Compactified Jacobians of nodal curves

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Dedicated to the memory of Professor Eckart Viehweg, who suddenly passed away a few days before this course was given.

Contents

1. Preliminaries

1.1. **Conventions and Notations.** We work over an algebraically closed field of any characteristic.

The word "curve" means projective scheme of pure dimension one. The genus of a curve X will always be the arithmetic genus $g_X := 1 - \chi(\mathcal{O}_X)$. For an irreducible curve, the geometric genus is defined as the genus of its normalization.

By a "family of curves" we mean a flat projective morphism $f : \mathcal{X} \to B$ such that for every $b \in B$ the fiber $X_b := f^{-1}(b)$ is a curve.

1.1.1. Let $Z \subset X$ be a subcurve. We denote by $Z^c := \overline{X \smallsetminus Z}$, and by $Z \cdot Z^c$ the Weil divisor

$$Z \cdot Z^c = \sum_{n \in Z \cap Z^c} n.$$

Also, $\delta_Z := \# Z \cap Z^c$, so that $\delta_Z = \deg_Z Z \cdot Z^c$.

1.1.2. A graph Γ is a finite one-dimensional simplex whose set of vertices (0-dimensional simplexes) and edges (1-dimensional simplexes) are denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively. Every edge e joins two verteces v, w, called the endpoints of e; if $v = w$ we say that e is a *loop*.

An orientation on Γ is the choice, for every edge $e \in E(\Gamma)$, of a starting vertex and an ending vertex between its endpoints.

The first Betti number of Γ is the integer $b_1(\Gamma) = rank_{\mathbb{Z}} H_1(\Gamma, \mathbb{Z})$. A connected graph is called a *tree* if its first Betti number is equal to 0

1.2. **Representable functors.** Let B and B' be two schemes over a base scheme S; we denote by $\mathrm{Hom}_S(B, F)$ the set of S-morphisms, i.e. S-regular maps, from B to B'. If $S = \operatorname{Spec} k$ for some field k we write $\mathrm{Hom}_{\operatorname{Spec} k}(B, F) = \mathrm{Hom}_k(B, F)$. If $S = \operatorname{Spec} \mathbb{Z}$ we simply omit the subscript: $\mathrm{Hom}_{\operatorname{Spec} \mathbb{Z}}(B, F) = \mathrm{Hom}(B, F)$.

We denote by SCH_S the category of schemes over a base scheme S and simply by SCH the category of all schemes (over $\operatorname{Spec} \mathbb{Z}$).

Definition 1.2.1. Let $\mathcal{F}$ be a contravariant functor from the category of schemes over S to that of sets

$$\mathcal{F} : \mathrm{SCH}_S \longrightarrow \mathrm{SET}.$$

A pair $(F, U_{\mathcal{F}})$, with $F \in \mathrm{SCH}_S$ and $U_{\mathcal{F}} \in \mathcal{F}(F)$, is said to *represent* $\mathcal{F}$ if for every scheme $B \in \mathrm{SCH}_S$ the natural map of sets

$$\mathrm{Hom}_S(B, F) \longrightarrow \mathcal{F}(B); \qquad \psi \mapsto \mathcal{F}(\psi)(U_{\mathcal{F}})$$

sending an S-morphism $\psi : B \to F$ to the image of $U_{\mathcal{F}}$ under the map

$$\mathcal{F}(\psi) : \mathcal{F}(F) \longrightarrow \mathcal{F}(B),$$

is a bijection.

Example 1.2.2. The prototypical example of a representable functor is the following. Let X be a scheme; define the functor $\mathcal{H}om(..., X) : \mathrm{SCH} \longrightarrow \mathrm{SET}$ which assigns to a scheme B the set $\mathrm{Hom}(B, X)$. It is easy to check that the pair (X, id_X) represents this functor.

Now, as we shall see, many interesting moduli functors, e.g. the moduli functor for smooth curves, are not representable. This motivates the following weakening of Definition 1.2.1.

Definition 1.2.3. Let $\mathcal{F} : \mathrm{SCH_S} \longrightarrow \mathrm{SET}$ be a functor as above. An S-scheme F is said to *coarsely represent* $\mathcal{F}$ if there exists a functor transformation

$$\Phi : \mathcal{F}(...) \longrightarrow \mathcal{H}om_S(..., F)$$

such that

(A) For every algebraically closed field k over which S is defined, the map

$$\Phi(\operatorname{Spec} k) : \mathcal{F}(\operatorname{Spec} k) \longrightarrow \operatorname{Hom}_S(\operatorname{Spec} k, F)$$

is a bijection.

(B) (Uniqueness) If N is an S-scheme and $\Psi : \mathcal{F}(...) \longrightarrow \mathcal{H}om_S(..., N)$ a functor transformation, then there exists a unique morphism $\pi : F \longrightarrow N$ such that the corresponding functor transformation $\Pi : \mathcal{H}om_S(..., F) \longrightarrow \mathcal{H}om_S(..., N)$ satisfies $\Psi = \Pi \circ \Phi$.

We say that a functor $\mathcal{F}$ is a moduli functor if for any scheme B the set $\mathcal{F}(B)$ parametrizes equivalence classes of geometrically defined objects. The main examples that we will consider are moduli functors for curves, and the Picard functors.

Example 1.2.4. The moduli functor for smooth curves is the functor $\mathcal{M}_g$ defined from the category of all schemes (i.e. $S = \operatorname{Spec} \mathbb{Z}$) which associates to a scheme B the set

$$\mathcal{M}_g(B) = \{\mathcal{C} \to B \text{ family of smooth curves of genus } g\}/_{\cong_B}$$

where $[\mathcal{C} \to B] \cong_B [\mathcal{C}' \to B]$ if there is a B-isomorphism between $\mathcal{C}$ and $\mathcal{C}'$, i.e. a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \cong\ } & \mathcal{C}' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

One of the first and most important results in classical moduli theory is the fact that there exists a coarse moduli scheme M_g for the functor $\mathcal{M}_g$.

Definition 1.2.5. If $\mathcal{F}$ is a representable moduli functor and F the scheme that represents it, then F is a called a *fine moduli scheme* and $U_{\mathcal{F}}$ the *universal family*.

If a moduli functor is coarsely represented by a scheme F, we say that F is a *coarse moduli scheme*.

1.3. **Rational points of moduli spaces.** Consider Definition 1.2.3 for some moduli functor; condition (A) means that the closed points of the moduli scheme parametrize the objects for which the moduli functor is defined. For example, if the functor is the moduli functor for smooth curves, $\mathcal{M}_g$, and M_g is its moduli scheme, the closed points of M_g over any algebraically closed field k are in bijection with the isomorphism classes of smooth curves of genus g over k. Let us explain this more precisely.

Let F be a scheme over a field k and let p be a point in F. We denote by $R(p)$ the residue field of p, that is, $R(p) = \mathcal{O}_{F,p}/m_p$. As F is defined over k there is always a natural injection

$$k \hookrightarrow R(p).$$

If the above injection is an isomorphism, the point p is said to be *rational over* k or a k*-rational point* of F. Denote

$$F(k) := \{p \in X : \ p \text{ is rational over } k\}$$

The notation $F(k)$ comes from a functorial perspective; in fact:

Lemma 1.3.1. *For every scheme F defined over a field k there is a natural identification:*

$$F(k) \cong \mathrm{Hom}_k(\mathrm{Spec}\, k, F).$$

Proof. Given a $\psi : \mathrm{Spec}\, k \longrightarrow F$, denote by p the image point of ψ. Then there is an induced natural homomorphism $\psi^* : \mathcal{O}_{F,p} \longrightarrow k$ which (after moding out by the maximal ideal of the left hand side) descends to a natural homomorphism of fields $R(p) \longrightarrow k$. As we observed above, $R(p)$ always contains k, hence the above morphism in an isomorphism.

Conversely, for every $p \in F(p)$ there is a natural isomorphism $R(p) \cong k$ (by definition). Then the quotient map $\mathcal{O}_{F,p} \longrightarrow R(p) = k$ determines a morphism $\mathrm{Spec}\, k \longrightarrow F$. ■

So, the elements of $\mathrm{Hom}_k(\mathrm{Spec}\, k, F)$ can be identified with the k-rational points of F. Thus, to say that a moduli problem admits a coarse moduli space F, is to say that for any algebraically cosed field k over which F is defined, the k-rational points of F are in bijection with the geometric objects parametrized by the functor $\mathcal{F}$, and defined over k.

Example 1.2.4 continued. Consider the moduli scheme M_g; it is is defined over $\mathrm{Spec}\, \mathbb{Z}$. For any algebraically closed field k the set of k-rational points of M_g is in bijection with the set of isomorphism classes of genus g smooth curves defined over k.

2. Lecture 1

2.1. **Moduli of stable curves.**

For further details about this part we refer to [DM69], [Gie82] and [HM98]

2.1.1. *Nodal curves.* Let X be a connected curve having only nodes (ordinary double points) as singularities. We will denote by γ the number of irreducible components of X, and $X = \cup_i^\gamma C_i$ the irreducible components decomposition of X; moreover we set $\delta = \# X_{\mathrm{sing}}$ and denote by

$$\nu : X^\nu = \sqcup_{i=1}^\gamma C_i^\nu \longrightarrow X$$

the normalization. We have the associated map of the structure sheaves

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_{X^\nu}$$

which yields the following exact sequence in cohomology:

(1)

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X^\nu, \mathcal{O}_{X^\nu}) \to k^\delta \to H^1(X, \mathcal{O}_X) \to H^1(X^\nu, \mathcal{O}_{X^\nu}) \to 0.$$

From this sequence we obtain a formula for the (arithmetic) genus of X, $g = h^1(X, \mathcal{O}_X)$

$$g = h^1(X^\nu, \mathcal{O}_{X^\nu}) + \delta - \gamma + 1 = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1 \quad (2)$$

where $g_i = h^1(C_i^\nu, \mathcal{O}_{C_i^\nu})$ is the geometric genus of C_i.

Let us now consider the sheaf of regular, never vanishing functions, i.e. the subsheaf of units in the structure sheaf, denoted $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$ for X

(similarly for other schemes); we have a cohomology sequence looking exactly as (1), by replacing $\mathcal{O}_X$ and $\mathcal{O}_{X^\nu}$ by $\mathcal{O}_X^*$ and $\mathcal{O}_{X^\nu}^*$ respectively, and k by k^*. Identifying $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$ and $\operatorname{Pic} X^\nu = H^1(X^\nu, \mathcal{O}_{X^\nu}^*)$ we get the following important short exact sequence

$$0 \longrightarrow (k^*)^{\delta-\gamma+1} \longrightarrow \operatorname{Pic} X \xrightarrow{\nu^*} \operatorname{Pic} X^\nu \longrightarrow 0 \tag{3}$$

where the map ν^* above denotes the pull-back of line bundles.

2.1.2. *Dual graph.* Let Y be a nodal curve, having δ_Y nodes, γ_Y irreducible components and c_Y connected components. The dual graph of Y is the graph Γ_Y whose vertices are identified with the irreducible components of Y and whose edges are identified with the nodes of Y; an edge joins two (possibly equal) vertices if the corresponding node is in the intersection of the corresponding irreducible components. So, Γ_Y has δ_Y edges, γ_Y vertices and c_Y connected components. Its first Betti number is

$$b_1(\Gamma_X) = \delta_Y - \gamma_Y + c_Y.$$

Lemma - Definition 2.1.1. *Let X be a connected nodal curve. The following conditions are equivalent.*

(1) Every node of X is separating ($X \smallsetminus n$ is disconnected $\forall n \in X_{sing}$).
(2) The dual graph of X is a tree.
(3) $\delta - \gamma + 1 = 0$.
(4) The pull back map induces an isomorphism $\operatorname{Pic} X \cong \operatorname{Pic} X^\nu$.

If the above conditions hold X is called of compact type.

Remark 2.1.2. For a nodal curve X the dualizing sheaf ω_X is invertible. Let $n_1, \ldots, n_\delta$ be the nodes of X and $p_i, q_i \in X^\nu$ be the branches of the node n_i for every $i = 1, \ldots, \delta$. Then we have

$$\nu^*\omega_X = \omega_{X^\nu}\big((\sum_1^\delta (p_i + q_i)\big).$$

2.1.3. *Stable curves and stabilization.* The original definition of stable curves, originating from ([MM64] and [DM69]), is usually given only for curves of genus $g \geq 2$, but it can be convenient to generalize it to all $g \geq 0$, so as to be able to define the "stabilization" of a curve ant genus (see Remark 2.1.8 for a different convention about stable curves of genus at most 1)

A connected nodal curve X of arithmetic genus $g \geq 0$ is called *stable* if each smooth rational component $E \subsetneq X$ meets $E^c = \overline{X \setminus E}$ in at least three points.

One easily checks that if $g = 0$ the only stable curve is $\mathbb{P}^1$, and if $g = 1$ a stable curve is either smooth, or irreducible with one node.

A nodal curve X of genus $g \geq 0$ is called *semistable* if each smooth rational component $E \subsetneq X$ meets E^c in at least two points.

Let $E \subsetneq X$ with $E \cong \mathbb{P}^1$. E will be called a *rational tail* if $\#E \cap E^c = 1$. E will be called an *destabilizing component* if $\#E \cap E^c \leq 2$.

Given any nodal connected curve X, the *stabilization* of X is defined as the curve $\overline{X}$ obtained as follows. If X is stable then $X = \overline{X}$, oherwise let $E \subsetneq X$ be a destabilizing component, we contract E to a point thereby obtaining a new curve X_1. If X_1 is stable we set $X_1 = \overline{X}$, otherwise we choose a destabilizing component of X_1 and contract it to a point. By iterating this process we certainly arrive at a stable curve $\overline{X}$. It is easy to check that $\overline{X}$ is unique up to isomorphism.

The stabilization of a non connected curve will be defined as the union of the stabilizations of its connected components.

Given a family of nodal curves, $f : \mathcal{X} \to B$, its stabilization is a family of stable curves $\overline{f} : \overline{\mathcal{X}} \to B$ such that for every $b \in B$ the fiber $\overline{X}_b$ of $\overline{f}$ is the stabilization of X_b.

Remark 2.1.3. Let X be a connected nodal curve of genus $g \geq 2$. X is *semistable* if and only if the dualizing sheaf ω_X has non-negative multidegree (i.e. nonnegative degree on every irreducible component of X.)

Proposition 2.1.4. *Let X be a connected nodal curve of genus $g \geq 2$. The following are equivalent.*

(1) X is stable.
(2) The dualizing sheaf ω_X has positive multidegree.
(3) ω_X is ample.
(4) X has a finite number of automorphisms.

Why are stable curves so important among all singular curves? Since every nodal curve has a uniquely determined stabilization, the above question is essentially equivalent to ask why nodal curves are so important. A simple natural answer is: Because nodes are the simplest type of singularities a curve can have. This is indeed true, but there is another, much less trivial fact which provides a stronger motivation. This is the so-called stable reduction theorem, which states the following.

Theorem 2.1.5 (Stable Reduction Theorem)**.** *Let B be smooth connected with $\dim B = 1$, and $b_0 \in B$ a fixed point. Let $f : \mathcal{X} \to B$ family of curves such that for every $b \in B \smallsetminus b_0$ the fiber of f over b is smooth. Then there exists a finite covering $\phi : B' \to B$ and a new family $h : \mathcal{Y} \to B'$ all of whose fibers are stable and such that on $\phi^{-1}(B \smallsetminus b_0) \subset B'$ the restriction of h is the base change of f.*

Moreover the family $h : \mathcal{Y} \to B'$ is uniquely determined by $\phi : B' \to B$.

We will not discuss this theorem in details, as we will not need it. But we do observe that the existence part holds also if we replace the word stable by the word semistable, or nodal. On the other hand the uniqueness trivially fails. To have uniqueness for families of semistable curves we need to require that the total space $\mathcal{Y}$ be a nonsingular minimal surface. If we do that, then the singular fibers will necessarily be semistable (a rational tail in the fibers would be contractible). Then uniqueness follows from the uniqueness of minimal models for surfaces.

The stable reduction theorem implies that the moduli space of stable curves (which we have not yet introduced, and whose construction was completed after the stable reduction theorem was proved) satisfies the valuative criterion for separation (uniqueness of stable limit) and properness (existence of the stable limit up to base change).

Indeed, let M be the moduli space of stable curves; assume only that M is a scheme whose closed points are in bijective correspondence with isomorphism classes of stable curves, and containing the moduli space of smooth curves as dense open subset. Let R be a DVR with function field K; let us prove that any rational map from $\operatorname{Spec} K$ to M extends to a regular map from the whole $\operatorname{Spec} R$ to M. Let $\phi : \operatorname{Spec} K \to M$ be a morphism. Then there exists an extension of DVR, $R \subset R'$, where R' is a DVR with quotient field K', such that the associated map $\phi' : \operatorname{Spec} K' \to M$ is the moduli map of a family of smooth curves over $\operatorname{Spec} K'$ which extends to a family of stable curves over $\operatorname{Spec} R'$ (this follows from the stable reduction theorem). But then ϕ' extends to a regular map $\phi' : \operatorname{Spec} R' \to M$. This of course implies that ϕ also extends to a regular map $\operatorname{Spec} R \to M$ (if M is not projective, let $\overline{M}$ be any completion; then ϕ extends to a map from $\operatorname{Spec} R$ to $\overline{M}$, and such an extension must be compatible with $\phi' : \operatorname{Spec} R' \to M \subset \overline{M}$).

2.1.4. *GIT-construction and properties of the scheme $\overline{M}_g$ and the stack $\overline{\mathcal{M}}_g$.* The moduli functor of stable curve is the contravariant functor defined analogously to $\mathcal{M}_g$, in example 1.2.4. $\overline{\mathcal{M}}_g$ associates to any scheme B the set

$$\overline{\mathcal{M}}_g(B) = \{f : \mathcal{X} \to B,\ f \text{ family of stable curves of genus } g\}/_{\cong}$$

where two families $f : \mathcal{X} \to B$ and $f' : \mathcal{X}' \to B$ are isomorphic if there exists a B-isomorphism between $\mathcal{X}$ and $\mathcal{X}'$.

If $\phi : B \to B'$ is a morphism, then the map $\overline{\mathcal{M}}_g(\phi) : \overline{\mathcal{M}}_g(B') \to \overline{\mathcal{M}}_g(B)$ is given by base change.

The fact that $\overline{\mathcal{M}}_g$ is coarsely representable was proved for the first time using Geometric Invariant Theory ([Gie82]), completing a program initiated by Deligne and Mumford ([DM69]) and [GIT]). The GIT construction has several advantages. As $\overline{M}_g$ is the geometric quotient of a smooth scheme, one obtains that $\overline{M}_g$ is a projective, reduced and normal variety having only finite quotient singularities. Moreover, it yields that the stack $\overline{\mathcal{M}}_g$ is a projective Deligne-Mumford stack.

Fact 2.1.6. *The moduli functor of genus g stable curves is coarsely represented by a projective integral, normal scheme $\overline{M}_g$, containing the moduli scheme of smooth curves M_g as a dense open subset. Moreover*

(1) If $g = 0$ then $\overline{M_0} = M_0$ is a point.
(2) If $g = 1$ then $\overline{M_1} \cong \mathbb{P}^1$ and $M_1 \cong \mathbb{A}^1$.
(3) If $g \geq 2$ then $\overline{M}_g$ has dimension $3g - 3$.
(4) The singularities of $\overline{M}_g$ are all of finite quotient type (more precisely, $\overline{M}_g$ is the quotient of a smooth scheme by a group acting with finite stabilizers).
(5) Denote by $\overline{M}_g^0 := \{X \in \overline{M}_g : \operatorname{Aut}(X) = \{id_X\}\}$. Then the singular locus of $\overline{M}_g$ is properly contained in $\overline{M}_g \smallsetminus \overline{M}_g^0$.

2.1.5. *Pointed stable curves.* Together with stable curves, it is quite convenient to consider stable curves with marked points.

Lemma - Definition 2.1.7. *Let X be a nodal connected curve of genus $g \geq 0$ and let $p_1, \ldots, p_n$ be n smooth points of X. We say that $(X; p_1, \ldots, p_n)$ is an* n-pointed stable curve *if the following equivalent conditions hold.*

(1) The line bundle $\omega_X(\sum_1^n p_i)$ has positive multidegree.

(2) *The group of authomorphisms of* X *which permute the points* $p_1, \dots, p_n$ *is finite.*
(3) *Let* $C \subset X$ *be an irreducible component.*
If $C \cong \mathbb{P}^1$ *then* $|C \cap C^c \cap \{p_1, \dots, p_n\}| \geq 3$.
If $p_a(C) = 1$, *then* $|C \cap C^c \cap \{p_1, \dots, p_n\}| \geq 1$.

Remark 2.1.8. We earlier made the convention that a smooth curve of genus 0 or 1 be stable. This convention was used to introduce the stablilization of a curve of any genus. On the other hand, a smooth curve of genus 0 or 1 is not stable as a pointed curve, according to the above definition. More generally, there exist no stable curves if $g = 0$ and $n \leq 2$, and if $g = 1$ and $n = 0$. !!

Fact 2.1.9. *If* $2g - 2 + n > 0$ *the moduli space of stable* n*-pointed curves of genus* g *is a projective integral variety, denoted* $\overline{M}_{g,n}$ *of dimension* $3g-3+n$.

2.2. **The boundary of $\overline{M}_g$.** This section concentrates on an explicit description of singular stable curves, in case $g \geq 2$.

Example 2.2.1. *Curves with one node.* We will now prove that *the closure in* $\overline{M}_g$ *of the locus of curves having one node has codimension one.* Since every singular stable curve can be obtained as the specialization of a stable curve with one node, this fact implies that the boundary, $\overline{M}_g \smallsetminus M_g$, of $\overline{M}_g$ is of pure dimension and

$$\operatorname{codim} \overline{M}_g \smallsetminus M_g = 1. \tag{4}$$

Suppose X is a stable curve of genus g with only node, n. Let $X^\nu \to X$ be the normalization. Then X is obtained from X^ν by identifying two points $p, q \in X^\nu$; the points p, q are called the *branches*, or the *gluing points*, of n.

Now there are two possibilities, either X is irreducible or it is reducible.

Suppose X is irreducible. Then X^ν is a smooth irreducible curve of genus $g-1$. For every smooth curve of genus $g-1$ and every pair of points p, q in it we obtain a stable curve of genus g by gluing together the two points. If $g - 1 \geq 2$ the curve X^ν has only finitely many automorphisms; therefore the family of uni-nodal curves of genus g having X^ν as nomalization has dimension 2. This yields that, if $g \geq 3$ the closure, Δ_0, of the locus of irreducible curves in $\overline{M}_g$ with one node has dimension

$$\dim \Delta_0 = \dim M_{g-1} + 2 = 3(g-1) - 3 + 2 = 3g - 4.$$

Moreover Δ_0 is irreducible, because so is M_{g-1}. If $g = 2$ a curve in Δ_0 is determined by a curve of genus 1, and a pair of points on it modulo the automorphism group of the curve, which has dimension 1. Therefore $\dim \Delta_0 = 1 + 2 - 1 = 2 = 3 \cdot 2 - 4$. so the above formula applies also if $g = 2$.

Let now $X^\nu = C_1 \cup C_2$ be reducible; let g_i be the genus of C_i; as X is stable $g_i \geq 1$ and we have $g_1 + g_2 = g$. Suppose that $g_i \geq 2$ for both i. Then for any pair C_1 and C_2 and any point $p_1 \in C_1$ and $p_2 \in C_2$ we have a stable curve of genus g

$$X = C_1 \sqcup C_2/_{p_1 = p_2}$$

Therefore the closure, Δ_{g_1}, of the locus of uni-nodal reducible curves in $\overline{M}_g$ having one component of genus g_1 (and the other of genus $g - g_1$ of course) has dimension

$$\dim \Delta_{g_1} = 3g_1 - 3 + 3g_2 - 3 + 2 = 3g - 4.$$

We leave it as an exercise to show that the same formula holds if $g_i = 1$ for one or both $i = 1, 2$. Since M_{g_i} is irreducible, we have that Δ_{g_1} is a prime divisor in $\overline{M}_g$.

In conclusion, we have that the closure in $\overline{M}_g$ of the locus of curves with only one node is a union of irreducible divisors:

$$\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{\lfloor \frac{g}{2} \rfloor}.$$

2.2.1. *Combinatorial decomposition of* $\overline{M}_g$. To study curves with more nodes, we begin by endowing $\overline{M}_g$ with a combinatorial decomposition by means of the *weighted dual graph* of a stable curve. Let X be a stable curve of genus g, the weighted dual graph of X is the weighted graph (Γ_X, w_X) such that Γ_X is the dual graph of X, and w_X is the *weight function* on the set of irreducible components of X, $V(\Gamma_X)$, assigning to a vertex the geometric genus of the corresponding component. Hence

$$g = \sum_{v \in V(\Gamma_X)} w_X(v) + b_1(\Gamma_X);$$

we call the above number the genus of (Γ_X, w_X). To say that X is a stable curve is to say that every $v \in V(\Gamma_X)$ such that $w_X(v) = 0$ has valency at least 3. A weigthed graph with this property will be called a *stable weighted graph.*

Now, for any stable weighted graph (Γ, w) of genus g, we denote by $C_g(\Gamma, w)$ the locus of curves in $\overline{M}_g$ having (Γ, w) as dual weighted graph:

$$C_g(\Gamma, w) := \{X \in \overline{M_g} : (\Gamma_X, w_X) = (\Gamma, w)\}.$$

Then we have

$$\overline{M_g} = \coprod_{(\Gamma, w) \text{ stable of genus } g} C_g(\Gamma, w). \tag{5}$$

To better analyze the above decomposition, let us introduce a partial ordering on the set of weighted graphs.

Definition 2.2.2. Let (Γ, w) and (Γ', w') be two weighted graphs, we say that (Γ', w') is a *(weighted) contraction* of (Γ, w), and write $(\Gamma, w) \geq (\Gamma', w')$, if (Γ', w') is obtained as follows. There exists $S \subset E(\Gamma)$ such that Γ' is obtained from Γ by contracting every edge in S. Let

$$\sigma : \Gamma \to \Gamma'$$

be the (surjective) contraction map and let

$$\nu : V(\Gamma) \to V(\Gamma')$$

be the associated surjection, mapping $v \in V(\Gamma)$ to $\sigma(v) \in V(\Gamma')$. Then w' satisfies:

$$w'(v') = \sum_{v \in \nu^{-1}(v')} w(v) + \#\{\text{loops contained in } \sigma^{-1}(v')\} =$$

$$= \sum_{v \in \nu^{-1}(v')} \Big(w(v) + \#\{\text{loops in } S \text{ based at } v\} \Big). \tag{6}$$

Remark 2.2.3. Let $(\Gamma, w) \geq (\Gamma', w')$; then $g(\Gamma, w) = g(\Gamma', w')$.
Moreover if (Γ, w) is stable so is (Γ', w').

Proposition 2.2.4. *Consider the combinatorial decomposition of $\overline{M}_g$ given in (5). Then*

(1) $C_g(\Gamma', w') \subset \overline{C_g(\Gamma, w)}$ *if and only if* $(\Gamma', w') \geq (\Gamma, w)$.
(2) $C(\Gamma, w)$ *is irreducible of codimension equal to* $\#E(\Gamma)$.

Proof. Part (2) is proved in 2.2.8 ■

Here is a basic lemma, with a simple instructive proof.

Lemma 2.2.5. *A stable curve X has at most $3g-3$ nodes. Moreover if $\#X_{sing} = 3g-3$ then X has $2g-2$ irreducible components all of geometric genus* 0*; such curves are called* graph curves.

Proof. Since every stable curve can be specialized to a curve having all components of geometric genus 0, and the number of nodes grows under a nontrivial specialization, a curve having the maximal number of nodes must have all components of geometric genus 0. Let X be such a curve. Then every vertex of its dual graph must have valency at least 3 (by stability), therefore $\gamma \leq 2\delta/3$. We obtain

$$g = \delta - \gamma + 1 \geq \delta - 2\delta/3 + 1 = \delta/3 + 1.$$

Hence $\delta \leq 3g-3$; moreover if equality holds every vertex of Γ_X is 3-valent and hence and $\gamma = 2g-2$. ■

Example 2.2.6. *Graph curves in $\overline{M}_g$.* Let X be a graph curve; hence it has $3g-3$ nodes and $2g-2$ irreducible components, all of geometric genus 0, and every component C_i of X satisfies $\deg_{C_i} \omega_X = 1$.

As we have seen, the dual graph of a graph curve is 3-regular (every vertex has valency equal to 3), and its first Betti number is equal to g.

Let us show that *for every g there exist only finitely many graph curves.* Let X be a stable curve and $\nu : X^\nu \to X$ its normalization. Then X^ν is the disjoint union of $2g-2$ copies of $\mathbb{P}^1$, moreover, for every $C_i^\nu \subset X^\nu$ we have

$$\#C_i^\nu \cap \nu^{-1}(X_{\text{sing}}) = 3.$$

Now, on $\mathbb{P}^1$ a triple of points has no moduli, therefore the pointed curve $(X^\nu, \nu^{-1}(X_{\text{sing}}))$ is the same for every graph curve X. Therefore to determine X we just need to know how the points $\nu^{-1}(X_{\text{sing}})$ are glued. There are obviously finitely many ways of gluing these points, so we are done.

Let $\gamma(g)$ be the number of points of $\overline{M}_g$ parametrizing graph curves.

Problem 2.2.7. *Find a formula for $\gamma(g)$.*

It is very easy to compute $\gamma(g)$ for $g \leq 3$, indeed $\gamma(0) = \gamma(1) = 0$, $\gamma(2) = 2$ and $\gamma(3) = 5$. However, an answer to the above problem, i.e. an explicit formula for $\gamma(g)$, is not known.

Generalizing what we proved in the previous examples

Lemma 2.2.8. *Let $\delta \geq 1$ be an integer. The closure in $\overline{M}_g$ of the locus of curves with exactly δ nodes has pure codimension equal to δ.*

Proof. If $g \leq 1$ the statement follows from the explicit description of $\overline{M}_g$ given in 2.1.6. Assume $g \geq 2$. We will prove the lemma by showing that for every weighted graph (Γ, w) of genus g the set $C(\Gamma, w) \subset \overline{M}_g$ is irreducible of codimension $\delta = \#E(\Gamma)$. A curve $X \in C(\Gamma, w)$ is determined by, for every $i = 1, \ldots, \gamma$, a curve $C_i^\nu \in M_{g_i}$ for $i = 1, \ldots, \gamma$ with δ_i marked points in it, where

$$\delta_i := \#\nu^{-1}(X_{\text{sing}}) \cap C_i^\nu.$$

Indeed, once we have such γ pointed curves, the gluing data of $\nu^{-1}(X_{\text{sing}})$ are uniquely determined by the graph Γ.

In other words, as the the δ_i-pointed curve C_i^ν varies in the moduli space M_{g_i,δ_i}, we have a surjective morphism

$$M_{g_1,\delta_1} \times \ldots \times M_{g_\gamma,\delta_\gamma} \longrightarrow C(\Gamma, w).$$

This shows that $C(\Gamma, w)$ is irreducible. Notice that for every i such that $g_i = 0$ we have $\delta_i \geq 3$ (by the stability); for every i such that $g_i = 1$ we have $\delta_i \geq 1$ (because $g \geq 2$). Therefore for every i we have $2g_i - 2 + \delta_i > 0$, hence fact 2.1.9 applies. Since the above surjection has finite fibers, we get $\dim M_{g_i,\delta_i} = 3g_i - 3 + \delta_i$ for all $i = 1, \ldots, \gamma$. We conclude

$$\dim C(\Gamma, w) = \sum_{i=1}^{\gamma} (3g_i - 3 + \delta_i) = 3 \sum_{i=1}^{\gamma} g_i - 3\gamma + 2\delta$$

(since $\sum_{i=1}^{\gamma} \delta_i = 2\delta$). Now $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$ hence

$$\dim C(\Gamma, w) = 3g - 3\delta + 3\gamma - 3 - 3\gamma + 2\delta = 3g - 3 - \delta.$$

■

3. Lecture 2

3.1. **Picard functor and Picard scheme.**

3.1.1. *Generalized Jacobians of nodal curves.* Let X be a nodal curve; we denote by $J(X)$ the generalized jacobian of X. We shall always identify it with the moduli space, $\operatorname{Pic}^{\underline{0}} X$, of isomorphism classes of line bundles having multidegree $\underline{0}$ (degree 0 on every irreducible component):

$$J(X) = \operatorname{Pic}^{\underline{0}} X.$$

$J(X)$ is a commutative algebraic group with respect to tensor product, and a semiabelian variety, i.e. there is a canonical exact sequence

$$0 \longrightarrow (k^*)^b \longrightarrow J(X) \xrightarrow{\nu^*} J(X^\nu) = \prod_{i=1}^{\gamma} J(C_i^\nu) \longrightarrow 0 \tag{7}$$

where $b = b_1(\Gamma_X)$ (compare with (3)).

Remark 3.1.1. By Lemma 2.1.1 $J(X)$ is projective if and only if X is a curve of compact type. In such a case there is a natural isomorphism

$$J(X) \cong \prod_{i=1}^{\gamma} J(C_i)$$

saying that a line bundle on X is uniquely determined by its restrictions to the irreducible components of X.

The definition of generalized jacobian and Picard scheme generalizes to families of curves over any base scheme B. To every family of curves $f : \mathcal{X} \to B$ one associates the relative jacobian, $J_f \to B$. It is a group scheme over B whose fibers are the jacobians of the fibers of f.

Similarly, there is a relative Picard scheme, $\operatorname{Pic}_f \to B$, again a group scheme, whose fibers are the Picard schemes of the fibers of f. We have an injective morphism of B-schems $J_f \hookrightarrow \operatorname{Pic}_f$.

The Picard scheme is a moduli space, in the sense that it coarsely represents a certain functor.

3.1.2. *The Picard functor and the Picard scheme.* Fix a flat projective morphism

$$f : X \longrightarrow B$$

and consider the category SCH_B of schemes over B. We shall use the following notation: if T is an object in SCH_B, we set $X_T := X \times_B T$ and

$$f_T : X_T \longrightarrow T$$

the base change of f. The Picard functor $\mathcal{P}ic_f$ associated to the above $f : X \to B$ goes from SCH_B to the category of sets, and associates to any object $T \in \mathrm{SCH}_B$ the set

$$\mathcal{P}ic_f(T) = \{\text{equivalence classes of line bundles on } X_T\},$$

where we say that two line bundles $\mathcal{L}$ and $\mathcal{L}'$ on X_T are equivalent if there exists a line bundle M on T such that

$$\mathcal{L} \cong \mathcal{L}' \otimes f_T^* M$$

Notice that $\mathcal{P}ic_f(T)$ is a group under tensor product of line bundles.

The representability properties of the Picard functor were studied by Grothendieck. To say that $\mathcal{P}ic_f$ is coarsely representable is to say that there exists a B-scheme Pic_f with the following properties.

(1) For any B-scheme T and any line bundle $\mathcal{L}$ on X_T there exists a unique morphism, $\mu_{\mathcal{L}}$, the *moduli map* of $\mathcal{L}$,
$$\mu_{\mathcal{L}} : T \to \mathrm{Pic}_f\,.$$
The map $\mu_{\mathcal{L}}$ maps a point $t \in T$ to the isomorphism class of the restriction of $\mathcal{L}$ to the fiber of f_T over t. This property can be more tersely stated by requiring that there exists a map
$$\mu^T : \mathcal{P}ic_f(T) \longrightarrow \mathrm{Hom}_B(T, \mathrm{Pic}_f); \quad \mathcal{L} \mapsto \mu_{\mathcal{L}}. \tag{8}$$
(2) For every algebraically closed field k the above map $\mu^{\mathrm{Spec}\, k}$ is a bijection. In other words, the closed points of Pic_f are in bijection with isomorphism classes of line bundles on the fibers of f over the closed points of B.

(3) Finally, the moduli scheme Pic_f is uniquely determined up to isomorphism.

The following theorem summarizes the main results see [SGAb] and [GIT].

Theorem 3.1.2 (Grothendieck). *Let $f : X \longrightarrow B$ be a flat projective morphism with integral geometric fibers.*

(1) There exists a group scheme Pic_f *over B which coarsely represents $\mathcal{P}ic_f$.*

(2) For every scheme T over B the natural map
$$\mathcal{P}ic_f(T) \longrightarrow \mathrm{Hom}_B(T, \mathrm{Pic}_f), \quad \mathcal{L} \mapsto \mu_{\mathcal{L}}$$
is injective.

(3) If f admits a section, then Pic_f *is a fine moduli scheme for $\mathcal{P}ic_f$ (i.e. the above maps $\mathcal{P}ic_f(T) \to \mathrm{Hom}_B(T, \mathrm{Pic}_f)$ are isomorphisms).*

If $B = \mathrm{Spec}\, k$ where k is an algebraically closed field, and X is an integral projective variety over k, we find the classical Picard group of X, $\mathrm{Pic}\, X = \mathrm{Pic}_f$ where $f : X \to \mathrm{Spec}\, k$ is the structure map.

Remark 3.1.3. The injectivity of the map $\mathcal{P}ic_f(T) \longrightarrow \mathrm{Hom}_B(T, \mathrm{Pic}_f)$ says that if $\mathcal{L}$ and $\mathcal{L}'$ are two line bundles on $\mathcal{X}_T$ which agree on every fiber of f_T, then $\mathcal{L}$ and $\mathcal{L}'$ are equivalent, that is, they differ by the pull-back of some line bundle of T.

3.1.3. *Representability and Poincaré line bundles.* What prevents $\mathcal{P}ic_f$ from having a fine moduli space is the existence of non-modular maps $T \to \mathrm{Pic}_f$. In other words, a continuously varying family of line bundles on the fibers of f_T does not necessarily "glue together" to a line bundle on the total space X_T. The theorem says that such a "gluing" exists if f has a section.

Suppose that $\mathcal{P}ic_f$ is represented by a scheme $\mathrm{Pic}_f \to B$. What is the universal element $\mathcal{U} = \mathcal{U}_{\mathcal{P}ic_f} \in \mathcal{P}ic_f(\mathrm{Pic}_f)$ (cf. subsection 1.2)? By the definition of the Picard functor, $\mathcal{U}$ is a line bundle on $\mathcal{X}_{\mathrm{Pic}_f} = \mathcal{X} \times_B \mathrm{Pic}_f$ with the following property. First, note that for any B-scheme T and any $\mathcal{L} \in \mathrm{Pic}\, \mathcal{X}_T$ the moduli map $\mu_L : T \to \mathrm{Pic}_f$ obviously lifts to a map
$$\widehat{\mu_L} : \mathcal{X}_T \to \mathcal{X}_{\mathrm{Pic}_f}.$$

Now, the pull-back $\widehat{\mu_L}^*\mathcal{U}$ is a line bundle on $\mathcal{X}_T$ whose moduli map must coincide with $\mu_{\mathcal{L}}$. By Theorem 3.1.2 (2), we have that $\mathcal{L}$ and $\widehat{\mu_L}^*\mathcal{U}$ are isomorphic modulo tensoring with the pull-back of a line bundle on T.

A line bundle $\mathcal{U} \in \operatorname{Pic}(\mathcal{X} \times_B \operatorname{Pic}_f)$ with this universal property is called a *Poincaré line bundle*.

By applying the universal property to $\operatorname{Spec} k$ for any field k over which B is defined we have that for any k-rational point $b \in B$ and any $L \in \operatorname{Pic} X_b$ the restriction of $\mathcal{U}$ to $X_b \times \{[L]\}$ is isomorphic to L.

3.2. **The universal Picard variety over M_g.** Let $f_g : \mathcal{C}_g \to M_g$ be the universal family of curves of genus $g \geq 3$. So, for every $[C] \in M_g^0$ (i.e. for every C such that $\operatorname{Aut}(C) \cong \{0\}$) we have a canonical isomorphism $f_g^{-1}([C]) \cong C$, se we shall usually identify $f_g^{-1}([C]) = C$. We can apply the above Theorem 3.1.2 and obtain a so-called universal Picard scheme $\operatorname{Pic}_{f_g} \to M_g$; this Picard scheme has infinitely many components, one for each $d \in \mathbb{Z}$, denoted by P_g^d, such that P_g^d parametrizes line bundles of relative degree d. So, for every d we have

$$\pi_d : P_g^d \longrightarrow M_g$$

and for every $[C] \in M_g^0$ we have $\pi_d^{-1}([C]) \cong \operatorname{Pic}^d C$ (another common notation for P_g^d is $P_{d,g}$).

The scheme P_g^d above is called the "Universal Picard variety" or the "Universal Jacobian" of degree d.

Theorem 3.2.1. *Let $g \geq 3$ and let $f_g : \mathcal{C}_g \to M_g$ be the universal family of curves. There exists a Poincaré line bundle on $\mathcal{C}_g \times_{M_g} P_g^d$ if and only if $(d - g + 1, 2g - 2) = 1$.*

The theorem was proved by Mestrano and Ramanan in [MR85], assuming characteristic 0. The extension to any characteristic and to all stable curves, with applications to the stack setting, was obtained later (see [C05, Sect. 5]).

3.2.1. *Isomorphisms between universal Picard varieties.* If C is a smooth curve then for any integers d, e the varieties $\operatorname{Pic}^d C$ and $\operatorname{Pic}^e C$ are isomorphic in a non canonical way (see below). This isomorphism is not, in general, induced by an isomorphism between P_g^d and P_g^e. The situation is summarized by the following statement.

Proposition 3.2.2. *Assume $g \geq 2$ and pick $d, e \in \mathbb{Z}$.*

P_g^d and P_g^e are birational over M_g if and only if they are isomorphic over M_g, if and only if $d = \pm e + n(2g - 2)$ for some $n \in \mathbb{Z}$.

For the proof of this Proposition we need the simple Lemma 3.2.3. As the Picard group $\operatorname{Pic} C$ is commutative we shall use the additive notation: $L + M := L \otimes M$ for all $L, M \in \operatorname{Pic} C$.

Lemma 3.2.3. *Let C be a general smooth curve of genus $g \geq 1$ and $d, e \in \mathbb{Z}$. Let $\alpha : \operatorname{Pic}^d C \to \operatorname{Pic}^e C$ be an isomorphism. Then one of the following two possibilities occurs.*

1. *There exists $T \in \operatorname{Pic}^{e-d} C$ such that $\alpha(L) = L + T$ for every $L \in \operatorname{Pic}^d C$.*

(2) There exists $T' \in \operatorname{Pic}^{e+d} C$ *such that* $\alpha(L) = -L + T'$ *for every* $L \in \operatorname{Pic}^d C$.

Proof. Fix $L_0 \in \operatorname{Pic}^d C$ and $M_0 \in \operatorname{Pic}^e C$. Then we have two isomorphisms

$$u_{L_0} : \operatorname{Pic}^d C \longrightarrow \operatorname{Pic}^0 C; \quad L \mapsto L - L_0$$

and

$$u_{M_0} : \operatorname{Pic}^e C \longrightarrow \operatorname{Pic}^0 C; \quad M \mapsto M - M_0.$$

For any isomorphism α as in the statement we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Pic}^d C & \xrightarrow{\alpha} & \operatorname{Pic}^e C \\ \downarrow & & \downarrow \\ \operatorname{Pic}^0 C & \xrightarrow{\alpha_0} & \operatorname{Pic}^0 C \end{array}$$

where the vertical arrows are u_{L_0} and u_{M_0} and $\alpha_0 := u_{M_0} \alpha u_{L_0}^{-1}$. So, every arrow of the diagram is an isomorphism.

Recall now that the automorphism group of the Jacobian of a general curve of genus $g \geq 1$ is generated by the involution and by the translations. Therefore there exists $T_0 \in \operatorname{Pic}^0 C$ such that one of the two following cases occurs:

(a) α_0 is a translation, i.e. $\alpha_0(P) = P + T_0$ for every $P \in \operatorname{Pic}^0 C$.

(b) For every $P \in \operatorname{Pic}^0 C$ we have $\alpha_0(P) = -P + T_0$.

Suppose α_0 is as in (a). The diagram is commutative, hence for any $L \in \operatorname{Pic}^d C$ we have

$$L - L_0 + T_0 = \alpha_0 u_{L_0}(L) = u_{M_0} \alpha(L) = \alpha(L) - M_0.$$

Therefore $\alpha(L) = L - L_0 + T_0 + M_0$. Setting $T := T_0 + M_0 - L_0$ we have that case (1) occurs.

Suppose now we are in case (b). As before, the commutativity of the diagram yields

$$L_0 - L + T_0 = \alpha_0 u_{L_0}(L) = u_{M_0} \alpha(L) = \alpha(L) - M_0$$

hence $\alpha(L) = -L + L_0 + M_0 + T_0$ by taking $L_0 + M_0 + T_0 = T'$ we have that case (2) occurs. ■

Remark 3.2.4. From the proof of the Lemma it is clear that the assumption that C be a general curve can be replaced by the more precise assumption that C be a curve such that the automorphism group of its Jacobian is generated by the involution and by the translations.

Proof of Proposition 3.2.2. If $d = \pm e + n(2g-2)$ then P_g^d and P_g^e are isomorphic over M_g. The proof of this fact is precisely the same as that of [C94, Lemma 8.1].

Suppose now that there is an birational M_g-map $\alpha_g : P_g^d \dashrightarrow P_g^e$. Let $V \subset P_g^d$ be an open subset over which α_g is regular. Let $C \in M_g$ be a curve such that $\operatorname{Pic}^d C \cap V \neq \emptyset$ (identifying $\operatorname{Pic}^d C$ with the fiber of $P_g^d \to M_g$ over C). Denote by $\alpha_C : \operatorname{Pic}^d C \dashrightarrow \operatorname{Pic}^e C$ the restriction of α_C. Then, as

$\operatorname{Pic}^d C$ and $\operatorname{Pic}^e C$ are isomorphic to an abelian variety (namely $\operatorname{Pic}^0 C$) we have that α_C is an isomorphism:

$$\alpha_C : \operatorname{Pic}^d C \xrightarrow{\cong} \operatorname{Pic}^e C$$

(see [Mi08, Sect I.3]). Of course, the set of $C \in M_g$ for which the above discussion holds is a non empty open subset of M_g. By Lemma 3.2.3 there exists a non empty open subset $U \subset M_g$ such that for every $C \in U$ the isomorphisms α_C are all either as in part (1), or as in part (2) of Lemma 3.2.3.

Suppose first that (1) occurs, i.e. for all $C \in U$ there exists $T_C \in \operatorname{Pic}^{e-d} C$ such that for all $L \in \operatorname{Pic}^d C$ we have $\alpha_C(L) = L + T_C$. This enables us to define a rational section of $P_g^{e-d} \to M_g$, namely

$$M_g \dashrightarrow P_g^{e-d}; \qquad C \mapsto T_C.$$

Now, it is well known that the only rational sections of $\coprod_{d \in \mathbb{Z}} P_g^d \longrightarrow M_g$ are those given by powers of the relative dualizing sheaf (this is the strong Franchetta Conjecture, proved in caracteristic 0 by N. Mestrano in [Me87], and later, in positive characteristic, by [S02]). Therefore, as $P_g^{e-d} \to M_g$ has a section, we must have $e - d = n(2g-2)$ as claimed.

Suppose now that case (2) occurs; hence for all $C \in U$ there exists $T'_C \in \operatorname{Pic}^{e+d} C$ such that $\alpha_C(L) = -L + T'_C$ for all $L \in \operatorname{Pic}^d C$. Hence we have a rational section of $P_g^{e+d} \to M_g$, i.e.

$$M_g \dashrightarrow P_g^{e+d}; \qquad C \mapsto T'_C.$$

Arguing as in the previous case we conclude that $e + d = n(2g-2)$.

We thus proved that if P_g^d and P_g^e are birational as M_g schemes we have $d = \pm e + n(2g-2)$. By the first part of the proof this yields that P_g^d and P_g^e are actually isomorphic. The proof is complete. ■

Remark 3.2.5. Proposition 3.2.2 extends to the compactifications of P_g^d and P_g^e , by [C94, Lemma 8.1], whose proof (of the necessary part) relies on our Proposition.

3.3. **Non-separatedness and the degree class group.**

3.3.1. *One parameter families and the twisting operations.* Let $f : \mathcal{X} \to B$ be a one parameter family of curves having smooth fibers over every $b \in B$ with $b \neq b_0$, and singular fiber over b_0; we call X the special fiber and refer to f as a smoothing of X. To simplify matters we assume that $\mathcal{X}$ is nonsingular.

The Picard scheme $\operatorname{Pic}_f \to B$ decomposes into its connected components $\operatorname{Pic}_f^d \to B$, parametrizing line bundles of degree d. Now $\operatorname{Pic}_f^d \to B$ is an irreducible scheme, whose general fiber is non-canonically isomorphic to an abelian variety. The special fiber will, however, have infinitely many connected components, unless X is irreducible.

Let $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$ and set

$$\operatorname{Pic}^{\underline{d}} X := \frac{\{L \in \operatorname{Pic} X : \underline{\deg} L = \underline{d}\}}{\cong} \cong J(X)$$

the moduli space of line bundles of multidegree $\underline{d}$ on X. Then the special fiber of $\operatorname{Pic}^d_f \to B$ is

$$\coprod_{|\underline{d}|=d} \operatorname{Pic}^{\underline{d}} X.$$

Let us now ask: is $\operatorname{Pic}^d_f \to B$ separated? The answer is: no, unless X is irreducible.

To explain why, let $\mathcal{L} := \mathcal{O}_{\mathcal{X}}(D)$ and suppose that D is not a multiple of X; consider the trivial bundle $\mathcal{O}_{\mathcal{X}}$. It is clear that $\mathcal{L}$ and $\mathcal{O}_{\mathcal{X}}$ restrict to isomorphic line bundles on every fiber but X. Indeed the restiction of $\mathcal{L}$ to the special fiber is nontrivial, as its multidegree is different from $\underline{0}$ (because $D \neq mX$). This is a simple instance of the "non-separatedness" of the Picard functor.

We now define a subgroup $\operatorname{Tw}_f X$ of $\operatorname{Pic}^0 X$ as follows $\operatorname{Tw}_f X$ is the set of all line bundles of the form $\mathcal{O}_{\mathcal{X}}(D)_{|X}$ where D is a divisor on $\mathcal{X}$ supported on the closed fiber X. Elements of $\operatorname{Tw}_f X$ are called *twisters* (or f-twisters). Twisters are the reason why the Picard functor is not separated.

Remark 3.3.1. (1) $\operatorname{Tw}_f X$ is a discrete subgroup of $\operatorname{Pic}^0 X$.

(2) [EM02, Cor 6.9] Let $L \in \operatorname{Pic}^0 X$ be such that there exists a subcurve $Z \subset X$ for which we have the following two identities:

$$L_Z = \mathcal{O}_Z(-Z \cdot Z^c) \quad L_{Z^c} = \mathcal{O}_{Z^c}(Z \cdot Z^c)$$

(notation in 1.1.1.) Then L is a twister, i.e. there exists a regular smoothing $\mathcal{X} \to B$ such that $L = \mathcal{O}_{\mathcal{X}}(Z)_{|X}$

(3) By the previous observation f-twisters depend on the family f, unless X is of compact type. On the other hand their multidegree does not, i.e. let $f'; \mathcal{X}' \to B'$ be another regular smoothing of X, then

$$\underline{\deg}\, \mathcal{O}_{\mathcal{X}}(D)_{|X} = \underline{\deg}\, \mathcal{O}_{\mathcal{X},}(D)_{|X}.$$

For more details we refer to [C05].

3.3.2. *Multidegree classes and Degree Class Group.* By the previous remark twisters depend on two types of data:

(1) discrete data, given by the choice of $D = \sum n_i C_i$, with $n_i \in \mathbb{Z}$,
(2) continuous data, namely the choice of the family f.

We shall now focus on the discrete data. For every component C_i of X denote, if $j \neq i$

$$k_{i,j} := \#(C_i \cap C_j)$$

and

$$k_{i,i} = -\#(C_i \cap \overline{X \setminus C_i})$$

then it is clear that for every pair i, j and for every non-singular $\mathcal{X}$

$$\deg_{C_j} \mathcal{O}_{\mathcal{X}}(C_i) = k_{i,j}$$

Obviously we have that $k_{i,j} = k_{j,i}$ and that $\sum_{j=1}^{\gamma} k_{i,j} = 0$ for every fixed i .

Now, for every $i = 1, \ldots, \gamma$ set

$$\underline{c}_i := (k_{1,i}, \ldots, k_{\gamma,i}) \in \mathbb{Z}^\gamma$$

and

$$\mathbf{Z} := \{\underline{d} \in \mathbb{Z}^\gamma : \ |\underline{d}| = 0\}$$

so that $\underline{c}_i \in \mathbf{Z}$. Consider the sublattice Λ_X of $\mathbf{Z}$ spanned by them

$$\Lambda_X := < \underline{c}_1, \ldots, \underline{c}_\gamma > .$$

Thus, Λ_X is the lattice formed by the multidegrees of all twisters, inside the abelian group $\mathbf{Z}$ of multidegrees of degree 0.

It is easy to see that Λ_X has rank $\gamma - 1$, in fact any $\gamma - 1$ among the $\underline{c}_1, \ldots \underline{c}_\gamma$ are independent over $\mathbb{Z}$, whereas the following natural relation occur

$$\sum_1^\gamma \underline{c}_i = \underline{0}$$

(since $\sum_1^\gamma \underline{c}_i = \underline{\deg}_C \mathcal{O}_X(C) = \underline{0}$).

Definition 3.3.2. The group $\Delta_X := \mathbf{Z}/\Lambda_X$ is called the *degree class group* of X.

The previous definition comes from [C94]. The degree class group has other incarnations; most remarkably, it coincides with the group of components of the Néron model of the Jacobian of a family of nodal, generically smooth curves, with regular total space (due to A.Raynaud [R70], see also [C05] for more details).

Example 3.3.3. If X is irreducible $\mathbf{Z} = \Lambda_X = 0$, hence $\Delta_X = 0$.

If X is a curve of compact type, then again $\Delta_X = 0$. More generally, we have that $\Delta_X = 0$ if and only if X is treelike, i.e. every node lying in two different components is separating. We leave the proof as exercise.

Let $\underline{d}$ and $\underline{d}'$ be two multidegrees (so that $\underline{d}, \underline{d}' \in \mathbb{Z}^\gamma$); we define them to be equivalent if their difference is the multidegree of a twister, i.e.:

$$\underline{d} \equiv \underline{d}' \;\Leftrightarrow\; \underline{d} - \underline{d}' \in \Lambda_X.$$

Now the quotient of the set of multidegrees with fixed total degree by this equivalence relation is a finite set Δ_X^d:

$$\Delta_X^d := \frac{\{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = d\}}{\equiv}.$$

It is easy to see that the cardinality of Δ_X^d is independent of d, and of course $\Delta_X^0 = \Delta_X$.

3.3.3. *Combinatorial description of the Degree Class Group.* To better appreciate the combinatorial nature of the degree class group, let us follow Oda and Seshadri [OS79][??] with a short digression. Fix an orientation on a graph Γ (cf. 1.1.2), then we have the standard homology operators ∂ and δ:

$$\partial : C_1(\Gamma, \mathbb{Z}) \longrightarrow C_0(\Gamma, \mathbb{Z}), \quad e \mapsto v - w$$

where the edge e goes from the vertex v to the vertex w. Next

$$\delta : C_0(\Gamma, \mathbb{Z}) \longrightarrow C_1(\Gamma, \mathbb{Z}); \quad v \mapsto \sum e_v^+ - \sum e_v^-$$

where the first sum is over all the edges e_v^+ starting at v, and the second sum is over all the e_v^- v. Then we introduce the group

$$\frac{\partial C_1(\Gamma, \mathbb{Z})}{\partial \delta C_0(\Gamma, \mathbb{Z})}.$$

The theorem of Kirchoff-Trent states that *such a group is finite and its cardinality is equal to the complexity of* Γ.

Proposition 3.3.4. *For a nodal connected curve X with dual graph Γ_X we have*

$$\Delta_X \cong \frac{\partial C_1(\Gamma_X, \mathbb{Z})}{\partial \delta C_0(\Gamma_X, \mathbb{Z})}.$$

In particular $\#\Delta_X$ is equal to the complexity of the dual graph of X.

Proof. The proof is straightforward using the definitions of ∂ and δ. First we identify $\mathbf{Z} \cong \partial C_1(\Gamma_X, \mathbb{Z})$ (because Γ_X is connected). Then we have $\Lambda_X \cong \partial\delta C_0(\Gamma_X, \mathbb{Z})$, in fact if v_i is the vertex of Γ_X corresponding to the component C_i of X, then, by definition, one obtains $\partial\delta(v_i) = -\underline{c}_i$, for every $i = 1, \dots, \gamma$.

The rest follows from the theorem of Kirchoff-Trent. ∎

Example 3.3.5. If X is a curve made of two smooth components meeting at δ points, or if X is a "cycle" of δ smooth components, then Δ_X is the cyclic group of order δ.

3.3.4. *Boundary points.* What type of boundary points for the compactified Picard scheme can we use?

To answer this question, let us consider a curve X with a node n, its normalization X^ν, and the two branch points $p, q \in X^\nu$ over n. Assume that X^ν is connected, and pick $M \in \operatorname{Pic} X^\nu$; then the set of line bundles on X pulling back to M is a k^*. Let us try and complete this k^*. Any line bundle $L \in \operatorname{Pic} X$ pulling back to M is obtained by gluing the fiber M_p of M at p with the fiber of M at q, M_q; after fixing local trivializations for M, any such gluing is an isomorphism obtained by mapping $1 \in k = M_p$ to some $c \in k = M_q$ with $c \neq 0$. We denote by $L = L^{(c)}$ the line bundle on X corresponding to this gluing.

What happens when c goes to 0? The isomorphism $M_p \xrightarrow{\cong} M_q$ degenerates to the zero map. So the sections of M compatible with this map are the sections vanishing at q. We can interpret this as the fact that M degenerates to $M(-q)$. Unfortunately, in doing so we have lowered the degree, which is obviously a problem. To fix this problem, we choose to replace the curve X by the "blow up of X at n", i.e. by the nodal curve Y obtained by adding to X^ν a smooth rational component E joining p with q. Now the limit of $L^{(c)}$ as c goes to 0 is a line bundle $\widehat{L} \in \operatorname{Pic} Y$ whose restriction to X^ν is $M(-q)$ and whose restriction on E is $\mathcal{O}_E(1)$ (i.e. the only degree 1 line bundle on E). Now $\deg \widehat{L} = \deg M$. Moreover the automorphisms of Y which fix X^ν and act only on E (a k^* of them) act transitively on the set (again a k^*) of all line bundles on Y whith the same restrictions on X^ν and E. Therefore our limit point $\widehat{L}$ is uniquely determined up those automorphisms of Y which fix X^ν.

With a similar reasoning, if c tends to ∞ the limit of $L^{(c)}$ is a line bundle on Y whose restriction to X^ν is $M(-p)$ and whose restriction to E is again $\mathcal{O}_E(1)$.

4. Lecture 3

4.1. **Balanced line bundles.** The discussion in 3.3.4 served as motivation for the fact that to compactify the Picard scheme we will use line bundles on a simple type of semistable curves, the so-called quasistable curves.

Before we begin, let us point out that, although we will describe the compactified Picard scheme for stable curves, there is no real loss of generality in doing so.

Remark 4.1.1. Let Y be a connected nodal curves and X its stabilization, The compactified Picard scheme of Y is naturally identified to the compactified Picard scheme of X.

Definition 4.1.2. Let Y be a connected nodal curve of genus g.

If $g \geq 2$, we say that Y is *quasistable* if is semistable (i.e. it has no rational tails) and if two exceptional components do not intersect each other.

If $g = 1$ we say that it is quasistable if it is stable, or if it is the union of two copies of $\mathbb{P}^1$.

If X is the stabilization of Y we say that Y is a *quasistable curve of* X. If $g \geq 2$ we denote by $\epsilon(Y)$ the number of exceptional components of Y, and

$$Y_{\text{exc}} := \bigcup_{E \text{ exceptional comp. of } Y} E, \qquad \widetilde{Y} := \overline{Y \smallsetminus Y_{\text{exc}}}. \tag{9}$$

so, $\widetilde{Y}$ is a partial normalization of the stabilization of Y at $\epsilon(Y)$ nodes (see 4.1.3 for $g = 1$).

Remark 4.1.3. Let $g = 1$ and $Y = C_1 \cup C_2$ with $C_i \cong \mathbb{P}^1$ and $\#C_1 \cap C_2 = 2$. We fix a stabilization map $Y \to X$, which contracts one of the two components. We then use the convention that the non contracted component (the normalization of X) is not exceptional, whereas the contracted component is exceptional. With this convention, notation (9) makes sense also when $g = 1$.

Remark 4.1.4. Let X be a stable curve with δ nodes. Then the set of all quasistable curves of X has cardinality 2^δ.

Recall that a nodal curve Y of genus g has an invertible dualizing sheaf ω_Y of degree $2g-2$, such that that for every subcurve $Z \subset Y$ we have, with the notation of 1.1.1, $\omega_Y \otimes \mathcal{O}_Z = \omega_Z(Z \cdot Z^c)$ so that $\deg_Z \omega_Y = 2g_Z - 2 + \delta_Z$, where g_Z is the arithmetic genus of Z.

Definition 4.1.5. Let Y be a quasistable curve of genus $g \geq 2$ and let $L \in \operatorname{Pic}^d Y$.

1. We say that L is *balanced* if its multidegree $\underline{\deg}\, L$ is balanced. i.e. if the following two properties hold.
   - (a) For every exceptional component $E \subset Y$ we have $\deg_E L = 1$.
   - (b) For any subcurve $Z \subset Y$ we have

$$|\deg_Z L - \deg_Z \omega_Y \frac{d}{2g-2}| \leq \frac{\delta_Z}{2} \tag{10}$$

2. We say that L, or $\underline{\deg}\, L$, is *strictly balanced* if it is balanced and if strict inequality holds in (10) for every subcurve $Z \subsetneq Y$ such that $Z \cap Z^c \not\subset Y_{\text{exc}}$.

(3) We denote

$$B_d(Y) = \{\underline{d} : |\underline{d}| = d \text{ balanced on } Y\} \supset SB_d(Y) = \{\underline{d} : \text{ strictly balanced}\}$$

(4) A stable curve X is called *d-general* if $B_d(X) = SB_d(X)$.

Remark 4.1.6. A couple of observations which simplify the computations. First of all, if $\underline{d}$ is a multidegree on Y and $Z \subset Y$ a subcurve we denote by $\underline{d}_Z \in \mathbb{Z}$ the total degree of the restriction of $\underline{d}$ to Z.

(1) It is convenient to introduce the notation

$$m_Z(d) := \deg_Z \omega_Y \frac{d}{2g-2} - \frac{\delta_Z}{2}, \qquad M_Z(d) := \deg_Z \omega_Y \frac{d}{2g-2} + \frac{\delta_Z}{2}.$$

Now (10) reads as follows:

$$m_Z(d) \leq \underline{d}_Z \leq M_Z(d).$$

(2) Condition (10) holds on Z if and only if it holds on Z^c.
(3) Condition (10) holds on every subcurve if and only if it holds on every connected subcurve.

Remark 4.1.7. Let X be a stable curve. Then every multidegree class $[\underline{d}] \in \Delta_X^d$ has a balanced representative; the balanced representative is unique if and only if it is strictly balanced ([C94]).

Example 4.1.8. $d = 0$.
The inequality (10) gives

$$-\delta_Z/2 \leq \deg_Z L \leq \delta_Z/2.$$

Let $X = C_1 \cup C_2$ with C_i nonsingular and $\delta = \delta_{C_i} = 2$ for $i = 1, 2$. Call $X_{\text{sing}} = \{n_1, n_2\}$. The set of non stable, quasistabe curves of X, is made of three curves: two curves Y_1 and Y_2 obtained by blowing up the node n_i of X; the curve $\widehat{X}$ obtained by blowing up both nodes of X.

We have,

(1) $SB_0(X) = \{(0, 0)\}$ and $B_0(X) = \{(-1, 1), (0, 0), (1, -1)\}$
(2) $SB_0(Y_i) = \emptyset$ and $B_0(Y_i) = \{(-1, 0, 1), (0, -1, 1)\}$ for $i = 1, 2$.
(3) $SB_0(\widehat{X}) = \{(-1, -1, 1, 1)\} = B_0(\widehat{X})$

where we ordered the multidegrees so that the degrees on the exceptional components are at the end.

More generally, for every $\delta \geq 1$ one easily checks that X is 0-general if and only if δ is odd.

Example 4.1.9. $d = g - 1$.
Condition (10) yields

$$g_Z - 1 \leq \deg_Z L \leq g_Z - 1 + \delta_Z.$$

Therefore a stable curve is $(g-1)$-general if and only if it is irreducible.

By imposing condition (10) to Z and to Z^c it suffices to impose only one of the two above inequalities to every subcurve of Y. In other words, a line bundle L of degree $g - 1$ is balanced if for every subcurve Z we have

$$g_Z - 1 \leq \deg_Z L$$

and if strict inequality holds for exceptional components.

Let us show that

$$X_{\text{sep}} \neq \emptyset \Rightarrow SB_{g-1}(X) = \emptyset. \tag{11}$$

Let $n \in X_{\text{sep}}$ and write $X = Z \cup Z^c$ with $Z \cap Z^c = n$. Hence $g = g_Z + g_{Z^c}$. Suppose by contradiction that there exists $\underline{d} \in SB_{g-1}(X)$. Then we have $\underline{d}_Z \geq g_Z$ and $\underline{d}_{Z^c} \geq g_{Z^c}$; hence

$$g - 1 = \underline{d}_Z + \underline{d}_{Z^c} \geq g_Z + g_{Z^c} = g$$

a contradiction. The converse of (11) also holds, as we shall explain later.

Definition 4.1.10. The *strictly balanced Picard functor* associated to a stable curve X is the contravariant functor $\overline{\mathcal{P}_X^d}$ from the category of schemes to the category of sets defined as follows. For any scheme B the set $\overline{\mathcal{P}_X^d}(B)$ is the set of equivalence classes of pairs $(\mathcal{Y} \xrightarrow{f} B, \mathcal{L})$ where $f : \mathcal{Y} \to B$ is a family of quasistable curves having X as stabilization, and $\mathcal{L} \in \operatorname{Pic} \mathcal{Y}$ is a relatively degree d, stricty balanced line bundle.

For any morphism $\phi : B' \to B$ the map $\overline{\mathcal{P}_X^d}(B) \to \overline{\mathcal{P}_X^d}(B')$ maps $(\mathcal{Y} \to B, \mathcal{L})$ to $(\mathcal{Y} \times_B B' \to B', \mathcal{L}')$ where $\mathcal{L}'$ is the pull back of $\mathcal{L}$ via $\mathcal{Y} \times_B B' \to \mathcal{Y}$.

We define $(\mathcal{Y} \xrightarrow{f} B, \mathcal{L})$ and $(\mathcal{Y}' \xrightarrow{f'} B, \mathcal{L}')$ to be *equivalent* if the following facts hold.

(1) The stabilizations of f and f' coincide; let $\mathcal{X} \to B$ be their stabilization, and let $\mathcal{Y} \xrightarrow{\sigma} \mathcal{X} \to B$ and $\mathcal{Y}' \xrightarrow{\sigma'} \mathcal{X} \to B$ be the stabilization maps.
(2) There exists an isomorphism $\alpha : \mathcal{Y} \to \mathcal{Y}'$ which commutes with σ and σ' (i.e. α is an $\mathcal{X}$-isomorphism) and a line bundle $M \in \operatorname{Pic} B$ such that $\alpha^* \mathcal{L}' \cong \mathcal{L} \otimes f^* M$.

Remark 4.1.11. Let $L, L' \in \operatorname{Pic} Y$ be two strictly balanced line bundles on the quasistable curve Y. Let $\widetilde{L}$ and $\widetilde{L}'$ be the restrictions of L and L' to the curve $\widetilde{Y} \subset Y$ (the complement of the exceptional components, cf. 4.1.2). Then L and L' are equivalent if and only if $\widetilde{L} \cong \widetilde{L}'$. Indeed, $\widetilde{L} \cong \widetilde{L}'$ if and only if L and L' differ only in the gluing data on $\widetilde{Y} \cap Y_{\text{exc}}$. On the other hand the automorphisms of Y which fix $\widetilde{Y}$ act transitively on those gluing data, and fix the stabilization of Y. Therefore $\widetilde{L} \cong \widetilde{L}'$ if and only if L and L' are equivalent.

4.1.1. *The strictly balanced class of a balanced line bundle.* To begin with, observe that the terminology "strictly balanced" is not to be confused with the terminology "stably balanced" used in [CCC04] or in [C05]. They coincide for stable curves; in general stably balanced implies strictly balanced, but the converse fails, see example 4.1.13 below.

A balanced multidegree on a quasistable curve is called stably balanced if part (2) of Definition 4.1.5 is strengthened by requiring that strict inequality holds in (10) for every subcurve $Z \subsetneq Y$ such that $Z^c \not\subset Y_{\text{exc}}$.

The degree d compactified Picard scheme of X, denoted $\overline{P^d_X}$, is constructed as the GIT-quotient of a certain scheme V_X, containing only GIT-semistable points, by a certain group G. So that there is a quotient morphism

$$V_X \longrightarrow \overline{P^d_X} = V_X/G. \tag{12}$$

Equivalence classes of strictly balanced line bundles of degree d on quasistable curves of X correspond to the closed (and GIT-semistable) orbits of G on V_X.

Equivalence classes of balanced line bundles correspond to GIT-semistable orbits, i.e. to all the G-orbits in V_X, and equivalence classes of stably balanced line bundles correspond to the GIT-stable orbits in V_X. Therefore a stably balanced line bundle is strictly balanced, but the converse fails in the presence of GIT-semistable points which are not stable.

Of course, if there are some non stable GIT-semistable points, some points of the GIT-quotient parametrize more than one G-orbit. But every such point parametrizes a unique closed orbit. Therefore to every point in the compactified Picard scheme $\overline{P^d_X}$ there corresponds a unique equivalence class of strictly balanced line bundles.

Remark 4.1.12. By what we said, to every balanced line bundle L on a quasistable curve of X there corresponds a strictly balanced line bundle, L^{SB}, uniquely determined up to equivalence. Indeed the equivalence class of L^{SB}, written $[L^{SB}]$, corresponds to the unique closed orbit contained in the closure of the orbit of L in V_X.

We call $[L^{SB}]$ the *strictly balanced class* of L. A formula for $[L^{SB}]$ will be given in Lemma 4.2.4.

Example 4.1.13. Let $X = C_1 \cup C_2$ be a stable curve made of two smooth components meeting at two points and let $Y = C_1 \cup C_2 \cup E_1 \cup E_2$ be the quasistable curve of X obtained by blowing up both nodes of X. Let $d = 0$. Consider the strictly balanced multidegree on Y

$$\underline{d} := (-1, -1, +1, +1) \in SB_0(Y)$$

(the exceptional components are listed at the end). Now, $\underline{d}$ is not stably balanced, indeed the degree on C_1 is equal to the minimum allowed by (10) but, of course, C_1^c contains C_2.

4.2. **Compactified Picard shemes of nodal curves.** In the next statement we will use the following notation: If Y is a quasistable curve of X and $L \in \operatorname{Pic}^{\underline{d}} X$ we denote by $\widetilde{L}$ the restriction of L to $\widetilde{Y}$ (see 4.1.2) and by $\widetilde{\underline{d}}$ the multidegree of $\widetilde{L}$, so that $\widetilde{L} \in \operatorname{Pic}^{\widetilde{\underline{d}}} \widetilde{Y}$.

Theorem 4.2.1. *Let X be a stable curve of genus $g \geq 2$. There exists a reduced, connected projective scheme $\overline{P^d_X}$ of pure dimension g with the following properties*

1. *$\overline{P^d_X}$ is a coarse moduli scheme for the strictly balanced Picard functor. In particular, there exists a map*

$$\overline{\mathcal{P}^d_X}(B) \to \operatorname{Hom}(B, \overline{P^d_X}); \quad \mathcal{L} \mapsto \mu_{\mathcal{L}}.$$

The above map is a bijection if $B = \operatorname{Spec} k$ with k an algebraically closed field.

(2) *For any family $f : \mathcal{Y} \to B$ of quasistable curves of X, and any relatively balanced degree d line bundle $\mathcal{L}$ on $\mathcal{Y}$, there exists a morphism $\mu_{\mathcal{L}} : B \to \overline{P^d_X}$ such that, $\forall b \in B$, the point $\mu_{\mathcal{L}}(b)$ parametrizes the strictly balanced class of the restriction of $\mathcal{L}$ to $f^{-1}(b)$ (cf 4.1.12).*

(3) *Let P^d_X be the smooth locus of $\overline{P^d_X}$. If $X_{sep} = \emptyset$ there exists a canonical isomorphism*

$$\overline{P^d_X} \supset P^d_X \cong \coprod_{\underline{d} \in SB_d(X)} \operatorname{Pic}^{\underline{d}} X.$$

If $X_{sep} \neq \emptyset$ there exists an isomorphism $P^d_X \cong \coprod_{\underline{d}} \operatorname{Pic}^{\underline{d}} X$ where $\underline{d}$ varies in a finite set.

(4) *There is a canonical isomorphism*

$$\overline{P^d_X} \cong \coprod_{\substack{Y \text{ quasistable curve of } X \\ \underline{d} \in SB_d(Y)}} \operatorname{Pic}^{\widetilde{\underline{d}}} \widetilde{Y}.$$

(5) *The generalized Jacobian $J(X)$ acts on $\overline{P^d_X}$ by tensor product. Each stratum of the decomposition (4) is left invariant under this action.*

Let us explain part (5). A point λ in $\overline{P^d_X}$ corresponds to a pair (Y, L) where Y is a quasistable curve of X and L a strictly balanced line bundle on Y. Recall from Remark 4.1.11 that λ is determined by the restriction $\widetilde{L}$ of L to $\widetilde{Y} \subset Y$; in other words we can write $\lambda = [\widetilde{Y}, \widetilde{L}]$. The curve $\widetilde{Y}$ is a partial normalization of X, let $\tilde{\nu} : \widetilde{Y} \to X$ be the normalization map. For every $M \in J(X) = \operatorname{Pic}^{\underline{0}} X$ the action of M maps $[\widetilde{Y}, \widetilde{L}]$ to $[\widetilde{Y}, \widetilde{L} \otimes \tilde{\nu}^* M]$. Since $\underline{\deg}\, \widetilde{L} = \underline{\deg}\, \tilde{\nu}^* M$ by gluing the line bundle $\tilde{\nu}^* M$ to $\mathcal{O}_E(1)$ on each exceptional component of Y we get a strictly balanced line bundle. From this we obtain that the stabilizer of λ under this action is a torus:

$$\operatorname{Stab}_{J(X)}(\lambda) = \{M \in J(X) : \tilde{\nu}^* M = \mathcal{O}_{\widetilde{Y}}\} \cong (k^*)^t \tag{13}$$

with $t = b_1(\Gamma_X) - = b_1(\Gamma_{\widetilde{Y}})$.

Remark 4.2.2. Parts (3) and (4) of the theorem imply that the singular locus of $\overline{P^d_X}$ has codimension one. In particular $\overline{P^d_X}$ is not a normal scheme, unless X is of compact type, in which case $\overline{P^d_X}$ is nonsingular (and irreducible). In fact if X is a curve of compact type, for every d there exists a unique quaistable curve Y of X such that $SB_d(Y) \neq \emptyset$. We leave this claim as an exercise.

Example 4.2.3. If X is a reducible curve not of compact type the structure of $\overline{P^d_X}$ varies with d. For example, consider a curve $X = C_1 \cup C_2$ as in example 4.1.8 (with the same notation). If $\delta \geq 3$ is odd then $\overline{P^0_X}$ has δ irreducible components, one for every multidegree class. On the other hand $\overline{P^{g-1}_X}$ has only $\delta - 1$ irreducible components, as

$$SB_{g-1}(X) = \{(g_1, g_2 + \delta - 2), (g_1 + 1, g_2 + \delta - 3), \ldots, (g_1 + \delta - 2, g_2)\}.$$

On the other hand if $\delta = 1$, so that X is of compact type, $\overline{P^0_X}$ ha one irreducible component such that

$$\overline{P^0_X} \cong \operatorname{Pic}^{(0,0)} X = \operatorname{Pic}^0 C_1 \times \operatorname{Pic}^0 C_2.$$

What if $d = g-1$? Then $SB_{g-1}(X) = \emptyset$. Let Y be the blow up of X at its node, so that $Y = C_1 \cup C_2 \cup E$; we have $SB_{g-1}(Y) = \{(g_1 - 1, g_2 - 1, 1)\}$, hence

$$\overline{P^{g-1}_X} \cong \operatorname{Pic}^{(g_1-1, g_2-1, 1)} Y = \operatorname{Pic}^{g_1-1} C_1 \times \operatorname{Pic}^{g_2-1} C_2 \times \operatorname{Pic}^1 E$$

hence

$$\overline{P^{g-1}_X} \cong \operatorname{Pic}^{g_1-1} C_1 \times \operatorname{Pic}^{g_2-1} C_2.$$

4.2.1. *Constructing the strictly balanced class.* Given a balanced line bundle $L \in \operatorname{Pic}^{\underline{d}} Y$ we want to construct $[L^{SB}]$. This will be done in Lemma 4.2.4, to state which we need some preliminaries. We say that a node n of Y is *exceptional* if it lies in an exceptional component of Y, and denote by $Y^{\text{exc}}_{\text{sing}}$ the set of exceptional nodes of Y. We shall say that a connected subcurve $Z \subset Y$ is *$\underline{d}$-minimal* if $\underline{d}_Z = m_Z$ (notation in Rk. 4.1.6). Set

$$S(\underline{d}) := \{n \in Y_{\text{sing}} \smallsetminus Y^{\text{exc}}_{\text{sing}} : \exists Z \ \ \underline{d}\text{-minimal with } \ n \in Z \cap Z^c\}. \tag{14}$$

We let $\sigma : Y(\underline{d}) \to Y$ be the blow-up of $S(\underline{d})$, so that $Y(\underline{d})$ is a quasistable curve, and σ contracts one exceptional component, $E_n \subset Y(\underline{d})$, for every $n \in S(\underline{d})$. For every $\underline{d}$-minimal $Z \subset Y$ there exists a subcurve of $Y(\underline{d})$ mapping isomorphically to Z via σ; we abuse notation and call Z this curve. For every $n \in S(\underline{d})$ lying in $Z \cap Z^c$ there exists a unique point $p_n \in Z \subset Y(\underline{d})$ mapping to n; of course p_n is a node of $Y(\underline{d})$. Let

$$\nu : Y^\nu(\underline{d}) \longrightarrow Y(\underline{d})$$

be the normalization of $Y(\underline{d})$ at all nodes of the form p_n for all $n \in S(\underline{d})$. Finally, $\forall n \in S(\underline{d})$ we pick a smooth point $r_n \in Y^\nu(\underline{d})$ such that $\nu(r_n) \in E_n$. With the above set-up, we have the following

Lemma 4.2.4. *Let Y be a quasistable curve of a stable curve X and let $L \in \operatorname{Pic}^{\underline{d}} Y$ be balanced. Then the strictly balanced class $[L^{SB}] \in \overline{P^d_X}$ of L parametrizes (strictly balanced) line bundles $L^{SB} \in \operatorname{Pic} Y(\underline{d})$ whose pull-back to $Y^\nu(\underline{d})$ satisfies*

$$\nu^* L^{SB} = \nu^* \sigma^* L \otimes O_{Y^\nu(\underline{d})}\Big(\sum_{n \in S(\underline{d})} (r_n - p_n)\Big). \tag{15}$$

Proof. Recall that we have a GIT-quotient map $V_X \longrightarrow \overline{P^d_X} = V_X/G$, and that strictly balanced classes correspond to the closed orbits of the action of G on V_X We use [C94, Lemma 6.1] characterizing the closed orbits in V_X in terms of the multidegrees of the line bundles they parametrize. That Lemma states that L is strictly balanced if and only if for every $\underline{d}$-minimal $Z \subset Y$ we have that $Z \cap Z^c$ is contained in the union of the exceptional components of Y.

The curve $Y(\underline{d})$ has been constructed by blowing up every non-exceptional node lyig in $Z \cap Z^c$ for every $\underline{d}$-minimal curve. The pull-back of L to $Y(\underline{d})$ is not balanced, as it has degree 0 to the exceptional components E_n. To adjust $\sigma^* L$ to get a balanced (in fact, a strictly balanced) line bundle equivalent to L we need to tensor by $O_{Y^\nu(\underline{d})}(\sum_{n\in S(\underline{d})}(r_n - p_n))$ its pull back to $Y^\nu(\underline{d})$. Checking the details is easy, and we leave it to the reader. ■

Theorem 4.2.5. *Fix d and $g \geq 3$.*

(1) There exists an integral, normal, projective scheme $\overline{P_g^d}$ with a projective morphism, π, onto $\overline{M}_g$, with the following property. For every family of quasistable curves $f : \mathcal{Y} \to B$ of genus g and every relatively balanced, degree d line bundle on $\mathcal{Y}$ there exists a morphism $\mu_{\mathcal{L}} : B \to \overline{P_g^d}$ such that the diagram below is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\mu_{\overline{f}}} & \overline{M}_g \\ & {\scriptstyle \mu_{\mathcal{L}}} \searrow \quad \nearrow {\scriptstyle \pi} & \\ & \overline{P_g^d} & \end{array}$$

where $\mu_{\overline{f}}$ is the moduli map of the stabilization $\overline{f}$ of f.

(2) The following conditions are equivalent

(a) $(d - g + 1, 2g - 2) = 1$.

(b) Every $X \in \overline{M}_g$ is d-general.

(c) $\overline{P_g^d}$ is the geometric quotient of a smooth scheme by a group acting with finite stabilizers.

If these conditions hold, the morphism π is flat. Moreover $\overline{P_g^d}$ is the coarse moduli scheme of a Deligne-Mumford stack $\overline{\mathcal{P}}_{d,g}$ with a strongly representable morphism onto the moduli stack of stable curves $\overline{\mathcal{M}}_g$.

Condition (2a) is the same as that of Theorem 3.2.1 ([MR85]). The constructon of the scheme $\overline{P_g^d}$ was performed in [C94], using GIT along the same lines as the construction of $\overline{M}_g$. All of the above results, with the exception of the part concerning the associated stacks, were proved in [C94].

For the stack version and the relation of the above construction with Néron models of Jacobians see [C05]. More generally the Artin stack associated to the scheme $\overline{P_g^d}$ for every d is studied in details in [M08].

5. Lecture 4

5.1. **Linear series on nodal curves.** The central objects in the classical theory of linear series on curves are the so-called Brill-Noether varieties, which we will now define for any connected nodal curve Y. Fix two integers d and $r \geq 0$.

$$W^r_d(Y) := \{L \in \operatorname{Pic}^d Y : \ h^0(Y,L) \geq r+1\}.$$

Among the fundamental facts for line bundles on smooth curves which extend directly to nodal ones we recall

Theorem 5.1.1. *Let Y be a nodal connected curve of genus g*

(1) (Riemann-Roch) *For any $L \in \operatorname{Pic}^d Y$ we have*
$$h^0(Y,L) - h^1(Y,L) = d - g + 1.$$
(2) (Serre duality) *Let ω_X be the dualizing sheaf of Y; then*
$$h^0(Y,\omega_Y) = h^1(Y,\mathcal{O}_Y) = g.$$

Let us now state some other important facts which are easily seen to fail for reducible nodal curves.

Theorem 5.1.2. *Let C be a smooth curve of genus g and let $L \in \operatorname{Pic}^d C$.*

(1) If $d < 0$ then $h^0(C,L) = 0$.
(2) (Clifford theorem) *If $0 \leq d \leq 2g-2$ then $h^0(C,L) \leq d/2+1$. If $d = 0$ or $d = 2g-2$, then equality holds if and only if $L = \mathcal{O}_C$ and $L = \omega_C$.*
(3) (Riemann theorem) *If $d \geq 2g-1$ then $h^0(C,L) = d-g+1$.*

Example 5.1.3. Let $X = C_1 \cup C_2$ with $g_1 = g_2 = 1$. Suppose $\#C_1 \cap C_2 = 1$. Let $L \in \operatorname{Pic}^{-1} X$ be any line bundle of multidegree $(-3,2)$. Then, calling $p_i \in C_i$ the branches over the node, we have

$$h^0(X,L) = h^0(C_2, L_{|C_2}(-p_2) = 1$$

because $degL_{|C_2}(-p_2) = 1$. This shows that (1) of the above theorem fails.

Let $L \in \operatorname{Pic}^0 X$ have multidegree $(-3,3)$. Then, notation as above,

$$h^0(X,L) = h^0(C_2, L_{|C_2}(-p_2) = 2$$

showing that Clifford's therem fails.

We leave it as an exercise to show that Riemann theorem, part (3) above, fails for infinitely many multidegrees (and for any reducible curve).

As we have seen in the previous lectures, if one restricts the Picard functor to balanced, or strictly balanced, line bundles, the functor has better representability properties. Now we shall see that, with such a restriction, also the Brill-Noether theory is better behaved.

Theorem 5.1.4. *Let Y be a quasistable curve of genus g and let $L \in \operatorname{Pic}^d Y$ be a balanced line bundle.*

(1) If $d \leq 0$ then $h^0(Y,L) = 0$ unless $L = \mathcal{O}_Y$.
(2) (Riemann theorem) *If $d \geq 2g-1$ then $h^0(Y,L) = d-g+1$.*

(See [C08, Thm 2.2.1 and Lemma 4.4.1]) The situation for the Clifford's theorem is less clear. Let us summarize the known facts in a unique statement.

Theorem 5.1.5 (Generalized Clifford theorem)**.** *Let Y be a quasistable curve of genus g and let $L \in \operatorname{Pic}^d Y$ be balanced, with $0 \leq d \leq 2g-2$. Then $h^0(Y,L) \leq d/2+1$ if one of the following conditions holds*

(1) $d = 0$ or $d = 2g-2$. Moreover, in this case $h^0(Y,L) = d/2+1$ if and only if $L = \mathcal{O}_Y$ or $L = \omega_Y$.
(2) Y has at most two irreducible components.
(3) $d = 1, 2$ and Y is free from separating nodes.
(4) $d = 3, 4$ and Y is stable and free from separating nodes.

The last two parts are sharp, as shown by several counterexamples. Moreover there are counterexamples showing that for $d \geq 5$ the Clifford theorem fails for stable curves free from separating nodes; we refer to [C08, Sections 3 and 4] for more details.

The following question, having a strong combinatorial nature, is open:

Problem 5.1.6. *How does Clifford theorem generalize (to balanced line bundles) for any d with $0 < d < 2g-2$?*

5.2. **Theta divisor.**

5.2.1. *Theta divisor of a smooth curve.* We refer to [ACGH] for more details about this subsection. The theta divisor on a smooth curve C of genus $g \geq 1$ can be intrinsically defined in $\operatorname{Pic}^{g-1} C$ as follows:

$$\Theta(C) := W^0_{g-1}(C) \subset \operatorname{Pic}^{g-1} C.$$

Let us show that $\Theta(C)$ is an irreducible codimension-one subvariety of $\operatorname{Pic}^{g-1} C$. Consider the Abel map α_C^{g-1} in degree $g-1$, i.e. the morphism

$$\alpha_C^{g-1} : C^{g-1} \longrightarrow \operatorname{Pic}^{g-1} C; \qquad (p_1, \ldots, p_{g-1}) \mapsto \mathcal{O}_C(\sum_{i=1}^{g-1} p_i). \tag{16}$$

It is clear that the image of α_C^{g-1} coincides with $W^0_{g-1}(C)$, hence $\Theta(C)$ is an irreducible closed subscheme. Now the generic fiber of α_C^{g-1} has dimension 0, in fact it is well known that if $g \geq 3$

$$\dim W^1_{g-1}(C) = \begin{cases} g-3 & \text{if } C \text{ is hyperelliptic} \\ g-4 & \text{otherwise} \end{cases}$$

(see [ACGH, p.250]). On the other hand of $g = 1, 2$ it is trivial to check that $W^1_{g-1}(C) = \emptyset$.

The theta divisor turns out to be ample, and, which is perhaps its most important property, the isomorphism class of the pair $(\operatorname{Pic}^{g-1} C, \Theta(C))$ uniquely determines the isomorphism class of C:

Theorem 5.2.1 (Torelli theorem)**.** *Let C and C' be two smooth connected curves of genus $g \geq 1$. Then $(\operatorname{Pic}^{g-1} C, \Theta(C)) \cong (\operatorname{Pic}^{g-1} C', \Theta(C'))$ if and only if $C \cong C'$.*

See [T13] or [ACGH].

The geometry of the theta divisor encodes some important properties of the curve C and its linear series. A striking instance of this is the following

Theorem 5.2.2 (Riemann Singularity Theorem)**.** *Let $L \in \mathrm{Pic}^{g-1} C$ be a line bundle on a smooth curve C such that $h^0(C, L) \neq 0$; let $[L] \in \Theta(C)$ be the corresponding point. Then $mult_{[L]}\Theta(C) = h^0(C, L)$.*

See [ACGH, Chapt.VI]. In particular we have that the singular locus of $\Theta(C)$ coincides with $W^1_{g-1}(C)$. This is one of the facts used to prove the characterization of hyperelliptic curves given above.

5.2.2. *Compactified Picard scheme in degree $g-1$.* Now let us consider a stable curve X, and let us try to generalize the previous picture.

By what we said, it seems reasonable to restrict to the case $d = g - 1$. Moreover, we have seen that the structure of the compatified Picard scheme $\overline{P^d_X}$ varies with d in general. This is another good reason to concentrate on the case $d = g - 1$. Let us define the Theta divisor in $\overline{P^{g-1}_X}$ exactly as for smooth curves. Recall that a point in $\overline{P^{g-1}_X}$ parametrizes a unique pair $[Y, L]$ where Y is a quasistable curve of X and $L \in \mathrm{Pic}^{g-1} Y$ is a strictly balanced line bundle on Y. Now we set

$$\Theta(X) := \{[Y, L] \in \overline{P^{g-1}_X} \text{ such that } h^0(Y, L) \neq 0\}$$

where it is understood that Y is a quasistable curve of X and $L \in \mathrm{Pic}^{g-1} Y$ is strictly balanced.

Now recall that $\widetilde{Y}$ is the curve obtained by removing the exceptional components from Y, and $\widetilde{L}$ denotes the restriction to $\widetilde{Y}$ of an $L \in \mathrm{Pic}\, Y$. We already noticed that the pair $(\widetilde{Y}, \widetilde{L})$ uniquely determines the point $[Y, L]$ in the compactified Picard scheme. Moreover we have

Remark 5.2.3. Let L be a balanced line bundle on the quasistable curve Y, then $h^0(Y, L) = h^0(\widetilde{Y}, \widetilde{L})$ ([C07, Lemma 4.2.5])

Before continuing, we will show that, if $d = g - 1$, computations become easier and the definition of balanced line bundles can be generalized. We begin by a straitghtforward lemma.

Lemma 5.2.4. *Let Y be a quasistable curve of genus $g \geq 2$. $L \in \mathrm{Pic}^{g-1} Y$ is strictly balanced if and only if the following hold*

(1) $\deg_E L = 1$ for every connected $E \subset Y_{exc}$;
(2) $\deg_Z L > g_Z - 1$ for every $Z \subsetneq Y$ such that $Z \cap Z^c \subsetneq Y_{exc}$;
(3) $\deg_Z L = g_Z - 1$ for every Z with $Z \cap Z^c \subset Y_{exc}$ such that $Z \subsetneq Y_{exc}$.

Proof. The "if" part is clear. Conversely, assume L strictly balanced. The first two conditions hold by definition; we need to prove condition (3). Recall that, since L is balanced, for every Z we have $g_Z - 1 \leq \deg_Z L$ (see Example 4.1.9). Let $E_1, \ldots, E_{\delta_Z}$ be the exceptional components intersecting Z, and set $E_Z = E_1 \cup \ldots \cup E_{\delta_Z}$. Then L is balanced if and only if $\deg_{E_Z} L = \delta_Z$. If $E_Z = Z^c$ we have

$$g - 1 = \deg_Z L + \deg_{E_Z} L \geq g_Z - 1 + \delta_Z = g - 1$$

(since $g_Z = g - \delta_Z$); therefore equality holds and $\deg_Z L = g_Z - 1$. If $Z^c \subset Y_{\text{exc}}$ we are done.

Now suppose $Z^c \not\subset Y_{\text{exc}}$, i.e. $E_Z \subsetneq Z^c$; recall that we can assume that Z is connected (Remark 4.1.6). Let $W = E_Z^c$, so that Z is a connected component of W. By the previous part of the argument we know that L is (strictly) balanced if and only if $\deg_W L = g_W - 1$; notice that $g_W = \sum_{i=1}^c g_{W_i} + 1 - c$ where $W_1, \ldots, W_c$ are the connected components of W. We have

$$\deg_W L = g_W - 1 = \sum_{i=1}^{c}(g_{W_i} - 1) \leq \sum_{i=1}^{c} \deg_{W_i} L = \deg_W L$$

therefore equality must hold. Hence for every connected component W_i of W we have $\deg_{W_i} L = g_{W_i} - 1$. In particular this holds for Z. ■

Our goal is to characterize strictly balanced line bundles by disregarding the exceptional components. This would significantly simplify matters, aslo in view of remark 5.2.3. The previous lemma enables us to do that. First we need a definition.

Notice also that the inequalities defining balanced and strictly balanced line bundles in degree $g-1$ make sense for every g (not only $g \geq 2$, as in Definition 4.1.5).

Definition 5.2.5. Let Y be a connected nodal curve of genus $g \geq 0$ and $L \in \operatorname{Pic} Y$. We say that L, or its multidegree, is *stable* if $\deg L = g - 1$ and if for every proper subcurve $Z \subsetneq Y$ we have $\deg_Z L > g_Z - 1$.

If Y is not connected, we say that $L \in \operatorname{Pic} Y$ is stable if the restriction of L to every connected component of Y is stable (in particular, $\deg L = g_Y - 1$).

We denote by $\Sigma(Y)$ the set of stable multidegrees on Y.

Example 5.2.6. If X is a stable curve we have $\Sigma(X) = SB_{g-1}(X)$.

Let $Y = C_0 \cup C_1$ with $\#C_0 \cap C_1 = 2$, $C_0 \cong \mathbb{P}^1$ (i.e. C_0 is an exceptional component) and C_1 smooth of genus $g - 1 \geq 1$. Now $\Sigma(Y) = \{(0, g-1)\}$ whereas $SB_{g-1}(Y) = \{(1, g-2)\}$.

Recall that for a quasistable curve Y of genus g the subcurve $\widetilde{Y}$ is a (possibly non connected) curve, of genus $g - \epsilon(Y)$ (cf.4.1.2).

The following is a trivial consequence of Lemma 5.2.4

Corollary 5.2.7. *Let Y be a quasistable curve of genus $g \geq 2$. The following map is a bijection*

$$SB_{g-1}(Y) \longrightarrow \Sigma(\widetilde{Y}); \qquad \underline{d} \mapsto \underline{d}_{\widetilde{Y}}.$$

Now we will give a simpler description of $\overline{P_X^{g-1}}$ and, with it, state some properties of $\Theta(X)$. First, for any set of nodes of X, $S \subset X_{\text{sing}}$, we will denote by $Y_S \to X$ the normalization of X at S; thus Y_S is a nodal curve of genus $g - \#S$. From the previous discussion and Theorem 4.2.1 we obtain the following

Proposition 5.2.8. *Let X be a nodal connected curve of genus g.*

(a) There is a canonical isomorphism:

$$\overline{P_X^{g-1}} \cong \coprod_{\substack{\emptyset \subseteq S \subseteq X_{sing} \\ \underline{d} \in \Sigma(Y_S)}} \operatorname{Pic}^{\underline{d}} Y_S. \tag{17}$$

(b) $\Sigma(Y_S) \neq \emptyset$ *if and only if* Y_S *has no separating node.*

Now we turn to the Theta divisor. Let $\underline{d} = (d_1, \dots, d_\gamma) \in \Sigma(X)$ be a stable multidegree (in particular, X is free from separatig nodes). Consider the following rational map

$$\alpha_X^{\underline{d}} : C_1^{d_1} \times \ldots \times C_\gamma^{d_\gamma} \dashrightarrow \operatorname{Pic}^{\underline{d}} X \subset \overline{P_X^{g-1}}; \quad (p_1, \dots, p_{g-1}) \mapsto [\mathcal{O}_X(\sum_1^\gamma p_i)].$$

The above map is regular away from the preimages of the nodes of X. It is a generalization of the Abel map described in (16), indeed the product $C_1^{d_1} \times \ldots \times C_\gamma^{d_\gamma}$ is an irreducible component of X^{g-1} (notice that, as $\underline{d}$ is stable, $d_i \geq 0$ for each i).

Theorem 5.2.9. *Let X be a nodal connected curve of genus g*

(a) There is a canonical isomorphism compatible with (17)

$$\Theta(X) \cong \coprod_{\substack{\emptyset \subseteq S \subseteq X_{sing} \\ \underline{d} \in \Sigma(Y_S)}} W_{\underline{d}}(Y_S). \tag{18}$$

(b) $\Theta(X)$ *is a Cartier, ample divisor of* $\overline{P_X^{g-1}}$.

(c) Assume that X is free from separating nodes. Then for every $\underline{d} \in \Sigma(X)$ we have

$$\operatorname{Pic}^{\underline{d}} X \supset \overline{\operatorname{Im}_{\alpha_X^{\underline{d}}}} = W_{\underline{d}}(X).$$

In particular, every irreducible component of $\overline{P_X^{g-1}}$ *contains exactly one irreducible component of* $\Theta(X)$.

The previous statement summarizes results of [S94], [E97], [Ale04] and [C07].

5.3. **Torelli problem.** The classical Torelli Theorem 5.2.1 can be stated using moduli spaces. Namely, let A_g be the moduli scheme of principally polarized abelian varieties of dimension g (which exists and is a quasiprojective coarse moduli scheme); then the Torelli map

$$\mathrm{t}_g : M_g \to A_g$$

mapping a curve to its Jacobian, polarized by the theta divisor (everything up to isomorphism) is injective.

Now, we know how to compactify M_g using stable curves, and we have shown that every stable curve has a compactified Jacobian, endowed with an ample Cartier divisor generalizing the Theta divisor.

It is thus natural to ask: does the Torelli map extend from $\overline{M}_g$ to some good compactification of A_g? If so, is the extension injective? If it is not injective, what are its fibers?

The moduli space A_g has been shown, over the years, to have several interesting compactifications. We want to introduce one that has been costructed recently, and which has the advantage of having a straightforward modular description which ties in well with what we discussed so far.

To explain that, let us list some useful properties of our compactified Picard scheme. Let X be a connected nodal curve of genus g, $J(X)$ its generalized Jacobian, $\overline{P_X^{g-1}}$ is compactified Picard scheme in degree $g-1$ and $\Theta(X) \subset \overline{P_X^{g-1}}$ its Theta divisor. Recall that $J(X)$ is a *semiabelian variety* i.e. it fits in an exact sequence of algebraic groups whose kernel is a torus (called the *toric part* of $J(X)$) and whose cokernel and abelian variety (see (7)).

Fact 5.3.1. *Let X be a connected nodal curve of genus g. Then*

(a) $\overline{P_X^{g-1}}$ *is a connected, projective, reduced scheme of pure dimension equal to* $\dim J(X) = g$.

(b) $J(X)$ *acts on* $\overline{P_X^{g-1}}$ *with finitely many orbits. The stabilizer of every orbit is a subtorus of the toric part of* $J(X)$.

(c) $\overline{P_X^{g-1}}$ *is seminormal.*

(d) $\Theta(X)$ *is an ample, Cartier divisor of* $\overline{P_X^{g-1}}$ *which does not contain any* $J(X)$*-orbit, and such that* $h^0(\Theta(X)) = 1$.

Some of the above properties have already appeared. For (a) see Theorem 4.2.1. For (b), we have seen after Theorem 4.2.1 that each $J(X)$ orbit is canonically isomorphic to $\mathrm{Pic}^{\underline{d}} Y_S$ with $\underline{d} \in \Sigma(Y_S)$, and its stabilizer is the set of line bundles on X which pull back to O_{Y_S} (cf. (13) and Proposition 5.2.8). The seminormality of $\overline{P_X^{g-1}}$ is proved in [Ale04]. Part (d) has almost entirely been discussed before; the fact that $\Theta(X)$ does ot contain any orbit follows from the fact that $W_{\underline{d}}(Y_S) \subsetneq \mathrm{Pic}^{\underline{d}} Y_S$ for every stable $\underline{d}$ on Y_S (using the decompositions of $\overline{P_X^{g-1}}$ and $\Theta(X)$ given in Proposition 5.2.8 and Theorem 5.2.9)

A consequence of the above properties is that the pair $(\overline{P_X^{g-1}}, \Theta(X))$ together with the action of $J(X)$ is a so called *principally polarized stable semi-abelic* pair, in the sense of Alexeev [Ale02].

As proved in [Ale02], is the fact that principally polarized stable semi-abelic pairs admit a projective moduli space:

Theorem 5.3.2 (Alexeev)**.** *There exists a projective scheme* $\overline{A}_g^{mod}$ *which is a coarse moduli space for principally polarized stable semi-abelic pairs of dimension g. The main irreducible component of* $\overline{A}_g^{mod}$ *contains a dense open subset naturally isomorphic to* A_g.

See also [Bri07] for an expository description.

Now let us go back to the the Torelli map t_g. The largest subset of $\overline{M}_g$ admitting a regular map to A_g extending t_g is the locus of curves of compact type (see Lemma 2.1.1). Moreover t_g extends to a regular map

$$\overline{\mathrm{t}}_g : \overline{M}_g \longrightarrow \overline{A}_g^{\mathrm{mod}}; \qquad [X] \mapsto (\overline{P_X^{g-1}}, \Theta(X))$$

([Ale04]). Now we can ask how the Torelli theorem generalizes. It is not hard to see that if $g \geq 3$ $\overline{\mathrm{t}}_g$ cannot possibly be injective. Indeed, it has positive dimensional fibers over the locus of curves having a separating node (see [Nam80, Thm 9.30(vi)]). The natural question is now: is $\overline{\mathrm{t}}_g$ injective away from curves with a separating node?

This question, with a different formulation (i.e. using a different compactification of A_g) was explicitely asked by Namikawa.

The answer, obtained very recently, is: no. In [CV209] there is a precise description of the fibers of $\overline{t}_g$; in particular, although on the locus of curves free from separting nodes, $\overline{t}_g$ has finite fibers, it fails to be injective on that locus as soon as $g \geq 5$.

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