Moduli spaces in algebraic and tropical geometry: tropicalizations

Lectures by Lucia Caporaso

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Contents

1. Smooth curves and their moduli

1.1. **Smooth curves and their Picard group.** Unless otherwise stated, by *curve* we mean a reduced, connected, projective variety (not necessarily irreducible) of dimension one, defined over $\mathbb{C}$.

Let C be a smooth curve. The *Picard group*, $\text{Pic}(C)$, of C can be defined in various equivalent ways, and we shall use each time the most convenient one. First, it is the set of divisors on C modulo linear equivalence, in symbols

$$\text{Pic}(C) = \text{Div}(C)/\sim .$$

Next, it is the set of isomorphism classes of line bundles (equivalently, of invertible sheaves) on C

$$\text{Pic}(C) = \{\text{Line bundles on } C\}/\cong .$$

Line bundles and their isomorphism classes are denoted in the same way. For $D \in \text{Div}(C)$ we denote by $\mathcal{O}(D)$ the corresponding line bundle. As is well known, in $D \sim D'$ then $\mathcal{O}(D) \cong \mathcal{O}(D')$.

$\text{Pic}(C)$ is an abelian group, with the trivial bundle, $\mathcal{O}_C$, as neutral element. With divisors, for the group operation one uses the additive notation, i.e. $[D], [D'] \in \text{Pic}(C)$ with $D, D' \in \text{Div}(C)$ then $[D]+[D'] := [D+D']$, whereas for line bundles one uses the multiplicative notation: $L, L' \in \text{Pic}(C)$ then $LL' := L \otimes L'$.

We have a surjective homomorphism

$$\deg : \text{Pic}(C) \longrightarrow \mathbb{Z}$$

such that if $D = \sum_{p \in C} n_p p$ then $\deg([D]) = \deg D = \sum_{p \in C} n_p$. Its kernel is a remarkable subgroup, written

$$\mathrm{Pic}^0(C) = \{L \in \mathrm{Pic}(C) : \deg L = 0\},$$

also called the *Jacobian variety* of C and denoted by $\mathrm{Jac}(C)$. It is an abelian variety. i.e. a projective algebraic group.

The following is well known

Theorem 1.1.1. *Let C be a smooth curve, then*

$$\mathrm{Pic}(C) \cong \mathbb{Z} \Longleftrightarrow C \cong \mathbb{P}^1.$$

For any divisor $D \in \mathrm{Div}(C)$ the set of effective divisors linearly equivalent to D is written as follows

$$|D| := \{E \in \mathrm{Div}(C) : E \geq 0, E \sim D\}.$$

If $|D|$ is not empty, then it is identified with a projective space

$$|D| = \mathbb{P}^{r(D)} = \mathbb{P}(H^0(C, D))$$

where $H^0(C, D) = H^0(C, \mathcal{O}(D))$ is the vectors space of global sections of $\mathcal{O}(D)$. Its dimension is written $h^0(C, D)$ and, of course,

$$r(D) = h^0(C, D) - 1$$

so that

$$|D| = \emptyset \Longleftrightarrow h^0(C, D) = 0$$

If $|D| \neq \emptyset$ we have a regular map

$$\phi_D : C \longrightarrow \mathbb{P}^{r(D)}.$$

Let $r(D) \geq 1$ and suppose $|D|$ has no base points (i.e. $h^0(C, D - p) = h^0(C, D) - 1$ for every $p \in C$); then for any hyperplane $H \subset \mathbb{P}^{r(D)}$ (an effective divisor on $\mathbb{P}^{r(D)}$), the divisor on C given by the pull-back of H, satisfies

$$\phi_D^* H \in |D|;$$

conversely, every $E \in |D|$ is obtained in this way.

Remark 1.1.2. For the trivial line bundle, $\mathcal{O}_C$, we have $\deg \mathcal{O}_C = 0$ and $h^0(C, \mathcal{O}_C) = 1$. Moreover, these two conditions characterize $\mathcal{O}_C$ among line bundles on C.

Remark 1.1.3. Let us introduce the most important line bundle on a curve C, the *canonical line bundle*, denoted by K_C and defined as the dual of the tangent bundle, T_C, of C,

$$K_C := T_C^*.$$

The *genus* of C is defined as follows

$$g_C := h^0(C, K_C);$$

we have

$$\deg K_C = 2g_C - 2$$

and K_C is the unique line bundle on C satisfying these two conditions.

If $k = \mathbb{C}$ the genus defined above is equal to the topological genus of the surface over $\mathbb{R}$ underlying C. Let us write S_C for the real surface underlying C; this is a compact, connected and orientable topological manifold of dimension 2: compactness and connectedness follow from the definition. Orientability follows from the fact that, in the analytic topology, S_C is covered by open subsets holomorphic to open subsets of $\mathbb{C}$, and holomorphic maps are conformal, hence preserve the orientation. So the orientation of $\mathbb{C}$ induces an orientation on S_C.

Theorem 1.1.4. *(Riemann-Roch) For any $D \in \mathrm{Div}(C)$ we have*

$$h^0(C, D) - h^0(C, K_C - D) = \deg D - g_C + 1.$$

Equivalently, as by Serre's duality $h^0(C, K_C - D) = h^1(C, D)$,

$$h^0(C, D) - h^1(C, D) = \deg D - g_C + 1.$$

Corollary 1.1.5.
1. *If $\deg D \geq 2g - 1$ then $h^0(C, D) = \deg D - g_C + 1$.*
2. *If $\deg D \geq 2g + 1$ then ϕ_D is an embedding.*

1.2. **Moduli spaces in low genus.** We will denote by M_g the *moduli spaces* of smooth curves of genus g, to be fully defined soon. As a first approximation, let us view M_g as the set of isomorphism classes of curves of genus g.

M_0 consists of one element, by the following, whose proof is an exercise.

Proposition 1.2.1. *If C is a smooth curve of genus 0, then $C \cong \mathbb{P}^1$.*

If $g = 1$, the classical j-invariant gives is a bijection $\mathcal{M}_1 \leftrightarrow k$, hence one can endow M_1 with the structure of an algebraic variety, namely the affine line

$$M_1 = \mathbb{A}^1.$$

Example 1.2.2. Let $g = 2$. Now K_C has degree 2 and determines a morphism

$$\phi : C \longrightarrow |K_C| = \mathbb{P}^1$$

necessarily surjective of degree 2. Moreover, up to automorphisms of $\mathbb{P}^1$, the map ϕ is unique. We say that a point $p \in \mathbb{P}^1$ is a *branch point* if $|\phi^{-1}(p)| = 1$. Since ϕ has degree 2, the number of ramification points coincides with the degree of the ramification divisor of ϕ, which is given by the Riemann-Hurwitz formula

Theorem 1.2.3. *(Riemann-Hurwitz) Let $\psi : C \to D$ be a non constant map between two smooth curves C and D of respective genus g_C and g_D. Let $R \in \mathrm{Div}(C)$ be the ramification divisor of ψ. Then*

$$\deg R = 2g_C - 2 - (2g_D - 2)\deg\psi.$$

By the Riemann-Hurwitz formula the ramification divisor of our ϕ has degree 6, hence ϕ has exactly 6 branch points.

Conversely, given 6 points in $\mathbb{P}^1$ there exists a unique curve C endowed with a degree 2 map to $\mathbb{P}^1$

On the other hand, any 6-tuple of points in $\mathbb{P}^1$ can be written, up to a unique automorphisms of $\mathbb{P}^1$ as

$$\{0, 1, \infty, b_1, b_2, b_3\} : \quad b_i \in k \smallsetminus \{0, 1\},\ i = 1, 2, 3.$$

Denote by $\Delta \subset (k \smallsetminus \{0, 1\})^3$ the union of all diagonals, then we have a surjection

$$(k \smallsetminus \{0, 1\})^3 \smallsetminus \Delta \longrightarrow M_2$$

which maps (b_1, b_2, b_3) to the curve C having a degree-2 map to $\mathbb{P}^1$ ramified over $\{0, 1, \infty, b_1, b_2, b_3\}$. Let $U := (k \smallsetminus \{0, 1\})^3 \smallsetminus \Delta$, then one easily checks that U is an affine variety, and M_2 is the quotient of U by the action of the symmetric group S_6, hence M_2 is an affine variety.

1.3. **The moduli scheme of smooth curves.** We have seen that the set of isomorphism classes of genus 1 and 2 is endowed with a natural structure of algebraic variety, dictated by the geometry of the objects it parametrizes. On the other hand, this structure tells us something about the parametrized curves. It tells us that there is a 1-dimensional (resp. 3-dimensional) space of curves of genus 1 (resp. 2). It also tells us that such curves do not form a complete space! This will be an important point in the sequel.

Let us list some properties that one would hope a moduli scheme, M_g, for smooth curves of genus g satisfies.

(1) The points of M_g are in bijection with isomorphism classes of smooth curves of genus g.
(2) For every *family* $f : \mathcal{C} \to B$ of smooth curves of genus g (i.e. for every flat proper morphism of schemes such that for every closed point $b \in B$ the fiber $C_b = f^{-1}(b)$ is a smooth curve of genus g), the natural map

$$\mu_f : B \longrightarrow M_g; \qquad b \longmapsto \mu_f(b) = [C_b]$$

is a morphism of varieties (i.e. it is a regular map).

(3) Properties (1) and (2) determine M_g up to isomorphism.

(4) For any morphism $\phi : B \longrightarrow M_g$ there exists a family (as defined in (2)) of smooth curves $f : \mathcal{C} \to B$ such that $\phi = \mu_f$, and this family is unique up to B-isomorphisms, i.e. if $f' : \mathcal{C}' \to B$ is another family such that $\mu_{f'} = \phi$ then there is an isomorphism $\alpha : \mathcal{C} \to \mathcal{C}'$ such that $\alpha \circ f' = f$.

The first three properties are satisfied for all $g \geq 0$. The case $g = 0$ is trivial, so we omit it in the next statement

Theorem 1.3.1 (Mumford)**.** *For every $g \geq 1$ there exists an integral, normal, non projective, algebraic variety M_g which satisfies properties* (1)*,* (2) *and* (3)*, but not* (4)*. Moreover*
If $g = 1$ then $\dim M_1 = 1$.
If $g \geq 2$ then $\dim M_g = 3g - 3$.

To stress that property (4) does not hold, one says that M_g is a *coarse* (rather than *fine*) moduli space.

Remark 1.3.2. Property (4) cannot possibly be satisfied. In fact, both the existence part and the uniqueness part fail, and the obstruction lies in the existence of curves having non trivial automorphism group. More precisely, there exist morphisms $\phi : B \to M_g$ for which there does not exist a family of smooth curves over B whose moduli map is ϕ. And there exist families of smooth curves over the same scheme B which are not isomorphic over B but have the same moduli map.

1.4. **Stable curves.** It is very important to notice that M_g is not complete. This says that there are families of smooth curves which degenerate to singular ones. We will study the problem of completing M_g in a modular way, i.e. by constructing a projective scheme $\overline{M}_g$ which contains M_g as dense open subset, and which is itself a moduli space.

A celebrated solution to this problem, provided by Deligne and Mumford, consists in extending the set of smooth curves to the set of reduced (possibly reducible) curves having at most nodal as singularities, and having finitely many automorphisms.

Recall that a point p of a curve X is a *node* if, locally at p, the curve X is formally analytically isomorphic to a neigborhood of the origin of the plane curve Y of equation $xy = 0$, i.e. if the complete local ring of X at p is isomorphic to the complete local ring of Y at the origin.

Definition 1.4.1. A *stable curve* is a connected reduced curve X having at most nodes as singularities, and such that $\mathrm{Aut}(X)$ is finite.

Remark 1.4.2. *Stable curves in the sense of Deligne and Mumford* Smooth curves of genus at least 2 are stable, since they have finitely many automorphisms. Conversely, smooth curves of genus ≤ 1 have infinitely many automorphisms.

There is a notion of genus for a singular stable curve which generalizes the genus of a smooth curve; we shall postpone this definition for the moment.

It is a fact, due to Deligne, Mumford and Gieseker, that for every $g \geq 2$ stable curves admit a coarse moduli space, $\overline{M}_g$, which is a projective irreducible variety containg M_g as a dense open subset, i.e. we have the following.

Theorem 1.4.3 (Deligne-Mumford, Gieseker)**.** *For every $g \geq 2$ there exists an integral, normal, projective variety $\overline{M}_g$ of dimension $3g-3$ whose points are in bijection with the set of isomorphism classes of stable curves of genus g. Moreover M_g is an open subset of $\overline{M}_g$.*

In section 1.3 we defined some properties with respect to smooth curves. One easily checks that properties (1) - (4) make sense if we replace the word "smooth" by the word "stable". Then $\overline{M}_g$ satisfies properties (1), (2) and (3), but not (4). Hence we say that $\overline{M}_g$ is a coarse moduli space for stable curves.

1.5. **Curves with marked points.** We now extend our consideration to curves with marked points. A smooth curve with n points (or an n-pointed curve), written $(C; p_1, \ldots, p_n) = (C; \underline{p})$, is a smooth curve C together with n distinct points $p_i \in C$.

Two such curves, $(C; \underline{p})$ and $(C', \underline{p'})$ are isomorphic if there exists an isomorphism $\alpha : C \to C'$ such that $\alpha(p_i) = p'_i$.

We denote by $M_{g,n}$ the set of isomorphism classes of smooth n-pointed curves of genus g. As we shall see, $M_{g,n}$ has the structure of an algebraic variety and is a moduli space for smooth n-pointed curves of genus g.

Example 1.5.1. Let us study $M_{0,4}$, which is easily seen to be the first new case of positive dimension.

For any $(\mathbb{P}^1; p_1, \ldots, p_4)$ there exists a unique isomorphism α, of $\mathbb{P}^1$ mapping (p_1, p_2, p_3) to $(0, 1, \infty)$; then $\alpha(p_4) \in \mathbb{C} \smallsetminus \{0, 1\}$ and we thus have a bijection

$$M_{0,4} \longrightarrow \mathbb{C} \smallsetminus \{0, 1\}.$$

Which shows that $M_{0,4}$ can be given the structure of an affine variety namely $M_{0,4} = \mathbb{A}^1 \smallsetminus \{0, 1\}$.

Definition 1.5.2. Let $g, n \geq 0$. A *nodal n-pointed curve* of genus g, written $(X; p_1, \ldots, p_n) = (X; \underline{p})$, is a (connected) nodal curve X of arithmetic genus g, together with n distinct nonsingular points $p_i \in X \setminus X_{\text{sing}}$. A nodal n-pointed curve is *stable* if the set of automorphisms of X mapping p_i to p_i for all $i = 1, \ldots, n$ is finite.

Theorem 1.5.3. *Let $2g - 2 + n \geq 1$. There exists a projective, irreducible, normal, variety of dimension $3g - 3 + n$, denoted by $\overline{M}_{g,n}$, which is the coarse moduli space of n-pointed stable curves of genus g. The moduli space of nonsingular n-pointed curves of genus g is an open subset $M_{g,n} \subset \overline{M}_{g,n}$.*

Example 1.5.4. In case $g = 0$ and $n = 4$, then $\overline{M_{0,4}} \cong \mathbb{P}^1$.

Remark 1.5.5. The fact that $\overline{M}_{g,n}$ is not a fine moduli space follows from the existence of curves with nontrivial automorphisms (in fact if $g = 0$ then $M_{0,n}$ is a fine moduli space for $n \geq 4$). On the other hand there do exist finite covers of $\overline{M}_{g,n}$ which are fine moduli spaces of stable curves with some extra structure. In particular, such coverings are endowed with universal families of stable pointed curves whose moduli map to $\overline{M}_{g,n}$ coincides with the covering map.