ABEL MAPS OF GORENSTEIN CURVES

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Abstract. For a Gorenstein curve X and a nonsingular point $P \in X$, we construct Abel maps $A\colon X \to J_X^1$ and $A_P\colon X \to J_X^0$, where J_X^i is the moduli scheme for simple, torsion-free, rank-1 sheaves on X of degree i. The image curves of A and A_P are shown to have the same arithmetic genus of X. Also, A and A_P are shown to be embeddings away from rational subcurves $L \subset X$ meeting $\overline{X-L}$ in separating nodes. Finally we establish a connection with Seshadri's moduli scheme $U_X(1)$ for semistable, torsion-free, rank-1 sheaves on X, obtaining an embedding of $A(X)$ into $U_X(1)$.

Contents

1. Introduction

Fix an algebraically closed field k of any characteristic and let X be a connected, projective curve over k. If X is smooth, there is, for each integer $d \geq 1$, a natural map,

$$A^d\colon X^d \longrightarrow \mathrm{Pic}^d X,$$

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with $\mathrm{Pic}^d X$ denoting the Picard scheme parameterizing line bundles of degree d on X; the map sends $(P_1, \dots, P_d)$ to $[\mathcal{O}_X(P_1 + \cdots + P_d)]$. Of course A^d factors through a map

$$A^{(d)}\colon X^{(d)} \longrightarrow \mathrm{Pic}^d X,$$

where $X^{(d)}$ is the dth symmetric product of X. The map $A^{(d)}$ is called the degree-d *Abel map* of X.

Classical variants are the degree-d Abel maps with base point $P \in X$,

$$A^d_P\colon X^{(d)} \longrightarrow \mathrm{Pic}^0 X,$$

which is simply A^d composed with the translation map $\mathrm{Pic}^d X \to \mathrm{Pic}^0 X$, taking $[\mathcal{L}]$ to $[\mathcal{L} \otimes \mathcal{O}_X(-dP)]$, and the compositions of $A^{(d)}$ and $A^{(d)}_P$ with the duality isomorphisms $\lambda^d\colon \mathrm{Pic}^d_X \to \mathrm{Pic}^{-d}_X$, taking $[\mathcal{L}]$ to $[\mathcal{L}^*]$.

Much of the geometry of X is encoded in the Abel maps, since their fibers are the complete linear systems of X. For instance, the gonality of X is the smallest integer d such that $A^{(d)}$ is not an embedding. In particular, X is hyperelliptic if and only if $A^{(2)}$ is not an embedding.

The Abel maps behave naturally in families of smooth curves. As smooth curves degenerate to singular ones, we would like to understand how the Abel maps degenerate. So, how to construct Abel maps for singular curves in a natural way?

If X is integral, Altman and Kleiman [AltK80] defined, for each $d \geq 1$, a natural map

$$\beta^{(d)}\colon \mathrm{Hilb}^d X \longrightarrow J^{-d}_X,$$

where $\mathrm{Hilb}^d X$ is the Hibert scheme of X parameterizing length-d subschemes, and J^{-d}_X is the compactified Jacobian parameterizing torsion-free, rank-1 sheaves of degree $-d$ on X; the map sends $[Y]$ to $[\mathcal{I}_{Y/X}]$. Again, the fibers of $\beta^{(d)}$ are projective spaces. And, if X is smooth, then $\mathrm{Hilb}^d X = X^{(d)}$, $J^{-d}_X = \mathrm{Pic}^{-d} X$ and $\beta^{(d)} = \lambda^d \circ A^{(d)}$.

On the other hand, if a curve is reducible, the situation is more complex. The current knowledge is concentrated on the two extremes: $d = 1$ and $d = g - 1$. For $d = g - 1$, the image of A^d turns out to be the theta divisor. For work extending the construction and the properties of the theta divisor to singular curves we refer the reader to [So94] and [E97] for irreducible curves, and to the more recent [A04] and [C07] for nodal, possibly reducible, curves.

As for $d = 1$, Edixhoven [Ed98] constructed and studied rational Abel maps of nodal curves to Néron models. As Néron models are seldom complete, his maps are not defined everywhere. In [CE06] the compactifications of Picard schemes and Néron models constructed in [**?**] and in [C05], are used; it is shown that, if X is stable, there exists a globally defined map $\overline{\alpha^1_X}\colon X \to \overline{P^1_X}$, where $\overline{P^1_X}$ is the compactified Picard scheme parameterizing equivalence classes of degree-1 "semibalanced" line bundles on semistable curves having X as a stable model.

In this paper we extend the construction of $\overline{\alpha^1_X}$ to any G-stable curve X, that is, to any reduced curve X with Gorenstein singularities whose dualizing sheaf is ample. Also, we describe the image and the fibers of our Abel map. More precisely, for any reduced curve X with Gorenstein singularities, we consider the fine moduli schemes J_X^d, parameterizing simple, torsion-free, rank-1 sheaves of degree d on X, for all integers d. And we construct a map (cf. 5.2):

$$A\colon X \longrightarrow J_X^1.$$

The schemes J_X^d are quite large, not even Noetherian, but have an open subscheme of finite type, $J_X^{d,ss}$, parameterizing semistable sheaves (cf. 2.4). The map A is constructed in such a way that $A(X) \subseteq J_X^{1,ss}$, if X is G-stable (cf. Theorem 5.4).

If X has no separating nodes (nodes whose removal disconnects the curve), then A sends Q to $[\mathcal{I}_{Q/X}^*]$. In particular, if X is smooth we recover the classical degree-1 Abel map $A^{(1)}$. On the other hand, if X does admit a separating node N, then $\mathcal{I}_{N/X}^*$ is not simple, and thus not parameterized by J_X^1. So, for each $Q \in X$ we create a new sheaf $\mathcal{I}_Q^1$ out of $\mathcal{I}_{Q/X}^*$, by tensoring the latter with suitable so-called "twisters", along the same lines of what was done in [CE06], and let A send Q to $[\mathcal{I}_Q^1]$.

In Theorem 6.3 we prove that A contracts every smooth rational subcurve $L \subsetneq X$ meeting its complementary curve in separating nodes of X, and A is an embedding off these subcurves. Also, $A(X)$ has the same arithmetic genus of X and its singularities are those of X, together with ordinary singularities with linearly independent tangent lines.

Unfortunately, the schemes $J_X^{d,ss}$ are not, in general, separated. To get a separated scheme, two alternatives are possible: to use either smaller schemes or quotient schemes. We consider both.

For each nonsingular point $P \in X$ the scheme $J_X^{d,ss}$ has an open subscheme $J_X^{d,P}$, parameterizing sheaves that are P-quasistable (cf. 2.4), which is projective over k. If P is suitably chosen, and X is G-stable, then $A(X) \subseteq J_X^{1,P}$ by our Theorem 5.4.

On the other hand, we consider Seshadri's coarse moduli schemes $U_X(d)$ for equivalence classes of semistable, torsion-free, rank-1 sheaves on X of degree d (cf. 2.6). There are natural maps $\Phi^d\colon J_X^{d,ss} \to U_X(d)$, taking a semistable sheaf to its class. Our Theorem 7.2 says that Φ^1 restricts to an embedding on $A(X)$.

We also construct Abel maps with base points. More precisely, for each nonsingular point $P \in X$ we construct a natural map (cf. 4.5),

$$A_P\colon X \longrightarrow J_X^0,$$

with image in $J_X^{0,P}$ (cf. Theorem 4.8). If X has no separating nodes, then A_P sends Q to $[\mathcal{I}_{Q/X} \otimes \mathcal{O}_X(P)]$. So, if X is smooth then $A_P = \lambda^0 \circ A_P^{(1)}$. If X has separating nodes, A_P is constructed with the help of twisters, as done for A.

The map A_P has the same description as A. In fact, for a suitably chosen P, we may view A as the composition of A_P with the duality map $J_X^0 \to J_X^0$, taking $[\mathcal{I}]$ to $[\mathcal{I}^*]$ and the translation map $J_X^0 \to J_X^1$, taking $[\mathcal{I}]$ to $[\mathcal{I} \otimes \mathcal{O}_X(P)]$. So, up to these isomorphisms, A may be viewed as one of the A_P. (In fact, everywhere in the paper we prove properties first for A_P and then extend them to A.)

The biggest difference between A and the A_P comes when we consider their composition to Seshadri's moduli schemes: $\Phi^0 \circ A_P$ may actually collapse components of X that were not collapsed by A_P, as we point out in Remark 7.3.

We conclude with a few comments about closely related questions and further developements. First, A is not always a natural map. In fact, if X has a "splitting" node (i.e. a separating node splitting the curve in two equal genus subcurves) then A depends on the choice of one of these subcurves (cf. 5.2). The lack of naturality is a major hurdle to extend our construction to families of curves. In this respect, the map to Seshadri's moduli space, $\Phi^1 \circ A$, looks more natural, as it is independent of the above choice by Theorem 7.2.

On the other hand, the maps A_P are natural, and it seems possible to extend their construction to families of pointed curves, whereas the compositions $\Phi^0 \circ A_P$ do not behave well. We hope to deal with Abel maps for families in the future.

Second, we don't treat higher degree Abel maps. It seems possible to define them not on X^d, $X^{(d)}$ or Hilb^d_X, but on blowups of them. Very little is known, apart from the case of the degree-2 Abel map for a nodal curve with two components meeting at two points, constructed in [Co06].

Here is a layout of the paper: In Section 2, we introduce the moduli schemes $J^{d,ss}_X$ and $U_X(d)$, and the quotient maps $\Phi^d\colon J^{d,ss}_X \to U_X(d)$. In Section 3, we construct the Abel maps A and A_P when X has no separating nodes. In Section 4, we construct the maps A_P in general, and in Section 5 we do the same for the map A. In Section 6, we prove properties of the maps A and A_P, describing their images and fibers. Finally, in Section 7 we show that Φ^1 restricts to an embedding on $A(X)$.

2. Compactified Jacobians

All schemes are assumed to be locally of finite type over a fixed algebraically closed field k. A point of a scheme means a closed point. A *curve* is a reduced, projective scheme of pure dimension 1 over k. If Y is a curve, we let $g_Y := 1-\chi(\mathcal{O}_Y)$ and call g_Y the (arithmetic) *genus* of Y.

Throughout the paper, X denotes a connected curve, ω its dualizing sheaf, g its (arithmetic) genus and P a point on the nonsingular locus of X.

2.1. (*Preliminaries*) A reduced union of irreducible components of X, connected or not, is called a *subcurve*. If Y is a proper subcurve of X, let Y' denote the *complementary subcurve*, that is, the reduced union of all the irreducible components of X not contained in Y. The intersection $Y \cap Y'$ is a finite scheme; let δ_Y denote its length. Since X is connected, $\delta_Y > 0$. Also, observe that

$$g_Y \leq g. \tag{2.1.1}$$

Let $\mathcal{I}$ be a coherent sheaf on X. We say that $\mathcal{I}$ is *torsion-free* if its associated points are generic points of X. We say that $\mathcal{I}$ is *of rank* 1 if $\mathcal{I}$ is invertible on a dense open subset of X. And we say that $\mathcal{I}$ is *simple* if $\mathrm{End}(\mathcal{I}) = k$. Each line bundle on X is torsion-free of rank 1 and simple.

Suppose $\mathcal{I}$ is torsion-free of rank 1. We call $\deg(\mathcal{I}) := \chi(\mathcal{I}) - \chi(\mathcal{O}_X)$ the *degree* of $\mathcal{I}$. For each vector bundle F on X,

$$\chi(\mathcal{I} \otimes F) = \mathrm{rk}(F)\chi(\mathcal{I}) + \deg(F) = \mathrm{rk}(F)(\deg(\mathcal{I}) + 1 - g) + \deg(F). \tag{2.1.2}$$

For each subcurve Y of X, let $\mathcal{I}_Y$ denote the restriction of $\mathcal{I}$ to Y modulo torsion, that is, the image of the natural map

$$\mathcal{I}|_Y \longrightarrow \bigoplus_{i=1}^{m} (\mathcal{I}|_Y)_{\xi_i},$$

where $\xi_1, \dots, \xi_m$ are the generic points of Y. We let $\deg_Y(\mathcal{I})$ denote the degree of $\mathcal{I}_Y$, that is, $\deg_Y(\mathcal{I}) := \chi(\mathcal{I}_Y) - \chi(\mathcal{O}_Y)$.

Let Y be a proper subcurve of X. By the defining property of the dualizing sheaf, the kernel of the restriction map $\omega \to \omega|_{Y'}$ is the dualizing sheaf ω_Y of Y. Suppose X is Gorenstein. Then ω is a line bundle of degree $2g-2$, and it follows that

$$\chi(\omega|_Y) = \chi(\omega_Y) + \delta_Y.$$

Thus, by duality,

$$\deg(\omega|_Y) = \chi(\omega_Y) + \delta_Y - \chi(\mathcal{O}_Y) = -2\chi(\mathcal{O}_Y) + \delta_Y = 2g_Y - 2 + \delta_Y. \tag{2.1.3}$$

Definition 2.2. The curve X is called *G-stable* if X is Gorenstein of genus $g \geq 2$, and does not contain any smooth rational component L with $\delta_L \geq 2$. Equivalently, using (2.1.3), the curve X is G-stable if X is Gorenstein and ω is ample.

If the singularities of X are (ordinary) nodes, then X is G-stable if and only if it is stable, in the sense of Deligne and Mumford.

2.3. (*Semistable sheaves*) Let F be a vector bundle and $\mathcal{I}$ a torsion-free, rank-1 sheaf of degree d on the curve X. We call $\mathcal{I}$ *semistable* with respect to F if

1. $\chi(\mathcal{I} \otimes F) = 0$ and
2. $\chi(\mathcal{N} \otimes F) \geq 0$ for each nonzero quotient $\mathcal{N}$ of $\mathcal{I}$ different from $\mathcal{I}$.

We call $\mathcal{I}$ *stable* with respect to F if the inequality in Condition 2 is always strict.

By (2.1.2), Condition 1 is verified if the *slope* $\mu(F) := \deg(F)/\mathrm{rk}(F)$ satisfies $\mu(F) = g-1-d$. As for Condition 2, observe that all nonzero torsion-free quotients of $\mathcal{I}$ are of the form $\mathcal{I}_Y$ for a subcurve $Y \subseteq X$. So, Condition 2 holds if and only if

$$\chi(\mathcal{I}_Y \otimes F|_Y) \geq 0 \tag{2.3.1}$$

for each proper subcurve Y of X. For stability, we require strict inequalities.

We say that $\mathcal{I}$ is *P-quasistable* with respect to F if Inequality (2.3.1) holds for each proper subcurve $Y \subset X$, with equality only if $P \notin Y$. Clearly, this notion depends only on which component of X the point P lies.

Suppose X is Gorenstein. Define a vector bundle E_d on X as follows: If $g \geq 2$,

$$E_d := \mathcal{O}_X^{\oplus 2g-3} \oplus \omega^{\otimes g-1-d}; \tag{2.3.2}$$

if $g = 0$, set $E_d := \mathcal{O}_X \oplus \omega^{\otimes d+1}$; and if $g = 1$, define E_d only if $d = 0$, setting $E_0 := \mathcal{O}_X$. We call E_d the *canonical d-polarization* of X. Notice that E_d has slope $\mu(E_d) = g-1-d$.

We say that $\mathcal{I}$ is (canonically) stable, semistable or P-quasistable if $\mathcal{I}$ is so with respect to E_d. If $g \geq 2$ then, for each subcurve $Y \subseteq X$,

$$\begin{aligned}\chi(\mathcal{I}_Y \otimes E_d|_Y) =&(2g-2)\chi(\mathcal{I}_Y) + \deg(E_d|_Y)\\ =&(2g-2)\deg_Y(\mathcal{I}) + (2g-2)\chi(\mathcal{O}_Y) + (g-1-d)\deg(\omega|_Y)\\ =&(2g-2)\deg_Y(\mathcal{I}) + (1-g)(\deg(\omega|_Y) - \delta_Y) + (g-1-d)\deg(\omega|_Y)\\ =&(2g-2)\deg_Y(\mathcal{I}) + (g-1)\delta_Y - d\deg(\omega|_Y),\end{aligned}$$

where we used (2.1.2) and (2.1.3). Thus, $\mathcal{I}$ is semistable if and only if

$$\deg_Y(\mathcal{I}) \geq d\Big(\frac{\deg_Y(\omega)}{2g-2}\Big) - \frac{\delta_Y}{2} \tag{2.3.3}$$

for each proper subcurve Y of X. If $g = 0$, then an analogous computation can be done, and the condition for semistability is the same. Finally, if $g = 1$ and $d = 0$, then $\mathcal{I}$ is semistable if and only if

$$\deg_Y(\mathcal{I}) \geq -\frac{\delta_Y}{2} \tag{2.3.4}$$

for each proper subcurve Y of X. Notice that (2.3.3) and (2.3.4) are equal conditions if $d = 0$. We leave it to the reader to formulate the analogous conditions for when $\mathcal{I}$ is stable or P-quasistable.

2.4. (*The fine compactified Jacobians*) There exists a scheme J_X parameterizing torsion-free, rank-1, simple sheaves on the curve X; see [E01] Thm. B, p. 3048. More precisely, given a scheme T, a T-flat coherent sheaf $\mathcal{I}$ on $X \times T$ is called torsion-free (resp. rank-1, resp. simple) on $X \times T/T$ if $\mathcal{I}|_{X\times t}$ is torsion-free (resp. rank-1, resp. simple) for every $t \in T$. The scheme J_X represents the functor that associates to each scheme T the set of torsion-free, rank-1 simple sheaves on $X\times T/T$ modulo equivalence $\sim$. Two such sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are called equivalent, $\mathcal{I}_1 \sim \mathcal{I}_2$, if there is a line bundle N on T such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes p_2^*N$, where $p_2\colon X \times T \to T$ is the projection map.

If T is a connected scheme, and $\mathcal{I}$ is a torsion-free, rank-1 sheaf on $X\times T/T$, then $d := \deg(\mathcal{I}|_{X\times t})$ does not depend on the choice of $t \in T$; we say that $\mathcal{I}$ is a degree-d sheaf on $X \times T/T$. Then there is a natural decomposition

$$J_X = \coprod_{d\in\mathbf{Z}} J_X^d,$$

where J_X^d is the subscheme of J_X parameterizing degree-d sheaves. The schemes J_X^d are universally closed over k; see [E01] Thm. 32, (2), p. 3068. However, in general, the J_X^d are neither of finite type nor separated over k.

Let F be a vector bundle on X with integer slope, and set $d := g-1-\mu(F)$. By [E01] Prop. 34, p. 3071, the subschemes J_F^{ss} (resp. J_F^s, resp. J_F^P) of J_X^d parameterizing simple and semistable (resp. stable, resp. P-quasistable) sheaves on X with respect to F are open. If X is Gorenstein and $F = E_d$ (the canonical

d-polarization defined in 2.3), we write

$$J_X^{d,ss} := J_{E_d}^{ss}, \quad J_X^{d,s} := J_{E_d}^{s} \quad \text{and} \quad J_X^{d,P} := J_{E_d}^{P}.$$

By [E01] Thm. A, p. 3047, J_F^{ss} is of finite type and universally closed, J_F^s is separated and J_F^P is complete over k. Actually, J_F^P is projective; see [E07] Prop. 2.4.

2.5. (*The S-equivalence*) Let F be a vector bundle on X. For each semistable sheaf $\mathcal{I}$ on X with respect to F, there is a maximal filtration

$$\emptyset \subsetneqq Y_1 \subsetneqq Y_2 \subsetneqq \cdots \subsetneqq Y_{q-1} \subsetneqq X$$

of X by subcurves Y_i such that $\chi(\mathcal{I}_{Y_i} \otimes F|_{Y_i}) = 0$ for each $i = 1, \dots, q$, which we call a *Jordan–Hölder filtration.* There may be many Jordan–Hölder filtrations associated to $\mathcal{I}$, but the collection of subcurves

$$\mathfrak{S}(\mathcal{I}) := \{Y_1, \overline{Y_2 - Y_1}, \dots, \overline{Y_{q-1} - Y_{q-2}}, \overline{X - Y_q}\}$$

and the isomorphism class of the sheaf

$$\mathrm{Gr}(\mathcal{I}) := \mathcal{I}_{Y_1} \oplus \mathrm{Ker}(\mathcal{I}_{Y_2} \to \mathcal{I}_{Y_1}) \oplus \cdots \oplus \mathrm{Ker}(\mathcal{I}_{Y_{q-1}} \to \mathcal{I}_{Y_{q-2}}) \oplus \mathrm{Ker}(\mathcal{I} \to \mathcal{I}_{Y_q})$$

depend only on $\mathcal{I}$, by the Jordan–Hölder Theorem.

We say that two semistable sheaves $\mathcal{I}$ and $\mathcal{K}$ on X are *S-equivalent* if $\mathfrak{S}(\mathcal{I}) = \mathfrak{S}(\mathcal{K})$ and $\mathrm{Gr}(\mathcal{I}) \cong \mathrm{Gr}(\mathcal{K})$.

(For a higher rank semistable sheaf, a Jordan–Hölder filtration is a filtration of the sheaf. However, in rank 1, this filtration is induced by a filtration of X as above.)

2.6. (*The coarse compactified Jacobians*) Let $X_1, \dots, X_n$ be the irreducible components of the curve X, and $\mathfrak{a} = (a_1, \dots, a_n)$ a n-tuple of positive rational numbers summing up to 1. For each subcurve $Y \subseteq X$, set $a_Y := \sum_{X_i \subseteq Y} a_i$. According to Seshadri [S82] Déf. 9 and Remarques on p. 153, a torsion-free, rank-1 sheaf $\mathcal{I}$ on X is $\mathfrak{a}$-*semistable* if

$$\chi(\mathcal{I}_Y) \geq a_Y \chi(I)$$

for each proper subcurve Y of X. Also, $\mathcal{I}$ is $\mathfrak{a}$-*stable* if the inequalities are strict.

Seshadri's notions of semistability and stability are encompassed by ours. More precisely, for each integer d there is a vector bundle F on X such that $\mathfrak{a}$-semistability (resp. $\mathfrak{a}$-stability) for degree-d, torsion-free, rank-1 sheaves is equivalent to semistability (resp. stability) with respect to F; see [E99] Obs. 13, p. 584. In fact, any vector bundle F on X such that

$$\mu(F|_{X_i}) = a_i(g - 1 - d) \quad \text{for each } i = 1, \dots n \tag{2.6.1}$$

has this property.

Two $\mathfrak{a}$-semistable sheaves are called S-equivalent if they are S-equivalent in the sense of 2.5 for a (and hence any) vector bundle F on X satisfying (2.6.1).

In [S82] Thm. 15, p. 155, Seshadri shows that there is a scheme $U_X(\mathfrak{a}, d)$ corepresenting the functor that associates to each scheme T the set of T-flat coherent sheaves $\mathcal{I}$ on $X \times T$ such that $\mathcal{I}|_{X \times t}$ is $\mathfrak{a}$-semistable and of degree d for every $t \in T$, modulo the same equivalence $\sim$ of 2.4. Furthermore, $U_X(\mathfrak{a}, d)$ is projective and parameterizes S-equivalence classes of $\mathfrak{a}$-semistable sheaves.

Let F be any vector bundle on X satisfying (2.6.1), and set $J_X(\mathfrak{a}, d) := J_F^{ss}$. (The particular F is irrelevant.) Since J_F^{ss} is a fine moduli space, there is a naturally induced morphism

$$\Phi_{\mathfrak{a}}^d \colon J_X(\mathfrak{a}, d) \longrightarrow U_X(\mathfrak{a}, d) \tag{2.6.2}$$

sending $[\mathcal{I}]$ to the S-equivalence class of $\mathcal{I}$. We call $\Phi_{\mathfrak{a}}^d$ the *S-map*.

If X is G-stable (cf. 2.2), let $U_X(d) := U_X(\mathfrak{a}, d)$, where

$$a_i := \frac{\deg_{X_i}(\omega)}{2g-2} \quad \text{for each } i = 1, \dots, n.$$

(Notice that $a_1 + \cdots + a_n = 1$ because X is Gorenstein, and the a_i are positive because ω is ample.) Since, for each integer i,

$$\mu(E_d|_{X_i}) = a_i(g - 1 - d),$$

where E_d is the canonical d-polarization of X (cf. 2.3), the S-map (2.6.2) becomes

$$\Phi^d \colon J_X^{d,ss} \longrightarrow U_X(d). \tag{2.6.3}$$

3. Abel maps

Assume from now on until the end of the paper that X is Gorenstein.

Definition 3.1. A *separating node* of the curve X is a point N for which there is a subcurve Z such that $\delta_Z = 1$ and $Z \cap Z' = \{N\}$.

Being X Gorenstein, a separating node is indeed a node, by [Cat82], Prop. 1.10, p. 59.

3.2. (*Degree-*0 *Abel maps*) For each point Q on the curve X, its sheaf of ideals $\mathfrak{m}_Q$ is torsion-free of rank 1 and degree -1. Also, if Q is not a separating node, $\mathfrak{m}_Q$ is simple, as it follows from the discussion in [E01], Ex. 38, p. 3073.

Let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$, and put

$$\mathcal{I} := \mathcal{I}_\Delta \otimes p_1^* \mathcal{O}_X(P),$$

where $p_1 \colon X \times X \to X$ is the first projection. The sheaf $\mathcal{I}$ is flat over X and, for each $Q \in X$,

$$\mathcal{I}|_{X \times Q} = \mathfrak{m}_Q \otimes \mathcal{O}_X(P).$$

If X is free from separating nodes, then $\mathcal{I}$ defines a morphism

$$A_P \colon X \longrightarrow J_X^0; \quad Q \mapsto [\mathfrak{m}_Q \otimes \mathcal{O}_X(P)]. \tag{3.2.1}$$

We call A_P the *degree-*0 *Abel map* of X with base P.

Proposition 3.3. *Assume that the curve X is free from separating nodes. Then:*

(1) $A_P(X) \subseteq J_X^{0,P}$.
(2) *If $X \not\cong \mathbb{P}^1$ then A_P is an embedding.*

Proof. For each $Q \in X$, its sheaf of ideals $\mathfrak{m}_Q$ satisfies

$$\deg_Y(\mathfrak{m}_Q) = \begin{cases} -1, & \text{if } Q \in Y, \\ 0, & \text{if } Q \notin Y, \end{cases}$$

for each subcurve $Y \subseteq X$. If Y is proper, then $\delta_Y > 1$ by hypothesis, and hence $\deg_Y(\mathfrak{m}_Q \otimes \mathcal{O}_X(P)) \geq -\delta_Y/2$, with equality only if $P \notin Y$. So $\mathfrak{m}_Q \otimes \mathcal{O}_X(P)$ is P-quasistable, showing that $A_P(X) \subseteq J_X^{0,P}$.

Assume $X \not\cong \mathbb{P}^1$. It remains to show that A_P is an embedding. Since X is complete and $J_X^{0,P}$ is separated, the induced map $X \to J_X^{0,P}$ is proper. Thus, we need only show that A_P separates points and tangent vectors. Equivalently, we need only show that every fiber of A_P is either empty or schematically a point.

Let $Q \in X$ and put $L := A_P^{-1}([\mathfrak{m}_Q \otimes \mathcal{O}_X(P)])$. From [AltK80] Lemma 5.17, p. 88, it follows that L is isomorphic to an open subscheme of the projective space

$$\mathbb{P}(\mathrm{Hom}_X(\mathfrak{m}_Q, \mathcal{O}_X)),$$

the open subscheme parameterizing injective homomorphisms. However, since A_P is proper over $J_X^{0,P}$, the fiber L is complete, and thus L is a projective space.

We need to show that L is a point. Suppose otherwise, by contradiction. Thus, since L has dimension at most 1, we have $L \cong \mathbb{P}^1$.

Let Q_1 and Q_2 be distinct points of L on the nonsingular locus of X. Since L is a fiber of A_P, we have an isomorphism $\mathfrak{m}_{Q_1} \to \mathfrak{m}_{Q_2}$. This isomorphism is given by multiplication by a rational function h of X, whose only pole is Q_1 and whose only zero is Q_2, both with order 1. The function h is constant on all components of X other than L, because h has no zeros or poles there. Let $Z := L'$. Since X is not isomorphic to $\mathbb{P}^1$, we have $Z \neq \emptyset$. We claim that L intersects Z transversally. In fact, if L intersected Z nontransversally at a point R, then $h|_L - h(R)$ would vanish at R with order at least 2. This is not possible because $h|_L$ has degree 1.

Let $Z_1, \dots, Z_q$ denote the connected components of Z. Since X is connected, each Z_i intersects L. If $\#(Z_i \cap L) = 1$, then we would have $\delta_{Z_i} = 1$, as we already know that Z intersects L transversally. Thus, each Z_i intersects L in at least two points. But then $h|_L$ takes the same value on these two points of L. This is again not possible because $h|_L$ has degree 1. We have reached a contradiction. □

3.4. (*Degree*-1 *Abel maps*) As in 3.2, let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$. Then $\mathcal{I}_\Delta$ is flat over X and, for each $Q \in X$, the restriction $\mathcal{I}_\Delta|_{X\times Q}$ is isomorphic to the sheaf of ideals $\mathfrak{m}_Q$. Since X is Gorenstein, the dual sheaf

$$\mathcal{I}_\Delta^* := \mathcal{H}om_{X\times X}(\mathcal{I}_\Delta, \mathcal{O}_{X\times X})$$

is also flat over X, and $\mathcal{I}_\Delta^*|_{X\times Q} \cong \mathfrak{m}_Q^*$ for each $Q \in X$.

As mentioned in 3.2, the sheaf $\mathfrak{m}_Q$ is simple if X is free from separating nodes. Since X is Gorenstein,

$$\mathrm{Hom}_X(\mathfrak{m}_Q^*, \mathfrak{m}_Q^*) = \mathrm{Hom}_X(\mathfrak{m}_Q, \mathfrak{m}_Q).$$

So, if X is free from separating nodes, $\mathcal{I}^*_\Delta|_{X\times Q}$ is simple for every $Q\in X$, and thus $\mathcal{I}^*_\Delta$ defines a morphism

$$A\colon X\longrightarrow J^1_X;\quad Q\mapsto[\mathfrak{m}^*_Q]. \tag{3.4.1}$$

We call A the *degree-*1 *Abel map* of X.

Proposition 3.5. *Assume that the curve X is free from separating nodes. Then:*

1. *If $X\not\cong\mathbb{P}^1$ then A is an embedding.*
2. *If $g\geq 2$ then $A(X)\subseteq J^{1,ss}_X$.*
3. *If X is G-stable then $A(X)\subseteq J^{1,s}_X$.*

Proof. The map A is the composition of $A_P\colon X\to J^0_X$ followed by the duality map $\lambda\colon J^0_X\to J^0_X$, sending $[\mathcal{I}]$ to $[\mathcal{I}^*]$, and the translation $\tau\colon J^0_X\to J^1_X$ by P, sending $[\mathcal{I}]$ to $[\mathcal{I}\otimes\mathcal{O}_X(P)]$. Assume $X\not\cong\mathbb{P}^1$; then A_P is an embedding by Proposition 3.3, and since λ and τ are isomorphisms, also A is an embedding.

Let us now see that $A(X)\subseteq J^{1,ss}_X$ if $g\geq 2$. We claim that $\deg(\omega|_Y)\geq 0$ for every subcurve $Y\subseteq X$. Indeed, since the degree is additive, we need only check the claim when Y is irreducible. Then $\deg(\omega|_Y)\geq 0$ by (2.1.3), because $g_Y\geq 0$ and, by hypothesis, $\delta_Y\geq 2$. Thus, since ω has degree $2g-2$, and since $g\geq 1$ and $\delta_Y\geq 2$,

$$\deg(\omega|_Y)\leq 2(g-1)\leq\delta_Y(g-1). \tag{3.5.1}$$

Now, for each $Q\in X$, there is a natural inclusion $\mathcal{O}_X\to\mathfrak{m}^*_Q$, where $\mathfrak{m}_Q$ is the sheaf of ideals of Q. Thus

$$\deg_Y(\mathfrak{m}^*_Q)\geq 0 \tag{3.5.2}$$

for every subcurve $Y\subseteq X$, and hence (3.5.1) implies that $\mathfrak{m}^*_Q$ is semistable.

Finally, suppose that X is G-stable. Let Y be a proper subcurve of X. Because of (3.5.1) and (3.5.2), for the inequality

$$\deg_Y(\mathfrak{m}^*_Q)\geq\frac{\deg_Y(\omega)}{2g-2}-\frac{\delta_Y}{2}$$

to be an equality we would need that $\deg_Y(\omega)=\delta_Y(g-1)$. Since $\delta_Y\geq 2$, we would need that $\delta_Y=2$ and $\deg_{Y'}(\omega)=0$. But this is not possible because ω is ample. Thus $A(X)\subseteq J^{1,s}_X$. □

Remark 3.6. There are special cases where $J^{1,ss}_X=J^{1,s}_X$, for instance if

$$\frac{\deg_Y(\omega)}{2g-2}-\frac{\delta_Y}{2} \tag{3.6.1}$$

is not an integer for any proper subcurve $Y\subset X$. This will be the case when X is G-stable and g is odd. Indeed, suppose (3.6.1) is an integer. Using (2.1.3), we have

$$\frac{\deg_Y(\omega)}{2g-2}-\frac{\delta_Y}{2}=\frac{(2-g)\delta_Y-2\chi(\mathcal{O}_Y)}{2g-2}.$$

Thus, if g is odd, δ_Y must be even, and thus $(2g-2)$ divides $\deg_Y(\omega)$. However, as we saw in the proof of Proposition 3.5, this implies that $\deg_Y(\omega)=0$ or $\deg_{Y'}(\omega)=0$, which is not possible if ω is ample.

If X is a stable curve, in the sense of Deligne and Mumford, it follows from [CE06] Prop. 3.15 that $J_X^{1,ss} = J_X^{1,s}$, unless $X = Y_1 \cup Y_2$, where Y_1 and Y_2 are connected proper subcurves of the same genus intersecting at an odd number of points.

Remark 3.7. Since X is assumed Gorenstein, the dualizing map

$$\lambda\colon J_X \longrightarrow J_X; \qquad [\mathcal{I}] \mapsto [\mathcal{I}^*]$$

is well-defined and takes J_X^d isomorphically onto J_X^{-d}, for every integer d. Furthermore, given any vector bundle F on X, using duality, we have

$$\lambda(J_F^{ss}) = J_{F^\dagger}^{ss}, \quad \lambda(J_F^P) = J_{F^\dagger}^P \quad \text{and} \quad \lambda(J_F^s) = J_{F^\dagger}^s,$$

where $F^\dagger := F^* \otimes \omega$. In particular, since $\mu(E_d^\dagger|_Y) = \mu(E_{-d}|_Y)$ for each integer d and each subcurve $Y \subseteq X$, it follows that

$$\lambda(J_X^{d,ss}) = J_X^{-d,ss}, \quad \lambda(J_X^{d,P}) = J_X^{-d,P} \quad \text{and} \quad \lambda(J_X^{d,s}) = J_X^{-d,s}.$$

Thus, we could have defined A_P as sending Q to $[\mathcal{I}_{Q/X}^* \otimes \mathcal{O}_X(-P)]$, or A as sending Q to $[\mathcal{I}_{Q/X}]$. Apart from the fact that the latter map would have J_X^{-1} as target instead of J_X^1, all the conclusions would remain the same.

Essentially, the same observation applies to the twisted Abel maps to be defined in 4.5 and 5.2.

4. Twisted Abel maps of degree 0

4.1. (*Spines and tails*). A *tail* of X is a proper subcurve $Z \subset X$ with $\delta_Z = 1$. If Z is a tail, so is Z', and the unique point N of $Z \cap Z'$ is a separating node. In this case, we say that Z and Z' are the tails attached to N, and that N generates Z and Z'. Notice that a tail is connected (because X is). A tail is called a *P-tail* if it does not contain P. We denote by $\mathcal{T}(X)$ the set of all tails of X and by $\mathcal{T}_P(X)$ the set of all P-tails.

A connected subcurve Y of X is called a *spine* if every point in $Y \cap \overline{X-Y}$ is a separating node. In this case, each connected component Z of $\overline{X-Y}$ is a tail intersecting Y transversally at a single point on the nonsingular loci of Y and Z.

Let Y be a subcurve of X. If a singular point of Y is a separating node of X, then it is also a separating node of Y. Conversely, if Y is a spine then a separating node of Y is a separating node of X. As a consequence, if a subcurve Z of Y is a spine of X, then Z is a spine of Y; conversely, if Z is a spine of Y and Y is a spine of X, then Z is a spine of X.

If Y is a nonempty proper union of spines of X, then any connected component of Y or $\overline{X-Y}$ is a spine. Two intersecting spines with no common component intersect transversally at a separating node of X.

A q-tuple $\mathfrak{Z} := (Z_1, \dots, Z_q)$ of spines covering X, each two with no component in common, is called a *spine decomposition* of X. If Y is a spine of X, then Y and the connected components of $\overline{X-Y}$ form a spine decomposition of X.

The following two lemmas will be much used.

Lemma 4.2. *Let Z_1 and Z_2 be tails of the curve X. Then*

$$\text{either} \quad Z_1 \cup Z_2 = X \quad \text{or} \quad Z_1 \cap Z_2 = \emptyset \quad \text{or} \quad Z_1 \subseteq Z_2 \quad \text{or} \quad Z_2 \subsetneq Z_1.$$

Proof. This is [CE06] Lemma 4.3. □

Lemma 4.3. *Let $\mathfrak{Z} := (Z_1, \dots, Z_q)$ be a spine decomposition of X. Then there is an isomorphism*

$$u \colon J_X \longrightarrow J_{Z_1} \times \cdots \times J_{Z_q}$$

sending $[I]$ to $([I|_{Z_1}], \dots, [I|_{Z_q}])$. Furthermore, for each integer d,

$$u(J_X^d) = \bigcup_{d_1 + \cdots + d_q = d} J_{Z_1}^{d_1} \times \cdots \times J_{Z_q}^{d_q}.$$

Proof. This is [E07] Prop. 3.2. □

4.4. (*Twisters on tails.*) By Lemma 4.3, or [CE06] Lemma 4.4, for each tail Z of the curve X there is a unique, up to isomorphism, line bundle on X whose restrictions to Z and Z' are $\mathcal{O}_Z(-N)$ and $\mathcal{O}_{Z'}(N)$, where N is the separating node generating Z. Denote this line bundle by $\mathcal{O}_X(Z)$. We call it a *twister*.

For simplification, for each formal sum $\sum a_Z Z$ of tails Z with integer coefficients a_Z, set

$$\mathcal{O}_X(\textstyle\sum a_Z Z) := \bigotimes \mathcal{O}_X(Z)^{\otimes a_Z}.$$

If Z is a tail of X attached to the node N, and $f \colon \mathcal{X} \to S$ is a one-parameter regular smoothing of (X, N) (a flat, projective morphism of schemes such that S has dimension one, $X = f^{-1}(s)$ for a nonsingular $s \in S$, and $\mathcal{X}$ is smooth at N), then Z is a Cartier divisor of $\mathcal{X}$, satisfying $\mathcal{O}_{\mathcal{X}}(Z)|_X \cong \mathcal{O}_X(Z)$ while $\mathcal{O}_{\mathcal{X}}(Z)|_{f^{-1}(t)} = \mathcal{O}_{f^{-1}(t)}$ for each $t \in S \smallsetminus s$. So, $\mathcal{O}_X(Z)$, though nontrivial, is the limit of a family of trivial sheaves.

4.5. (*Degree*-0 *twisted Abel maps*). Let $Q \in X$. If Q is not a separating node, let $\mathcal{M}_Q$ be the sheaf of ideals $\mathfrak{m}_Q$ of Q. Notice that $\mathcal{M}_Q$ is simple. If Q is a separating node, let Z be the P-tail generated by Q (so that $P \notin Z$) and let $\mathcal{M}_Q$ be the unique line bundle on X such that

$$\mathcal{M}_Q|_Z \cong \mathcal{O}_Z(-Q) \quad \text{and} \quad \mathcal{M}_Q|_{Z'} \cong \mathcal{O}_{Z'}.$$

(That $\mathcal{M}_Q$ exists and is unique follows from Lemma 4.3.) Again, $\mathcal{M}_Q$ is simple.

Define a map

$$A_P \colon X \longrightarrow J_X^0; \quad Q \mapsto [\mathcal{I}_Q] \tag{4.5.1}$$

where

$$\mathcal{I}_Q := \mathcal{M}_Q \otimes \mathcal{O}_X(P) \otimes \mathcal{O}_X(-\sum_{Z \in \mathcal{T}_P(X)\,;\, Z \ni Q} Z), \tag{4.5.2}$$

the sum running over all tails Z of X containing Q but not P. Since $\mathcal{M}_Q$ is simple, so is $\mathcal{I}_Q$, and hence A_P is well-defined. We call A_P the *degree*-0 *(twisted) Abel map* of X with base P. If X has no separating nodes then (4.5.1) coincides with (3.2.1). We will see in Theorem 4.8 that, in any case, A_P is a morphism of schemes.

Lemma 4.6. *Keep the notation of 4.5. Let W be a spine of X, and define*

$$B\colon W \to J^0_W; \quad Q \mapsto [\mathcal{I}_Q|_W].$$

Then B is a well-defined map and the following three statements hold:

(1) *If $P \in W$, then B is the degree-0 Abel map of W with base P.*
(2) *If $P \in W'$, then B is the degree-0 Abel map of W with base N, where N is the unique point of $W \cap W'$ on the same connected component of W' as P.*
(3) *In any case, the isomorphism class of $\mathcal{I}_Q|_{W'}$ does not depend on $Q \in W$.*

Proof. We will use induction on δ_W. Suppose first that $\delta_W = 1$, i.e., that W is a tail. Let N be the separating node generating W.

Let $Q \in W$. Let $Z_1, \dots, Z_n$ be the P-tails of X containing Q. It follows from Lemma 4.2 that either $Z_i \subset Z_j$ or $Z_j \subset Z_i$ for each distinct i and j. Thus, we may assume that

$$Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n.$$

By definition, $\mathcal{I}_Q = \mathcal{M}_Q \otimes \mathcal{K}_Q$, where $\mathcal{K}_Q := \mathcal{O}_X(P) \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n)$.

Suppose first that $P \in W'$. Then $W \in \mathcal{T}_P(X)$. As $W \ni Q$, we have that $W = Z_i$ for a certain i. The tails $Z_1, \dots, Z_{i-1}$ are also tails of W; in fact, they are all the N-tails of W containing Q. And $Z_i, \dots, Z_n$ are all the P-tails of X containing W. So

$$\begin{aligned}
\mathcal{K}_Q|_W \cong& \mathcal{O}_W(N) \otimes \mathcal{O}_W(-\sum_{Y \in \mathcal{T}_N(W)\,;\, Y \ni Q} Y)\\
\mathcal{K}_Q|_{W'} \cong& \mathcal{O}_{W'}(P) \otimes \mathcal{O}_X(-\sum_{Y \in \mathcal{T}_P(X)\,;\, Y \supseteq W} Y)|_{W'}.
\end{aligned}$$

Notice that $\mathcal{K}_Q|_{W'}$ does not depend on Q.

Since $Q \in W$ and $P \in W'$, we have that $\mathcal{M}_Q|_{W'} \cong \mathcal{O}_{W'}$, and hence $\mathcal{I}_Q|_{W'}$ does not depend on Q. In addition, if $Q = N$ or Q is not a separating node of X, then Q is not a separating node of W, and $\mathcal{M}_Q|_W$ is the sheaf of ideals of Q in W. On the other hand, if Q is a separating node of X different from N, then Q is a separating node of W; and if Y is the N-tail of W generated by Q, then Y is the P-tail of X generated by Q, and hence

$$\mathcal{M}_Q|_Y \cong \mathcal{O}_Y(-Q) \quad \text{and} \quad \mathcal{M}_Q|_{\overline{W-Y}} \cong \mathcal{O}_{\overline{W-Y}}.$$

In any case, it follows that $[\mathcal{I}_Q|_W]$ is the image of Q under the degree-0 Abel map of W with base N. This finishes the proof of the lemma in the case where $P \in W'$

Now, suppose $P \in W$. We claim that either $Z_i \cap W' = \emptyset$, or $Z_i \supseteq W'$, for each integer i. Indeed, Z_i and W' are both P-tails. If $Z_i \cap W' \neq \emptyset$, then either $Z_i \supseteq W'$ or $Z_i \subseteq W'$ by Lemma 4.2. However, if $Z_i \subseteq W'$, since $Q \in Z_i$ and $P \notin W'$, we have that $W' = Z_j$ for some $j \geq i$. But, since $Q \in W$ as well, it follows that $Q = N$ and $i = j = 1$. But then $Z_i = W'$, proving the claim. Notice from our reasoning above that $W' = Z_i$ for a certain i if and only if $Q = N$.

If $Z_i \cap W' = \emptyset$, then Z_i is tail of W. And if $Z_i \supsetneq W'$, then $\overline{Z_i - W'}$ is a tail of W. On the other hand, let Y be a tail of W. If $N \notin Y$, then Y is a tail of X as well, with $Y \cap W' = \emptyset$. And if $N \in Y$, then $Y \cup W'$ is a tail of X. Thus

$$\mathcal{K}_Q|_W \cong \mathcal{O}_W(P) \otimes \mathcal{O}_W(-\sum_{Y \in \mathcal{T}_P(W)\,:\,Y \ni Q} Y) \otimes \mathcal{L}|_W,$$

and $\mathcal{K}_Q|_{W'} \cong \mathcal{L}|_{W'}$, where $\mathcal{L} := \mathcal{O}_X(-W')$ if $Q = N$ and $\mathcal{L} := \mathcal{O}_X$ otherwise.

Notice that $\mathcal{M}_Q|_{W'} \cong \mathcal{O}_{W'}$ unless $Q = N$, in which case $\mathcal{M}_Q|_{W'} \cong \mathcal{O}_{W'}(-N)$. In any case, $\mathcal{I}_Q|_{W'}$ is trivial, whence independent from Q. In addition, $\mathcal{M}_Q|_W$ is the sheaf of ideals Q in W, if Q is not a separating node of X. On the other hand, if $Q = N$ then $\mathcal{M}_Q|_W = \mathcal{O}_W$ and $\mathcal{L}|_W \cong \mathcal{O}_W(-N)$. And if Q is a separating node of X different from N, then Q is a separating node of W; and if Y is the P-tail of W generated by Q, then either Y or $Y \cup W'$ is the P-tail of X generated by Q, depending on whether N is on Y or not, and whence

$$\mathcal{M}_Q|_Y \cong \mathcal{O}_Y(-Q) \quad \text{and} \quad \mathcal{M}_{\overline{W-Y}} \cong \mathcal{O}_{\overline{W-Y}}.$$

In any case, it follows that $[\mathcal{I}_Q|_W]$ is the image of Q under the degree-0 Abel map of W with base P, finishing the proof of the lemma when W is a tail.

Now, suppose $\delta_W > 1$. Let Z be a connected components of W', and $Y := \overline{X - Z}$. Then Y is a tail. By induction, the map

$$C\colon Y \to J^0_Y; \quad Q \mapsto [\mathcal{I}_Q|_Y]$$

is well-defined, and is the degree-0 Abel map of Y with base P if $P \in Y$, and base N' if $P \notin Y$, where N' is the separating node of X generating Y. Also the isomorphism class of $\mathcal{I}_Q|_Z$ does not depend on $Q \in Y$.

Note that W is a spine of Y and $\#W \cap \overline{Y - W} = \delta_W - 1$. So, by induction, B is well-defined. Furthermore, the isomorphism class of $\mathcal{I}_Q|_{\overline{Y-W}}$ does not depend on $Q \in W$. Since neither does the isomorphism class of $\mathcal{I}_Q|_Z$, and $\overline{Y - W}$ does not intersect Z, we obtain (3).

By induction, if $P \in W$, and hence $P \in Y$, then B is the degree-0 Abel map of W with base P. If $P \notin W$, there are two cases to consider: If $P \notin Y$, that is, if $P \in Z$, then B is the degree-0 Abel map of W with base N'; and if $P \in Y$, then P belongs to a connected component of $\overline{X - W}$ other than Z, and thus B is the degree-0 Abel map of W with base the point of intersection of this component and W. □

Definition 4.7. A smooth rational component L of the curve X is called a *separating line* if L is a spine.

Theorem 4.8. *The degree-0 Abel map A_P* (defined in (4.5.1)) *is a morphism of schemes. Furthermore:*

(1) $A_P(X) \subseteq J_X^{0,P}$.
(2) *If X contains no separating lines, then A_P is an embedding.*

Proof. If X has no separating nodes, we proved the statements in Proposition 3.3. So, let W be a tail of X. We will assume, by induction, that the theorem holds for curves with less separating nodes than X. Assume, without loss of generality, that $P \in W'$. Let N be the separating node generating W. Notice that the separating nodes of X are N and those of W and W'. Thus W and W' have fewer separating nodes than X.

By Lemma 4.6, under the identification $J_X = J_W \times J_{W'}$ given by Lemma 4.3,

$$A_P|_W = (A_1, B)\colon W \longrightarrow J^0_W \times J^0_{W'}, \tag{4.8.1}$$

where A_1 is the degree-0 Abel map of W with base N and B is constant, and

$$A_P|_{W'} = (B', A_2)\colon W' \longrightarrow J^0_W \times J^0_{W'}, \tag{4.8.2}$$

where A_2 is the degree-0 Abel map of W' with base P and B' is constant. By induction, $A_P|_W$ and $A_P|_{W'}$ are morphisms of schemes. Then so is A_P, because W and W' intersect transversally at nonsingular points.

Now, let Q be a point of X. Let $Z_1, Z_2 \dots, Z_n$ be the P-tails of X containing Q. Using Lemma 4.2, as in the proof of Lemma 4.6, we may assume that

$$Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n.$$

Let $\mathcal{K} := \mathcal{O}_X(-\sum Z_i)$. Keep the notation of 4.5. We want to show that $\mathcal{I}_Q$ is P-quasistable. Let Y be a connected proper subcurve of X. It is enough to show that either $\deg_Y(\mathcal{I}_Q) \geq 0$, or

$$P \notin Y \quad \text{and} \quad \deg_Y(\mathcal{I}_Q) \geq -1 \geq -\delta_Y/2. \tag{4.8.3}$$

By [CE06] Lemma 4.8, we have $\deg_Y(\mathcal{K}) \geq -1$. Suppose first that $\deg_Y(\mathcal{K}) = -1$. By the same lemma, $Y \subseteq Z_1'$.

Suppose $Q \in Y$. Then $Q \in Y \cap Y'$ and Q is the separating node generating Z_1. So $\mathcal{I}_Q|_Y \cong \mathcal{O}_Y(P) \otimes \mathcal{K}|_Y$ if $P \in Y$ and $\mathcal{I}_Q|_Y \cong \mathcal{K}|_Y$ otherwise. If $P \in Y$, then $\deg_Y(\mathcal{I}_Q) = 0$. On the other hand, if $P \notin Y$, then $\delta_Y \geq 2$ and $\deg_Y(\mathcal{I}_Q) = -1$; so (4.8.3) holds.

Now, suppose $Q \notin Y$. Then $\deg_Y(\mathcal{I}_Q) = 0$ if $P \in Y$ and $\deg_Y(\mathcal{I}_Q) = -1$ if $P \notin Y$. We need only show now that, if Y is tail, then $P \in Y$. Indeed, if Y is a tail, then Y is generated by the same node that generates a Z_j, for a certain j, again by [CE06] Lemma 4.8. Since $Y \subseteq Z_1'$, we have $Y = Z_j'$, and hence $Y \ni P$.

Suppose now that $\deg_Y(\mathcal{K}) \geq 0$. Then $\deg_Y(\mathcal{I}_Q) \geq -1$. If $Q \notin Y$ or $P \in Y$, then $\deg_Y(\mathcal{I}_Q) \geq 0$. Now, suppose $Q \in Y$ and $P \notin Y$. If $\delta_Y \geq 2$, then (4.8.3) holds. On the other hand, if Y is a tail, then $Y = Z_j$ for a certain j, and hence $\deg_Y(\mathcal{K}) = 1$. In this case, $\deg_Y(\mathcal{I}_Q) = 0$.

At any rate, $\mathcal{I}_Q$ is P-quasistable. Since this holds for each $Q \in X$, the map A_P factors through $J^{0,P}_X$.

Finally, assume X contains no separating lines. First, observe that a separating node of W is a separating node of X. So, given a smooth, rational component L of W, since

$$L \cap L' \subseteq (L \cap \overline{W - L}) \cup \{N\},$$

not all points of $L \cap \overline{W - L}$ are separating nodes of W. By induction, A_1 is an embedding. By the same reasoning, A_2 is also an embedding. Hence $A_P|_W$ and $A_P|_{W'}$ are embeddings.

Now, let $Q_1 \in W$ and $Q_2 \in W'$ and assume that $A_P(Q_1) = A_P(Q_2)$. Then

$$A_1(Q_1) = B'(Q_2) \quad \text{and} \quad B(Q_1) = A_2(Q_2).$$

Since A_1 is injective, and $B'(W') = \{A_1(N)\}$, we have that $Q_1 = N$. Also, since A_2 is injective and $B(W) = \{A_2(N)\}$, we have $Q_2 = N$. Hence $Q_1 = Q_2$. It follows that A_P is injective.

Also, since $A_P|_W$ and $A_P|_{W'}$ are immersions, so is A_P everywhere but possibly at N. But A_P is an immersion also at N, because $A_P|_W$ and $A_P|_{W'}$ are immersions at N, and, under the identification $J_X = J_W \times J_{W'}$ given by Lemma 4.3, they take the tangent spaces of W and W' at N into the linearly independent subspaces $T_{J_W,A_1(N)} \oplus 0$ and $0 \oplus T_{J_{W'},A_2(N)}$ of $T_{J_X,A_P(N)}$, respectively.

Thus A_P separates points and tangent vectors. Since X is complete, and A_P factors through $J_X^{0,P}$, which is separated, A_P is an embedding. □

5. Twisted Abel maps of degree 1

5.1. (*Small tails and splitting nodes.*) We set now a rule that associates to every separating node N of the curve X exactly one of the two tails that N generates; we shall call the chosen tail the *small tail* generated by N, and denote it by Z_N. To do this, let Z and Z' be the two tails generated by N; then $g_Z + g_{Z'} = g$. There are two cases:

(1) If $g_Z \neq g_{Z'}$, let Z_N be the one between Z and Z' having smaller genus. Thus $g_{Z_N} < g/2$.
(2) If $g_Z = g_{Z'} = g/2$, make an arbitrary choice between Z and Z', and set it equal to Z_N. In this case we call N a *splitting node* of X.

We denote by $\mathcal{ST}(X) \subset \mathcal{T}(X)$ the set of all small tails of X. By definition, there is a bijection between the set of separating nodes of X and $\mathcal{ST}(X)$. Observe that, if X has a splitting node, $\mathcal{ST}(X)$ depends upon the choice made in (2) above.

5.2. (*Degree-*1 *twisted Abel maps*). For each $Q \in X$ define a torsion-free, rank-1 sheaf $\mathcal{N}_Q$ on X as follows. If Q is not a separating node, let $\mathcal{N}_Q$ be the sheaf of ideals of Q. If Q is a separating node, let $Z \in \mathcal{ST}(X)$ be the unique small tail attached to it, and let $\mathcal{N}_Q$ be the unique line bundle on X such that

$$\mathcal{N}_Q|_Z \cong \mathcal{O}_Z(-Q) \quad \text{and} \quad \mathcal{N}_Q|_{Z'} \cong \mathcal{O}_{Z'}.$$

(That $\mathcal{N}_Q$ exists and is unique follows from Lemma 4.3.) Note that $\mathcal{N}_Q$ is simple.

Define a map

$$A\colon X \longrightarrow J_X^1 \tag{5.2.1}$$

sending a point Q of X to $[\mathcal{I}_Q^1]$, where

$$\mathcal{I}_Q^1 := (\mathcal{N}_Q)^* \otimes \mathcal{O}_X(\sum_{Z\in\mathcal{ST}(X)\,:\,Z\ni Q} Z), \tag{5.2.2}$$

the sum running over all small tails of X containing Q. Since X is Gorenstein, and $\mathcal{N}_Q$ is simple, so is $\mathcal{I}_Q^1$, and hence A is well-defined. We call A the *degree-*1 *Abel map* of X. If X has no separating nodes then (5.2.1) coincides with (3.4.1). But if X admits a splitting node N, then the definition of A depends on the choice of the small tail Z_N associated to N (see 5.1). We will see in Theorem 5.4 that A is a morphism of schemes.

Lemma 5.3. *If the curve X is G-stable, then the following statements hold:*

(1) *If Z_1 and Z_2 are tails of X with $Z_1 \subsetneqq Z_2$, then $g_{Z_1} < g_{Z_2}$.*
(2) *The curve X has at most one splitting node* (defined in 5.1).
(3) *There is a point Q on the nonsingular locus of X such that $\mathcal{ST}(X) = \mathcal{T}_Q(X)$.*

Proof. We prove Statement (1). As X is G-stable, ω is an ample line bundle. So

$$2g_{Z_1} - 1 = \deg_{Z_1}(\omega) < \deg_{Z_2}(\omega) = 2g_{Z_2} - 1,$$

and hence $g_{Z_1} < g_{Z_2}$.

As for Statement (2), suppose X has a splitting node Q, and let Z be a tail it generates. Thus $g_Z = g_{Z'} = g/2$. By contradiction, assume X has another splitting node, N; we may assume $N \in Z$. Then N generates two tails of Z, one of which is a tail of X. So Z contains properly a tail of X of genus $g/2 = g_Z$, contradicting Statement (1).

Finally, let us prove the last statement. We may assume that X has a tail, hence a small tail, hence a maximal small tail, Z. Let Q be a point on the nonsingular locus of X lying on the irreducible component of Z' containing $Z \cap Z'$. We claim that $\mathcal{ST}(X) = \mathcal{T}_Q(X)$. Indeed, let W be a tail of X. Suppose first that $Q \in W$. Then $W \cap Z \neq \emptyset$ and $W \not\subseteq Z$. If $W \cup Z = X$ then $W \supseteq Z'$, and since Z' is not small, neither is W. If instead $W \cup Z \neq X$, then $Z \subsetneqq W$, by Lemma 4.2, and thus W is again not small, by the maximality of Z. Conversely, suppose W is not small. Then W' is. But then, as we have just proved, $Q \notin W'$, and so $Q \in W$. □

Theorem 5.4. *The degree-*1 *Abel map A* (defined in (5.2.1)) *is a morphism of schemes. Furthermore:*

(1) *If X has no separating lines, then A is an embedding.*
(2) *If X is G-stable and P does not lie on any small tail of X, then $A(X) \subseteq J_X^{1,P}$.*
(3) *If X is G-stable then $A(X) \subseteq J_X^{1,ss}$.*

Proof. By Lemma 5.3, there is a point Q on the nonsingular locus of X such that $\mathcal{ST}(X) = \mathcal{T}_Q(X)$, or equivalently, such that Q does not lie on any small tail of X. So Statement (3) follows from (2). Also, A is a morphism of schemes because, as in the proof of Proposition 3.5, it is the composition of A_Q, the degree-0 Abel map of X with base Q, with the duality map and the translation-by-Q map. Furthermore, if X contains no separating lines, since the dualizing and translation maps are isomorphisms, and since A_Q is an embedding by Theorem 4.8, also A is an embedding.

Let us prove Statement (2). Let $Q \in X$, and let $Z_1, Z_2 \dots, Z_n$ be the small tails of X containing Q. Using Lemma 4.2, as in the proof of Lemma 4.6, we may assume that

$$Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n.$$

By hypothesis, $P \notin Z_n$.

Keep the notation of 5.2. Let W be a connected proper subcurve of X. We need to show that

$$\deg_W(\mathcal{I}_Q^1) \geq \frac{\deg_W(\omega)}{2g-2} - \frac{\delta_W}{2}, \tag{5.4.1}$$

with equality only if $P \notin W$.

Let $\mathcal{K} := \mathcal{O}_X(\sum Z_i)$. By [CE06] Lemma 4.8, we have $\deg_W(\mathcal{K}) \geq -1$. Suppose first that $\deg_W(\mathcal{K}) = -1$. Then $\deg_W(\mathcal{I}_Q^1) \geq -1$. Also, by the same lemma, there is a unique integer j such that the separating node generating Z_j is contained in $W \cap W'$. In addition, $W \subseteq Z_j$, and in particular, $W \subseteq Z_n$. So $P \notin W$. Since $W \subseteq Z_n$, and since Z_n is a small tail,

$$\deg(\omega|_W) \leq \deg(\omega|_{Z_n}) \leq g - 1.$$

So, if $\delta_W \geq 3$ then (5.4.1) holds. On the other hand, suppose $\delta_W = 1$. Since $W \cap W'$ contains the separating node of Z_j, and $W \subseteq Z_j$, we must have $W = Z_j$. In this case, $\deg_W(\mathcal{I}_Q^1) = 0$, and hence (5.4.1) holds as well.

Now, suppose $\delta_W = 2$. We need to show that $\deg_W(\mathcal{I}_Q^1) \geq 0$. Since W contains the separating node generating Z_j, and $W \subseteq Z_j$, we have that $W = \overline{Z_j - Y}$ for a certain tail Y of X properly contained in Z_j. Since Z_j is a small tail, so is Y. If $Y = Z_i$ for a certain $i < j$, then $W \cap W'$ would also contain the separating node generating Z_i, which is not possible. So, since Y is a small tail, $Q \notin Y$. Thus $Q \in W$. If $Q \notin W \cap W'$ then $\deg_W(\mathcal{N}_Q^*) = 1$, and hence $\deg_W(\mathcal{I}_Q^1) = 0$. On the other hand, if $Q \in W \cap W'$, since $Q \notin Y$, it follows that Q is the separating node generating Z_j. Then $j = 1$ and $\deg_W(\mathcal{N}_Q^*) = 1$ as well, implying that $\deg_W(\mathcal{I}_Q^1) = 0$.

The upshot is that (5.4.1) holds and $P \notin W$ if $\deg_W(\mathcal{K}) = -1$. Now, suppose $\deg_W(\mathcal{K}) \geq 0$. Then $\deg_W(\mathcal{I}_Q^1) \geq 0$. If W is not a tail, then (5.4.1) holds, and the inequality is strict because $W \neq X$, and hence $\deg_W(\omega) < 2g - 2$. Suppose now that W is a tail. There are two cases to consider: $Q \in W$ and $Q \notin W$.

Suppose first that $Q \in W$. If $W \not\subseteq Z_1'$, then $\deg_W(\mathcal{N}_Q^*) = 1$, and hence $\deg_W(\mathcal{I}_Q^1) \geq 1$. In this case, Inequality (5.4.1) holds and is strict. On the other hand, if $W \subseteq Z_1'$, since $Q \in Z_1$, we get that Q is a separating node and $W = Z_1'$. In this case,

$$\deg_W(\mathcal{I}_Q^1) = \deg_W(\mathcal{K}) = 1,$$

and thus Inequality (5.4.1) holds as well and is strict.

Now, suppose $Q \notin W$. Then $W' \ni Q$. If $P \in W$, then W' is a P-tail, and hence a small tail. Since $W' \ni Q$, we have that $W' = Z_i$ for a certain integer i, and hence $\deg_W(\mathcal{K}) = 1$. Again, Inequality (5.4.1) holds and is strict. On the other hand, if $P \notin W$, then W is a small tail, and hence $\deg_W(\omega) \leq g - 1$. Since $\deg_W(\mathcal{I}_Q^1) \geq 0$, Inequality (5.4.1) holds.

At any rate, $\mathcal{I}_Q^1$ is P-quasistable. Since this holds for every $Q \in X$, it follows that $A(X) \subseteq J_X^{1,P}$. □

6. Properties of the Abel maps

Lemma 6.1. *The curve X has genus $g = 0$ if and only if every irreducible component of X is a separating line.*

Proof. By induction on the number of irreducible components of X. If X is irreducible, $g = 0$ implies that X is smooth; so the lemma holds trivially. Suppose now that X is reducible, and let L be an irreducible component of X and $Z_1, \dots, Z_n$ the connected components of L'. Since X is Gorenstein and L is irreducible, L is a separating line if and only if $g_L = 0$ and $\text{length}(L \cap Z_i) = 1$ for each $i = 1, \dots, n$ or, equivalently, $g_L = 0$ and

$$\text{length}(L \cap Z_1) + \cdots + \text{length}(L \cap Z_n) = n. \tag{6.1.1}$$

Consider the cohomology sequence associated to the natural exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_L \oplus \mathcal{O}_{Z_1} \oplus \cdots \oplus \mathcal{O}_{Z_n} \longrightarrow \mathcal{O}_{L\cap Z_1} \oplus \cdots \oplus \mathcal{O}_{L\cap Z_n} \longrightarrow 0. \tag{6.1.2}$$

If $h^1(X, \mathcal{O}_X) = g = 0$ then $g_L = h^1(L, \mathcal{O}_L) = 0$ and (6.1.1) holds, whence L is a separating line.

The converse is immediate, as every irreducible component of X is isomorphic to $\mathbb{P}^1$ and every singularity is a separating node (see Example 9.8 in [C05]). □

Definition 6.2. A *separating tree of lines* of the curve X is a spine of (arithmetic) genus 0. Equivalently, using Lemma 6.1, a separating tree of lines of X is a connected union of separating lines of X.

Theorem 6.3. *Let A and A_P be the Abel maps of the curve X. Assume $g > 0$. Set $B := A$ or $B := A_P$, and let $\widetilde{X} := B(X)$. Then the following statements hold:*

1. *Let S be the union of all separating lines of X. Let $Y_1, \dots, Y_n$ be the connected components of S'. Then $B|_{Y_i}$ is an embedding for each $i = 1, \dots, n$.*
2. *For any two distinct points $Q_1, Q_2 \in X$, we have that $B(Q_1) = B(Q_2)$ if and only if Q_1 and Q_2 belong to the same separating tree of lines.*
3. *Let L be a maximal separating tree of lines of X, and let $R \in J_X$ for which $B(L) = \{R\}$. Let $N_1, \dots, N_\delta$ be the points of $L \cap L'$. Then $\widetilde{X}$ has an ordinary δ-fold singularity at R, with linearly independent tangent lines equal to the images of the differentials $d_{N_i}B : T_{X,N_i} \to T_{J_X,R}$.*
4. *$\widetilde{X}$ is a curve of arithmetic genus g.*

Proof. Assume first that $B := A_P$. For each $Q \in X$, let $\mathcal{I}_Q$ be the simple, torsion-free, rank-1 sheaf on X such that $B(Q) = [\mathcal{I}_Q]$.

We prove Statement (1). First, observe that each Y_i is a spine, and contains no separating lines. Since Y_i is a spine, using Lemma 4.3, it is enough to show that the map

$$B_i \colon Y_i \to J_{Y_i}; \quad Q \mapsto [\mathcal{I}_Q|_{Y_i}]$$

is an embedding. But, by Lemma 4.6, the map B_i is a degree-0 Abel map. And, since Y_i contains no separating lines, Theorem 4.8 implies that B_i is an embedding.

Consider Statement (2), keeping the notation of Statement (1). Suppose first that Q_1 and Q_2 belong to the same separating tree of lines. Since the tree is connected, to show that $B(Q_1) = B(Q_2)$ we may assume that Q_1 and Q_2 lie on the smooth locus of X and on the same separating line, L. But then $\mathcal{I}_{Q_1}|_L \cong \mathcal{I}_{Q_2}|_L$ and $\mathcal{I}_{Q_1}|_{L'} \cong \mathcal{I}_{Q_2}|_{L'}$, as is easily seen. Since L is spine, Lemma 4.3 yields $\mathcal{I}_{Q_1} \cong \mathcal{I}_{Q_2}$.

Suppose now that Q_1 and Q_2 do not belong to the same tree of separating lines. We must prove that $B(Q_1) \neq B(Q_2)$. As we have just seen that B is constant along separating lines, we may assume that $Q_1, Q_2 \in Y_1 \cup \cdots \cup Y_n$. Let $L_1, \ldots, L_m$ be the connected components of S. Since X is connected, there are a positive integer t, and integers $i_1, \ldots, i_t \in \{1, \ldots, n\}$ and $j_1, \ldots, j_{t-1} \in \{1, \ldots, m\}$ such that $Q_1 \in Y_{i_1}$ and $Q_2 \in Y_{i_t}$, while $Y_{i_\ell} \cap L_{j_\ell} \neq \emptyset$ and $L_{j_\ell} \cap Y_{i_{\ell+1}} \neq \emptyset$ for each $\ell = 1, \ldots, t-1$. Choose t minimum; then $Y_{i_\ell} \neq Y_{i_t}$ for every $\ell < t$. We will show that $B(Q_1) \neq B(Q_2)$ by induction on t.

If $t = 1$ then Q_1 and Q_2 belong to the same Y_i, and hence $B(Q_1) \neq B(Q_2)$ by Statement (1). Suppose now that $t \geq 2$. And suppose that $B(Q_1) = B(Q_2)$, by contradiction. In particular, using Lemma 4.3, we have $\mathcal{I}_{Q_1}|_{Y_{i_t}} \cong \mathcal{I}_{Q_2}|_{Y_{i_t}}$. But, since Y_{i_t} is a spine, it follows from Lemma 4.6 that $\mathcal{I}_{Q_1}|_{Y_{i_t}} \cong \mathcal{I}_{Q}|_{Y_{i_t}}$ for each Q on the same connected component of Y'_{i_t} as Q_1.By connectedness, there will be such Q on $L_{i_{t-1}} \cap Y_{i_t}$. Since $\mathcal{I}_Q|_{Y_{i_t}} \cong \mathcal{I}_{Q_2}|_{Y_{i_t}}$, and since B_{i_t} is an embedding, as we saw in the proof of Statement (1), it follows that $Q_2 \in L_{i_{t-1}}$. Since $L_{i_{t-1}}$ is a separating tree of lines, $B(Q_2) = B(M)$ for any chosen $M \in Y_{i_{t-1}} \cap L_{i_{t-1}}$. Since $B(Q_1) = B(M)$, it follows by induction that $M = Q_1$. But then Q_1 and Q_2 are on the same separating tree of lines, namely $L_{i_{t-1}}$, reaching a contradiction.

Finally, we prove Statements (3) and (4). We proceed by induction on the number of separating nodes of X. If zero, then (3) is vacuous and (4) follows from (1).

Now, let L be a maximal tree of separating lines of X, and $R \in J_X$ such that $B(L) = \{R\}$. Notice that $L \neq X$ because $g > 0$. Let $Z_1, \ldots, Z_n$ be the connected components of L' and $N_1, \ldots, N_\delta$ the points in $L \cap L'$, Since L is a spine with genus 0, we have that $n = \delta$ and

$$g = g_{Z_1} + \cdots + g_{Z_n}. \tag{6.3.1}$$

Also, up to reordering the Z_i, we may assume that Z_i is generated by N_i for each $i = 1, \ldots, n$. Notice that, for each i, no separating line of Z_i contains N_i, because otherwise its union with L would be a tree of separating lines of X larger than L itself. Also, notice that each Z_i has less separating nodes than X.

Since $(L, Z_1, \ldots, Z_n)$ is a spine decomposition of X, by Lemma 4.3 there is a natural isomorphism

$$u = (u_L, u_1, \ldots, u_n)\colon J_X \longrightarrow J_L \times J_{Z_1} \times \cdots \times J_{Z_n}.$$

For each $i = 1, \ldots, n$, let $\widetilde{Z}_i := B(Z_i)$. By Lemma 4.6, as Q moves on Z_i, the images $u_L(B(Q))$ and $u_j(B(Q))$, for $j \neq i$, remain constant. Thus $\widetilde{Z}_i \cong u_i(B(Z_i))$. However, by the same Lemma 4.6, the composition $u_i \circ B|_{Z_i}$ is a degree-0 Abel map. So, by induction, $\widetilde{Z}_i$ has genus g_{Z_i}.

Now, since N_i is not contained in a separating line of Z_i, it follows from Statement (1) that $d_{N_i}B$ is injective on T_{Z_i,N_i}, and hence $\widetilde{Z}_i$ is nonsingular at R, with $T_{\widetilde{Z}_i,R} = d_{N_i}B(T_{Z_i,N_i})$. Since B contracts L, by Statement (2), it follows that $T_{\widetilde{Z}_i,R}$ is the whole image of $d_{N_i}B$.

Using u to make the identification

$$T_{J_X,R} = T_{J_L,u_L(R)} \oplus T_{J_{Z_1},u_1(R)} \oplus \cdots \oplus T_{J_{Z_n},u_n(R)},$$

we may view $T_{\widetilde{Z}_i,R}$ as a subspace of

$$0 \oplus 0 \oplus \cdots \oplus 0 \oplus T_{J_{Z_i},u_i(R)} \oplus 0 \oplus \cdots \oplus 0$$

for each $i = 1, \dots, n$. Thus the $T_{\widetilde{Z}_i,R}$ are linearly independent subspaces of $T_{J_X,R}$. Since R lies on the nonsingular loci of all the $\widetilde{Z}_i$, and the $\widetilde{Z}_i$ cover $\widetilde{X}$, it follows that $\widetilde{X}$ has a n-fold singularity at R, with the $T_{\widetilde{Z}_i,R}$ for tangent lines, finishing the proof of Statement (3).

As for Statement (4), since the $\widetilde{Z}_i$ intersect transversally at R, we have exact sequences of the form:

$$0 \to \mathcal{O}_{\tilde{Z}_1}(-R) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Z}_2\cup\cdots\cup\tilde{Z}_n} \to 0,$$

$$0 \to \mathcal{O}_{\tilde{Z}_2}(-R) \to \mathcal{O}_{\tilde{Z}_2\cup\cdots\cup\tilde{Z}_n} \to \mathcal{O}_{\tilde{Z}_3\cup\cdots\cup\tilde{Z}_n} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{O}_{\tilde{Z}_{n-1}}(-R) \to \mathcal{O}_{\tilde{Z}_{n-1}\cup\tilde{Z}_n} \to \mathcal{O}_{\tilde{Z}_n} \to 0.$$

Computing Euler characteristics, and using that $\widetilde{Z}_i$ has genus g_{Z_i} for $i = 1, \dots, n$,

$$\chi(\mathcal{O}_{\widetilde{X}}) = -g_{Z_1} - g_{Z_2} \cdots - g_{Z_{n-1}} + 1 - g_{Z_n} = 1 - g,$$

where the last equality is (6.3.1). Thus $\widetilde{X}$ has genus g.

Finally, choosing a nonsingular point Q of X away from all small tails of X, we have that $A\colon X \to J_X^1$ is obtained from A_Q by taking duals and then translating by $\mathcal{O}_X(Q)$. Since these operations are isomorphisms, all of the statements proved above for A_Q hold for A. □

7. Abel maps to Seshadri's compactified Jacobian

Recall that for each point Q of the curve X we defined the simple, torsion-free, rank-1, degree-1 sheaf $\mathcal{I}_Q^1$, and let $A(Q) := [\mathcal{I}_Q^1]$; see 5.2. When X has a splitting node N, the definition of $\mathcal{I}_Q^1$ and that of A depend on the choice of one of the two tails generated by N (the small tail Z_N).

If X is G-stable, Theorem 5.4 says that $\mathcal{I}_Q^1$ is semistable with respect to the canonical 1-polarization E_1 (cf. 2.3.2). Its Jordan–Hölder filtrations (cf. 2.5) are easy to describe.

Lemma 7.1. *Assume the curve X is G-stable, and let $Q \in X$. Then $\mathcal{I}_Q^1$ is stable if and only if X admits no splitting node. If X has a splitting node N, and Z_N is the small tail generated by N, then*

$$\emptyset \subsetneqq Z_N \subsetneqq X$$

is the unique Jordan–Hölder filtration of $\mathcal{I}_Q^1$.

Proof. As mentioned before the statement, $\mathcal{I}_Q^1$ is semistable by Theorem 5.4. So, given a proper subcurve $Y \subset X$, we have

$$\deg_Y(\mathcal{I}_Q^1) \geq \frac{\deg_Y(\omega)}{2g-2} - \frac{\delta_Y}{2}. \tag{7.1.1}$$

We must show that equality holds in (7.1.1) if and only if there is a splitting node N and $Y = Z_N$. For this, we may assume that Y is connected.

Since X is G-stable,

$$0 < \frac{\deg_Y(\omega)}{2g-2} < 1.$$

Also, as seen in the proof of Theorem 5.4, a consequence of [CE06] Lemma 4.8, we have $\deg_Y(\mathcal{I}^1_Q) \geq -1$. Hence, since δ_Y is a positive integer, equality in (7.1.1) may hold only in two cases:

(1) $\deg_Y(\mathcal{I}^1_Q) = -1$, while $\deg_Y(\omega) = g-1$ and $\delta_Y = 3$.
(2) $\deg_Y(\mathcal{I}^1_Q) = 0$, while $\deg_Y(\omega) = g-1$ and $\delta_Y = 1$.

We claim that the first case is not possible. Indeed, suppose by contradiction that it occurs. Since $\delta_Y = 3$, we have $\deg_Y(\omega) = 2g_Y + 1$, which implies that $g_Y = g/2 - 1$. But since $\deg_Y(\mathcal{I}^1_Q) = -1$, it follows that

$$\deg_Y(\mathcal{O}_X(\sum_{Z \in \mathcal{ST}(X)\,;\, Z \ni Q} Z)) = -1.$$

So, by [CE06] Lemma 4.8, we have that $Y \subseteq Z$ for a certain small tail Z. Now, $Y \neq Z$ because $\delta_Y = 3$. Then, since ω is ample, $\deg_Z(\omega) > \deg_Y(\omega)$. On the other hand, since Z is a small tail, $\deg_Z(\omega) \leq g-1$, and hence $\deg_Z(\omega) \leq \deg_Y(\omega)$, reaching a contradiction.

Now, suppose the second case occurs. Then Y is a tail of genus $g/2$. So X has a splitting node N, and we must show that $Y = Z_N$. Suppose by contradiction that $Y \neq Z_N$ or, in other words, that Y' is a small tail. There are two cases to consider: $Q \in Y$ and $Q \in Y'$. If $Q \in Y'$, then Y' is the largest small tail containing Q, and hence $\deg_Y(\mathcal{I}^1_Q) = 1$, a contradiction.

Suppose now that $Q \in Y$. Since $\deg_Y(\mathcal{I}^1_Q) \neq 1$, there is a small tail Z of X containing Q. For each such Z, we have that $Z \not\supseteq Y$, because otherwise the smallness of Z would imply that $Y = Z$, and Y is not small. Also, $Z \not\supseteq Y'$, because otherwise the smallness of Z would imply that $Z = Y'$, and thus $Q \in Y \cap Y'$. But in this case, Y' would be the unique small tail containing Q, implying that $\deg_Y(\mathcal{I}^1_Q) = 1$, a contradiction. By Lemma 4.2, the only possibility left is that $Z \subsetneq Y$. But since this must hold for each small tail Z containing Q, we would get $\deg_Y(\mathcal{I}^1_Q) = 1$ again, a contradiction.

Conversely, let N be a splitting node, and suppose $Y = Z_N$. We must show that $\deg_Y(\mathcal{I}^1_Q) = 0$. If $Q \in Y$, then Y is the largest small tail containing Q, and hence $\deg_Y(\mathcal{I}^1_Q) = 0$. On the other hand, if $Q \notin Y$, then all small tails containing Q are strictly contained in Y', and hence $\deg_Y(\mathcal{I}^1_Q) = 0$ as well. □

Theorem 7.2. *Assume that the curve X is G-stable. Let $A\colon X \to J^1_X$ be the degree-*1 *Abel map* (cf. 5.2) *and $\Phi^1\colon J^{1,ss}_X \to U_X(1)$ the S-map* (cf. (2.6.3)). *Then the following two statements hold:*

(1) *$\Phi^1 \circ A$ is independent of the choice of a small tail of genus $g/2$.*
(2) *Φ^1 restricts to a closed embedding on $A(X)$.*

Proof. Suppose that X admits a splitting node N (unique by Lemma 5.3). Let Z be a tail generated by N. Then A depends on whether we choose Z or Z' as small tail or, in other words, whether we set $Z_N := Z$ or $Z_N = Z'$. Let $Q \in X$. Let $\mathcal{I}$ (resp.

$\mathcal{I}'$) be the torsion-free, rank-1 sheaf on X such that $A(Q)=[\mathcal{I}]$ (resp. $A(Q)=[\mathcal{I}']$), when $Z_N=Z$ (resp. $Z_N=Z'$). By Lemma 7.1, we have $\mathfrak{S}(\mathcal{I})=\mathfrak{S}(\mathcal{I}')=\{Z,Z'\}$. Also, $\mathcal{I}'|_Z\cong\mathcal{I}|_Z\otimes\mathcal{O}_Z(N)$ and $\mathcal{I}'|_{Z'}\cong\mathcal{I}|_{Z'}\otimes\mathcal{O}_{Z'}(-N)$, and hence

$$\mathrm{Gr}(\mathcal{I})=\mathcal{I}|_Z\oplus(\mathcal{I}|_{Z'}\otimes\mathcal{O}_{Z'}(-N))\cong(\mathcal{I}'|_Z\otimes\mathcal{O}_Z(-N))\oplus\mathcal{I}'|_{Z'}=\mathrm{Gr}(\mathcal{I}').$$

So $\mathcal{I}$ and $\mathcal{I}'$ are S-equivalent, and thus $\Phi^1(A(Q))$ is independent of the choice of Z_N.

As for the second statement, we need only show that $\Phi^1|_{A(X)}$ separates points and tangent vectors. For each $Q\in X$, let $\mathcal{I}^1_Q$ be the torsion-free, rank-1 sheaf on X such that $A(Q)=[\mathcal{I}^1_Q]$. To show that $\Phi^1|_{A(X)}$ separates points, we need to show that for each $Q,Q'\in X$,

$$\mathfrak{S}(\mathcal{I}^1_Q)=\mathfrak{S}(\mathcal{I}^1_{Q'})\text{ and }\mathrm{Gr}(\mathcal{I}^1_Q)\cong\mathrm{Gr}(\mathcal{I}^1_{Q'})\quad\text{if and only if}\quad\mathcal{I}^1_Q\cong\mathcal{I}^1_{Q'}.\tag{7.2.1}$$

Now, if X contains no splitting node then Lemma 7.1 asserts that $\mathcal{I}^1_Q$ and $\mathcal{I}^1_{Q'}$ are stable, and thus (7.2.1) holds trivially. On the other hand, if X has a splitting node N, then Lemma 7.1 implies that $\mathfrak{S}(\mathcal{I}^1_Q)=\{Z,Z'\}$ and

$$\mathrm{Gr}(\mathcal{I}^1_Q)=\mathcal{I}^1_Q|_Z\oplus(\mathcal{I}^1_Q|_{Z'}\otimes\mathcal{O}_{Z'}(-N)),$$

where $Z:=Z_N$. An analogous description holds for $\mathcal{I}^1_{Q'}$. So $\mathfrak{S}(\mathcal{I}^1_Q)=\mathfrak{S}(\mathcal{I}^1_{Q'})$, and $\mathrm{Gr}(\mathcal{I}^1_Q)\cong\mathrm{Gr}(\mathcal{I}^1_{Q'})$ if and only if $\mathcal{I}^1_Q|_Z\cong\mathcal{I}^1_{Q'}|_Z$ and $\mathcal{I}^1_Q|_{Z'}\cong\mathcal{I}^1_{Q'}|_{Z'}$, which, by Lemma 4.3, occurs if and only if $\mathcal{I}^1_Q\cong\mathcal{I}^1_{Q'}$. So (7.2.1) holds.

Finally, for any $Q\in X$, Lemma 7.1 implies that $\mathfrak{S}(\mathcal{I}^1_Q)$ is a spine decomposition, whether X admits a splitting node or not. So, given any nonzero $v\in T_{J_X,A(Q)}$, it follows from [E07] Lemma 3.11 and Prop. 4.3 that there is a subscheme $\Theta\subseteq U_X(a,d)$ such that $d\Phi_{A(Q)}(v)\notin T_{\Theta,\Phi(A(Q))}$. Thus $\Phi|_{A(X)}$ separates tangent vectors. □

Remark 7.3. In general, it is not true that $\Phi^0\colon J^{0,ss}_X\to U_X(0)$ restricts to a closed embedding on $A_P(X)$. For a simple example, suppose X is a nodal curve with four irreducible components: two of them, X_1 and X_2, smooth and rational, meeting at two points, N_1 and N_2; the third, X_3, nonrational, meeting only X_1 at a single point N_3, and the fourth, X_4, also nonrational, meeting only X_2 at a single point N_4. From its description, X is stable. Also, it contains no separating lines, so A_P is an embedding by Theorem 4.8. Suppose further that $g_{X_3}=g_{X_4}$. Then the only small tails of X are X_3 and X_4.

Suppose $P\in X_1$. For each $Q\in X$, let $\mathcal{I}_Q$ be the torsion-free, rank-1 sheaf on X such that $A_P(Q)=[\mathcal{I}_Q]$ (cf. 4.5). Let $Q\in X_2-\{N_1,N_2,N_4\}$. Given a proper subcurve Y of X, we have that $\deg_Y(\mathcal{I}_Q)=-\delta_Y/2$ if and only if $Y=X_2\cup X_4$. Thus $\mathfrak{S}(\mathcal{I}_Q)=\{X_1\cup X_3,X_2\cup X_4\}$ and

$$\mathrm{Gr}(\mathcal{I}_Q)=\mathcal{O}_{X_2\cup X_4}(-Q)\oplus\mathcal{O}_{X_1\cup X_3}(P-N_1-N_2).$$

Now, since X_2 is rational, Lemma 4.3 yields that, as Q moves on $X_2-\{N_1,N_2,N_4\}$, the isomorphism class of $\mathcal{O}_{X_2\cup X_4}(-Q)$ does not change. Therefore, the composition $\Phi^0\circ A_P$ contracts X_2.

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