta} p_i), L_2)].$$

3. ABEL MAPS AND THETA DIVISORS

3.1. Preliminary analysis.

3.1.1. The smooth case. Let C be a smooth curve. For every integer $d \geq 1$ the d -th Abel map is defined as follows

$$\alpha_C^d : C^d \longrightarrow \text{Pic}^d C; \quad (p_1, \dots, p_d) \mapsto \mathcal{O}_C(\sum p_i);$$

α_C^d is a regular map. It is well known that Abel maps are defined more generally for any family of smooth curves over any scheme. Moreover the image of the d -th Abel map in $\text{Pic}^d C$ is equal to the variety $W_d(C) := \{L \in \text{Pic}^d C : h^0(C, L) \neq 0\}$,

$$(10) \quad \text{Im } \alpha_C^d = W_d(C), \quad \text{and } \dim W_d(C) = \min\{d, g\}.$$

The situation is particularly interesting if $1 \leq d \leq g - 1$; then $W_d(C)$ is a proper subvariety of $\text{Pic}^d C$. Moreover, for any nonnegative integer r , the loci in $W_d(C)$ where the fiber dimension of the d -th Abel map is at least r are the Brill-Noether varieties $W_d^r(C)$ (so that $W_d(C) = W_d^0(C)$). The geometry of line bundles and linear series on a smooth curve C is encoded in the varieties $W_d^r(C)$; see [ACGH] for the general theory, from the time of Riemann to the twentieth century.

How does this picture extend to singular curves?

3.1.2. Naive approach for singular curves. Fix a degree $d \geq 1$ and a curve X . One may define a rational map

$$(11) \quad \tilde{\alpha}_X^d : \begin{array}{ccc} X^d & \dashrightarrow & \text{Pic}^d X \\ (p_1, \dots, p_d) & \mapsto & \mathcal{O}_X(\sum_1^d p_i) \end{array}$$

which is regular if all the p_i are nonsingular points of X . The above definition is the simple minded extension of the smooth curve case, and it turns out to be non-satisfactory, unless X is irreducible (see Proposition 3.1.3). To be more precise, denote by $X = \cup_{i=1}^{\gamma} C_i$ the irreducible component decomposition of X and for any $\underline{d} = (d_1, \dots, d_{\gamma}) \in \mathbb{Z}^{\gamma}$ such that $|\underline{d}| = d$ and $\underline{d} \geq 0$ (i.e. $d_i \geq 0, \forall i$), set

$$(12) \quad X^{\underline{d}} := C_1^{d_1} \times \dots \times C_{\gamma}^{d_{\gamma}};$$

more generally, for any permutation σ of the set $\{1, \dots, \gamma\}$, let

$$(13) \quad X_{\sigma}^{\underline{d}} := C_{\sigma(1)}^{d_1} \times \dots \times C_{\sigma(\gamma)}^{d_{\gamma}}.$$

Thus the $X_{\sigma}^{\underline{d}}$ are the irreducible components of $X^{\underline{d}}$. If σ is the identity we often omit it (as in (12)). Now let

$$(14) \quad \alpha_X^{\underline{d}} : X^{\underline{d}} \dashrightarrow \text{Pic}^{\underline{d}} X$$

(respectively, $\alpha_{X,\sigma}^{\underline{d}} : X_{\sigma}^{\underline{d}} \dashrightarrow \text{Pic}^{\underline{d}} X$) be the restriction of $\tilde{\alpha}_X^{\underline{d}}$ to $X^{\underline{d}}$ (respectively, to $X_{\sigma}^{\underline{d}}$). These maps are of course defined only if every d_i is nonnegative. To simplify matters, whenever they are not defined, we shall set $\text{Im } \alpha_{X,\sigma}^{\underline{d}} = \emptyset$.

We define for any curve X and any multidegree \underline{d}

$$(15) \quad W_{\underline{d}}(X) := \{L \in \text{Pic}^{\underline{d}} X : h^0(X, L) \geq 1\} \subset \text{Pic}^{\underline{d}} X$$

and $W_d(X) = \coprod_{|\underline{d}|=d} W_{\underline{d}}(X)$. In analogy with 3.1.1 we ask what is the relation between $\alpha_X^{\underline{d}}$ and $W_{\underline{d}}(X)$. Here is where the first type of pathologies appears, as the following result demonstrates.

For any component C_i of X denote by g_i its arithmetic genus and by $\delta_i = \#(C_i \cap \overline{X \setminus C_i})$. We leave it to the reader to find the analogue for $\alpha_{X,\sigma}^{\underline{d}}$.

Proposition 3.1.3. *Let X be a curve of genus $g \geq 2$ and $\underline{d} \in \mathbb{Z}^{\gamma}$ with $|\underline{d}| = d \geq 1$.*

(1) *If there exists an i such that $d_i \geq g_i + \delta_i$, then $\dim W_{\underline{d}}(X) = g$ and $\dim \text{Im } \alpha_X^{\underline{d}} \leq d - 1$.*

(2) *Assume that $|\underline{d}| = g - 1$.*

(a) *If \underline{d} is not semistable, then $\dim W_{\underline{d}}(X) = g$ and $\dim \text{Im } \alpha_X^{\underline{d}} \leq g - 2$.*

(b) *If \underline{d} is semistable, then $\dim \text{Im } \alpha_X^{\underline{d}} = \dim W_{\underline{d}}(X) = g - 1$.*

Proof. Part (a) is a special case of Part (1). Part (b) combines some results of Beauville, namely Lemma 2.1 and Prop. 2.2 in [B77], with some of [C07] (Prop. 3.6).

Let us prove part (1). Assume that $d_i \geq g_i + \delta_i$; then for every $L_i \in \text{Pic}^{d_i} C_i$ we have $h^0(C_i, L_i) \geq \delta_i + 1$. Therefore for every $L \in \text{Pic}^{\underline{d}} X$, L admits some global section that does not vanish on C_i . Hence $h^0(X, L) \geq 1$ and $W_{\underline{d}}(X) = \text{Pic}^{\underline{d}} X$. This proves the first statement of part (1).

For the second, suppose that $\alpha_X^{\underline{d}}$ is defined (i.e. that $\underline{d} \geq 0$). Call $C'_i = \overline{X \setminus C_i}$ the complementary curve of C_i and let $\nu_i : C_i \amalg C'_i \rightarrow X$ be the normalization of X at $C_i \cap C'_i$. Let $L \in \text{Im } \alpha_X^{\underline{d}}$ and set $M := \nu_i^* L$, so that M is determined by a pair of line bundles $L_i \in \text{Pic } C_i$ and $L'_i \in \text{Pic } C'_i$. We have that $h^0(C_i, L_i) \geq \delta_i + 1$ (by what we said before) and $h^0(C'_i, L'_i) \geq 1$ because $L \in \text{Im } \alpha_X^{\underline{d}}$. Therefore $h^0(M) \geq \delta_i + 1 + 1 = \delta_i + 2$; this implies that

$$h^0(X, L) \geq h^0(M) - \delta_i \geq 2.$$

We conclude that the fibers of the map α_X^d have dimension at least 1, and hence that $\dim \text{Im } \alpha_X^d \leq \dim X^d - 1 = d - 1$ as wanted. \blacksquare

3.2. Abel maps in degree $g - 1$.

3.2.1. Defining Abel maps by specialization. Proposition 3.1.3 indicates that dealing with Abel maps of reducible curves is a delicate matter and more sophisticated tools may be needed. Therefore, we shall now define Abel maps for a curve X after a smoothing f of X is given, and using compactified Picard schemes. In other words, we shall add some variational data, defining Abel maps for a so called “enriched curve”, represented by a pair (X, f) . Furthermore we shall use a compactified Picard scheme as the image space of our Abel maps. This will give us a better behaved object, yet one that may (and will) depend on the the choice of the smoothing, and on the choice of the compactified Picard scheme.

3.2.2. Abel maps in degree $g - 1$. We first deal with the case $d = g - 1$, so that we have a canonical choice for the compactified Picard scheme (see 2.4.2).

Let X and $f : \mathcal{X} \rightarrow B$ be a curve and a regular smoothing for it. Denote by $\mathcal{X}^d = \mathcal{X} \times_B \dots \times_B \mathcal{X}$ the d -th fibered power over B . Classically, the d -th Abel map is the moduli map associated to the universal effective divisor on $\mathcal{X}^d \times_B \mathcal{X}$ (see below). We shall approach the problem in the same way. So consider the natural projection

$$\mathcal{X}^g = \mathcal{X}^{g-1} \times_B \mathcal{X} \xrightarrow{\pi} \mathcal{X}^{g-1}$$

(above and throughout this section, all schemes, maps and products are over B). The map π has $g-1$ tautological rational sections $\sigma_i(p_1, \dots, p_{g-1}) = (p_1, \dots, p_{g-1}; p_i)$ which determine a line bundle \mathcal{E} on the smooth locus of \mathcal{X}^g , i.e. introducing the (Weil) divisor

$$E = \sum_1^{g-1} \overline{\sigma_i(\mathcal{X}^{g-1})}$$

we define

$$\mathcal{E} = \mathcal{O}_{\mathcal{X}^g}(E)$$

which is locally free on an open subset of \mathcal{X}^g . Now, the π -relative multidegree of \mathcal{E} varies with the irreducible components of \mathcal{X}^{g-1} . Indeed, with the notation of (12), let $X^{\underline{d}} \subset X^{g-1} \subset \mathcal{X}^{g-1}$ be an irreducible component (so that $\underline{d} \geq 0$), then for a generic point $t \in X^{\underline{d}}$ we have

$$\underline{\deg} \mathcal{E}|_{\pi^{-1}(t)} = \underline{d}.$$

More generally, if $t \in X_{\sigma}^{\underline{d}}$ we have $\underline{\deg} \mathcal{E}|_{\pi^{-1}(t)} = (d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(\gamma)})$. Observe that the restriction of \mathcal{E} over $\text{Spec } K$ is the so-called universal effective divisor on $\mathcal{X}_K^{g-1} \times \mathcal{X}_K$, whose moduli map $\mathcal{X}_K^{g-1} \rightarrow \text{Pic}^{g-1} \mathcal{X}_K$ is the $g - 1$ -th Abel map of \mathcal{X}_K . So, we would like to complete this, associating to \mathcal{E} a map $\mathcal{X}^{g-1} \dashrightarrow \overline{P}_f^{g-1}$. With the functorial description of \overline{P}_f^{g-1} in mind (see 2.4.11), the question we need an answer for is: is \underline{d} semistable?

To better explain how to proceed we concentrate on our leading example.

Example 3.2.3. Assume that $X = X_{\delta}$ (cf. 2.1.5), so that we have a simple description of $\Sigma^{ss}(X)$ and hence of \overline{P}_X^{g-1} . The irreducible component decomposition

of X^{g-1} is

$$X^{g-1} = \bigcup_{l=1}^{g-1} (C_1^l \times C_2^{g-1-l} \cup C_2^l \times C_1^{g-1-l})$$

so that if $t \in C_1^l \times C_2^{g-1-l}$, then $\deg \mathcal{E}_{|\pi^{-1}(t)} = (l, g-1-l)$ while if $t \in C_2^l \times C_1^{g-1-l}$, then $\deg \mathcal{E}_{|\pi^{-1}(t)} = (g-1-l, l)$. Now we claim that for every $l = 0, \dots, g-1$ there exists an integer $a(l) \in \mathbb{Z}$ such that

$$(16) \quad (l, g-1-l) + (-a(l)\delta, a(l)\delta) \in \Sigma^{ss}(X).$$

Indeed, $(-a(l)\delta, a(l)\delta) \in \Lambda_X$, so $(l, g-1-l) + (-a(l)\delta, a(l)\delta) \equiv (l, g-1-l)$; as every multidegree class has a semistable representative the claim is proved.

If $(l, g-1-l)$ is semistable we choose $a(l) = 0$. Note that $a(l)$ may not be unique, but this will turn out to be irrelevant. Indeed, $a(l)$ is not unique if and only if $(l, g-1-l) + (-a\delta, +a\delta)$ is strictly semistable. From the description given in 2.4.11 one sees that the choice of $a(l)$ plays no role.

Now we define

$$(17) \quad \mathcal{L}_{g-1} = \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}^g} \left(\sum_{l=1}^{g-1} a(l) (C_1^l \times C_2^{g-1-l} \times C_1 + C_2^{g-1-l} \times C_1^l \times C_1) \right)$$

which is locally free over the smooth locus of \mathcal{X}^{g-1} . Let t be a smooth point of X^{g-1} ; by construction, if either $t \in C_1^l \times C_2^{g-1-l}$ or $t \in C_2^{g-1-l} \times C_1^l$ we have that

$$\deg(\mathcal{L}_{g-1})_{|\pi^{-1}(t)} = (l - a(l)\delta, g-1-l + a(l)\delta)$$

which is semistable, by (16). Therefore there exists a canonical rational map $\bar{\mu}_{\mathcal{L}_{g-1}} : \mathcal{X}^{g-1} \dashrightarrow \overline{P_f^{g-1}}$ (see 2.4.11).

We denote $\alpha_f^{g-1} := \bar{\mu}_{\mathcal{L}_{g-1}}$ and call it the $g-1$ -th Abel map associated to f . The restriction of α_f^{g-1} to the closed fiber is the rational map

$$(18) \quad \alpha_{f,X}^{g-1} : X^{g-1} \dashrightarrow \overline{P_X^{g-1}}.$$

This is the definition we were aiming at; so we call $\alpha_{f,X}^{g-1}$ the $g-1$ -th Abel map of X associated to f . By construction, $\alpha_{f,X}^{g-1}$ is regular at (p_1, \dots, p_{g-1}) for every $p_1, \dots, p_{g-1} \in X \setminus X_{\text{sing}}$.

We just gave the definition in the special case of a curve with two components. The general case can be dealt with using the very same procedure, paying quite a price in terms of notation. Rather than going through this, we prefer to deal with a problem that arises immediately.

3.2.4. Naturality. We now consider the following question:

Does $\alpha_{f,X}^{g-1} : X^{g-1} \dashrightarrow \overline{P_X^{g-1}}$ depend on the choice of the smoothing f ?

We shall say that $\alpha_{f,X}^{g-1}$ is *natural* if it is independent of the choice of f , i.e. if for every regular smoothings f and f' of X , we have

$$\alpha_{f,X}^{g-1} = \alpha_{f',X}^{g-1}.$$

We have defined $\alpha_{f,X}^{g-1}$ only for a vine curve $X = X_\delta$ (cf. 2.1.5), so we shall focus on this simple yet interesting case. See [B06] for the general result, valid for all stable curves. With the notation of Example 2.1.5, we have

Proposition 3.2.5. *Let $X = X_\delta$. If $\delta = 1$ then $\alpha_{f,X}^{g-1}$ is natural.*

Assume $\delta \geq 2$, then $\alpha_{f,X}^{g-1}$ is natural if and only if

$$\delta \geq g - 1 \text{ and } \{g_1, g_2\} \neq \{0, 2\}.$$

Equivalently, $\alpha_{f,X}^{g-1}$ is natural if and only if $g_i \leq 1$ for $i = 1, 2$.

Proof. The map α_f^{g-1} is defined as the moduli map $\bar{\mu}_{\mathcal{L}_{g-1}}$, and \mathcal{L}_{g-1} is a so called “twist” of $\mathcal{E} = \mathcal{O}(E)$ (i.e. \mathcal{L}_{g-1} and \mathcal{E} differ only over the closed point of B). It is clear that the restriction of \mathcal{E} to the fibers over X^{g-1} is independent of f (indeed $\mathcal{E}_{|\pi^{-1}(p_1, \dots, p_{g-1})} = \mathcal{O}_X(p_1 + \dots, p_{g-1})$).

Now consider the other factor $\mathcal{T} := \mathcal{O}_{\mathcal{X}^g} \left(\sum_{l=0}^{g-1} a(l) (C_1^l \times C_2^{g-1-l} \times C_1 + C_2^{g-1-l} \times C_1^l \times C_1) \right) = \mathcal{L}_{g-1} \otimes \mathcal{E}^{-1}$ of (17). The restriction of \mathcal{T} to the fibers of π is a twister. Now recall that, if $\delta \geq 2$, a nontrivial twister on X_δ depends on the choice of \mathcal{X} (see remark 2.2.3). On the other hand if $\delta = 1$, a twister on X is uniquely determined by its multidegree. Hence if $\delta = 1$ the map $\alpha_{f,X}^{g-1}$ does not depend on f .

Assume from now on $\delta \geq 2$. By what we said, $\alpha_{f,X}^{g-1}$ is natural iff the map is not twisted, iff $\mathcal{T} = 0$. Now, $\mathcal{T} = 0$ iff the multidegree $(l, g-1-l)$ is semistable for every $l = 0, \dots, g-1$ (by 3.2.2).

Assume $\delta \geq g-1$. Since $g = g_1 + g_2 + \delta - 1$, this is equivalent to $g_1 + g_2 \leq 2$. There are thus four cases to consider:

- 1: $g_1 = g_2 = 0$;
- 2: $g_1 = 0, g_2 = 1$;
- 3: $g_1 = g_2 = 1$ and
- 4: $g_1 = 0, g_2 = 2$.

In the first three cases, i.e. when $g_i \leq 1$ for $i = 1, 2$, one checks that $(l, g-1-l)$ is indeed semistable for every $l = 0, \dots, g-1$, so $\mathcal{T} = 0$.

In case 4 the multidegree $(g-1, 0)$ is unstable, hence \mathcal{T} is nontrivial over the component $C_1^{g-1} \subset X^{g-1} \subset \mathcal{X}^{g-1}$ and hence $\alpha_{f,X}^{g-1}$ is not natural.

Conversely, assume $\delta \leq g-2$. We use examples 2.2.7 and 2.4.8. We have that δ is equal to the number of multidegree classes. Therefore the set $\{(l, g-1-l), l = 0, \dots, g-1\}$, having cardinality $g \geq \delta + 2$, contains at least two pairs of equivalent multidegrees corresponding to two different classes. Hence at least one of such pairs contains a nonsemistable multidegree, so the map α_f^{g-1} is twisted (i.e. $\mathcal{T} \neq 0$) and $\alpha_{f,X}^{g-1}$ does depend on f . ■

3.3. Abel maps of arbitrary degree. Now we briefly illustrate what is known for a general $d \geq 1$.

3.3.1. Abel maps of degree 1. If $d = 1$ the picture is rather well understood by [AK80], [EGK00] for integral curves, and by [CE07] for reducible ones.

With an approach similar to what we used in case $d = g-1$, one constructs a map $\alpha_X^1 : X \rightarrow \overline{P}_X^1$ which turns out to be natural and, more remarkably, regular. If X is irreducible the choice of \overline{P}_X^1 is not an issue (see 2.3.6). In case X is reducible \overline{P}_X^1 can be chosen to be either the compactified Picard scheme constructed in [C94], or the one constructed in [E01]. We refer to the above mentioned papers for details.

For a smooth curve C (or a family of smooth curves) a fundamental fact is that the degree-1 Abel map is a closed embedding. Moreover, recall that, after a point

$p_0 \in C$ is chosen, we can “translate” α_C^1 by composing it with the isomorphism $\text{Pic}^1 C \rightarrow \text{Pic}^0 C$ mapping $[L]$ to $[L(-p_0)]$. In this way we get the “Abel-Iacobi” map

$$\alpha_{p_0} : C \hookrightarrow \text{Pic}^0 C = J(C)$$

mapping $p \in C$ to $[\mathcal{O}(p - p_0)]$. Recall that α_{p_0} is a closed embedding, endowed with a universal property with respect to mappings to Abelian varieties; namely every map $h : C \rightarrow A$ of C to an Abelian variety A factors uniquely through α_{p_0} , i.e. $h = h' \circ \alpha_{p_0}$ for a unique $h' : A \rightarrow J(C)$. In particular $\alpha_{p_0}(C)$ generates $J(C)$ as a group.

What if X is singular? Then the degree 1-Abel map of X , and its translates mapping to the compactified jacobian (see above), turn out to be closed embeddings, unless it cannot possibly be so for simple reasons. More precisely, it is not hard to see that if X contains a smooth rational component L that is attached to its complement only in separating nodes of X , the degree-1 Abel map must contract L to a point (note that $\text{Pic}^d L$ is a point for all d). These are the only cases where the map fails to be an embedding. In particular, if X is free from separating nodes, its degree-1 Abel map is a closed embedding (see [CCE07]).

3.3.2. Abel maps of irreducible curves. The situation is more subtle for higher d , where not much is known about compactifying Abel maps, even when there is no problem in defining them simply as rational maps. For example, for irreducible curves there exists up to isomorphism a unique compactified Picard scheme and one can proceed as we did for $d = g - 1$. Now it is easy to see that naturality is not an issue (see below). So consider the rational map

$$\alpha_X^d : X^d \dashrightarrow \overline{P_X^d}$$

mapping (p_1, \dots, p_d) to $[\mathcal{O}_X(\sum p_i)]$ if all the p_i are smooth points of X . It is known that for $d \geq 2$ there is no hope that the map be regular; indeed one needs to modify X^d (blowing it up) to extend it. The explicit description of such a modification is known in very few cases, for $d = 2$; see [Co06].

A different approach which replaces X^d with the Hilbert scheme of length d subscheme on X is pursued in [EK05].

3.3.3. Naturality for arbitrary d . Let us now consider reducible curves. In the previous section we have dealt with the case $d = g - 1$, considered Abel maps as rational maps, and have seen that there are very few curves for which this map is natural (see Proposition 3.2.4)

For arbitrary $d \geq 2$, the issue is complicated by the fact that, as we said, there exist different compactified Picard schemes. Anyways, after some choice for $\overline{P_X^d}$ is made, one can work by specialization and produce a rational map

$$(19) \quad \alpha_{f,X}^d : X^d \dashrightarrow \overline{P_X^d}$$

similarly to what we did in case $g - 1$. We omit any explicit definition to avoid choosing a specific compactified Picard scheme. In fact, the point of this section is precisely to describe some facts that depend only on the type of compactified Picard scheme, and that can be proved without using the details of any specific construction. Namely, what about naturality of $\alpha_{f,X}^d$?

We shall concentrate on compactified Picard schemes of N-type. Observe that if \overline{P}_f^d is of Néron type, the existence of a canonical map

$$\alpha_f^d : \mathcal{X}^d \dashrightarrow \overline{P}_f^d$$

is guaranteed by the Néron mapping property. Indeed, denote by $\dot{\mathcal{X}}^d \subset \mathcal{X}^d$ the smooth locus of $\mathcal{X}^d \rightarrow B$. The generic fiber \mathcal{X}_K^d of $\dot{\mathcal{X}}^d \rightarrow B$ has its own Abel map $\alpha_{\mathcal{X}_K}^d : \mathcal{X}_K^d \rightarrow \text{Pic}^d \mathcal{X}_K$. The Néron mapping property yields a unique map

$$N(\alpha_{\mathcal{X}_K}^d) : \dot{\mathcal{X}}^d \longrightarrow N_f^d.$$

Now, as \overline{P}_f^d is of Néron type, we can consider the map $n_f^{-1} : N_f^d \rightarrow \overline{P}_f^d$ (cf. Definition 2.3.5). Composing, we get a regular map

$$\dot{\mathcal{X}}^d \xrightarrow{N(\alpha_{\mathcal{X}_K}^d)} N_f^d \xrightarrow{n_f^{-1}} \overline{P}_f^d$$

whose restriction to X^d is the rational map $\alpha_{f,X}^d$ introduced in (19).

We need to introduce a combinatorial invariant:

Definition 3.3.4. Let X be a (connected, nodal) curve and Γ_X its dual graph. The *essential graph* of X is the graph $\overline{\Gamma}_X$ obtained from Γ_X by eliminating every loop and by contracting every separating edge to a point.

The “edge connectivity” of $\overline{\Gamma}_X$, (i.e. the minimal number of edges that one needs to remove to disconnect $\overline{\Gamma}_X$) will be called the *essential connectivity* of X and denoted $\epsilon(X)$.

Example 3.3.5. If $X = X_\delta$ with $\delta \geq 2$ we have that $\Gamma_X = \overline{\Gamma}_X$; if $\delta = 1$ then $\overline{\Gamma}_X$ is a point. We have

$$\epsilon(X_\delta) = \begin{cases} \delta & \text{if } \delta \geq 2 \\ +\infty & \text{if } \delta = 1. \end{cases}$$

If X is irreducible, $\overline{\Gamma}_X$ is a point and $\epsilon(X) = +\infty$.

The following follows immediately from theorem 1.5 in [C06].

Proposition 3.3.6. *Let X be a curve and \overline{P}_X^d be a compactified Picard scheme of Néron-type. Let $\alpha_{f,X}^d : X^d \dashrightarrow \overline{P}_X^d$ be the Abel map associated to a regular smoothing f of X . Then $\alpha_{f,X}^d$ is natural only if $d < \epsilon(X)$.*

Remark 3.3.7. Using Theorem 2.3.9 (which ensures that every curve admits some compactified Picard scheme on N-type) one sees that the locus in \overline{M}_g of curves that fail to admit a natural Abel map in degree $d \geq 2$ is quite large, i.e. it has codimension equal to 2. Indeed, by Proposition 3.3.6 and Example 3.3.5, the curve X_δ with $\delta = 2$ does not admit any natural d -th Abel map, unless $d = 1$.

3.3.8. Abel maps for families over a higher dimensional base. Let $h : \mathcal{C} \rightarrow S$ be a family of smooth curves over any scheme S . As we mentioned in 3.1.1, for every $d \geq 1$ there exists a (relative) d -th Abel map

$$\alpha_h^d : \mathcal{C}^d \rightarrow \text{Pic}_h^d = \text{Pic}_{\mathcal{C}/S}^d$$

where \mathcal{C}^d denotes the fibered power over S and Pic_h^d the relative Picard scheme in degree d (a smooth projective scheme over S , as all fibers of h are smooth).

In particular we can apply that to the universal family of smooth curves, and ask whether the construction extends to stable curves.

More precisely, assume $g \geq 3$ and let $h_g : \mathcal{C}_g \rightarrow M_g^0$ (respectively $\overline{h}_g : \overline{\mathcal{C}}_g \rightarrow \overline{M}_g^0$) be the universal family of smooth (respectively stable) curves of genus g over the moduli space of automorphism-free curves. Let $\mathcal{C}_g^d \rightarrow M_g^0$ and $\overline{\mathcal{C}}_g^d \rightarrow \overline{M}_g^0$ be the d -th fibered powers over the respective bases. Of course \overline{h}_g is not a smooth morphism, so, let us introduce its smooth locus, denoted $\widetilde{h}_g : \widetilde{\mathcal{C}}_g \rightarrow \overline{M}_g^0$ and its d -th fibered power

$$\overline{\mathcal{C}}_g^d \supset \widetilde{\mathcal{C}}_g^d \longrightarrow \overline{M}_g^0.$$

Consider now the d -th Abel map for the universal smooth curve

$$(20) \quad \alpha_{h_g}^d : \mathcal{C}_g^d \longrightarrow \mathrm{Pic}_{h_g}^d = \mathrm{Pic}_g^d,$$

where $\mathrm{Pic}_g^d = \mathrm{Pic}_{\mathcal{C}_g/M_g^0}^d$ is a standard quick notation for the universal Picard variety over M_g^0 . Assume now that d is such that every $X \in \overline{M}_g$ has a compactified degree- d Picard scheme of N-type, and that such compactified Picard schemes glue together over \overline{M}_g ; by Theorem 2.3.9 this amounts to assume that $(d - g + 1, 2g - 2) = 1$.

Recall that there exists a compactification $\overline{P}_{d,g}^{Ner} \rightarrow \overline{M}_g$ for Pic_g^d , which is the moduli scheme for the stack $\overline{\mathcal{P}}_{d,g}^{Ner}$ (see 2.3.9). So we ask: does the map (20) extend to a regular map $\widetilde{\mathcal{C}}_g^d \longrightarrow \overline{P}_{d,g}^{Ner}$?

From Proposition 3.3.6, one sees that, if $d \geq 2$, the answer is *no*.

By contrast, in case $d = 1$ the answer is *yes*; in fact E. Esteves proved that the map $\alpha_{h_g}^1$ extends to a regular map over the whole of $\overline{\mathcal{C}}_g$; and, more generally, that this holds in its stack version ([E07]).

Finally, a similar reasoning works if $d = g - 1$. In such a case we need Proposition 3.2.5 rather than Proposition 3.3.6, and the scheme $\overline{P}_{g-1,g}$ (see (7)) parametrizing compactified Picard schemes in degree $g - 1$. Again we obtain that *there exists no regular map from $\widetilde{\mathcal{C}}_g^{g-1}$ to $\overline{P}_{g-1,g}$* .

3.4. The theta divisor of \overline{P}_X^{g-1} .

3.4.1. Theta divisor of a smooth curve. Let us consider a smooth curve C of genus $g \geq 2$. Using the set up of 3.1.1, the locus of effective line bundles in $\mathrm{Pic}^{g-1} C$ is a divisor (by (10)), called the *theta divisor* of C , and denoted

$$\Theta(C) := W_{g-1}(C) \subset \mathrm{Pic}^{g-1} C.$$

Many properties of the curve C are encoded in the geometry of $\Theta(C)$. For example, assume $g \geq 4$, then C is hyperelliptic iff $\dim \Theta(C)_{\mathrm{sing}} = g - 3$ (where $\Theta(C)_{\mathrm{sing}}$ is the singular locus of $\Theta(C)$); if C is not hyperelliptic, then $\dim \Theta(C)_{\mathrm{sing}} = g - 4$; in both cases $\Theta(C)_{\mathrm{sing}}$ has pure dimension and it is irreducible if C is hyperelliptic.

On the other hand $\Theta(C)_{\mathrm{sing}}$ is precisely described in terms of special line bundles on C (a line bundle is called “special” if its space of global sections has dimension higher than expected). Indeed, the Riemann singularity theorem states that for every $L \in \Theta(C)$, the multiplicity of $\Theta(C)$ at L is equal to $h^0(C, L)$. In particular we get

$$\Theta(C)_{\mathrm{sing}} = W_{g-1}^1(C) = \{L \in \mathrm{Pic}^{g-1} C : h^0(C, L) \geq 2\}.$$

Observe that, as we saw in (10), $\Theta(C)$ is the image of the $(g-1)$ -th Abel map, therefore it is irreducible. Finally, recall that $\Theta(C)$ is a principal polarization on $\text{Pic}^{g-1} C$ (see [ACGH]).

3.4.2. Theta divisor of a generalized jacobian. Now suppose that X is singular. We may define, using the notation (15),

$$\widetilde{\Theta}(X) := \{L \in \text{Pic}^{g-1} X : h^0(X, L) \geq 1\} = \coprod_{|\underline{d}|=g-1} W_{\underline{d}}(X).$$

We know already, by Proposition 3.1.3, that $\widetilde{\Theta}(X)$ has only finitely many components of the right dimension (i.e. of dimension $g-1$); indeed if \underline{d} is not semistable we have $W_{\underline{d}}(X) = \text{Pic}^{\underline{d}} X$. So $\widetilde{\Theta}(X)$ is not a divisor, if X is reducible.

Assume that \underline{d} is semistable, then $W_{\underline{d}}(X)$ has dimension equal to $g-1$; moreover the same holds for the image of the Abel map $\alpha_{\underline{d}}^X$. Since the Abel map is only a rational map, let us denote its closure by $A_{\underline{d}}(X) := \overline{\text{Im } \alpha_{\underline{d}}^X} \subset \text{Pic}^{\underline{d}} X$. We now ask what the relation between $W_{\underline{d}}(X)$ and $A_{\underline{d}}(X)$ is. Are they equal (as for a smooth curve)? If \underline{d} is strictly semistable, the answer is *no*, as we shall see in the next example.

Example 3.4.3. Let $X = X_{\delta} = C_1 \cup C_2$ and consider the strictly semistable multidegree (notation in Example 2.1.5)

$$\underline{d} = (g_1 - 1, g_2 - 1 + \delta).$$

$W_{\underline{d}}(X)$ has dimension $g-1$ (by Prop. 3.1.3). We shall prove that $W_{\underline{d}}(X)$ has two different irreducible components, one of which (necessarily by Prop. 3.1.3) coincides with $A_{\underline{d}}(X)$.

Consider the normalization $\nu : X^{\nu} = C_1 \amalg C_2 \rightarrow X$. For any $L \in \text{Pic}^{\underline{d}} X$ denote by $M = \nu^* L$. Since X^{ν} is disconnected, M is uniquely determined by its restrictions L_1 and L_2 to C_1 and C_2 . So, pick a pair $(L_1, L_2) \in \text{Pic}^{(g_1-1, g_2-1+\delta)} X^{\nu}$. If $h^0(C_1, L_1) \neq 0$, i.e. if $L_1 \in \Theta(C_1)$, then (as every $L_2 \in \text{Pic}^{g_2-1+\delta} C_2$ has $h^0(C_2, L_2) \geq \delta$ by Riemann-Roch)

$$h^0(X^{\nu}, M) = h^0(C_1, L_1) + h^0(C_2, L_2) \geq \delta + 1.$$

Therefore for every $L \in \text{Pic} X$ such that $\nu^* L = (L_1, L_2) = M$ we have that $h^0(X, L) \geq h^0(X^{\nu}, M) - \delta \geq 1$. We conclude that $W_{\underline{d}}(X)$ contains a closed subset W_1 given as

$$W_1 = (\nu^*)^{-1}(\Theta(C_1) \times \text{Pic}^{g_2-1+\delta} C_2),$$

where

$$\nu^* : \text{Pic}^{\underline{d}} X \longrightarrow \text{Pic}^{g_1-1} C_1 \times \text{Pic}^{g_2-1+\delta} C_2$$

is the pull-back map. The fibers of ν^* are irreducible of dimension $\delta-1$ (by 2.1.3), hence W_1 is irreducible of dimension

$$\dim W_1 = (g_1 - 1) + g_2 + (\delta - 1) = g - 1,$$

so that W_1 is an irreducible component of $W_{\underline{d}}(X)$.

Now suppose that $h^0(C_1, L_1) = 0$. Call q_1, \dots, q_{δ} the points of C_2 mapping to the nodes of X . If $h^0(C_2, L_2(-q_1 - \dots - q_{\delta})) \neq 0$, every L such that $\nu^* L = (L_1, L_2)$ lies in $W_{\underline{d}}(X)$. The locus $D \subset \text{Pic}^{g_2-1+\delta} C_2$ of line bundles L_2 such that $h^0(C_2, L_2(-q_1 - \dots - q_{\delta})) \neq 0$ is a so-called *translate* of $\Theta(C_2)$. Indeed, consider

the isomorphism $u : \text{Pic}^{g_2-1} C_2 \longrightarrow \text{Pic}^{g_2-1+\delta} C_2$ mapping N to $N(+q_1 + \dots + q_\delta)$. Then D is equal to $u(\Theta(C_2))$. In particular D is irreducible of dimension $g_2 - 1$. We just constructed a second irreducible component W_2 of $W_{\underline{d}}(X)$:

$$W_2 = (\nu^*)^{-1}(\text{Pic}^{g_1-1} C_1 \times D).$$

Arguing as for W_1 , we see that W_2 is irreducible of dimension $g_1+g_2-1+\delta-1 = g-1$.

We conclude that $W_{\underline{d}}(X) = W_1 \cup W_2$ as wanted (we leave it to the reader to show that in $W_{\underline{d}}(X)$ there is nothing else).

The previous example had \underline{d} strictly semistable. If \underline{d} is stable, $W_{\underline{d}}(X)$ turns out to be irreducible (see [C07]) and equal to $A_{\underline{d}}(X)$.

3.4.4. Compactifying the Theta divisor. The history follows a similar pattern as for compactified Picard schemes and Abel maps, with the case of irreducible curves being solved much earlier. As we mentioned (cf. Theorem 2.4.1), the case of an irreducible curve was treated in [So94] and [E97], where it is proved that, on the compactified jacobian of X , there exists a Cartier, ample divisor $\Theta(X)$ and that, just as for smooth curves, $3\theta(X)$ is very ample (in the latter paper). For a reducible X , the situation remained open for some time (despite some breakthroughs in [B77]) partly because of the diversity of the existing compactified Picard schemes. The proof of the fact that $\overline{P_X^{g-1}}$ (the canonical one, see 2.4.2) has a theta divisor which is Cartier and ample is due to [A104]. A new element in his construction is the use of semiabelian group actions, which also yields a placement of the pair $(\overline{P_X^{g-1}}, \Theta(X))$ within the degeneration theory of principally polarized abelian varieties (see [A104] section 5, in particular 5.4).

Comparing to the smooth case, to the rich picture we partly sketched in 3.4.1, the subject opens up now with a variety of interesting issues and unsolved problems. Indeed, not much is known about the geometry of the theta divisor of a singular curve and about its interplay with the geometry of the curve and its jacobian.

3.4.5. The Theta divisor of $\overline{P_X^{g-1}}$. The definition of $\Theta(X)$ is given by the non vanishing of h^0 . To do this properly we must say what the points of $\overline{P_X^{g-1}}$ parametrize. There are at least two good options: semistable torsion free sheaves of rank 1 and degree $g-1$ (see [A104] for example); or stable line bundles on the partial normalizations of X . We will use the second option, for consistency with Proposition 2.4.9, and because this description enables us to describe examples quite easily.

We stated in Proposition 2.4.9 that a point in $\overline{P_X^{g-1}}$ corresponds naturally to a line bundle on some normalization of X , and we described an instance of this correspondence in 2.4.10. We shall here use the same notation. Let us denote by ℓ a point in $\overline{P_X^{g-1}}$ and by $L \in \text{Pic}^{\underline{d}} X_S^\nu$ the corresponding point (recall that X_S^ν is the normalization of X at $S \subset X_{\text{sing}}$) so that $\ell = \epsilon_S^{\underline{d}}(L)$. We set

$$(21) \quad h^0(\ell) := h^0(X_S^\nu, L),$$

now we define

$$(22) \quad \Theta(X) := \{\ell \in \overline{P_X^{g-1}} : h^0(\ell) \neq 0\}.$$

The fact that the above set-theoretic definition coincides with the definition of the divisor $\Theta(X)$ studied in [So94], [E97] and [A104] is shown in [C07]. As we said, description (22) is good to easily give examples as we are going to do next, consistently with the stratification of Proposition 2.4.9.

Example 3.4.6. If X is irreducible with δ nodes, and $g \geq 1$, then $\Theta(X)$ is irreducible and

$$(23) \quad \Theta(X) \cong W_d(X) \prod_{i=1}^{\delta-1} \left(\prod_{\substack{S_i \subset X_{\text{sing}} \\ \#S_i=i}} W_{d-i}(X_{S_i}^\nu) \right) \prod \Theta(X^\nu).$$

Each stratum $W_{d-i}(X_{S_i}^\nu)$ is irreducible of dimension $g - 1 - i$.

A characterization of stable curves X such that $\Theta(X)$ is irreducible is [C07], 4.2.2. Reducible curves whose theta divisor is irreducible do exist. Here is what happens for the vine curve X_δ .

Example 3.4.7. Let $X = X_\delta$ (as in 2.1.5).

If $\delta = 1$, then $\Theta(X)$ is reducible:

$$(24) \quad \Theta(X_1) \cong \Theta(C_1) \times \text{Pic}^{g_2-1} C_2 \cup \Theta(C_2) \times \text{Pic}^{g_1-1} C_1.$$

If $\delta = 2$ then $\Theta(X)$ is irreducible and

$$(25) \quad \Theta(X_2) \cong W_{(g_1, g_2)}(X) \prod \left(\Theta(C_1) \times \text{Pic}^{g_2-1} C_2 \cup \Theta(C_2) \times \text{Pic}^{g_1-1} C_1 \right).$$

If $\delta \geq 3$, then $\Theta(X)$ is reducible, having $\delta - 1$ irreducible components.

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