ON ABEL MAPS OF STABLE CURVES

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Contents

1. Introduction

We construct Abel maps for a stable curve X. Namely, for each one-parameter deformation of X to a smooth curve, having regular total space, and each $d \geq 1$, we construct by specialization a map $\alpha_X^d : \dot{X}^d \to \overline{P_X^d}$, where $\dot{X} \subseteq X$ is the smooth locus, and $\overline{P_X^d}$ is the coarse moduli scheme for equivalence classes of degree-d "semibalanced" line bundles on semistable curves having X as a stable model. For $d = 1$, we show that α_X^1 extends to a map $\overline{\alpha_X^1} : X \to \overline{P_X^1}$, and does not depend on the choice of the deformation. Finally, we give a precise description of when $\overline{\alpha_X^1}$ is injective.

The theory of Abel maps for smooth curves goes back to the 19th century. In the modern language, let C be a smooth projective curve, and $\operatorname{Pic}^d C$ its degree-d Picard variety parametrizing line bundles of degree d on C. For each $d > 0$ there exists a remarkable morphism, often called the d-th *Abel map*:

$$\begin{array}{ccc} C^d & \longrightarrow & \operatorname{Pic}^d C \\ (p_1, \dots, p_d) & \mapsto & \mathcal{O}_C(\sum p_i). \end{array}$$

This map has been extensively studied and used in the literature. For $d = 1$, after the choice of a "base" point on C, it gives the Abel–Jacobi embedding $C \hookrightarrow \operatorname{Pic}^1 C \cong \operatorname{Pic}^0 C$ (unless $C \cong \mathbb{P}^1$). For an interesting historic survey see [K04] or [K05].

What about Abel maps for singular curves? Abel maps were constructed for all integral curves in [AK], and further studied in [EGK00], [EGK02] and [EK05]. In [AK], it is shown that the first Abel map of an integral singular curve is an embedding into its compactified Picard scheme. However almost nothing is known for reducible curves, not even when they are stable. This lack of knowledge appears all the more regrettable because of the importance of stable curves in moduli theory.

In the present paper we construct Abel maps for stable curves. As we see it, Abel maps should satisfy the following natural properties. First, they should have a geometric meaning. More explicitly, recall that for a smooth

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curve C the d-th Abel map is the "moduli map" defined by a natural line bundle on $C^d \times C$; see 2.5. We want a similar property to hold for singular curves as well.

Second, Abel maps should vary continuously in families. In particular, given a one-parameter family of smooth curves specializing to a singular curve, we expect the d-th Abel maps of the smooth fibers to specialize to the d-th Abel map of the singular fiber.

Both requirements turn out to be nontrivial. In order to address the second one we view stable curves as limits of smooth ones. So, let X be a stable curve and $f : \mathcal{X} \to B$ be a family of curves over a local one-dimensional regular base B, with regular total space $\mathcal{X}$, smooth generic fiber and X as closed fiber. We observe that there exists a canonical way to partially extend the d-th Abel map of the generic fiber of f, by using the Néron model N_f^d of the degree-d Picard scheme of that fiber. The Néron mapping property yields a close relative of the d-th Abel map of X, defined on the nonsingular locus $\dot{X}^d \subseteq X^d$, which we call the d-th *Abel–Néron map* of X; see 2.6. The target of this map is the closed fiber of N_f^d, rather than the Picard scheme of X.

Néron models appeared first in [N64]; good references in a more modern language are [R70] and [BLR]. The great advantage of Abel–Néron maps is their naturality, obtained directly from the universal property of Néron models. However, they have two major drawbacks. First, they do not have any a-priori modular interpretation. Second, they are not defined on the whole X^d.

To attack these problems we consider the geometric compactified Picard scheme introduced in [C94] and further studied in [C05]. If X is suitably general, more precisely "d-general" (Definition 3.6), there exists a proper B-scheme $\overline{P_f^d}$ that is a coarse moduli space for equivalence classes of degree-d "semibalanced" line bundles on semistable curves having the fibers of f as stable models; see 3.8. These are line bundles whose multidegree satisfies certain inequalities; see Definition 3.2. It is shown in [C05] that $\overline{P_f^d}$ contains N_f^d as a dense open subscheme. Thus not only does $\overline{P_f^d}$ give a geometrically meaningful description of N_f^d, but also a completion of it; alternatively see (6) and (14).

So, assume for now that X is d-general. Let $\overline{P_X^d}$ be the closed fiber of $\overline{P_f^d}$. It does not depend on the choice of f, as it is explained in 3.8; see also [C05], Section 5. Because N_f^d sits inside $\overline{P_f^d}$, we obtain our *d-th Abel map* $\alpha_X^d : \dot{X}^d \to \overline{P_X^d}$ (see Theorem 3.10).

The map α_X^d is modular but an explicit description for it is hard to exhibit in full generality; we do this only for curves with two components (see Proposition 3.12). The case $d = 1$ turns out to be easier. By means of Theorem 4.6 we give an explicit description of the line bundle defining α_X^1. Using this description, we show in Corollary 4.10 that α_X^1 does not depend on the choice of f, a remarkable property not to be expected in general for $d > 1$; see Remark 3.14. (More precisely, this property holds for d smaller than a certain invariant of the graph of X; see [C06].)

Using our modular description of α^1_X, we construct a completion for it as a regular map $\overline{\alpha^1_X} : X \to \overline{P^1_X}$ (Theorem 5.5). Finally, we prove that $\overline{\alpha^1_X}$ is as close as it can be to an injection (see Proposition 5.9 for the precise statement).

Finally, suppose that X is not 1-general; then g is even and $[X]$ lies in a proper closed subset of $\overline{M}_g$ (see Proposition 3.15). Then $\overline{P^1_f}$ fails to contain N^1_f; nevertheless our existence results do extend, suitably modified (see 5.10), whereas uniqueness and injectivity results (like Proposition 5.9) may fail. In this case the setup is significantly more complicated for standard technical reasons (presence of non-GIT-stable points, or of nonfine moduli spaces.) This is why we chose to first work under the assumption of 1-generality, and to later indicate, in 5.10 and 5.13, how to modify proofs and statements to include the special case.

Constructing Abel maps for reducible curves presents difficulties not found for integral curves, due to the lack of natural, separated target spaces. The use of Néron models as target spaces is not new in the literature: in [E98] Abel–Jacobi maps for nodal curves were studied by means of the Néron mapping property, similarly to what we do here with our Abel–Néron maps. However, Néron models are seldom proper and thus we cannot expect Abel maps to Néron models to be complete. In this framework, our contribution is that of bringing compactified Picard schemes into the picture. This enables us to compactify Néron models and hence to obtain a target space into which complete Abel maps might be considered. In fact, we prove that α^1_X extends over the whole of X. For $d \geq 2$ the completion problem is more subtle and still open. A preliminary study, as well as the case of irreducible curves, shows that to complete α^d_X one would need to blow up the source, i.e. to blow up X^d.

Our paper is organized as follows. Section 2 is devoted to preliminaries of various types. In Section 3 we describe degree-d Abel–Néron maps to the compactified Picard scheme. In Section 4 we establish the modular description of the Abel–Néron map in degree 1, and show that it is independent of the choice of the deformation. Finally, in Section 5 we construct the completed degree-1 Abel map, give it a modular description and study when it is injective. Finally, from 5.10 to the end of the paper, we explain how to handle the special case of non-1-general curves.

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2. Néron models of Picard schemes

2.1. *Setup.* We work over a fixed algebraically closed field k. All schemes are assumed locally of finite type over k, unless stated otherwise.

For us, a *curve* is a reduced and connected projective scheme of dimension 1. Mostly, we will deal with *nodal curves*, that is, curves whose only singularities are nodes.

A *regular pencil* (*of curves*) is a flat projective morphism $f : \mathcal{X} \to B$ between connected, regular schemes such that $\dim B = 1$, every geometric fiber of f is a curve, and f is smooth over a dense open subscheme of B.

We call a regular pencil $f : \mathcal{X} \to B$ *local* if $B = \operatorname{Spec} R$, where R is a discrete valuation ring over k having k as residue field. If X is the closed fiber, we will also say that f is a *regular smoothing of* X.

For each regular pencil $f : \mathcal{X} \to B$ we shall let $K := k(B)$, the field of rational functions of B, and denote by $\mathcal{X}_K$ the generic fiber of f. Notice that $\mathcal{X}_K$ is a smooth curve over K.

Given any morphism $f : \mathcal{X} \to B$, and any integer $d \geq 1$, let $f_d : \mathcal{X}^d_B \to B$ denote the d-th fibered power of $\mathcal{X}$ over B. If f is a regular pencil, we denote the open subset of $\mathcal{X}^d_B$ where f_d is smooth by $\dot{\mathcal{X}}^d_B$; so

$$\dot{\mathcal{X}}^d_B := \mathcal{X}^d_B \smallsetminus \operatorname{Sing}(f_d)$$

If f is a local regular pencil, let $\dot{X}^d$ denote the closed fiber of $\dot{\mathcal{X}}^d_B \to B$; so

$$\dot{X}^d = \{(p_1, \ldots, p_d) : p_i \in X \smallsetminus X_{\text{sing}}\}.$$

Given any morphism $f : \mathcal{X} \to B$ and any B-scheme T, the base change of f to T is denoted $f_T : \mathcal{X}_T \to T$.

2.2. *The relative Picard scheme.* Let $f : \mathcal{X} \to B$ be a regular pencil, and d an integer. The closed fibers of f are geometric, by our general assumption, and the general fiber is smooth. Thus the irreducible components of the fibers of f are geometrically irreducible. By a theorem of Mumford's, [BLR], Thm. 2, p. 210, the (relative) Picard scheme Pic_f of f exists, and is locally of finite type over B.

Furthermore, Pic_f is formally smooth over B by [BLR], Prop. 2, p. 232, whence smooth over B by [BLR], Prop. 6, p. 37.

Let Pic^d_f be the degree-d Picard scheme of f, the open subscheme of Pic_f parametrizing line bundles of relative degree d. Given any B-scheme S and any line bundle $\mathcal{L}$ on $\mathcal{X}_S$ of f_S-relative degree d, there is a *moduli map* associated to $\mathcal{L}$,

$$\mu_{\mathcal{L}} : \begin{array}{ccc} S & \longrightarrow & \operatorname{Pic}^d_f \\ s & \mapsto & L_s, \end{array} \tag{1}$$

where $L_s \in \operatorname{Pic}^d X_s$ is the restriction of $\mathcal{L}$ to the fiber $X_s := f_S^{-1}(s)$. The map $\mu_{\mathcal{L}}$ determines $\mathcal{L}$ up to tensoring with pullbacks of line bundles from S. Notice that to a map $S \to \operatorname{Pic}^d_f$ there does not necessarily correspond a line bundle on $\mathcal{X}_S$, though the line bundle will exist, for instance, if f admits a section; see [BLR], Prop. 4, p. 204.

2.3. *Néron models of Picard schemes.* Let $f : \mathcal{X} \to B$ be a regular pencil, and d an integer. Recall that a basic characteristic (and a drawback for various applications) of the Picard scheme Pic^d_f is that it is not separated over B, if f has reducible special fibers. One way to fix this is to introduce the Néron model:

$$N^d_f := \mathrm{N}(\operatorname{Pic}^d \mathcal{X}_K).$$

The Néron model is a smooth, separated (possibly not proper) scheme of finite type over B with generic fiber equal to $\operatorname{Pic}^d \mathcal{X}_K$, which satisfies a fundamental mapping property that uniquely determines it. Namely, for every smooth B-scheme Z each map $Z_K \to \operatorname{Pic}^d \mathcal{X}_K$ extends uniquely to a map $Z \to N^d_f$; see [BLR], Def. 1, p. 12.

The existence of N_f^d for any regular pencil f is likely well known. Since this result is fundamental for our work, but we could not find the precise statement to refer to, we sketch a proof of it using results in [BLR]. First, assume that f is local, that is, B is the spectrum of a discrete valuation ring R. Then there is a Néron model of $\operatorname{Pic}^d \mathcal{X}_K$ over B, which is equal to Pic_f^d if f is smooth. Indeed, since $\operatorname{Pic}^d \mathcal{X}_K$ is a $\operatorname{Pic}^0 \mathcal{X}_K$-torsor, by descent theory we may assume that R is a strictly Henselian ring; see [BLR], Cor. 3, p. 158. In this case, f admits a section through its smooth locus, by [BLR], Prop. 5, p. 47. This section can be used to produce a B-isomorphism $\operatorname{Pic}_f^d \to \operatorname{Pic}_f^0$. We may thus assume that $d = 0$. In this case, there is a Néron model of $\operatorname{Pic}^0 \mathcal{X}_K$ over B because $\operatorname{Pic}^0 \mathcal{X}_K$ is an Abelian variety over K; see [BLR], Cor. 2, p. 16. Furthermore, if f is smooth, Pic_f^0 is an Abelian B-scheme, whence is the Néron model of $\operatorname{Pic}^0 \mathcal{X}_K$ over B by [BLR], Prop. 8, p. 15.

Now, consider the general case. Let $U \subseteq B$ be the largest open subscheme over which f is smooth, and set $h := f|_{f^{-1}(U)}$. As we saw above, Pic_h^d restricts to the Néron model of $\operatorname{Pic}^d \mathcal{X}_K$ locally around each point of U. Then Pic_h^d is the Néron model of $\operatorname{Pic}^d \mathcal{X}_K$ over U by [BLR], Prop. 4, p. 13. Finally, the local existence of Néron models of $\operatorname{Pic}^d \mathcal{X}_K$ around each point of B, and the existence of the Néron model over the dense open subscheme $U \subseteq B$ imply the (global) existence of the Néron model over B, by [BLR], Prop. 1, p. 18.

Since Pic_f^d is smooth over B, a first consequence of the mapping property of the Néron model N_f^d is the existence of a canonical B-morphism

$$q_f : \operatorname{Pic}_f^d \longrightarrow N_f^d \tag{2}$$

which is the identity on the generic fiber.

Assume now that the geometric fibers of f are nodal. Let X be a closed fiber of f. In the description of the Néron model, and also in our paper, the following subgroup $\operatorname{Tw}_f X \subseteq \operatorname{Pic}^0 X$ of (isomorphism classes of) distinguished line bundles plays an important role:

$$\operatorname{Tw}_f X := \frac{\{\mathcal{O}_{\mathcal{X}}(D)|_X : D \in \operatorname{Div} \mathcal{X} \text{ with } \operatorname{Supp} D \subset X\}}{\cong} \subset \operatorname{Pic}^0 X.$$

The divisors D appearing above are simply sums with integer coefficients of the components of X, which are Cartier divisors of $\mathcal{X}$ because $\mathcal{X}$ is regular. Line bundles in $\operatorname{Tw}_f X$ are called *twisters*. Here is a useful observation:

$$\forall\, T, T' \in \operatorname{Tw}_f X, \qquad T = T' \Leftrightarrow \underline{\deg}\, T = \underline{\deg}\, T', \tag{3}$$

where $\underline{\deg}$ denotes the multidegree of a line bundle on X. More precisely, let $X = \cup_{i=1}^{\gamma} C_i$ be the decomposition into irreducible components, then the multidegree $\underline{\deg}\, L$ of $L \in \operatorname{Pic} X$ is defined as $\underline{\deg}\, L := (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L)$.

Since twisters are specializations of the trivial line bundle of the generic fiber, $\mathcal{O}_{\mathcal{X}_K}$, all of them must be identified in any separated quotient of Pic_f^0. In particular, $q_f(T) = q_f(\mathcal{O}_X)$ for each $T \in \operatorname{Tw}_f X$.

We shall now identify multidegrees that differ by multidegrees of twisters. Let γ be the number of irreducible components of X, and set

$$\Lambda_X := \{\underline{\deg}\, T : T \in \operatorname{Tw}_f X\} \subseteq \mathbb{Z}^{\gamma}.$$

Define now an equivalence relation "≡" on multidegrees by setting

$$\underline{d} \equiv \underline{d}' \Leftrightarrow \underline{d} - \underline{d}' \in \Lambda_X.$$

The set of multidegree classes $\underline{d} + \Lambda_X$ with fixed total degree $|\underline{d}| := \sum d_i$ equal to d is denoted by Δ_X^d. Thus

$$\Delta_X^d := \frac{\{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = d\}}{\equiv}. \tag{4}$$

It is well known that Δ_X^0 is a finite group, a purely combinatorial invariant of X, called the degree class group of X in [C94], but known before as the group of connected components of N_f^0; see [BLR], Thm. 1 on p. 274 or [R70], Thm. 8.1.2 on p. 64. In addition, for each d there are a (noncanonical) bijection between the set of connected components of N_f^0 and that of N_f^d, and a (nonunique) bijection $\Delta_X^0 \to \Delta_X^d$, obtained by summing with any multidegree $\underline{d}$ with $|\underline{d}| = d$.

For each $\delta \in \Delta_X^d$, let $\underline{d}$ be any multidegree representing δ, and set

$$\operatorname{Pic}_f^\delta := \operatorname{Pic}_f^{\underline{d}} \subset \operatorname{Pic}_f^d, \tag{5}$$

where $\operatorname{Pic}_f^{\underline{d}}$ parametrizes line bundles with fixed multidegree $\underline{d}$ on X. The particular choice of representative $\underline{d}$ is not important; see [C05], 3.9.

Assume now that f is a regular smoothing of X. At this point we are able to describe the Néron model of $\operatorname{Pic}^d \mathcal{X}_K$:

$$\mathrm{N}_f^d \cong \frac{\coprod_{\delta \in \Delta_X^d} \operatorname{Pic}_f^\delta}{\sim_K}, \tag{6}$$

where $\sim_K$" denotes the gluing along the generic fiber, equal to $\operatorname{Pic}^d \mathcal{X}_K$; see [C05], Lemma 3.10.

Let N_X^d denote the closed fiber of N_f^d. Observe that N_X^d is a disjoint union of finitely many copies of the generalized Jacobian of X: picking a representative $\underline{d}^\delta$ for each class $\delta \in \Delta_X^d$, we have

$$N_X^d \cong \coprod_{\delta \in \Delta_X^d} \operatorname{Pic}^{\underline{d}^\delta} X.$$

Although the above isomorphism is not canonical, we see that the scheme structure of N_X^d does not depend on f. The closed points of N_X^d are in 1-1 correspondence with the degree-d line bundles on X modulo twisters. In particular, for $d = 0$, we have $q_f^{-1}(q_f(\mathcal{O}_X)) = \operatorname{Tw}_f X$.

2.4. *Néron maps.* Let $f : \mathcal{X} \to B$ be a regular pencil. Let T be a B-scheme, and $\mathcal{L}$ a line bundle on $\mathcal{X}_T$ of relative degree d over T. Let $\mu_{\mathcal{L}} : T \to \operatorname{Pic}_f^d$ be the moduli map of $\mathcal{L}$, defined in 2.2. Consider the composition:

$$\overline{\mu_{\mathcal{L}}} : T \xrightarrow{\mu_{\mathcal{L}}} \operatorname{Pic}_f^d \xrightarrow{q_f} N_f^d.$$

We call $\overline{\mu_{\mathcal{L}}}$ the *Néron map of* $\mathcal{L}$. Notice that $\mathcal{L}$ is certainly not determined by its Néron map, not even modulo pullbacks of line bundles on T. In fact, if $D \subset \mathcal{X}$ is a Cartier divisor entirely supported on a closed fiber of f, then $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}_T}(D_T)$ has the same Néron map as $\mathcal{L}$, because $N_f^d \to B$ is separated.

2.5. *Abel–Néron maps.* Let us recall the precise definition of the Abel map of a smooth curve, using the set up of [GIT], Section 6, pp. 118, 119.

Let $h : \mathcal{C} \to S$ be a smooth curve over a scheme S, i.e. a smooth morphism whose fibers are curves. For each integer $d \geq 1$, let $\mathcal{C}_S^d$ be the d-th fibered power of $\mathcal{C}$ over S. There is a canonical S-morphism

$$\begin{array}{ccc} \mathcal{C}_S^d & \longrightarrow & \operatorname{Pic}_h^d \\ C_s^d \ni (p_1, \dots, p_d) & \mapsto & \mathcal{O}_{C_s}(p_1 + \cdots + p_d), \end{array} \tag{7}$$

defined over each $s \in S$ by taking a d-tuple of points of the fiber C_s to the line bundle associated to their sum, which we shall call the *d-th Abel map* of h. Recall that the above map is the moduli map of a natural line bundle on $\mathcal{C}_S^d \times_S \mathcal{C}$, namely the one associated to the Cartier divisor $\sum_1^d S_i$, where each S_i is the image of the i-th natural section σ_i of the first projection $\mathcal{C}_S^d \times_S \mathcal{C} \to \mathcal{C}_S^d$, given by

$$\sigma_i(p_1, \dots, p_d) = ((p_1, \dots, p_d), p_i). \tag{8}$$

We may apply this construction to a regular pencil $f : \mathcal{X} \to B$. First of all, since $\mathcal{X}_K$ is smooth over K, we may consider the d-th Abel map of $\mathcal{X}_K$:

$$\alpha_K^d : \mathcal{X}_K^d \longrightarrow \operatorname{Pic}^d \mathcal{X}_K. \tag{9}$$

The above map extends to a map $\dot{\mathcal{X}}_B^d \to \operatorname{Pic}_f^d$. Indeed, the extension is the moduli map of the line bundle associated to the Cartier divisor E^d of $\dot{\mathcal{X}}_B^d \times_B \mathcal{X}$, where E^d is the sum of the images of the d natural sections $\sigma_1, \dots, \sigma_d$ given by (8) of the first projection $\dot{\mathcal{X}}_B^d \times_B \mathcal{X} \to \dot{\mathcal{X}}_B^d$. Composing with the map q_f of (2), we obtain the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}_B^d \times_B \mathcal{X}}(E^d)$ (see 2.4), which is also an extension of α_K^d.

The first simple but crucial observation is the following (well-known) fact:

Lemma - Definition 2.6. *Let $f : \mathcal{X} \to B$ be a regular pencil. For each integer $d \geq 1$ there exists a unique morphism, which we call the d-*th Abel–Néron map of *f,*

$$\mathrm{N}(\alpha_K^d) : \dot{\mathcal{X}}_B^d \longrightarrow \mathrm{N}(\operatorname{Pic}^d \mathcal{X}_K) = N_f^d$$

whose restriction to the generic fiber is α_K^d. The map $\mathrm{N}(\alpha_K^d)$ is the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}_B^d \times_B \mathcal{X}}(E^d + D)$ for every Cartier divisor D of $\dot{\mathcal{X}}_B^d \times_B \mathcal{X}$ supported on any finite number of closed fibers of $\dot{\mathcal{X}}_B^d \times_B \mathcal{X} \to B$.

Proof. The existence and uniqueness of an extension of α_K^d is a straightforward consequence of the Néron mapping property, [BLR], Def. 1, p. 12: since $\dot{\mathcal{X}}_B^d$ is smooth over B and has generic fiber $\mathcal{X}_K^d$, the Abel map α_K^d admits a unique extension $\mathrm{N}(\alpha_K^d) : \dot{\mathcal{X}}_B^d \to \mathrm{N}(\operatorname{Pic}^d \mathcal{X}_K)$.

Since also the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}_B^d \times_B \mathcal{X}}(E^d + D)$, for any D as described in the lemma, extends α_K^d, the last statement follows from the fact that N_f^d is separated over B. ∎

3. Abel maps to balanced Picard schemes

We want to give a modular interpretation of the Abel–Néron maps and, at the same time, study the problem of completing them. To do this we shall use some results of [C05], where Néron models are glued together over the moduli space of stable curves and are endowed with a geometrically meaningful completion. Our moduli problem is centered around Definition 3.2 below. First, we recall a few concepts.

3.1. Let X be a nodal curve of arithmetic genus $g \geq 2$. Denote by ω_X its canonical, or dualizing bundle. For each proper subcurve $Z \subsetneq X$, always assumed to be complete, let $Z' := \overline{X \smallsetminus Z}$ and $k_Z := \# Z \cap Z'$. Also, let $w_Z := \deg_Z \omega_X$. If Z is connected, denote by g_Z its arithmetic genus, and recall that

$$w_Z = 2g_Z - 2 + k_Z, \tag{10}$$

a well-known identity that can be proved using adjunction.

We call X *semistable* (resp. *stable*) if $k_Z \geq 2$ (resp. $k_Z \geq 3$) for each smooth rational component Z of X. Those Z for which $k_Z = 2$ are called *exceptional*. A semistable curve is called *quasistable* if two exceptional components never meet each other. If X is semistable, then it has a *stable model*, that is, a stable curve $\overline{X}$ and a map $X \to \overline{X}$ contracting all exceptional components. We may also say that X *is semistable over* $\overline{X}$.

If X is semistable, it follows from (10) that $w_Z \geq 0$ for each subcurve $Z \subseteq X$, with equality if and only if Z is a union of exceptional components.

A *family of semistable* (resp. *stable*, resp. *quasistable*) *curves* is a flat, projective map $f : \mathcal{X} \to B$ whose geometric fibers are semistable (resp. stable, resp. quasistable) curves. A line bundle of degree d on such a family $f : \mathcal{X} \to B$ is a line bundle on $\mathcal{X}$ whose restriction to each fiber has degree d.

Definition 3.2. Let X be a semistable curve of arithmetic genus $g \geq 2$, and let $L \in \operatorname{Pic}^d X$.

(i) We say that L and its multidegree $\underline{\deg} L$ are *semibalanced* if for each connected proper subcurve $Z \subsetneq X$ the *Basic Inequality* below holds:

$$m_Z(d) \leq \deg_Z L \leq M_Z(d), \tag{11}$$

where

$$M_Z(d) := \frac{d w_Z}{2g-2} + \frac{k_Z}{2}$$

and

$$m_Z(d) := \begin{cases} M_Z(d) - k_Z & \text{if } Z \text{ is not an exceptional component,} \\ 0 & \text{if } Z \text{ is an exceptional component.} \end{cases}$$

(ii) We call L and $\underline{\deg} L$ *balanced* if they are semibalanced and if for each exceptional component $E \subset X$ we have

$$\deg_E L = 1.$$

(iii) We call L and $\underline{\deg} L$ *stably balanced* if they are balanced and if for each connected proper subcurve $Z \subsetneq X$ such that $\deg_Z L = m_Z(d)$ the complement Z' is a union of exceptional components.

(iv) We denote by B_X^d the set of balanced multidegrees on X, and by $\tilde{B}_X^d$ the subset of stably balanced ones (in [C05] B_X^d denotes the semibalanced multidegrees).

(v) A line bundle (of degree d) on a family of semistable curves is called semibalanced (balanced or stably balanced) if its restriction to each geometric fiber of the family is.

Remark 3.3. Definition 3.2 is 4.6 of [C05], which originates from [C94], Section 3.1 and from [CCC04], 5.1.1. There the Basic Inequality is

$$\left|\deg_Z L - \frac{dw_Z}{2g-2}\right| \leq \frac{k_Z}{2},$$

and the extra condition for when Z is an exceptional component is imposed separately. For convenience, we presented a set of inequalities, (11), that includes the exceptional cases. Abusing the terminology, we still call (11) the Basic Inequality.

We mention some basic, useful consequences of the definition.

(A) If X is stable, then (i) and (ii) coincide, i.e. a semibalanced line bundle is always balanced.

(B) There can only exist a balanced line bundle on a semistable curve X if X is quasistable. Indeed, let $Z \subset X$ be a connected chain of exceptional components. If L is a semibalanced line bundle on X, then L has degree 0 on every component of Z but possibly one, where L may have degree 1. However, if L is balanced, L cannot have degree 0 on any component of Z. Then $Z \cong \mathbb{P}^1$ and $\deg_Z L = 1$.

(C) To check whether L is semibalanced it is enough to check whether $\deg_Z L \geq m_Z(d)$ for every connected proper subcurve $Z \subsetneq X$, and $\deg_Z L \geq 0$ if Z is exceptional. Indeed, let Z be such a subcurve, and let $Y_1, \ldots, Y_n$ denote the connected components of Z'. By hypothesis,

$$\deg_{Y_i} L \geq m_{Y_i}(d) \geq \frac{dw_{Y_i}}{2g-2} - \frac{k_{Y_i}}{2}$$

for each i. Since $k_{Y_1} + \cdots + k_{Y_n} = k_{Z'} = k_Z$, summing up the above inequalities we get

$$\deg_{Z'} L \geq \frac{dw_{Z'}}{2g-2} - \frac{k_Z}{2}.$$

Now, since $\deg_Z L + \deg_{Z'} L = d$ and $w_Z + w_{Z'} = 2g-2$, we get

$$\deg_Z L = d - \deg_{Z'} L \leq d - \frac{dw_{Z'}}{2g-2} + \frac{k_Z}{2} = \frac{dw_Z}{2g-2} + \frac{k_Z}{2}.$$

3.4. In [C05], Lemma 4.4, it is proved that each multidegree class has a semibalanced representative. More precisely, fix an integer d, and let X be a stable curve. Recall the notation in (4) and 3.2 (iv). Then Lemma 4.4 of [C05] implies that the natural map below is surjective (square brackets denoting classes):

$$\begin{array}{rccc} [\ldots]: & B_X^d & \longrightarrow & \Delta_X^d \\ & \underline{d} & \mapsto & [\underline{d}] \end{array} \tag{12}$$

We shall say that X is "d-general" if the map (12) is bijective; see Definition 3.6 below

3.5. The moduli problem for balanced line bundles was introduced and studied in [C94] to compactify the universal Picard scheme over M_g. That compactification was constructed as a GIT-quotient. We do not need to recall all the details of this construction here. However we shall recall that there are morphisms

$$H_d \xrightarrow{\pi_d} H_d/G \xrightarrow{\phi_d} \overline{M}_g$$

where H_d is an open subscheme of a suitable Hilbert scheme, acted upon by an algebraic group G, the map π_d is a GIT-quotient map, and ϕ_d is a surjective, projective morphism. The quotient scheme H_d/G, denoted by $\overline{P}_{d,g}$, is integral and projective. The fiber of ϕ_d over a general smooth curve X is exactly $\operatorname{Pic}^d X$; see [C94], Thm. 6.1.

Denote by $U \subseteq \overline{P}_{d,g}$ the nonempty open subscheme over which π_d restricts to a geometric GIT-quotient, i.e. all fibers over points in U are G-orbits, and all stabilizers are finite and reduced.

Definition 3.6. Let X be a stable curve of arithmetic genus $g \geq 2$. We say that X is *d-general* if any of the following equivalent conditions hold:

(i) $\phi_d^{-1}(X) \subset U$.
(ii) The class map (12) of 3.4 is bijective.
(iii) Every balanced line bundle on X is stably balanced, i.e. $\tilde{B}_X^d = B_X^d$.

The equivalence of these three conditions follows from [C94] Lemma 6.1.

It is known that *all stable curves of genus g are d-general if and only if the integers d and g satisfy* $(d-g+1, 2g-2) = 1$; see [C94], Prop 6.2.

3.7. We need to recall when two semibalanced line bundles are defined to be equivalent. Let X be a stable curve, and X_1 and X_2 two semistable curves having X as stable model. For each $i = 1, 2$ let L_i be a semibalanced line bundle on X_i. Let Y_i be the semistable curve obtained by contracting all exceptional components of X_i where L_i has degree 0. Then there is a unique line bundle M_i on Y_i whose pullback to X_i is L_i. Since L_i is semibalanced, M_i is balanced and Y_i is quasistable. Let $F_i \subseteq Y_i$ be the union of all the exceptional components of Y_i, and let $\widetilde{Y_i} := F_i'$, the complementary subcurve of F_i in Y_i. Then L_1 *and* L_2 *are equivalent if there is an isomorphism* $Y_1 \to Y_2$ *such that* $M_1|_{\widetilde{Y_1}} \cong M_2|_{\widetilde{Y_2}}$ *under the identification given by it.* Notice that M_i is equivalent to L_i for $i = 1, 2$.

Thus, every equivalence class includes always a balanced line bundle N on a quasistable curve Y. The quasistable curve Y is unique, but N is not. What is unique is the restriction of N to $\tilde{Y}$, the complementary subcurve of the union F of all the exceptional components of Y. The quasistable curve Y and $N|_{\tilde{Y}}$ determine the equivalence class. The restriction $N|_F$ is also unique, since a balanced line bundle must have degree 1 on every exceptional component. So, our equivalence relation disregards the gluing data of the bundles over the points in $\tilde{Y} \cap F$.

If $\mathcal{X} \to B$ is a family of semistable curves, then two semibalanced line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X} \to B$ are called equivalent if and only if their restrictions to every geometric fiber of $\mathcal{X} \to B$ are equivalent in the sense explained above.

3.8. Let d and g be integers, with $g \geq 2$. Assume first that $d-g+1$ and $2g-2$ are coprime, so that every stable curve of arithmetic genus g is d-general. Then the construction summarized in 3.5 can be improved, by considering stacks. More precisely, there exist two (modular) Deligne–Mumford stacks $\overline{\mathcal{P}}_{d,g}$ and $\mathcal{P}_{d,g}$, each one equipped with a natural, strongly representable morphism to $\overline{\mathcal{M}}_g$. (To tie in with 3.5, we mention that $\overline{\mathcal{P}}_{d,g}$ is the quotient stack $[H_d/G]$.) The following properties hold; see [C05], Section 5 for details:

(A) For each (d-general) stable curve X, denote by P^d_X and $\overline{P^d_X}$ the fibers of $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ over X. Since $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are strongly representable over $\overline{\mathcal{M}}_g$, both P^d_X and $\overline{P^d_X}$ are quasiprojective schemes.

The first, P^d_X, is the fine moduli scheme of degree-d balanced line bundles on X. The second, $\overline{P^d_X}$, is the coarse moduli scheme of equivalence classes of degree-d semibalanced line bundles on semistable curves having X as stable model; see 3.7. Actually, $\overline{P^d_X}$ is not far from being a fine moduli scheme; see (C) below.

P^d_X lies naturally inside $\overline{P^d_X}$ as an open and dense subscheme.

(B) Let $f : \mathcal{X} \to B$ be any family of (d-general) stable curves of genus g, and consider the schemes

$$P^d_f := B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g} \quad \text{and} \quad \overline{P^d_f} := B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{P}}_{d,g}.$$

(That these are indeed schemes follows, again, from the fact that the maps $\mathcal{P}_{d,g} \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{P}}_{d,g} \to \overline{\mathcal{M}}_g$ are strongly representable.) We have a natural inclusion $P^d_f \subset \overline{P^d_f}$. An explicit description of P^d_f, for when f is a local regular pencil, is (13) below. As for $\overline{P^d_f}$, the following fact holds: to each triple $(T, \mathcal{Y} \to T, \mathcal{L})$ where T is a B-scheme, $\mathcal{Y} \to T$ is a family of semistable curves having $f_T : \mathcal{X}_T \to T$ as stable model, and $\mathcal{L}$ is a semibalanced line bundle of degree d on $\mathcal{Y} \to T$, there corresponds a moduli map

$$\hat{\mu}_{\mathcal{L}} : T \longrightarrow \overline{P^d_f},$$

taking each geometric point t of T to the equivalence class of the restriction of $\mathcal{L}$ to the (geometric) fiber of $\mathcal{Y}$ over t. We call $\hat{\mu}_{\mathcal{L}}$ the *moduli map* of $\mathcal{L}$.

The image of $\hat{\mu}_{\mathcal{L}}$ is contained in P^d_f if and only if $\mathcal{L}$ has degree 0 on every exceptional component of every geometric fiber of $\mathcal{Y} \to T$.

(C) The scheme P^d_f is a fine moduli scheme; see [C05], Cor. 5.14 and Rmk. 5.15. Also, $\overline{P^d_f}$ is not far from being a fine moduli scheme. In fact, it is endowed with a quasiuniversal pair $(\mathcal{Z} \to \overline{P^d_f}, \mathcal{N})$, where $\mathcal{Z} \to \overline{P^d_f}$ is a family of quasistable curves having $f_{\overline{P^d_f}} : \mathcal{X}_{\overline{P^d_f}} \to \overline{P^d_f}$ as stable model, and $\mathcal{N}$ is a balanced line bundle of degree d on $\mathcal{Z} \to \overline{P^d_f}$ that has a role similar to that of a Poincaré bundle. Indeed, for each triple $(T, \mathcal{Y} \to T, \mathcal{L})$ where T is a B-scheme, $\mathcal{Y} \to T$ is a family of semistable curves with stable model $f_T : \mathcal{X}_T \to T$, and $\mathcal{L}$ is a semibalanced line bundle of degree d on $\mathcal{Y} \to T$, there is a map $\mathcal{Y} \to \mathcal{Z}$ such that the diagram of maps below is commutative,

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\hat{\mu}_{\mathcal{L}}} & \overline{P^d_f}, \end{array}$$

and such that $\mathcal{L}$ is equivalent to the pullback of $\mathcal{N}$ to $\mathcal{Y}$; see 3.7. (The map $\mathcal{Y} \to \mathcal{Z}$ is certainly not uniquely determined, which is why we call the pair $(\mathcal{Z} \to \overline{P^d_f}, \mathcal{N})$ quasiuniversal.) Furthermore, if $\mathcal{Y} \to T$ is a family of quasistable curves, and $\mathcal{L}$ is balanced, then the map $\mathcal{Y} \to \mathcal{Z}$ can be chosen such that the above diagram is a fibered product diagram.

Remark 3.9. If $(d-g+1, 2g-2) \neq 1$, almost everything in 3.8 works over the open subset of $\overline{M}_g$ parametrizing d-general stable curves. For a proof, it suffices to argue exactly as for Theorem 5.9 in [C05], after replacing $\overline{\mathcal{M}}_g$ with the substack of d-general curves, and the stacks $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ with the corresponding substacks (over d-general curves).

The only assertion in 3.8 that does not hold is the existence of a "Poincaré" line bundle, in (C), which will never be used in this paper.

We are ready to go back to the study of Abel maps.

Theorem 3.10. *Let $f : \mathcal{X} \to B$ be a regular pencil of d-general stable curves. Then there exists a canonical map*

$$\alpha^d_f : \dot{\mathcal{X}}^d \longrightarrow P^d_f$$

which restricts to the d-th Abel map on the generic fiber.

Proof. We may glue local extensions of the d-th Abel map of $\mathcal{X}_K$ because they are unique. Thus we may assume f is local; let X be the closed fiber of f. In this case, the explicit description of P^d_f is (see [C05], Cor. 5.14)

$$P^d_f = \frac{\coprod_{\underline{d} \in B^d_X} \mathrm{Pic}^{\underline{d}}_f}{\sim_K}, \tag{13}$$

where, as in 2.3, "$\sim_K$" means gluing over the generic fiber.

As we know from (6) in 2.3, N^d_f is described in a very similar way. Indeed, by [C05], Thm. 6.1, we have a canonical isomorphism

$$\epsilon^d_f : P^d_f \xrightarrow{\cong} N^d_f. \tag{14}$$

To describe it precisely is straightforward: for each $\underline{d} \in B^d_X$ the restriction of ϵ^d_f to $\mathrm{Pic}^{\underline{d}}_f$ is the natural isomorphism

$$\mathrm{Pic}^{\underline{d}}_f \xrightarrow{\cong} \mathrm{Pic}^{[\underline{d}]}_f \subset N^d_f,$$

restricting to the identity on the generic fibers. The isomorphism ϵ^d_f is completely described because, since X is d-general, the class map $B^d_X \to \Delta^d_X$ is bijective, by Definition 3.6. To conclude, use Lemma 2.6 and (14) to define $\alpha^d_f := (\epsilon^d_f)^{-1} \circ \mathrm{N}(\alpha^d_K)$. ∎

We call α_f^d the *d-th Abel map* of f. The natural problem now is to describe α_f^d as the moduli map of a balanced line bundle on $\pi : \dot{\mathcal{X}}^d \times_B \mathcal{X} \to \dot{\mathcal{X}}^d$. Since P_f^d is a fine moduli scheme, this should be possible. In fact, the proof of [C94], Prop. 4.1, p. 621, can be used to produce an algorithm for determining the necessary twisters we need to tensor $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$ with to get the balanced line bundle.

However the explicit description of this line bundle turns out to be difficult to find in general. In Section 4 we will give it for $d = 1$. And in the next subsection, 3.11, we will give it for every d in a special case.

3.11. *Two-component curves.* Let X be a stable curve with only two irreducible components, C_1 and C_2. Let g be the arithmetic genus of X and $\delta := \#C_1 \cap C_2$. Set (see 3.2)

$$m := \lceil m_{C_1}(d) \rceil = \left\lceil \frac{dw_{C_1}}{2g-2} - \frac{\delta}{2} \right\rceil$$

where $\lceil x \rceil$ denotes the ceiling of a real number x, that is, the smallest integer not smaller than x. The set B_X^d of balanced multidegrees of X satisfies

$$B_X^d \supseteq \{(m, d-m), (m+1, d-m-1), \dots, (m+\delta-1, d-m-\delta+1)\}, \tag{15}$$

with equality if and only if $m_{C_1}(d)$ is not integer, if and only if X is d-general.

For each integer a define $r(a)$ to be the integer determined by the following two conditions:

$$0 \leq r(a) < \delta \quad \text{ and } \quad a - m \equiv r(a) \mod \delta.$$

Using this notation we have:

Proposition 3.12. *Let X be a stable curve with exactly two irreducible components, C_1 and C_2. Let $\delta := \#C_1 \cap C_2$. For each regular smoothing $f : \mathcal{X} \to B$ of X, let*

$$\mathcal{L}^{(d)} := \mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}\Big(E^d + \sum_{a=0}^{d} \frac{a-m-r(a)}{\delta} C_1^a \times C_2^{d-a} \times C_1\Big)$$

where, abusing notation, we view

$$C_1^a \times C_2^{d-a} \times C_1 \subset X^d \times X \subset \mathcal{X}^d \times_B \mathcal{X}$$

as a Cartier divisor of $\dot{\mathcal{X}}^d \times_B \mathcal{X}$, by restriction. Then $\mathcal{L}^{(d)}$ is balanced on $f_{\dot{\mathcal{X}}^d} : \dot{\mathcal{X}}^d \times_B \mathcal{X} \to \dot{\mathcal{X}}^d$. Furthermore, if X is d-general, then the d-th Abel map $\alpha_f^d : \dot{\mathcal{X}}^d \to P_f^d$ is the moduli map of $\mathcal{L}^{(d)}$.

Proof. Since each $C_1^a \times C_2^{d-a} \times C_1$ is supported over the closed point of B, the line bundle $\mathcal{L}^{(d)}$ coincides with $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$ over the generic point of B. This implies that $\mathcal{L}^{(d)}$ and $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$ have the same Néron map.

If X is d-general, α_f^d is defined and coincides with the Néron map of $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$ on $\mathcal{X}_K^d$. On the other hand, if the line bundle $\mathcal{L}^{(d)}$ is balanced on $\pi : \dot{\mathcal{X}}^d \times_B \mathcal{X} \to \dot{\mathcal{X}}^d$, there is an associated moduli map $\hat{\mu}_{\mathcal{L}^{(d)}} : \dot{\mathcal{X}}^d \to P_f^d$ by 3.8 (B). Since $\hat{\mu}_{\mathcal{L}^{(d)}}$ coincides with the Néron map of $\mathcal{L}^{(d)}$ on $\mathcal{X}_K^d$, it follows that $\hat{\mu}_{\mathcal{L}^{(d)}} = \alpha_f^d$. Thus, it suffices to prove that $\mathcal{L}^{(d)}$ is balanced.

To verify this, we must compute the multidegree of the restriction of $\mathcal{L}^{(d)}$ to every singular fiber of $\pi : \dot{\mathcal{X}}^d \times \mathcal{X} \longrightarrow \dot{\mathcal{X}}^d$. Of course, all singular fibers lie over $\dot{X}^d$. So, let $p \in \dot{X}^d$, and denote by X_p the fiber $\pi^{-1}(p)$. The point p determines a unique pair of nonnegative integers (a_0, b_0) such that $a_0 + b_0 = d$ and $p \in C_1^{a_0} \times C_2^{b_0}$. Then, identifying X_p with X in the natural way,

$$(E^d \cdot C_1, E^d \cdot C_2) = (a_0, b_0) = (a_0, d - a_0). \tag{16}$$

To compute the intersection degrees with C_1 and C_2 of the remaining summands defining $\mathcal{L}^{(d)}$, notice first that, since $p \in C_1^{a_0} \times C_2^{b_0}$, the only nonzero degrees come from the summand indexed by $a = a_0$. Now, using

$$(C_1^{a_0} \times C_2^{b_0} \times C_1) \cdot (C_1, C_2) = (-\delta, \delta)$$

and (16), we get

$$\underline{\deg}_{X_p} \mathcal{L}^{(d)} = (\deg \mathcal{L}^{(d)}|_{C_1}, \deg \mathcal{L}^{(d)}|_{C_2}) = (m + r(a_0), d - m - r(a_0)),$$

which is balanced because $0 \leq r(a_0) < \delta$; see (15). ■

Example 3.13. *Let X be a "split" curve of arithmetic genus g, that is, $X = C_1 \cup C_2$ with $C_i \cong \mathbb{P}^1$ and $\#C_1 \cap C_2 = g+1$. Then, for each $d = 1, \dots, g$ and any regular smoothing $f : \mathcal{X} \to B$ of X, the map α_f^d is the moduli map of $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$. In particular, given $p_1, \dots, p_d \in \dot{X}$ we have, independently of f,*

$$\alpha_f^d(p_1, \dots, p_d) = \mathcal{O}_X(p_1 + \dots + p_d).$$

A split curve is d-general if and only if $d \equiv g \mod 2$. Actually, the conclusion of Example 3.13 is valid regardless of X being d-general, because at any rate $\mathcal{O}_{\dot{\mathcal{X}}^d \times_B \mathcal{X}}(E^d)$ is stably balanced on $f_{\dot{\mathcal{X}}^d}$.

Remark 3.14. The case of split curves is in a sense special. In general, we should expect the restriction $\alpha_f^d|_{\dot{X}^d}$ of the d-th Abel map of Proposition 3.12 to depend on the choice of smoothing f. For a simple concrete example of this dependence, consider the case $d = 2$ and $\delta = 2$. Then X is stable and 2-general if C_1 and C_2 have distinct positive arithmetic genera. Indeed, X is stable because the genera of C_1 and C_2 are positive. Moreover, since they are distinct, $w_{C_i}/(g-1)$ is never an integer, for any $i = 1, 2$. So the Basic Inequality is always strict, and hence X is 2-general.

Suppose C_1 has smaller genus. Then $m = 0$, and thus $r(0) = 0$, $r(1) = 1$, but $r(2) = 0$. Let $p, q \in C_1 \setminus C_1 \cap C_2$. Then

$$\alpha_f^2(p, q) = \mathcal{O}_X(p + q) \otimes \mathcal{O}_{\mathcal{X}}(C_1)|_X.$$

Now, as f varies through all smoothings of X, the restriction $\mathcal{O}_{\mathcal{X}}(C_1)|_X$ varies through all line bundles restricting to $\mathcal{O}_{C_1}(-r_1 - r_2)$ and $\mathcal{O}_{C_2}(r_1 + r_2)$, where r_1 and r_2 are the nodes of X. So $\alpha_f^2(p, q)$ depends on the choice of f.

Denote by Σ_g^d the locus in $\overline{M}_g$ of curves that are not d-general. Then Σ_g^d is a proper closed subset of $\overline{M}_g$; in fact, with the notation of 3.5, Σ_g^d is the image via ϕ_d of the complement of U. As we mentioned in 3.5, Σ_g^d is empty if and only if $(d - g + 1, 2g - 2) = 1$. Since the rest of our paper is devoted to Abel maps for $d = 1$, we conclude this section by describing the locus of curves that are not 1-general.

Proposition 3.15. *Let $g \geq 2$.*

(i) *If g is odd then Σ_g^1 is empty.*

(ii) *If g is even then Σ_g^1 is the closure in $\overline{M}_g$ of the locus of curves X such that $X = C_1 \cup C_2$, with C_1 and C_2 smooth of the same genus and $\#C_1 \cap C_2$ odd.*

Proof. If g is odd then

$$(1-g+1, 2g-2) = (g-2, g-1) = 1.$$

So Part (i) follows; see 3.6.

For Part (ii), let X be a stable curve. Suppose first that X has the description given in (ii). Then

$$\Big(\frac{1-\delta}{2}, \frac{1+\delta}{2}\Big) \in B_X^1 \smallsetminus \tilde{B}_X^1,$$

where $\delta := k_{C_1} = k_{C_2}$. So X is in Σ_g^1. Since Σ_g^1 is closed in $\overline{M}_g$, it contains the closure of the locus defined in (ii).

Suppose now that X is in Σ_g^1, i.e. there is a line bundle L on X such that $\underline{\deg}\, L \in B_X^1 \smallsetminus \tilde{B}_X^1$. Then there is a connected, proper subcurve $Z \subsetneq X$ such that either $m_Z(1)$ or $M_Z(1)$ is equal to $\deg_Z L$. Then both $m_Z(1)$ and $M_Z(1)$ are integers. Thus

$$\frac{w_Z}{w} + \frac{k_Z}{2} \in \mathbb{Z} \tag{17}$$

where $w := \deg \omega_X = 2g-2$. Now, since X is stable,

$$0 < \frac{w_Z}{w} < 1. \tag{18}$$

In particular, $\frac{w_Z}{w}$ is never an integer, and thus (17) implies that k_Z is odd. Since k_Z is odd, (17) and (18) immediately yield $\frac{w_Z}{w} = \frac{1}{2}$.

If Z' is connected, then, since $\frac{w_{Z'}}{w} = \frac{1}{2}$ as well, we have $g_Z = g_{Z'}$. Since both Z and Z' are limits of smooth curves, X lies in the closure of the locus described in (ii).

Thus it remains to show that Z' is connected. Let $Z'_1, \dots, Z'_m$ be the connected components of Z'. Notice that

$$k_{Z'_1} + \cdots + k_{Z'_m} = k_{Z'} = k_Z. \tag{19}$$

Suppose by contradiction that $m > 1$. Then

$$0 < \frac{w_{Z'_i}}{w} < \frac{1}{2} \tag{20}$$

for each i. Since $\underline{\deg}\, L \in B_X^1$, we have

$$\frac{w_{Z'_i}}{w} - \frac{k_{Z'_i}}{2} \leq \deg_{Z'_i} L \leq \frac{w_{Z'_i}}{w} + \frac{k_{Z'_i}}{2}$$

for each i. Using (20), we get

$$\frac{1-k_{Z'_i}}{2} \leq \deg_{Z'_i} L \leq \frac{k_{Z'_i}}{2}$$

for each i. Summing up, and using (19), we get

$$\frac{m-k_Z}{2} \leq \deg_{Z'} L \leq \frac{k_Z}{2}.$$

Now, since $\deg L = 1$, we must have

$$\frac{2-k_Z}{2} \leq \deg_Z L \leq \frac{2-m+k_Z}{2}. \tag{21}$$

Suppose first that $\deg_Z L = M_Z(1)$, that is, $\deg_Z L = (1+k_Z)/2$. Then (21) implies $1+k_Z \leq 2-m+k_Z$, and hence $m \leq 1$, a contradiction.

Finally, suppose $\deg_Z L = m_Z(1)$, that is, $\deg_Z L = (1-k_Z)/2$. Then (21) implies $2-k_Z \leq 1-k_Z$, a contradiction as well. ∎

4. Geometric interpretation of the first Abel map

The following diagram represents the families we shall deal with in this section, starting from a regular pencil of stable curves $f : \mathcal{X} \to B$:

$$\begin{array}{ccccc} \dot{\mathcal{X}} \times_B \mathcal{X} & \overset{\subset}{\longrightarrow} & \mathcal{X}^2_B & \longrightarrow & \mathcal{X} \\ {\scriptstyle\dot{\pi}}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \dot{\mathcal{X}} & \overset{\subset}{\longrightarrow} & \mathcal{X} & \longrightarrow & \overline{P^d_f}, \end{array}$$

where π is the projection onto the first factor.

We denote by $\Delta \subset \mathcal{X}^2_B$ the diagonal. Its restriction to $\dot{\mathcal{X}} \times_B \mathcal{X}$ is a Cartier divisor. Denote by $\mathcal{O}_{\dot{\mathcal{X}}\times_B\mathcal{X}}(\Delta)$ the associated line bundle. We may view $\mathcal{O}_{\dot{\mathcal{X}}\times_B\mathcal{X}}(\Delta)$ as a family of degree-1 line bundles on the fibers of $\dot{\pi}$. Recall that the first Abel map of the generic fiber of f is the moduli map of the restriction of $\mathcal{O}_{\dot{\mathcal{X}}\times_B\mathcal{X}}(\Delta)$; see 2.5. We want to interpret the first Abel map α^1_f, defined in Theorem 3.10, as the moduli map of a balanced line bundle on $\dot{\pi}$, which will necessarily be a (possibly trivial) twist of $\mathcal{O}_{\dot{\mathcal{X}}\times_B\mathcal{X}}(\Delta)$.

In fact, we shall see that $\mathcal{O}_{\dot{\mathcal{X}}\times_B\mathcal{X}}(\Delta)$ fails to be balanced over points of a singular fiber X of f only if X has a separating node. To fix this, we will tensor $\mathcal{O}_{\dot{\mathcal{X}}\times_B\mathcal{X}}(\Delta)$ by twisters supported on so-called "tails".

We need a few preliminary results which hold for any curve X, possibly having singularities other than nodes. For the sake of future applications of the techniques developed in this paper, from now until 4.5, and in 4.8, 4.11 and 4.13, we shall be in this more general situation, i.e. X will be any (reduced, connected and projective) curve over an algebraically closed field.

Let r be a node of X and $X^\nu_r \to X$ be the normalization of X at r only. If X^ν_r is not connected, r is called a *separating node* of X.

Definition 4.1. Let X be a curve of arithmetic genus g. A proper subcurve $Q \subsetneq X$ will be called a *tail* of X if Q intersects the complementary subcurve Q' in a separating node r of X. We say that Q is *attached to* r or that r *generates* Q. A tail Q of X will be called *small* if $g_Q < g/2$ and *large* if $g_Q > g/2$. Let

$$\mathcal{Q}(X) := \{Q \subset X : Q \text{ is a small tail of } X\}.$$

If X has no separating node, for instance if X is smooth, then $\mathcal{Q}(X) = \emptyset$.

If r is a separating node of X, then X^ν_r has two connected components, isomorphic to the two tails generated by r; hence every tail is connected.

For every tail $Q \subset X$ we have that $g = g_Q + g_{Q'}$. So, at least one of the two tails attached to a separating node has arithmetic genus at most $g/2$.

If the curve X is stable and 1-general, it follows from Proposition 3.15 that no tail of X can have genus equal to $g/2$, or in other words that every tail of X is either small or large.

Remark 4.2. Let r be a separating node of X generating the tails Q and Q'. If $Z \subset X$ is a connected subcurve not containing r, then Z is entirely contained in either Q or Q'.

Lemma 4.3. *Let X be a curve and Q_1 and Q_2 two tails of X. Then*

$$Q_1 \cup Q_2 = X \quad \text{or} \quad Q_1 \cap Q_2 = \emptyset \quad \text{or} \quad Q_1 \subseteq Q_2 \quad \text{or} \quad Q_2 \subseteq Q_1.$$

Proof. For each $i = 1, 2$ let r_i be the separating node of X generating Q_i. If $r_1 = r_2$ then either $Q_1 = Q_2'$, and hence $Q_1 \cup Q_2 = X$, or $Q_1 = Q_2$. So we may assume that $r_1 \neq r_2$.

Thus $r_1 \notin Q_2$ or $r_1 \notin Q_2'$. Suppose first that $r_1 \notin Q_2$. Since Q_2 is connected, either $Q_2 \subset Q_1$ or $Q_2 \subset Q_1'$ by Remark 4.2. If $Q_2 \subset Q_1'$ then $Q_1 \cap Q_2 \subseteq Q_1 \cap Q_1' = \{r_1\}$, and hence $Q_1 \cap Q_2 = \emptyset$ because $r_1 \notin Q_2$. So either $Q_2 \subset Q_1$ or $Q_1 \cap Q_2 = \emptyset$.

The case where $r_1 \notin Q_2'$ is treated similarly. In this case, either $Q_2' \subset Q_1'$, and hence $Q_1 \subset Q_2$, or $Q_1' \cap Q_2' = \emptyset$, and hence $Q_1 \cup Q_2 = X$. ■

Lemma 4.4. *Let X be a curve, and Q a tail of X. Then, for any two line bundles L_1 on Q and L_2 on Q', there is, up to isomorphism, a unique line bundle L on X such that $L|_Q \cong L_1$ and $L|_{Q'} \cong L_2$.*

Proof. Let r be the separating node of X to which Q is attached. For each isomorphism $\mu : L_1|_{\{r\}} \to L_2|_{\{r\}}$, let L be the kernel of the composition

$$\phi_\mu : L_1 \oplus L_2 \longrightarrow L_1|_{\{r\}} \oplus L_2|_{\{r\}} \xrightarrow{\tilde{\mu}} L_2|_{\{r\}},$$

where $\tilde{\mu} := (-\mu, 1)$. Since ϕ_μ is surjective, $L \neq L_1 \oplus L_2$. Also, since μ is an isomorphism, $L \neq L_1(-r) \oplus L_2$ and $L \neq L_1 \oplus L_2(-r)$. Since r is a node of X, it follows that L is a line bundle, and $L|_Q \cong L_1$ and $L|_{Q'} \cong L_2$.

Conversely, if N is a line bundle on X for which there are isomorphisms $\lambda_1 : N|_Q \to L_1$ and $\lambda_2 : N|_{Q'} \to L_2$, then N is the kernel of ϕ_μ, where $\mu := \lambda_2|_{\{r\}} \circ \lambda_1^{-1}|_{\{r\}}$.

Finally, if $\mu' : L_1|_{\{r\}} \to L_2|_{\{r\}}$ is another isomorphism, the kernel of ϕ_μ is carried isomorphically to the kernel of $\phi_{\mu'}$ by the automorphism

$$(a, 1) : L_1 \oplus L_2 \longrightarrow L_1 \oplus L_2,$$

where a is the unique scalar such that $\mu = a\mu'$. ■

4.5. *Twisters on tails.* Let X be a curve. By Lemma 4.4, for each tail Q of X there is a unique, up to isomorphism, line bundle on X whose restrictions to Q and Q' are $\mathcal{O}_Q(-r)$ and $\mathcal{O}_{Q'}(r)$, where r is the separating node of X generating Q. Denote this bundle by $\mathcal{O}_X(Q)$.

For each formal sum $\sum a_Q Q$ of tails Q with coefficients $a_Q \in \mathbb{Z}$, set

$$\mathcal{O}_X(\textstyle\sum a_Q Q) := \bigotimes \mathcal{O}_X(Q)^{\otimes a_Q}.$$

If X is a nodal curve, and a closed fiber of a regular pencil $f : \mathcal{X} \to B$, then

$$\mathcal{O}_{\mathcal{X}}(\textstyle\sum a_Q Q)|_X \cong \mathcal{O}_X(\sum a_Q Q). \tag{22}$$

So *twisters supported on tails do not depend on the chosen regular pencil.* To check (22) it is enough to observe that, for each tail Q of X, since

$$\mathcal{O}_{\mathcal{X}}(Q)|_Q \cong \mathcal{O}_Q(-r) \text{ and } \mathcal{O}_{\mathcal{X}}(Q)|_{Q'} \cong \mathcal{O}_Q(r),$$

Lemma 4.4 implies that $\mathcal{O}_{\mathcal{X}}(Q)|_X \cong \mathcal{O}_X(Q)$.

To state the main result of this section we need some notation, similar to the one used in Proposition 3.12. Let $f : \mathcal{X} \to B$ be a regular pencil. Let Z be a subcurve of X, where $X \subset \mathcal{X}$ is a singular fiber of f. Then Z is a divisor of $\mathcal{X}$. Now, the restriction π_Z of the first projection $\pi : \mathcal{X}^2_B \to \mathcal{X}$ over Z is the trivial family

$$\mathcal{X}^2_B \supset Z \times X \xrightarrow{\pi_Z} Z.$$

Thus, for any other subcurve $Z_1 \subseteq X$, the product $Z \times Z_1$ can be viewed as a Weil divisor of $\mathcal{X}^2_B$. Now, since the open subscheme $\dot{\mathcal{X}} \times_B \mathcal{X} \subset \mathcal{X}^2_B$ is regular, the restriction of $Z \times Z_1$ to it is a Cartier divisor. Let $\mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(Z \times Z_1)$ denote the associated line bundle. Using this notation, we have an explicit description of the map $\alpha^1_f : \dot{\mathcal{X}} \longrightarrow P^1_f$ defined in Theorem 3.10.

Theorem 4.6. *Let $f : \mathcal{X} \to B$ be a regular pencil of stable curves. Then the line bundle*

$$\mathcal{L}^{(1)} := \mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}\Big(\Delta + \sum_{\substack{b \in B \\ Q \in \mathcal{Q}(X_b)}} Q \times Q\Big).$$

is balanced on $\dot{\pi} : \dot{\mathcal{X}} \times_B \mathcal{X} \to \dot{\mathcal{X}}$. Furthermore, assume that the fibers of f are 1-general. Then the following two statements hold.

(i) *The morphism $\alpha^1_f : \dot{\mathcal{X}} \to P^1_f$ is the moduli map of $\mathcal{L}^{(1)}$.*

(ii) *If $\mathcal{M}$ is a balanced line bundle on $\dot{\pi} : \dot{\mathcal{X}} \times_B \mathcal{X} \to \dot{\mathcal{X}}$ having α^1_f as moduli map, then $\mathcal{M} \cong \mathcal{L}^{(1)}$, up to pullbacks from $\dot{\mathcal{X}}$.*

Remark 4.7. But for Part (ii), the theorem generalizes to curves that are not 1-general. See 5.10 and Remark 5.13.

Proof. The divisor $\sum Q \times Q$ is entirely supported on a union of closed fibers of $\mathcal{X} \times_B \mathcal{X} \to B$. By Lemma 2.6, the Néron maps of $\mathcal{L}^{(1)}$ and $\mathcal{O}(\Delta)$ are equal. Now, if the fibers of f are 1-general, α^1_f agrees with the Néron map of $\mathcal{O}(\Delta)$ on $\mathcal{X}_K$. On the other hand, if $\mathcal{L}^{(1)}$ is balanced on $\dot{\pi} : \dot{\mathcal{X}} \times_B \mathcal{X} \to \dot{\mathcal{X}}$, there is an associated moduli map $\hat{\mu}_{\mathcal{L}^{(1)}} : \dot{\mathcal{X}} \to P^1_f$ by 3.8 (B). Since $\hat{\mu}_{\mathcal{L}^{(1)}}$ coincides with the Néron map of $\mathcal{L}^{(1)}$ on $\mathcal{X}_K$, it follows that $\hat{\mu}_{\mathcal{L}^{(1)}} = \alpha^1_f$. Thus, to prove Part (i) it is enough to prove that $\mathcal{L}^{(1)}$ is balanced on $\dot{\pi}$.

To prove Part (ii), it is also enough to show that $\mathcal{L}^{(1)}$ is balanced, since P^1_f is a fine moduli scheme; see 3.8 (C).

Let us now prove that $\mathcal{L}^{(1)}$ is indeed balanced. We need only check this on each singular fiber of f, whence we may assume f is local. Let X be the closed fiber. It suffices to consider the singular fibers of the first projection

$$\dot{\pi} : \dot{\mathcal{X}} \times_B \mathcal{X} \longrightarrow \dot{\mathcal{X}},$$

which are all isomorphic to X. Let $p \in X$ be a nonsingular point, and set $L_p^{(1)} := \mathcal{L}^{(1)}|_{\dot{\pi}^{-1}(p)}$. Then

$$L_p^{(1)} \cong \mathcal{O}_X(p) \otimes \mathcal{O}_X(\sum_{\substack{Q \in \mathcal{Q}(X) \\ p \in Q}} Q). \tag{23}$$

We conclude by Lemma 4.9 (ii), observing that, since X is stable, a semibalanced line bundle on X is necessarily balanced; see 3.3 (A). ∎

The next two lemmas are needed to finish the proof of Theorem 4.6.

Lemma 4.8. *Let X be a curve, and Z a connected, proper subcurve. Let*

$$Q_1 \subset Q_2 \subset \cdots \subset Q_{n-1} \subset Q_n \tag{24}$$

be a chain of tails of X, and let r_i be the separating node of X generating Q_i for each $i = 1, \ldots, n$. Then

$$-1 \leq \deg_Z \mathcal{O}_X(\sum Q_i) \leq 1.$$

Furthermore, the extremes are attained if and only if there is a unique j such that $r_j \in Z \cap Z'$. In this case, the lower bound is attained if $Z \subseteq Q_j$, and the upper bound is attained if $Z \subseteq Q'_j$.

Proof. If $r_\ell \notin Z$ for any $\ell = 1, \ldots, n$, then $\deg_Z \mathcal{O}_X(\sum Q_i) = 0$. Suppose now that Z contains at least one r_ℓ. Let i and j be the smallest and greatest integers such that $r_i \in Z$ and $r_j \in Z$, respectively. Since Z is connected, Z contains as well all the irreducible components of X containing $r_{i+1}, \ldots, r_{j-1}$. In particular, $r_\ell \notin Z \cap Z'$ for any $\ell = i+1, \ldots, j-1$.

If $Z \cap Z'$ contains both r_i and r_j or neither of them, $\deg_Z \mathcal{O}_X(\sum Q_i) = 0$. If $Z \cap Z'$ contains r_i but not r_j, then $\deg_Z \mathcal{O}_X(\sum Q_i) = 1$ and $Z \subseteq Q'_i$. At last, if $Z \cap Z'$ contains r_j but not r_i, then $\deg_Z \mathcal{O}_X(\sum Q_i) = -1$ and $Z \subseteq Q_j$. ∎

Lemma 4.9. *Let X be a semistable curve, and p a nonsingular point. Then the following two statements hold.*

(i) *The line bundle $\mathcal{O}_X(p)$ is semibalanced if and only if p does not belong to any small tail of X.*
(ii) *The line bundle*

$$L_p^{(1)} := \mathcal{O}_X(p) \otimes \mathcal{O}_X(\sum_{\substack{Q \in \mathcal{Q}(X) \\ p \in Q}} Q)$$

is semibalanced.

Proof. The "if part" of (i) is a consequence of (ii), as the sum of tails in (ii) is zero when p does not belong to any small tail. As for the "only-if part", recall from 3.1 that

$$w_Z = 2g_Z - 2 + k_Z \tag{25}$$

for every connected proper subcurve $Z \subset X$. In particular,

$$w_Q < g - 1 \text{ for every small tail } Q \text{ of } X. \tag{26}$$

So, if p is contained in a small tail Q, then

$$M_Q(1) < 1 = \deg_Q \mathcal{O}_X(p).$$

Hence the Basic Inequality (11) is not satisfied for Q. So $\mathcal{O}_X(p)$ is not semibalanced.

We need only prove (ii) now. First, since X is semistable, $w_Z \geq 0$ for every subcurve $Z \subseteq X$; see 3.1. As a consequence,

(27) $\quad w_{Z_1} \leq w_{Z_2}$ for all subcurves Z_1 and Z_2 of X with $Z_1 \subseteq Z_2$.

Let $Q_1, \dots, Q_n$ be the small tails of X containing p, and $r_1, \dots, r_n$ their generating nodes. Since $w_{Q_i} + w_{Q_j} < 2g-2$ by (26), we have $Q_i \cup Q_j \neq X$ for each i and j. By Lemma 4.3, up to reordering, we may assume that

$$Q_1 \subset Q_2 \subset \cdots \subset Q_{n-1} \subset Q_n.$$

Let $N := \mathcal{O}_X(\sum_1^n Q_i)$; so $L_p^{(1)} = \mathcal{O}_X(p) \otimes N$. Let Z be any connected, proper subcurve of X. Then $\deg_Z N \geq -1$ by Lemma 4.8, and hence $\deg_Z L_p^{(1)} \geq -1$. As pointed out in Remark 3.3 (C), we need only show that $\deg_Z L_p^{(1)} \geq m_Z(1)$, and $\deg_Z L_p^{(1)} \geq 0$ if Z is exceptional.

First, suppose $\deg_Z N = -1$. By Lemma 4.8, there is j such that $r_j \in Z$ and $Z \subseteq Q_j$. By (26) and (27),

$$w_Z \leq w_{Q_j} < g-1, \tag{28}$$

and hence $m_Z(1) \leq 0$. Thus, if $p \in Z$,

$$\deg_Z L_p^{(1)} = 0 \geq m_Z(1).$$

Suppose $p \notin Z$. Then $Z \neq Q_j$. Since $r_j \in Z$, either $k_Z \geq 3$ or Z is a tail of Q_j. Now, if Z were a tail of Q_j, then $\overline{Q_j \setminus Z}$ would be a tail of X contained in Q_j, whence a small tail. Since $p \in \overline{Q_j \setminus Z}$, we would have $\overline{Q_j \setminus Z} = Q_i$ for some $i < j$, or $Z = \overline{Q_j \setminus Q_i}$. But then $\deg_Z N = 0$, a contradiction. Thus $k_Z \geq 3$. In particular, Z is not an exceptional component of X. It follows now from (28) that $m_Z(1) < -1$, and hence

$$\deg_Z L_p^{(1)} \geq -1 > m_Z(1).$$

Second, suppose $\deg_Z N \geq 0$. Then $\deg_Z L_p^{(1)} \geq 0$. By (27) we have that $w_Z \leq w_X = 2g-2$. So, if Z is not a large tail of X, then $m_Z(1) \leq 0$, and hence

$$\deg_Z L_p^{(1)} \geq 0 \geq m_Z(1).$$

On the other hand, suppose that Z is a large tail. At any rate, $m_Z(1) \leq 1/2$. Thus, if $p \in Z$,

$$\deg_Z L_p^{(1)} \geq 1 \geq 1/2 \geq m_Z(1). \tag{29}$$

Finally, suppose $p \notin Z$. Then p lies on Z', which is a small tail of X. Thus $Z' = Q_j$ for some j, and hence $Z = Q_j'$. It follows that $\deg_Z N = 1$, and hence (29) holds as well. ∎

Let X be a 1-general stable curve. Let $\dot{X} := X \setminus X_{\text{sing}}$. For any regular smoothing f of X, let

$$\alpha_X^1 := \alpha_f^1|_{\dot{X}} : \dot{X} \longrightarrow P_X^1.$$

The notation is not ambiguous by the following consequence of Theorem 4.6.

Corollary 4.10. *Let X be a 1-general stable curve. Then α^1_X does not depend on f. In fact, for each nonsingular point $p \in X$ we have*

$$\alpha^1_X(p) = \mathcal{O}_X(p) \otimes \mathcal{O}_X(\sum_{\substack{Q\in\mathcal{Q}(X)\\ p\in Q}} Q).$$

Proof. The expression of $\alpha^1_X(p)$ follows from (23) in the proof of Theorem 4.6. By 4.5 the map α^1_X does not depend on f. ■

If X is free from separating nodes then α^1_X is injective. This follows immediately from Lemma 4.13 below. The same lemma will be used in the proof of Proposition 5.9, a more general and precise statement. For the lemma and the proposition, the definition below is used.

Definition 4.11. Let X be a curve. A rational, smooth component C of X is called a *separating line* if C intersects $\overline{X \setminus C}$ in separating nodes of X. More generally, a connected subcurve $Z \subseteq X$ of arithmetic genus 0 is called a *separating tree of lines* if Z intersects $\overline{X \setminus Z}$ in separating nodes of X.

4.12. Let X be a curve, and $Z \subsetneq X$ a proper connected subcurve such that Z intersects $\overline{X \setminus Z}$ in separating nodes of X. Then the connected components of $\overline{X \setminus Z}$ are tails of X. In addition, if r is a separating node of Z, then r is a separating node of X.

A curve of arithmetic genus 0 is a curve of compact type, i.e. a nodal curve with every node separating, whose irreducible components are smooth and rational. So, if Z is a separating tree of lines, every node of Z is a separating node of Z, and hence of X. It follows that every connected subcurve of Z is also a separating tree of lines. In particular, every irreducible component of Z is a separating line.

We shall later need the following lemma.

Lemma 4.13. *Let X be a curve, and p and q distinct nonsingular points of X. Let $C \subseteq X$ be the irreducible component containing p. Then there is an isomorphism $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$ if and only if C contains q and is a separating line of X.*

Proof. Assume first that C contains q and is a separating line of X. Since C is smooth and rational, $\mathcal{O}_C(p) \cong \mathcal{O}_C(q)$. We may thus assume $C \neq X$. Since C meets $C' := \overline{X \setminus C}$ in separating nodes, applying Lemma 4.4 a few times, we can show that a line bundle on X is uniquely determined by its restrictions to C and to C'. Since $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ restrict to isomorphic line bundles on C and to the trivial line bundle on C', it follows that $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$.

Conversely, suppose $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$. Since $\mathcal{O}_X(p)$ has degree 1 on C, so has $\mathcal{O}_X(q)$, and hence $q \in C$ as well. Now, since $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$, in particular $\mathcal{O}_C(p) \cong \mathcal{O}_C(q)$. Since $p \neq q$, it follows from [AK], Thm. 8.8, p. 108 that $C \cong \mathbb{P}^1$.

If $C = X$ we are done. Suppose thus that $C \neq X$, and let $C' := \overline{X \setminus C}$. Also, suppose by contradiction that $C \cap C'$ is not made of separating nodes of X. Then there is a connected subcurve $Z \subseteq C'$ such that $C \cap Z$ is a scheme of length at least 2.

Since X is connected, the restriction $\tau : H^0(X, \mathcal{O}_X(p)) \to H^0(C, \mathcal{O}_C(p))$ is injective. But it is not surjective. Indeed, if a nonconstant $\sigma \in H^0(C, \mathcal{O}_C(p))$ could be extended to $\tilde{\sigma} \in H^0(X, \mathcal{O}_X(p))$, then $\tilde{\sigma}$ would have to be constant on Z and hence σ would be constant on $C \cap Z$. Since $C \cong \mathbb{P}^1$, this is impossible, σ being a nonconstant section of $\mathcal{O}_{\mathbb{P}^1}(1)$ So τ is injective, but not surjective, and hence $h^0(X, \mathcal{O}_X(p)) = 1$. Since $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$, it follows that $p = q$, an absurd. ■

5. Completing the first Abel map

The main result of this section is Theorem 5.5. We shall prove it first in a simpler case in Proposition 5.2, where we have a neater statement concerning the modularity, see 5.3.

As in Section 4, certain basic results of this section hold in more generality for curves having singularities other than nodes. Apart from the notation set in 5.1 below, these results are concentrated in 5.4.

5.1. Fix a nodal curve X. For each node $r \in X$, let $\nu_r : X_r^\nu \to X$ be the partial normalization of X at r, and denote by $\hat{X}_r$ the connected nodal curve obtained by adding to X_r^ν a smooth rational curve E_r connecting the two points of $\nu_r^{-1}(r)$. Thus

$$\hat{X}_r = X_r^\nu \cup E_r$$

and there is a natural surjection $\sigma_r : \hat{X}_r \to X$ such that $\sigma_r(E_r) = \{r\}$ and $(\sigma_r)_{|X_r^\nu} = \nu_r$ (so that σ_r is an isomorphism away from E_r). The nodal curve $\hat{X}_r$ will be considered up those automorphisms of E_r that fix the two attaching points $E_r \cap X_r^\nu$.

Assume now that X is a 1-general stable curve, and let r be a node of X. Let $\overline{r} \in E_r$ be any point distinct from the attaching points. If X has no separating nodes, then the line bundle $\mathcal{O}_{\hat{X}_r}(\overline{r}) \in \operatorname{Pic}^1 \hat{X}_r$ is balanced by Lemma 4.9, and hence determines a point of $\overline{P_X^1} \smallsetminus P_X^1$; see 3.8 (B). This point does not depend on the choice of $\overline{r}$ because, in any case, the restriction of $\mathcal{O}_{\hat{X}_r}(\overline{r})$ to X_r^ν is trivial; see 3.7. Thus we shall denote it by ℓ_r.

Proposition 5.2. *Let $f : \mathcal{X} \to B$ be a regular pencil of 1-general stable curves free from separating nodes. Then $\alpha_f^1 : \dot{\mathcal{X}} \to P_f^1$ extends to an injection*

$$\overline{\alpha_f^1} : \mathcal{X} \longrightarrow \overline{P_f^1}$$

such that $\overline{\alpha_f^1}(r) = \ell_r \in \overline{P_X^1}$ for each node r of each closed fiber X of f.

Remark 5.3. More precisely, the proof will show that $\overline{\alpha_f^1}$ is the moduli map of the line bundle $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$, where $\mathcal{Y} \to \mathcal{X}_B^2$ is a partial resolution of singularities, and $\tilde{\Delta}$ is the proper transform in $\mathcal{Y}$ of the diagonal Δ; see 5.4.

Proof. Denote by $\rho : \mathcal{Y} \to \mathcal{X}_B^2$ the partial resolution of singularities described in 5.4, from where we take some of the properties mentioned below. The map ρ is an isomorphism away from the points (r, r) for $r \in \mathcal{X} \smallsetminus \dot{\mathcal{X}}$. On the other hand, if $r \in \mathcal{X} \smallsetminus \dot{\mathcal{X}}$, then $\rho^{-1}(r, r)$ is a copy of $\mathbb{P}^1$. In addition, composing ρ with the first projection π,

$$\mathcal{Y} \xrightarrow{\rho} \mathcal{X}_B^2 \xrightarrow{\pi} \mathcal{X},$$

we obtain a family of quasistable curves $\mathcal{Y} \to \mathcal{X}$ having $\pi : \mathcal{X}^2_B \to \mathcal{X}$ as stable model.

For each closed fiber X of f, and each $r \in X_{\text{sing}} \subset \mathcal{X}$, let Y_r be the fiber of $\pi \circ \rho$ over r. Then $Y_r = \hat{X}_r$, where $\hat{X}_r$ is as defined in 5.1. On the other hand, each fiber of $\pi \circ \rho$ over $\dot{\mathcal{X}}$ is the same as the corresponding fiber of π.

Let $\tilde{\Delta} \subset \mathcal{Y}$ be the proper transform of Δ. By Property 5.4 (B), the map ρ restricts to an isomorphism between $\tilde{\Delta}$ and Δ. Also, $\tilde{\Delta}$ meets each fiber $Y_r = \hat{X}_r$ of $\pi \circ \rho$ over any $r \in \mathcal{X} \smallsetminus \dot{\mathcal{X}}$ transversally at a nonsingular point $\overline{r}$ contained in the exceptional component E_r.

To prove that α^1_f extends, we prove two claims: first, that $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$ is balanced on $\pi \circ \rho : \mathcal{Y} \to \mathcal{X}$, so it induces a morphism $\overline{\alpha^1_f} : \mathcal{X} \to \overline{P^1_f}$, its moduli map; and second, to show that $\overline{\alpha^1_f}$ extends α^1_f, that the restriction of $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$ to the each fiber of $\pi \circ \rho$ over $\dot{\mathcal{X}}$ is isomorphic to the corresponding restriction of $\mathcal{L}^{(1)}$, whose moduli map is α^1_f by Theorem 4.6.

We may now assume that f is local. Let X be its closed fiber. For each $r \in X_{\text{sing}}$, since $\tilde{\Delta}$ intersects Y_r transversally at $\overline{r}$, we have

$$\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})|_{Y_r} \cong \mathcal{O}_{\hat{X}_r}(\overline{r}), \tag{30}$$

which is balanced by Lemma 4.9. In addititon, for each nonsingular point $p \in X$ we have

$$\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})|_{Y_p} \cong \mathcal{O}_X(p)$$

which is balanced and isomorphic to the corresponding restriction of $\mathcal{L}^{(1)}$, also by Lemma 4.9. Therefore $\mathcal{O}_{\mathcal{Y}}(\tilde{\Delta})$ induces a moduli map

$$\overline{\alpha^1_f} : \mathcal{X} \longrightarrow \overline{P^1_f}$$

which extends α^1_f. Notice that (30) also shows that $\overline{\alpha^1_f}(r) = \ell_r$ for each $r \in X_{\text{sing}}$.

To show that $\overline{\alpha^1_f}$ is injective it suffices to consider singular points of X, by Lemma 4.13 and by the fact that $\overline{\alpha^1_f}(r) \in \overline{P^1_X} \smallsetminus P^1_X$ for every node $r \in X$. Now, if $r \in X_{\text{sing}}$, then $\overline{\alpha^1_f}(r)$ represents a balanced line bundle on $\hat{X}_r$. Hence, two different nodes r and r' of X are mapped to two points of $\overline{P^1_X}$ corresponding to balanced line bundles on different quasistable curves, namely $\hat{X}_r$ and $\hat{X}_{r'}$. Thus $\overline{\alpha^1_f}(r) \neq \overline{\alpha^1_f}(r')$; see 3.7. ■

5.4. *Resolution of singularities.* In the proof of Proposition 5.2 we used a partial resolution of singularities of $\mathcal{X}^2_B$ which we are now going to describe in detail, and in more generality.

Let $f : \mathcal{X} \to B$ be a regular pencil. The threefold $\mathcal{X}^2_B$ is singular at the points (r_1, r_2), where r_1 and r_2 are (not necessarily distinct) singular points of the same closed fiber of f.

Let X be a closed fiber of f, and r_1 and r_2 nodes of X. Since f is regular, locally around r_i the surface $\mathcal{X}$ is formally equivalent to the surface in $\mathbb{A}^3$ given by the equation $x_i y_i = t$, where t denotes a local parameter of B at the closed point covered by X. Pulling back these local equations to $\mathcal{X}^2_B$ under the two projection maps $\mathcal{X}^2_B \to \mathcal{X}$, and abusing of the same notation, we get that $\mathcal{X}_B^2$ is formally equivalent, locally around (r_1, r_2), to the threefold in $\mathbb{A}^5$ given the equations

$$\begin{cases} x_1 y_1 = t, \\ x_2 y_2 = t. \end{cases}$$

If $r_1 = r_2$, then the diagonal $\Delta \subset \mathcal{X}_B^2$ contains (r_1, r_2), and we may assume that it is given locally around (r_1, r_2) by

$$\begin{cases} x_1 y_1 = t, \\ x_2 = x_1, \\ y_2 = y_1. \end{cases}$$

Locally around (r_1, r_2) we may eliminate t, and view $\mathcal{X}_B^2$ as the cone $C \subset \mathbb{A}^4$ over the smooth quadric in $\mathbb{P}^3$ given by $x_1 y_1 = x_2 y_2$. Also, if $r_1 = r_2$, we may view Δ as the plane $D \subset \mathbb{A}^4$ given by $x_2 = x_1$ and $y_2 = y_1$. Notice that C is singular only at the origin. To resolve this singularity we need only blow up a plane in C containing the origin. Any plane will do, but let us blow up the plane given by $x_1 = x_2 = 0$. The blowup is the nonsingular threefold $\tilde{C} \subset \mathbb{P}^1 \times \mathbb{A}^4$ given by the equations

$$\begin{cases} \xi_2 x_1 = \xi_1 x_2, \\ \xi_1 y_1 = \xi_2 y_2, \end{cases}$$

where ξ_1, ξ_2 are homogeneous coordinates of $\mathbb{P}^1$. The blowup $\gamma : \tilde{C} \to C$ is isomorphic to C away from the origin. In addition, the fiber F over the origin is given by $x_1 = x_2 = y_1 = y_2 = 0$, and hence is isomorphic to $\mathbb{P}^1$.

The exceptional divisor E of the blow up $\tilde{C}$ is given by $x_2 = 0$ where $\xi_2 \neq 0$, and $x_1 = 0$ where $\xi_1 \neq 0$. In particular, $F \subset E$. Now, since $\xi_2 x_1 = \xi_1 x_2$, summing the divisor given by $\xi_1 = 0$ to E we get the principal divisor given by $x_1 = 0$. Thus $E \cdot F = -1$.

Suppose $r_1 = r_2$. Then $\gamma^{-1}(D)$ is given by $x_1(\xi_1 - \xi_2) = y_2(\xi_1 - \xi_2) = 0$ where $\xi_1 \neq 0$, and by $x_2(\xi_1 - \xi_2) = y_1(\xi_1 - \xi_2) = 0$ where $\xi_2 \neq 0$. Thus $\gamma^{-1}(D)$ is the union of the Cartier divisor given by $\xi_1 = \xi_2$ and the fiber F. The strict transform $\tilde{D}$ of D is thus a Cartier divisor intersecting F transversally at a point.

For $i = 1, 2$, let $\phi_i : \tilde{C} \to \mathbb{A}^2$ be the composition of γ with the projection onto the plane with coordinates x_i, y_i. The fiber of ϕ_1 over the origin is given by $x_1 = y_1 = \xi_1 x_2 = \xi_2 y_2 = 0$. It is the union of F and the affine lines N_1, given by $x_1 = y_1 = \xi_1 = y_2 = 0$, and N_2, given by $x_1 = y_1 = \xi_2 = x_2 = 0$. The lines N_1 and N_2 do not meet, and F intersects each N_i transversally at a single point. Also, ϕ_2 maps N_1 and N_2 isomorphically onto the lines $y_2 = 0$ and $x_2 = 0$, respectively.

The exceptional divisor E contains N_2, and intersects N_1 transversally. Since $\xi_1 \neq 0$ on N_2, we have $E \cdot N_2 = 0$. If $r_1 = r_2$, the strict transform $\tilde{D}$ does not meet either N_1 or N_2, and intersects F transversally.

(An analogous description holds if we reverse the roles of ϕ_1 and ϕ_2.)

We will now consider the global picture. Let $\mathcal{I}_\Delta$ denote the ideal sheaf of the diagonal $\Delta \subset \mathcal{X}_B^2$, and let $\check{\mathcal{I}}_\Delta$ denote the dual sheaf, i.e.

$$\check{\mathcal{I}}_\Delta := Hom(\mathcal{I}_\Delta, \mathcal{O}_{\mathcal{X}_B^2}).$$

Since $\mathcal{I}_\Delta$ is a sheaf of ideals, $\check{\mathcal{I}}_\Delta$ is a sheaf of fractional ideals of $\mathcal{X}^2_B$. A piece of notation: for each open subscheme $U \subseteq \mathcal{X}^2_B$ and each sheaf of fractional ideals $\mathcal{M}$ of U, consider its powers $\mathcal{M}^n$, and form the associated sheaf of Rees algebras:

$$\mathcal{R}(\mathcal{M}) := \mathcal{O}_U \oplus \mathcal{M} \oplus \mathcal{M}^2 \oplus \cdots \oplus \mathcal{M}^n \oplus \cdots .$$

Set $\mathcal{Y} := \mathrm{Proj}_{\mathcal{X}^2_B}(\mathcal{R}(\check{\mathcal{I}}_\Delta))$, and let $\rho : \mathcal{Y} \to \mathcal{X}^2_B$ be the structure map.

We may view ρ as a blowup. In fact, for any open subscheme $U \subseteq \mathcal{X}^2_B$ over which there is an embedding $\iota : \check{\mathcal{I}}_\Delta|_U \to \mathcal{L}$ into an invertible sheaf $\mathcal{L}$, we may view $\rho : \rho^{-1}(U) \to U$ as the blowup of U along the closed subscheme $V \subseteq U$ whose sheaf of ideals $\mathcal{I}_{V|U}$ satisfies $\iota(\check{\mathcal{I}}_\Delta|_U) = \mathcal{I}_{V|U}\mathcal{L}$. In other words, ι induces an isomorphism over U:

$$\mathrm{Proj}_U(\mathcal{R}(\mathcal{I}_{V|U})) \longrightarrow \mathrm{Proj}_U(\mathcal{R}(\check{\mathcal{I}}_\Delta)|_U).$$

In the same vein, for each invertible sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_U$ we have that $Hom(\mathcal{I}_\Delta|_U, \mathcal{J}) = \check{\mathcal{I}}_\Delta|_U \mathcal{J}$, and hence we obtain a canonical isomorphism over U:

$$\mathrm{Proj}_U(\mathcal{R}(\check{\mathcal{I}}_\Delta)|_U) \longrightarrow \mathrm{Proj}_U(\mathcal{R}(Hom(\mathcal{I}_\Delta|_U, \mathcal{J}))).$$

Since $\mathcal{I}_\Delta$ is invertible away from the points (r, r) for $r \in \mathcal{X} \smallsetminus \dot{\mathcal{X}}$, it follows from the above description that ρ is an isomorphism away from these same points. In addition, around the points (r, r), where r is a node of a closed fiber of f, the map ρ is formally equivalent to the blowup described above, because

$$\mathrm{Hom}_A((x_1 - x_2, y_1 - y_2), (x_1 - x_2)) = (x_1, x_2), \text{ where } A := \frac{k[[x_1, x_2, y_1, y_2]]}{(x_1y_1 - x_2y_2)}.$$

Then all of the properties above, verified locally, yield global properties of ρ. Indeed, assume that the fibers of f are nodal. (It would actually be enough to assume that the fibers are Gorenstein.) Then, recalling that $\pi : \mathcal{X}^2_B \to \mathcal{X}$ denotes the first projection, the following statements hold:

(A) The composition

$$\pi \circ \rho : \mathcal{Y} \longrightarrow \mathcal{X}$$

is a family of curves whose fiber Y_r over a point r of a closed fiber X of f is X, if r is nonsingular, and $\hat{X}_r$, described in 5.1, if r is a node.

(B) Let $\tilde{\Delta} \subset \mathcal{Y}$ denote the proper transform of Δ. For each node r of each closed fiber X of f, the transform $\tilde{\Delta}$ intersects the fiber Y_r transversally at a point lying in the exceptional component $E_r = \rho^{-1}(r, r)$.

(C) Let Q be a tail of a closed fiber X of f, and r the node of X generating Q. Let

$$\widetilde{Q^2} := \rho^{-1}(Q \times Q).$$

Then $\widetilde{Q^2}$ is a Cartier divisor of $\mathcal{Y}$ containing E_r. Furthermore,

$$\widetilde{Q^2} \cdot E_r = -1, \quad \widetilde{Q^2} \cdot \hat{Q} = -1, \quad \text{and} \quad \widetilde{Q^2} \cdot \hat{Q}' = 1,$$

where, using the notation in 5.1, $\hat{Q} := \sigma_r^{-1}(Q)$ and $\hat{Q}' := \overline{\hat{X}_r \smallsetminus \hat{Q}}$, i.e. $\hat{Q}$ is the tail of $\hat{X}_r$ mapping to Q and containing E_r, and $\hat{Q}'$ is the complementary tail.

We may now generalize Proposition 5.2.

Theorem 5.5. *Let $f : \mathcal{X} \to B$ be a regular pencil of 1-general stable curves. Then there exists a morphism*

$$\overline{\alpha^1_f} : \mathcal{X} \longrightarrow \overline{P^1_f}$$

extending $\alpha^1_f : \dot{\mathcal{X}} \to P^1_f$. If r is a node of a closed fiber X of f, then $\overline{\alpha^1_f}(r) \in P^1_X$ if and only if r is a separating node of X.

Remark 5.6. The result extends to curves that are not 1-general. See 5.10 and 5.13.

Proof. As in the proof of Theorem 3.10 we may work locally around each singular fiber. So, assume f is local, and let X denote its closed fiber.

The new difficulty with respect to Proposition 5.2 is that, if X has separating nodes, α^1_f is the moduli map of a nontrivial "twist" of the diagonal by Theorem 4.6, and thus the same must hold for its completion. Fortunately, however, the divisors we need for the "twist" are already present in the partial resolution of singularities $\rho : \mathcal{Y} \to \mathcal{X}^2_B$ described in 5.4.

Namely, let $Q_1, \dots, Q_m$ be all the small tails of X. Let $\tilde{\Delta} \subset \mathcal{Y}$ be the strict transform of Δ, and set $\widetilde{Q^2_i} := \rho^{-1}(Q_i \times Q_i)$ for $i = 1, \dots, m$. As seen in 5.4, all the $\widetilde{Q^2_i}$ and $\tilde{\Delta}$ are Cartier divisors. Define the line bundle

$$\mathcal{M} := \mathcal{O}_{\mathcal{Y}}\Big(\tilde{\Delta} + \widetilde{Q^2_1} + \cdots + \widetilde{Q^2_m}\Big). \tag{31}$$

We claim that $\mathcal{M}$ is semibalanced on the composition $\pi \circ \rho : \mathcal{Y} \to \mathcal{X}$ of ρ with the first projection $\pi : \mathcal{X}^2_B \to \mathcal{X}$. Once the claim is proved, we may let $\overline{\alpha^1_f} : \mathcal{X} \longrightarrow \overline{P^1_f}$ be the moduli map of $\mathcal{M}$; see 3.8 (B).

To prove the claim, first observe that ρ is an isomorphism over $\dot{\mathcal{X}} \times_B \mathcal{X}$, whence

$$\mathcal{M}|_{\dot{\mathcal{X}} \times_B \mathcal{X}} \cong \mathcal{O}_{\dot{\mathcal{X}} \times_B \mathcal{X}}(\Delta + Q_1 \times Q_1 + \cdots + Q_m \times Q_m),$$

which is balanced, by Theorem 4.6, and defines α^1_f. Thus, once $\mathcal{M}$ is shown to be semibalanced, we have that $\overline{\alpha^1_f}|_{\dot{\mathcal{X}}} = \alpha^1_f$.

Now, let $r \in X_{\text{sing}}$. The fiber $Y_r := (\pi \circ \rho)^{-1}(r)$ is equal to $\hat{X}_r$ by Property 5.4 (A). Also, $\tilde{\Delta}$ intersects Y_r transversally at a point $\overline{r}$ of the exceptional component $E_r = \rho^{-1}(r, r)$, by Property 5.4 (B).

For each $i = 1, \dots, m$, let r_i be the separating node of X generating Q_i. Let

$$\hat{Q}_i := \sigma_r^{-1}(Q_i) \subset \hat{X}_r,$$

and let $\hat{Q}'_i$ be its complement in $\hat{X}_r$. Then $\hat{Q}_i$ is a small tail of $\hat{X}_r$ dominating Q_i, and containing E_r if and only if $r \in Q_i$. If $r = r_i$ then also $\overline{\hat{Q}_i \setminus E_r}$ is a small tail of $\hat{X}_r$. These are all the small tails of $\hat{X}_r$: the subcurves $\hat{Q}_1, \dots, \hat{Q}_m$, together with $\overline{\hat{Q}_i \setminus E_r}$ in case $r = r_i$.

For each $i = 1, \dots, m$, the subscheme $Q_i \times Q_i \subset \mathcal{X}^2_B$ is a Cartier divisor away from (r_i, r_i). Identifying Y_r with $\hat{X}_r$, we claim that

$$\mathcal{O}_{\mathcal{Y}}(\widetilde{Q^2_i})|_{Y_r} \cong \begin{cases} \mathcal{O}_{\hat{X}_r} & \text{if } r \notin Q_i, \\ \mathcal{O}_{\hat{X}_r}(\hat{Q}_i) & \text{if } r \in Q_i. \end{cases} \tag{32}$$

In fact, if $r \notin Q_i$, then $\widetilde{Q_i^2}$ does not meet Y_r, and hence (32) holds. Suppose now that $r \in Q_i$. Recall that $\hat{Q}_i$ is a tail of $\hat{X}_r$. Let s_i denote its generating node. If $r \neq r_i$ then, since $Q_i \times Q_i$ is a Cartier divisor of $\mathcal{X}_B^2$ at (r, r_i), we have

$$\text{(33)} \qquad \mathcal{O}_{\mathcal{Y}}(\widetilde{Q_i^2})|_{\hat{Q}_i} \cong \mathcal{O}_{\hat{Q}_i}(-s_i) \quad \text{and} \quad \mathcal{O}_{\mathcal{Y}}(\widetilde{Q_i^2})|_{\hat{Q}_i'} \cong \mathcal{O}_{\hat{Q}_i'}(s_i).$$

The same restrictions are achieved with $\mathcal{O}_{\hat{X}_r}(\hat{Q}_i)$. Thus (32) follows from Lemma 4.4. Finally, if $r = r_i$ then (33) still holds, by Property 5.4 (C), and hence (32) follows in the same way. The proof of (32) is complete.

Now, notice that Q_i contains r if and only if $\hat{Q}_i$ contains $\overline{r}$. In addition, if $r = r_i$ then $\overline{r} \notin \overline{\hat{Q}_i \smallsetminus E_r}$. Since $\hat{Q}_1, \ldots, \hat{Q}_m$, and $\overline{\hat{Q}_i \smallsetminus E_r}$ if $r = r_i$, are all the small tails of $\hat{X}_r$, it follows from (32) that

$$\mathcal{M}|_{Y_r} \cong \mathcal{O}_{\hat{X}_r}(\overline{r}) \otimes \mathcal{O}_{\hat{X}_r}\Big(\sum_{\substack{Q \in \mathcal{Q}(\hat{X}_r) \\ \overline{r} \in Q}} Q\Big),$$

which is semibalanced by Lemma 4.9. Our claim is proved, and thus we finish the proof of the existence of $\overline{\alpha_f^1}$.

To prove the second statement of the theorem, it suffices to prove that for any node $r \in X$ we have

$$\deg_{E_r} \mathcal{M} = \begin{cases} 1 & \text{if } r \text{ is not separating,} \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, notice that, if $r \neq r_i$ then $\widetilde{Q_i^2} \cdot E_r = 0$, whereas if $r = r_i$ then $\widetilde{Q_i^2} \cdot E_r = -1$ by Property 5.4 (C). Since at any rate $\tilde{\Delta} \cdot E_r = 1$, the degree of $\mathcal{M}|_{E_r}$ is 1, unless $r = r_i$ for some i, in which case the degree is 0. Since X is 1-general, each separating node of X generates a small tail, and hence is equal to r_i for some i. So $\overline{\alpha_f^1}(r) \in P_X^1$ if and only if r is a separating node. ∎

Example 5.7. Let X be a curve of compact type with two components, C_1 and C_2. Then C_1 and C_2 are smooth, and $C_1 \cap C_2 = \{r\}$, where r is the unique node of X. Assume $g_{C_1} < g_{C_2}$. Then $\hat{X}_r = C_1 \cup E \cup C_2$ and $\mathcal{Q}(\hat{X}_r) = \{C_1, C_1 \cup E\}$, where $E = \mathbb{P}^1$. The line bundle $\mathcal{M}$ in the proof of Theorem 5.5, whose moduli map is $\overline{\alpha_f^1}$, satisfies

$$\mathcal{M} = \mathcal{O}_{\mathcal{Y}}\Big(\tilde{\Delta} + \widetilde{C_1^2}\Big).$$

In this case, it is easy to describe the completed Abel map. First notice that there is a canonical isomorphism $\overline{P_X^1} \cong \operatorname{Pic}^0 C_1 \times \operatorname{Pic}^1 C_2$, essentially by Lemma 4.4. Hence, a point $\ell \in \overline{P_X^1}$ is represented by a pair (L_1, L_2) with $L_i \in \operatorname{Pic} C_i$. For $i = 1, 2$ let $q_i \in C_i$ lying above r. Then

$$\overline{\alpha_f^1}(p) = \begin{cases} (\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_{C_2}(q_2)) & \text{if } p \in C_1, \\ (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(p)) & \text{if } p \in C_2. \end{cases}$$

In particular, $\overline{\alpha_f^1}(r) = (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(q_2))$. Thus, composing $\overline{\alpha_f^1}|_{C_1}$ with the projection $\operatorname{Pic}^0 C_1 \times \operatorname{Pic}^1 C_2 \to \operatorname{Pic}^0 C_1$ we obtain the classical Abel–Jacobi map of C_1 with base point q_1, i.e.

$$\begin{array}{ccc} C_1 & \longrightarrow & \operatorname{Pic}^0 C_1 \\ p & \mapsto & \mathcal{O}_{C_1}(p-q_1). \end{array}$$

The analogous composition for C_2 gives the first Abel map $C_2 \to \operatorname{Pic}^1 C_2$.

Let X be a 1-general stable curve, and f a regular smoothing of X. The restriction $\overline{\alpha^1_f}|_{\dot{X}}$ coincides with $\alpha^1_f|_{\dot{X}}$, whence does not depend on f by Corollary 4.10. So $\overline{\alpha^1_f}|_X$ does not depend on f either, and we may set $\overline{\alpha^1_X} := \overline{\alpha^1_f}|_X$.

5.8. *Explicit description of the complete Abel map.* Let X be a 1-general stable curve and $p \in X$. We shall now explicitely describe $\overline{\alpha^1_X}(p)$, in formulas (34) and (35) below, following the proof of Theorem 5.5.

First some notation. Let $P_1, \ldots, P_m$ be all the small tails of X containing p. (The unusual naming of the tails using "P" rather than "Q" is to match the notation of the proof of Proposition 5.9 below.) By Lemma 4.3 we can write

$$P_m \subset P_{m-1} \subset \cdots \subset P_2 \subset P_1.$$

Set $Z_i := \overline{P_i - P_{i+1}}$ for each $i = 1, \ldots, m-1$ and $Z_m := P_m$, so that $P_1 = \cup_1^m Z_i$. Also, put $Q := P'_m$, a large tail of X. Hence

$$X = P_1 \cup Q = \cup_1^m Z_i \cup Q.$$

Let $r_1, \ldots, r_m$ be the separating nodes of X generating $P_1, \ldots, P_m$. Notice that $Z_i \cap Z'_i = \{r_i, r_{i+1}\}$ if $i = 1, \ldots, m-1$, and $Z_m \cap Z'_m = \{r_m\}$. Therefore each of the Z_i and Q meets the complementary curve in separating nodes of X. Hence, by iterated use of Lemma 4.4, to give a line bundle on X it suffices to give its restrictions to all the Z_i and to Q.

We are now ready to describe $\overline{\alpha^1_X}(p)$ if p is a nonsingular point or a separating node of X (in which case of course $p = r_m$). Recall that by Theorem 5.5, $\overline{\alpha^1_X}(p)$ corresponds to a line bundle on X. We have

(34) $\overline{\alpha^1_X}(p) = \{\mathcal{O}_Q(r_1), \mathcal{O}_{Z_1}(r_2 - r_1), \ldots, \mathcal{O}_{Z_{m-1}}(r_m - r_{m-1}), \mathcal{O}_{Z_m}(p - r_m)\}.$

Now, suppose that p is a nonseparating node of X. Then we know that $\overline{\alpha^1_X}(p)$ corresponds to a line bundle on $\hat{X}_p$. Let $E \subset \hat{X}_p$ be the exceptional component of $\hat{X}_p$, and let $\widetilde{Z_m}$ denote the normalization of Z_m at p only. Keeping the above notation we have

$$\hat{X}_p = Q \cup Z_1 \cup \cdots \cup Z_{m-1} \cup \widetilde{Z_m} \cup E.$$

Now, recall from 3.7 that $\overline{\alpha^1_X}(p)$ is uniquely determined by a line bundle L, of degree 0, on the complementary curve of E; that is, arguing as above, by the string of the restrictions of L to Q, Z_1, $\ldots$, Z_{m-1}, $\widetilde{Z_m}$. We have

(35) $\overline{\alpha^1_X}(p) = \{\mathcal{O}_Q(r_1), \mathcal{O}_{Z_1}(r_2 - r_1), \ldots, \mathcal{O}_{Z_{m-1}}(r_m - r_{m-1}), \mathcal{O}_{\widetilde{Z_m}}(-r_m)\}$

Proposition 5.9. *Let X be a 1-general stable curve. Let p and q be distinct points of X. Then $\overline{\alpha^1_X}(p) = \overline{\alpha^1_X}(q)$ if and only if p and q belong to the same separating tree of lines of X.*

A similar result for the Abel–Jacobi map to the (degree-0) Jacobian is proved by B. Edixhoven in [E98], Prop. 9.5. His statement (necessarily) excludes the case where p or q is a nonseparating node, since there the target space of the map is a noncompactified Néron model.

Proof. Suppose first that p and q belong to a separating tree of lines $Z \subset X$. Since Z is connected, to prove that $\overline{\alpha^1_X}(p) = \overline{\alpha^1_X}(q)$ it is enough to consider the case where p and q are nonsingular points of X in the same irreducible component C of Z. Now, C is a separating line of X; see 4.12. Thus $\mathcal{O}_X(p) \cong \mathcal{O}_X(q)$ by Lemma 4.13. Since p and q lie on the same component, it follows that $\alpha^1_X(p) = \alpha^1_X(q)$, and hence $\overline{\alpha^1_X}(p) = \overline{\alpha^1_X}(q)$.

Conversely, suppose $\overline{\alpha^1_X}(p) = \overline{\alpha^1_X}(q)$. We claim that p and q are nonsingular points or separating nodes of X. Indeed, suppose by contradiction, and without loss of generality, that p is a nonseparating node of X. Then $\overline{\alpha^1_X}(p) \in \overline{P^1_X} \setminus P^1_X$ by Theorem 5.5. Since $\overline{\alpha^1_X}(p) = \overline{\alpha^1_X}(q)$, it follows from Theorem 5.5 as well that q is also a nonseparating node of X. However, $\overline{\alpha^1_X}(p)$ and $\overline{\alpha^1_X}(q)$ correspond to balanced line bundles on different quasistable curves, $\hat{X}_p$ and $\hat{X}_q$; see 5.8. So $\overline{\alpha^1_X}(p) \neq \overline{\alpha^1_X}(q)$; see 3.7. The contradiction proves the claim.

Since p and q are nonsingular or separating nodes of X, both $\overline{\alpha^1_X}(p)$ and $\overline{\alpha^1_X}(q)$ are line bundles on X. Let $P_1, \dots, P_m$ be the small tails containing p and $Q_1, \dots, Q_n$ the small tails containing q. (We may have $m = 0$ or $n = 0$.) It follows from Lemma 4.3, as in the proof of Theorem 4.6, that, up to reordering the tails,

$$P_m \subset P_{m-1} \subset \cdots \subset P_2 \subset P_1 \quad \text{and} \quad Q_n \subset Q_{n-1} \subset \cdots \subset Q_2 \subset Q_1.$$

Set $P_0 := Q_0 := X$. Let $r_1, \dots, r_m$ be the separating nodes of X generating $P_1, \dots, P_m$, and $s_1, \dots, s_n$ those generating $Q_1, \dots, Q_n$. In addition, set $P_{m+1} := Q_{n+1} := \emptyset$, and put $r_{m+1} := p$ and $s_{n+1} := q$.

We may assume $m \leq n$, without loss of generality. Let i be the largest nonnegative integer such that $i \leq m$ and $P_j = Q_j$ for $j = 0, 1, \dots, i$. Then also $r_j = s_j$ for $j = 1, \dots, i$. We claim that $P_{i+1} \cap Q_{i+1} = \emptyset$. Indeed, if $i = m$ then P_{i+1} is already empty. Suppose $i < m$. If $P_{i+1} \subseteq Q_{i+1}$, then Q_{i+1} is a small tail containing p. And since

$$P_{i+1} \subseteq Q_{i+1} \subset Q_i = P_i,$$

we have $P_{i+1} = Q_{i+1}$, contradicting the maximality of i. In a similar way, $Q_{i+1} \not\subseteq P_{i+1}$. Since $P_{i+1} \cup Q_{i+1} \neq X$, because P_{i+1} and Q_{i+1} are small tails, it follows from Lemma 4.3 that $P_{i+1} \cap Q_{i+1} = \emptyset$, proving our claim. In particular, $r_{i+1} \neq s_{i+1}$.

As $P_i = Q_i$, we may consider $Y := \overline{P_i \setminus (P_{i+1} \cup Q_{i+1})}$. As P_{i+1} and Q_{i+1} do not meet, their union cannot be P_i, a connected subcurve of X. Thus Y is a subcurve of X. It is also connected, being either equal to, or a tail of, $\overline{P_i \setminus P_{i+1}}$, which in turn is either equal to, or a tail of, P_i, a tail of X.

Since $Y \subseteq \overline{P_i \setminus P_{i+1}}$, the restriction of $\overline{\alpha^1_X}(p)$ to Y is $\mathcal{O}_Y(r_{i+1} - r_i)$; see 5.8. Analogously, $\overline{\alpha^1_X}(q)$ restricts to $\mathcal{O}_Y(s_{i+1} - s_i)$. Since $\overline{\alpha^1_X}(p) = \overline{\alpha^1_X}(q)$ and $r_i = s_i$, it follows that $\mathcal{O}_Y(r_{i+1}) \cong \mathcal{O}_Y(s_{i+1})$. Since $r_{i+1} \neq s_{i+1}$, by Lemma 4.13 applied to the curve Y, we see that r_{i+1} and s_{i+1} are contained in a separating line C of Y. Since $Y \cap Y'$ is made of separating nodes of X, so is $C \cap C'$; see 4.12. In other words, C is a separating line of X.

For $\ell = 1, \ldots, m-i$ let $Y_\ell := \overline{P_{i+\ell} - P_{i+\ell+1}}$. Then $\overline{\alpha^1_X}(p)$ restricts to $\mathcal{O}_{Y_\ell}(r_{i+\ell+1} - r_{i+\ell})$. On the other hand, since $Y_\ell \subset \overline{Q_i \setminus Q_{i+1}}$ for each ℓ, but neither $s_i \in Y_\ell$ nor $s_{i+1} \in Y_\ell$, we have that $\overline{\alpha^1_X}(q)$ restricts to the trivial bundle $\mathcal{O}_{Y_\ell}$. Applying Lemma 4.13 to the curve Y_ℓ, we get that $r_{i+\ell}$ and $r_{i+\ell+1}$ are contained in a separating line C^p_ℓ of Y_ℓ. As before, C^p_ℓ is also a separating line of X.

Similarly, for each $\ell = 1, \ldots, n-i$ the points $s_{i+\ell}$ and $s_{i+\ell+1}$ are contained in a separating line C^q_ℓ of X. The union of all the separating lines, namely,

$$C^p_{m-i}, \ldots, C^p_2, C^p_1, C, C^q_1, C^q_2, \ldots, C^q_{n-i},$$

is a separating tree of lines containing p and q. ■

To give a finer description of the map $\overline{\alpha^1_X}$, in particular to characterize when it is a closed embedding, requires further techniques, such as the theory of theta functions from [E01]. We left this out to keep the paper to a reasonable length.

5.10. *Curves that are not* 1*-general.* We conclude by discussing the case of stable curves which are not 1-general. Recall that such curves form a proper closed subset of $\overline{M}_g$, nonempty if and only if g is even, and their combinatorial structure is described in Proposition 3.15. What kind of complications occur for curves that are not 1-general, or not d-general?

The stack $\overline{\mathcal{P}}_{d,g}$ introduced in 3.8 in the case $(d-g+1, 2g-2) = 1$ is constructed as a quotient stack, i.e. $\overline{\mathcal{P}}_{d,g} := [H_d/G]$; notation as in 3.5. The same definition can be given for each d, obtaining in this way a quotient stack $[H_d/G]$. However, when non-d-general curves appear, this stack presents some pathologies.

More precisely, recall from 3.5 that the scheme-theoretic quotient H_d/G is endowed with a natural surjective morphism $\phi_d : H_d/G \to \overline{M}_g$. The open subset of $\overline{M}_g$ over which the quotient map $\pi_d : H_d \to H_d/G$ is a geometric quotient is exactly the locus of d-general curves. The problem is that, as soon as $\pi_d : H_d \to H_d/G$ fails to be a geometric quotient, the following pathologies occur:

(i) $[H_d/G]$ fails to be a Deligne–Mumford stack.
(ii) The natural map of stacks $[H_d/G] \to \overline{\mathcal{M}}_g$ fails to be representable.
(iii) Néron models are not parametrized by $[H_d/G]$.

However, when studying Abel maps, we can still obtain some results. Since the stack $\overline{\mathcal{P}}_{d,g}$ behaves badly, let us consider the scheme $\overline{P}_{d,\,g} := H_d/G$ introduced in 3.5. As we mentioned above, there is always a surjective morphism $\phi_d : \overline{P}_{d,\,g} \to \overline{M}_g$. By [C94], Thm. 6.1, p. 641, $\overline{P}_{d,\,g}$ is an integral projective scheme. It is also normal, being a GIT-quotient of H_d, which is nonsingular by [C94], Lemma 2.2, p. 609.

Although $\overline{P}_{d,\,g}$ is not a coarse moduli space, not even away from curves with nontrivial automorphisms, $\overline{P}_{d,\,g}$ does satisfy useful functorial properties. Thus, let $f : \mathcal{X} \to B$ be a regular pencil of stable curves. Let $B \to \overline{M}_g$

be the associated map, and define

$$\overline{P^d_f} := B \times_{\overline{M}_g} \overline{P}_{d,g}.$$

If X is a closed fiber of f, denote by $\overline{P^d_X}$ the corresponding fiber of $\overline{P^d_f}$ over B. As mentioned above, $\overline{P^d_f}$ may fail to contain the Néron model N^d_f. However, a functorial property holds: the moduli property given in 3.8 (B) holds exactly as stated. More precisely, to any semibalanced line bundle $\mathcal{L}$ on a family of semistable curves $\mathcal{Y} \to T$ having $\mathcal{X}_T \to T$ as stable model, where T is any B-scheme, we can associate a canonical moduli map $\hat{\mu}_{\mathcal{L}} : T \longrightarrow \overline{P^d_f}$; see [C94], Prop. 8.1, p. 653.

The main weakness, when nongeneral curves are present, is that different balanced line bundles on the same quasistable, or even stable, curve may be mapped to the same point in $\overline{P^d_f}$. Let us give an example of this behavior with regard to Abel maps.

Example 5.11. Let $X = C_1 \cup C_2$ be a curve of compact type as in Example 5.7. However, assume now that C_1 and C_2 have the same genus. Then X is not 1-general. As before, let r be the unique node of X, and let q_i be the point of C_i lying over r for $i = 1, 2$.

We shall now exhibit three nonequivalent balanced line bundles that correspond to the same point of $\overline{P^1_X}$. Notice that, by Lemma 4.4, to give a line bundle on a curve of compact type is equivalent to give a line bundle on each irreducible component of the curve.

Let $p \in C_1 \smallsetminus \{q_1\}$. Our first line bundle is $L_1 \in \operatorname{Pic} X$ corresponding to the pair

$$(\mathcal{O}_{C_1}(p), \mathcal{O}_{C_2}).$$

Let $Y := \hat{X}_r$; so $Y = C_1 \cup E \cup C_2$, where E is the exceptional component. Our second line bundle is $L_2 \in \operatorname{Pic} Y$ corresponding to the triple

$$(\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_E(q_E), \mathcal{O}_{C_2}),$$

where q_E is any point of E. Finally, our third line bundle is $L_3 \in \operatorname{Pic} X$ corresponding to the pair

$$(\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_{C_2}(q_2)).$$

We leave out the proof that L_1, L_2 and L_3 correspond to the same point of $\overline{P^1_X}$, referring to [C94], 7.2, Example 2, p. 645 for more details.

The above example shows that, if $f : \mathcal{X} \to B$ is a regular pencil with a non-1-general fiber X, then $\overline{P^1_f}$ and $\overline{P^1_X}$ are not coarse moduli schemes for balanced line bundles. However, we can still get a map $\overline{\alpha^1_f} : \mathcal{X} \to \overline{P^1_f}$ restricting to the classical Abel map of $\mathcal{X}_K$, by using our modular interpretation of the Abel map. In fact, essentially the same line bundle $\mathcal{M}$ given in (31) can be used to produce a moduli map $\overline{\alpha^1_f} : \mathcal{X} \to \overline{P^1_f}$. Most of the results in Sections 4 and 5 hold, provided we change one definition, as explained in 5.12 below.

5.12. *Small tails, again.* Let X be a stable curve of arithmetic genus g. Taking into account the case where X is not 1-general, we need to adjust the definition of the set $\mathcal{Q}(X)$ in 4.1. Suppose that X has a separating node that generates two tails Q and Q' of equal genus. It is easy to see that, if such a node exists, then it is unique. (A 1-general curve will never admit such a node by Proposition 3.15.) We must add to the set $\mathcal{Q}(X)$ of small tails of X either Q or Q', thus making an arbitrary choice between Q and Q', which nonetheless turns out to be completely irrelevant.

So, $\mathcal{Q}(X)$ *is defined as the set of all small tails of* X *together with one tail of genus* $g/2$*, if any such tail exists.*

Remark 5.13. The following results of the paper hold with essentially the same proof, as long as we use the modified definition of $\mathcal{Q}(X)$ for stable curves X of 5.12:

(i) Theorem 4.6, excluding Part (ii).
(ii) Corollary 4.10.
(iii) Theorem 5.5.

What will certainly fail is the possibility to interpret the Abel map in a unique way. In other words, if $f : \mathcal{X} \to B$ is a regular pencil, an extension $\overline{\alpha^1_f} : \mathcal{X} \to \overline{P^1_f}$ of the Abel map of $\mathcal{X}_K$ is obtained as the moduli map of a semibalanced line bundle, however the line bundle is not uniquely determined.

References

[AK] A. Altman, S. Kleiman. *Compactifying the Picard scheme.* Adv. Math. **35** (1980), 50–112.

[BLR] S. Bosch, W Lüktebohmert, M. Raynaud: Néron models. E.M.G. (21) Springer, Berlin, 1990..

[C94] L. Caporaso. *A compactification of the universal Picard variety over the moduli space of stable curves.* J. of the Amer. Math. Soc. **7** (1994), 589–660.

[C05] L. Caporaso. *Néron models over moduli of stable curves.* Amer. J. of Math., to appear. Also available at math.AG/0502171.

[C06] L. Caporaso. *Naturality of Abel maps.* Manuscripta Math., to appear. Also available at math.AG/0611477.

[CCC04] L. Caporaso, C. Casagrande, M. Cornalba. *Moduli of roots of line bundles on curves.* Trans. Amer. Math. Soc., to appear. Also available at AG/0404078.

[E98] B. Edixhoven. *On Néron models, divisors and modular curves.* J.Ramanujan Math. Soc. **13**, no. 2 (1998), 157–194.

[E01] E. Esteves. *Compactifying the relative Jacobian over families of reduced curves.* Trans. Amer. Math. Soc. **353** (2001), 3045–3095.

[EGK00] E. Esteves, M. Gagné, S. Kleiman. *Abel maps and presentation schemes.* Comm. Algebra **28**, no. 12 (2000), 5961–5992.

[EGK02] E. Esteves, M. Gagné, S. Kleiman. *Autoduality of the compactified Jacobian.* J. London Math. Soc. **65**, no. 3 (2002), 591–610.

[EK05] E. Esteves, S. Kleiman. *The compactified Picard scheme of the compactified Jacobian.* Adv. Math. **198**, no. 2 (2005), 484–503.

[K04] S. Kleiman. *What is Abel's theorem anyway?* In: The legacy of Niels Henrik Abel, 395–440, Springer, Berlin, 2004.

[K05] S. Kleiman. *The Picard scheme.* In: Fundamental algebraic geometry, 235–321, Mathematical Surveys and Monographs, vol. 123, Amer. Math. Soc., Providence, 2005.

[GIT] D. Mumford, J. Forgarty, F. Kirwan: Geometric Invariant Theory. (First edition 1965) Third edition E.M.G. 34 Springer, 1994.

[N64] A. Néron. *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux.* Inst. Hautes Études Sci. Publ. Math. **21** (1964).

[R70] M. Raynaud. *Spécialisation du foncteur de Picard.* Inst. Hautes Études Sci. Publ. Math. **38** (1970), 27–76.

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