Tropical and algebraic curves: comparing their divisor theory

Lucia Caporaso
Università Roma 3

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Curves and graphs of genus $g \geq 2$

1. A *weighted graph* (G, w) of genus g is a finite connected graph with a weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$
   $g := b_1(G) + \sum_{v \in V(G)} w(g)$
   (G, w) is *stable* if no vertex of weight 0 has valency ≤ 2.
2. A *tropical curve* is a metric weighted graph $\Gamma = (G, w, \ell)$ where $\ell : E(G) \to \mathbb{R}_{>0}$. Γ is *pure* if $w = 0$
   (G, w) is the *combinatorial type* of Γ.
3. A (Deligne-Mumford) *stable curve* is a connected, projective, nodal algebraic curve C, such that every smooth rational component of C meets its complementary curve in at least 3 points.
   To a stable curve C we associate its *weighted dual graph*, (G_C, w_C). C is stable if and only if so is (G_C, w_C).

Moduli of tropical and algebraic curves of genus $g \geq 2$.

$M_g^{\rm trop}$ = Moduli of tropical curves of genus g

Connected / Not compact / $\dim M_g^{\rm trop} = 3g-3$

$\overline{M_g}$ = Moduli of stable curves of genus g

Irreducible / Projective / $\dim \overline{M_g} = 3g-3$

These two moduli spaces have a canonical partition

$$\overline{M_g} = \bigsqcup_{(G,w)} \mathcal{C}(G,w), \qquad M_g^{\rm trop} = \bigsqcup_{(G,w)} \mathcal{C}^{\rm trop}(G,w)$$

both unions over all stable weighted graphs (G,w) of genus g.

Partitions of $\overline{M_g}$ and M_g^{trop}

$$\overline{M_g} = \bigsqcup_{(G,w)} \mathcal{C}(G,w), \qquad M_g^{\text{trop}} = \bigsqcup_{(G,w)} \mathcal{C}^{\text{trop}}(G,w)$$

both unions over all stable weighted graphs (G,w) of genus g.

$$\mathcal{C}(G,w) := \{C \in \overline{M_g} \ \text{ whose dual graph is } \ (G,w)\}$$

$$\mathcal{C}^{\text{trop}}(G,w) := \{\Gamma \in M_g^{\text{trop}} \ \text{ whose combinatorial type is } \ (G,w)\}$$

Both partitions have a poset structure under inclusion of closures. The obvious correspondence between the two partitions reverses the inclusions.

Divisors

X = graph / tropical curve / algebraic curve.

The group of *divisors* of X, $\mathrm{Div}(X)$, is the free abelian group generated by the points of X.

- Classical theory (e.g. classification, projective models) of nonsingular algebraic curves is based on their divisor theory.
- Weightless graphs have a divisor theory with strong similarities with the algebraic [Baker-Norine 07]. Riemann-Roch theorem and Clifford inequality hold.
- Pure tropical curves as well [Gathmann-Kerber, Mikhalkin-Zharkov 08]. Riemann-Roch holds.
- Weighted graphs and tropical curves as well [Amini-C soon]. Riemann-Roch holds.

The following are well defined in all contexts above.

1. Degree of a divisor and effective divisors. $\mathrm{Div}^d(X)$ is the set of divisors of degree d.
2. Linear equivalence "$\sim$" on $\mathrm{Div}(X)$. It preserves the degree.
   $\mathrm{Pic}(X) := \mathrm{Div}(X)/\sim, \qquad \mathrm{Pic}^d(X) := \mathrm{Div}^d(X)/\sim$
   - $\mathrm{Pic}^d(G, w)$ is a finite set of cardinality independent of d.
   - • $\mathrm{Pic}^d(C)$ is an algebraic variety independent of d.

   $\mathrm{Pic}^d(C)$ may fail to be quasi-projective and/or complete:
   $\mathrm{Pic}^d(C)$ is projective if and only if G_C is a tree.
3. Rank of a divisor, $r(D)$, invariant under linear equivalence.
   - $r(D) = h^0(C, D) - 1$.
4. Canonical divisor $K_X \in \mathrm{Div}(X)$.
   - $K_\Gamma = \sum_{p \in \Gamma} \big((2w(p) - 2 + \mathrm{val}(p)\big) p$

General Riemann-Roch formula.

Theorem (Riemann-Roch)

Let X be a graph/tropical curve/algebraic curve of genus g. Then

$$r(D) - r(K_X - D) = \deg D - g + 1$$

for every $D \in \mathrm{Div}(X)$.

If C is an algebraic curve, by "genus" we mean arithmetic genus.

Let C be an alg. curve and (G, w) its dual graph. Then $V(G)$ is the set of irreducible components of C. There is an obvious map

$$\begin{array}{ccc} \mathrm{Div}(C) & \stackrel{\tau}{\longrightarrow} & \mathrm{Div}(G,w) \\ D & \mapsto & \tau(D) = \sum_{v \in V(G)} \deg D_{|v} v \end{array} \qquad (1)$$

and under this map we have

$$\tau(K_C) = K_{(G,w)}.$$

Remark: there is no relation between $r(D)$ on C and $r(\tau(D))$ on G.
Example: $r(\tau(D)) \leq \deg D$ whereas $r(D)$ is unbounded (with $\deg D$ fixed).

Remark: *Divisor theory on singular curves is known to be very complex.*
Example: Clifford inequality fails non-trivially for reducible curves.

Brill-Noether loci

Definition of Brill-Noether loci W^r_d

$$W^r_d(X) = \{D \in \mathrm{Div}^d(X) : \; r(D) \geq r\}/_\sim$$

where $d, r \in \mathbb{Z}_{\geq 0}$.

If $X = C$ alg.curve then $W^r_d(C)$ is an algebraic variety whose geometry reflects some properties of C.
For example, dim $W^1_{g-1}(C)$ characterizes when C is hyperelliptic.

Whenever $W^r_d(C) \subsetneq \mathrm{Pic}^d(C)$ the elements of $W^r_d(C)$ are called *special*.

Empty Brill-Noether loci

Brill-Noether number: $\rho(g,d,r) := g-(r+1)(g-d+r)$

Theorem (Brill-Noether Theorem)

If $\rho(g,d,r)<0$ then $W^r_d(C)=\emptyset$ for the general curve C of genus g (i.e. for every C in an open subset of $\overline{M_g}$).

[Griffitths-Harris 80] Several other proofs.

Question: *What about graphs and tropical curves?*

Fact [Cools-Draisma-Payne-Robeva 10]: *There exist graphs and tropical curves of genus g such that W^r_d is empty.*

Problems:

1. *Characterize such graphs.*
2. *Study these loci in M_g^{trop}. Are they closed?* See [LPP] about this.

Non-empty Brill-Noether loci

Theorem (Existence Theorem)

If $\rho(g, d, r) \geq 0$ then $W^r_d(C) \neq \emptyset$ for every algebraic curve C of genus g.

[Kempf, Kleimann-Laksov $\sim$ 72]

Question: *What about graphs and tropical curves?*
Fact [Baker 09]: *The Existence theorem holds for tropical curves.*
Fact [C11]: *The Existence theorem holds for graphs.*
Both follow from the Specialization Lemma + the Existence Theorem for algebraic curves.

Problem: *Find a purely combinatorial proof of the Existence Theorem for graphs [C11].*

Special divisors: from algebraic curves to graphs

$$\overline{M^r_{g,d}} := \mathrm{cl}\{C \in M_g : \quad W^r_d(C) \neq \emptyset\} \subset \overline{M_g}$$

Theorem (C)

Let $C \in \mathcal{C}(G, w) \subset \overline{M_g}$. Assume that $C \in \overline{M^r_{g,d}}$.
Then there exists a tropical curve $\Gamma \in C^{\mathrm{trop}}(G, w) \subset M_g^{\mathrm{trop}}$ such that $W^r_d(\Gamma) \neq \emptyset$.
If $r = 1, d = 2$ (i.e. C is hyperelliptic), then $W^1_2(G, w) \neq \emptyset$.

Proof is based on the generalization of the Specialization Lemma to the case of weighted graphs.

Special divisors: from tropical to algebraic

Does the existence of special divisors on a graph (G, w) implies the existence of special divisors on some stable curve $C \in \mathcal{C}(G, w)$?
With the exception of hyperelliptic graphs, the answer seems to be NO.
The case $r = 1$ (d-gonal curves) can be dealt with Harris-Mumford admissible coverings, which gives a complete answer for all d.

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