MODULI THEORY AND ARITHMETIC OF ALGEBRAIC VARIETIES[1]

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Contents

1 Introduction and Preliminaries

1.1 Summary

This paper surveys a few applications of algebro-geometric moduli theory to issues concerning the distribution of rational points in algebraic varieties.

A few well known arithmetic problems with their expected answers (the so-called "diophantine conjectures") are introduced in section 2, explaining their connection with a circle of ideas, whose goal is to find a unifying theme in analytic, differential, algebraic and arithmetic geometry.

In section 3 we illustrate how, applying the modern moduli theory of algebraic varieties, the diophantine conjectures imply uniform boundedness of rational points on curves, which is also open. The crucial tool is an algebro-geometric

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theorem (the "Correlation" or "fibered power" theorem) about families of varieties of general type.

A series of enumerative results about rational points on curves is described in section 4.1.

Finally, we examine related issues over function fields. This is an appropriate thing to do, in fact, as we explain in section 2.3, part of the motivation for the diophantine conjectures came from parallel results over function fields.

We show how other aspects of moduli theory are used to obtain that various uniform boundedness phenomena do hold over function fields.

1.2 Conventions

We work in characteristic zero. The following notation will be fixed throughout. K is a number field and V an algebraic variety defined over K; V is projective, reduced, irreducible, and of positive dimension. The set of K-rational points of V is denoted by $V(K)$.

When considering V over an extension of K, for example over $\mathbb{C}$, we will continue to denote it by V.

All varieties are assumed to be connected. The word "curve" means projective variety of dimension 1. The word "family" stands for a projective morphism of integral varieties, often denoted by $f : X \longrightarrow B$; the fiber over a point $b \in B$ is denoted by $X_b := f^{-1}(b)$.

A complex function field, denoted by $L = \mathbb{C}(B)$, is the field of rational functions of an integral variety B, which can be assumed to be projective, to fix ideas.

Whenever dealing with number fields and function fields simultaneoulsly, we will use the special notation F to denote either type of field.

The symbols g, d, s, q stand for nonnegative integers, and $g \geq 2$, unless otherwise specified.

1.3 Kodaira dimension and varieties of general type

Given an abstract algebraic variety V, consider its rational images in projective space, $\phi : V \dashrightarrow \mathbb{P}^r$ where ϕ is a rational map. To such data one canonically associates a line bundle L on V: $L := \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Viceversa, given a "sufficiently nice" line bundle M on V one constructs a rational mapping

$$\phi_M : V \dashrightarrow \mathbb{P}(H^0(V, M)^*).$$

Here are two definitions to clarify this set-up (see [KM98] for more details).

Definition 1. Let L be a line bundle on V

1. L is *big* if for some positive integer k the map ϕ_{L^k} gives an embedding in projective space of a non-empty open subset of V.

2. L is *ample* if for some positive integer k the map ϕ_{L^k} gives an embedding of V in projective space.

Assume now that V is nonsingular and consider its canonical line bundle, K_V. If, for a given integer $n \geq 1$, the rational mapping

$$\phi_{K_V^n} : V \dashrightarrow \mathbb{P}(H^0(V, K_V^n)^*),$$

is defined, it will be called the *n-canonical mapping* of V.

An important birational invariant is the *Kodaira dimension* $\kappa(V)$

$$\kappa(V) := \max_{n\geq 1}\{\dim \mathrm{Im}\phi_{K_V^n}\}$$

It might very well happen that K_V and its powers have no sections (for example, this will be the case for V birational to projective space); in such a case the Kodaira dimension of V is set to be equal to $-\infty$ (so that κ is additive with respect to products).

Of course we always have $\kappa(V) \leq \dim V$. We shall be particularly interested in the case when equality holds, that is, K_V is big.

Definition 2. 1. A nonsingular variety V is *of general type* if $\kappa(V) = \dim V$

2. A variety is of general type if a desingularization of it is of general type.

Part (2) is well defined, since the Kodaira dimension is a birational invariant.

On the other hand, the definition presents a subtlety that will play a crucial role in the sequel (see section 3.3). Let V be a (possibly singular) variety possessing the dualizing line bundle ω_V. Recall that ω_V is the unique line bundle making Serre duality valid and, if V is nonsingular, $\omega_V = K_V$. Now the warning is: to say that a singular variety V is of general type is not the same as saying that ω_V is big. Consider in fact the following

Example 1.3.1. $V \subset \mathbb{P}^2$ is an irreducible curve of degree 4 having 3 nodes. Then the dualizing sheaf of V can be computed by the adjunction formula,

$$\omega_V = K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(V) \otimes \mathcal{O}_V = \mathcal{O}_V(1)$$

hence ω_V is the line bundle defining the embedding of V in $\mathbb{P}^2$, in particular ω_V is big. On the other hand, there is a birational morphism $r : \mathbb{P}^1 \longrightarrow V$; to obtain it, consider the $\mathbb{P}^1$ of all conics C_t, with $t \in \mathbb{P}^1$, passing through the 3 nodes of V and any other fixed point of V, then by Bézout's theorem C_t meets V in one remaining point, thus determining the morphism r. Therefore $\mathbb{P}^1$ is the desingularization of V, hence $\kappa(V) = -\infty$ and V is not of general type. The point is that although there are plenty of differential forms that are regular on the complement of the three nodes of V (which follows from ω_V being ample), the pull-back of these forms via the morphism r will not extend to regular forms on the whole of $\mathbb{P}^1$. The singularities of V are, in a suitable sense, too complicated (with the appropriate terminology we shall say that they are not "canonical", see section 3.3).

Some more useful examples.

Example 1.3.2. A curve is of general type if and only if its genus is at least 2; if the curve is singular, by "genus" we mean the genus of its normalization.

Example 1.3.3. Smooth complete intersections of high enough degree in projective space are of general type; more precisely if $V \subset \mathbb{P}^r$ is the complete intersection of c hypersurfaces $H_1, ..., H_c$ with $\deg H_i = d_i$ (and $\dim V = r - c$) then V is of general type if and only if $\sum d_i \geq r + 2$ (this fact follows directly from the adjunction formula).

Example 1.3.4. If V is of general type and G is a subgroup of automorphisms of V (so that G is finite), then V/G needs not be of general type, but for m large enough the quotient V^m/G by the diagonal action of G on V^m is of general type. Notice that this is true regardless of V being smooth (see [CHM97] corollary 4.1).

Conjecturally, the property of being of general type should be strictly related to the analytic, differential, and diophantine structure of a variety.

1.4 Moduli of algebraic varieties

Fix a polynomial $h(x) \in \mathbb{Q}[x]$. Denote by M_h the moduli scheme of smooth algebraic varieties whose canonical bundle is ample and has Hilbert polynomial equal to $h(x)$.

The simplest case is when h has degree 1, then our moduli space is non-empty if there is an integer $g \geq 2$ such that

$$h(x) = (2g - 2)x - g + 1$$

and M_h will be the moduli space of curves of fixed genus g, which is usually denoted by M_g.

For the main properties and the construction of such moduli spaces we refer to the book of E. Viehweg [Vi95].

We need to recall here the fact that such spaces M_h are not complete, and in very few cases modular completions for them exist. One of such cases is M_g, which is an integral, normal quasiprojective variety, admitting a modular compactification, $\overline{M_g}$, which is the moduli space of Deligne-Mumford stable curves; $\overline{M_g}$ is a normal, integral, projective variety.

If V is a smooth algebraic variety as above, we denote by $[V] \in M_h$ the point of the moduli scheme corresponding to V.

Recall that if $f : X \longrightarrow B$ is family of nonsingular varieties as above, then there exists a unique regular morphism, called the *classifying morphism*

$$\begin{array}{rccc} \gamma_f : & B & \longrightarrow & M_h \\ & b & \mapsto & [X_b] \end{array}$$

More generally, if $X \longrightarrow B$ is a family such that the locus $U \subset B$ where the fiber is smooth is open, we shall continue to denote by $\gamma_f : B \dashrightarrow M_h$ the rational map whose restriction to U is the classifying morphism.

Definition 3.

1. A family $X \longrightarrow B$ is *isotrivial* if there exists a dense open subset $U \subset V$ over which all fibers are isomorphic.
2. A family has *maximal variation of moduli* if there is a non-empty open subset $U \subset B$ such for every $u_0 \in U$ the set $\{u \in U : X_u \cong X_{u_0}\}$ is finite.

In particular, a family of generically smooth varieties for which there exists a moduli scheme is isotrivial if and only if its classifying map is constant, and it has maximal variation of moduli if and only if its classifying map is generically finite.

In fact, moduli theory allows us to measure how much a family varies in moduli.

Definition 4. Let $f : X \longrightarrow B$ be a family of generically smooth varieties of general type. Then the *variation of moduli of the family*, denoted $Var(f)$ is the dimension of the image of its classifying map, that is, the number

$$Var(f) := \dim \operatorname{Im} \gamma_f$$

Such a definition may appear to make sense only if there exists a moduli scheme (e.g. if the canonical bundle is ample), but in fact it works more generally, as stated (see section 2 of [Kol87]).

2 From geometry to arithmetic

We shall assume throughout this section that V is nonsingular.

2.1 The one dimensional case

For curves the picture is very suggestive: the algebro geometric condition of being of general type (that is, of having genus at least 2) can be characterized in terms of

(1) topology
or (2) complex analysis
or (3) differential geometry
or (4) arithmetic.

More precisely, here are some well known fundamental facts.

1. The universal covering of V is the open unit disc (Uniformization).
2. If $f : \mathbb{C} \longrightarrow V$ is a holomorphic map, then f is constant (Picard-Borel theorem).
3. V carries a metric of constant negative curvature.
4. $V(K)$ is finite for any number field K (Mordell conjecture - Faltings theorem).

Conversely, each of the properties listed above implies that the curve is of general type. In fact, if V has genus 0 then V is the Riemann sphere. Hence V is simply connected, it contains copies of $\mathbb{C}$ as algebraic open subsets and it carries a metric of constant positive curvature. Finally it is easy to see that V has infinitely many rational points over a sufficiently large number field.

If V has genus 1, then $V = \mathbb{C}/\Gamma$ is a complex torus and the quotient map $\mathbb{C} \longrightarrow \mathbb{C}/\Gamma$ is the universal covering (with the intrinsic identifications $\mathbb{C} \cong H^0(V, K_V)^*$ and $\Gamma \cong H_1(V, \mathbb{Z})$); moreover V carries a flat metric. If V contains a K rational point, then $V(K)$ can be given a group structure. The theorem of Mordell-Weil states that the set of K-rational points is a finitely generated abelian group $V(K) = \mathbb{Z}^r \oplus H$ with H a finite group. It is well known that for any curve of genus 1 (or for any abelian variety) there exists a finite extension F of K, such that $V(F)$ is infinite.

Additionally, in the genus 1 case, natural uniformity problems arise, such as: (1) How can H vary? (2) Is r uniformly bounded, or for any N there is an elliptic curve for which $r > N$?

Question (1) has a definite answer thanks to a theorem of Mazur ('76) (for $K = \mathbb{Q}$) showing that there exist exactly sixteen finite groups (including the trivial group) that do occur as the torsion part of $V(K)$. This has been generalized over all number fields by Merel [Me97] who proved that the number of torsion points of an elliptic curve defined over a degree d extension of $\mathbb{Q}$ is uniformly bounded. The second question is open.

2.2 Higher dimensional varieties

Thus, in the world of curves there is a general dichotomy between curves that are of general type and curves that are not. It is natural to ask if something similar occurs for varieties of higher dimension. So, let V be a variety of general type; what can one say about V from a complex analytic, differential geometric, diophantine point of view? With the one-dimensional case in mind, recall the analytic terminology:

Definition 5. A variety V is

1. *Brody hyperbolic* if any holomorphic map $f : \mathbb{C} \longrightarrow V$ is constant.
2. *pseudo-hyperbolic* if the locus of all images of holomorphic, non-constant maps $f : \mathbb{C} \longrightarrow V$ is not Zariski dense.

It is clear that Brody hyperbolicity is not birationally invariant, which motivates the second part of the above definition.

Turning now to differential geometry: given the hyperbolic distance on the open unit disc one defines the Kobayashi semi-distance d_{kob} on V (see [L86] for details). Then

Definition 6. A variety V is *Kobayashi hyperbolic* when $d_{kob}(x, y) = 0$ implies $x = y$, that is, if d_{kob} is a distance on V.

It turns out that a holomorphic map is always distance decreasing with respect to the Kobayashi semi-distance, moreover, the Kobayashi semi-distance is identically 0 on $\mathbb{C}$. Hence if V is Kobayashi hyperbolic, then V is Brody hyperbolic.

It is an important theorem of Brody that the converse is true (provided that V is compact). Hence the analytic and the differential geometric notions of hyperbolicity coincide for projective varieties.

The expected link with algebraic geometry is

Conjecture (Lang [L86]). *V is hyperbolic if and only if V and all of its subvarieties are of general type*

We briefly describe two non trivial examples of varieties of general type which are not hyperbolic. The first (of Mike Artin) is a smooth surface of degree 5 in $\mathbb{P}^3$ (whose canonical bundle is very ample, by 1.3.3) having a plane section C with 6 nodes, so that C is a rational curve. The second is a whole class of examples: the Fermat's hypersurfaces of degree at least $r+2$ in $\mathbb{P}^r$. They always contain lines, and, again by 1.3.3, they are of general type.

Kobayashi suggested that such examples should be exceptional, by conjecturing (in [Kob70]) that a generic hypersurface of high enough degree in $\mathbb{P}^{r+1}$ is hyperbolic; it is known that, for such a conjecture to hold in dimension $r \geq 2$, the degree would have to be at least $2r+1$. In '98 one important step was taken by Demailly-El Goul [DE00], who proved the conjecture true for very generic surfaces in $\mathbb{P}^3$ of degree at least 42; here the terms "generic" and "very generic" mean that the exceptional locus is a finite, respectively countable, union of proper closed algebraic subsets of the moduli space of hypersurfaces.

The significance of such a Theorem is better understood when compared with a previous famous result of Clemens and Xu, stating that a very generic surface in $\mathbb{P}^3$ of degree at least 5 contains no algebraic curves of genus less than 2, leaving the case of transcendental curves open.

The birational analogue is the following well known

Conjecture (Green-Griffiths [GG79]). *If V is of general type then V is pseudo hyperbolic.*

If the above statement were true, it would imply that, in a variety of general type, the union of all curves of genus less than 2 is not Zariski-dense. In particular, a surface of general type would contain only finitely many curves of genus 0 and 1. This is known to be true for surfaces satisfying the inequality $c_1^2 > c_2$; Bogomolov proved this for algebraic curves ([B77]), his method was extended to holomorphic curves by McQuillan ([Mc98]). Some more progress about the above conjecture in dimension 2 is in [LM97].

2.3 Diophantine conjectures.

What about arithmetic properties of varieties of general type?

The main conjectures on the distribution of rational points in varieties of general type have been partly motivated by parallel results over function fields.

To start with, the analogue of the Mordell conjecture for function fields was proved by Manin [M63] about twenty years before the proof over number fields (by Faltings). More precisely, let L be the function field of a projective variety B over the complex numbers. Manin showed that a curve defined over L, having genus at least 2, either has finitely many L-rational points, or it is birationally a product and all but finitely many of its L-points correspond to constant sections.

Besides Manin's, there have been other approaches to the geometric Mordell problem. We are mostly interested in one of them which on the one hand highlights the analogies with the number field case; on the other hand establishes a link with moduli theory. We begin with some notation

Definition 7. Let F be a number field or a function field $F = \mathbb{C}(B)$.

1. Denote by $\mathcal{C}_g(F)$ the set of F-isomorphism classes of nonisotrivial, nonsingular, projective curves of genus g defined over F.

2. Let S be either (if F is a number field) a finite set of places of F or, if $F = \mathbb{C}(B)$, a closed subset of B. Denote by $\mathcal{C}_g(F, S)$ the subset of $\mathcal{C}_g(F)$ parametrizing curves having good reduction outside of S

(In the aritmetic case the nonisotriviality condition is of course useless.)

During the ICM in 1962, Shafarevich conjectured that *If F is either a number field or the function field of a complex curve, then $\mathcal{C}_g(F, S)$ is finite.*

The geometric case was obtained by Parshin (if S is empty) and Arakelov (in general) (see [P68], [Ar71]); they proved that if B is a smooth complex curve and $S \subset V$ a finite subset, then there exist only finitely many non-isotrivial families of smooth curves of genus $g \geq 2$ over $B - S$.

The connection with the Mordell problem was at the same time established by Parshin, in [P68], where, by the celebrated "Kodaira-Parshin trick", he showed that the Shafarevich conjecture implies the Mordell conjecture, both for function fields and for number fields:

Lemma 8 (Kodaira-Parshin trick). *Let F be a number field or the function field of a complex curve. Assume that the set $\mathcal{C}_g(F, S)$ is finite for every F and S as above. Then for every $C \in \mathcal{C}_g(F)$ the set $C(F)$ is finite*

The number field version of the Shafarevich conjecture (and hence of the Mordell conjecture) was proved by Faltings in [F83].

Going back to the Theorem of Manin, it turns out ([M63] proposition 5) that the case of higher dimensional bases B can be reduced, by induction on the dimension, to the case $\dim B = 1$, to which we therefore restrict ourselves. Then the second part of the theorem says that, if the family is birational to a product $B \times C_0$, where C_0 is a curve of genus at least 2 defined over $\mathbb{C}$, then there can only be finitely many non-constant sections $B \to B \times C_0$. This is a straightforward consequence of a classical theorem of De Franchis ([DF13]), according to which there can only be finitely many non-constant morphisms from B to C_0, provided that C_0 has genus 2 or more.

In '75 Kobayashi and Ochiai generalized De Franchis Theorem to complex varieties of any dimension:

Theorem 9 ([KO75]). *Let Y and Z be projective varieties of the same dimension, with Y of general type. Then there exist finitely many dominant rational maps from Z to Y.*

A generalization of Manin's theorem to varieties of higher dimension was obtained by Noguchi[N81]:

Theorem 10 ([N81]). *Let X be smooth of general type defined over a function field L. Assume that X has ample cotangent bundle. Then either $X(L)$ is not Zariski-dense, or X is definable over $\mathbb{C}$ and all but finitely many of its L-rational points are defined over $\mathbb{C}$.*

Clearly, if X has ample cotangent bundle, then X is of general type and so are all of its subvarieties. The converse is false.

Such a set of results and trends inspired the following speculations in the arithmetic setting. Let K be a number field and V a variety defined over K.

Conjecture (Weak form). *If V is of general type, then $V(K)$ is not Zariski-dense in V.*

This statement is attributed to E. Bombieri, S. Lang and P. Vojta, and often referred to as the "Weak Lang conjecture". The next stronger form is due to Lang.

Conjecture (Strong form). *If V is of general type, then there is a proper, closed subset V_0 of V such that, for any number field K, all but finitely many K-rational points of V lie in V_0.*

The above conjectures are known to be true only for curves ([F83]) and subvarieties of abelian varieties ([F94]).

We will talk more about them (namely, about their implications) in the sequel.

2.4 When are rational points Zariski dense?

Consider varieties of low Kodaira dimension and look for geometric conditions that should ensure density of rational points over some number field.

We start with some terminology:

Definition 11. Let V be a variety defined over a number field K; rational points of V are *potentially dense* if there exists a finite extension K' of K such that $V(K')$ is Zariski dense in V.

According to the diophantine conjectures, rational points should not be potentially dense on varieties admitting a dominant map to a variety of general type. What about the rest of the world? A result of Colliot-Thélène, Skorobogatov, Swinnerton-Dyer [CSS97] shows that it is wrong to conjecture that rational points are potentially dense everywhere else. They prove (Section 2 in [CSS97]) the following:

Theorem 12. *[CSS97] Let $p : X \longrightarrow \mathbb{P}^1$ be a fibration with X smooth, irreducible and projective and such that the generic fiber of p is smooth. Assume that all objects above are defined over the number field K. If p has at least 5 double (i.e. non reduced) fibers, then the K-rational points of X must all lie in a finite union of fibers.*

In particular, rational points are not potentially dense on X. Explicit examples of such fibrations are found already in dimension 2, with X a hyperelliptic surface which does not admit a dominant map to a variety of general type (Proposition 3.1 in [CSS97]).

When are then rational points potentially dense?

Obvious or well known cases where potential density holds are rational, unirational and abelian varieties. As ventured by J. Harris and Y. Tschinkel in [HT98], natural candidates are varieties whose canonical bundle is "negative". They propose

Conjecture ([HT98]). *Rational points are potentially dense on*

1. *all Fano varieties.*

2. *(stronger form) all varieties V such that $-K_V$ is nef.*

where a smooth variety V is called "Fano" if $-K_V$ is ample, and a line bundle L is called "nef" if for every curve C in V, $\deg_C L \geq 0$.

As a first piece of evidence, they show that rational points are potentially dense on smooth quartic hypersurfaces of dimension at least 3. The case of dimension 2, quartic surfaces in $\mathbb{P}^3$, is not yet completely understood: in [HT98] potential density is proved only under the additional assumption that the quartic surface contains a line ℓ. If this happens the surface has a fibration in elliptic curves, given by all planes containing ℓ, whose intersection with the surface is given by ℓ union a varying residual plane cubic; this is what allows them to prove density of rational points.

This example is quite typical: in almost all cases in which potential density is known, the method of proof involves exhibiting a dense collection of subvarieties having a dense subset of rational points over the same number field.

This is done in [BT00] and [BT98] for elliptic K3 surfaces and for Enriques surfaces by means of elliptic fibrations, and in [Si91] for K3's having infinitely many automorphisms.

Part 2 of the above conjecture is a special case of Conjecture III_A in [C01], which proposes that *rational points are potentially dense on V if and only if V is special* (in the sense of F. Campana: Definition 2.1 in [C01]). The relation comes from the prediction that varieties with nef anticanonical bundle ought to be special (see [C01], 2.2).

To conclude: not surprisingly, the results in dimension 2 and 3 owe a good deal to the corresponding classification theorems; we summarize them as follows.

Dimension 2: potential density is known to hold for all surfaces which are not of general type, with the exception of K3's with finite automorphism group and without elliptic fibrations, which case is not known. However, it is worth noticing that for every K3 surface, potential density has been proved for its symmetric products of suitable order ([HasT]).

In dimension 3, potential density is known for all Fano threefolds, with the exception of double covers of $\mathbb{P}^3$ ramified in a smooth sextic surface ([HT98], [BT99]). It is unknown for the remarkable case of Calabi-Yau varieties.

3 Moduli theory and uniformity conjectures

3.1 Consequences of the diophantine conjectures

If the diophantine conjectures (stated in 2.3) were true, they would have a big impact about the distribution of rational points on curves. To explain how, let us begin by applying them to a special case. Let $f : X \to B$ be a family of curves of genus at least 2, assume that B is a nonsingular curve and that everything is defined over K. Faltings Theorem ensures that for every point $b \in B$, the fiber X_b of f over b has a finite set of K-rational points; consider the union F_K of all such sets:

$$F_K := \cup_{b \in B(K)} X_b(K)$$

Now ask whether the restriction of f to F_K has bounded fibers over B; notice that F_K might very well be dense in X.

In the special case in which X is of general type, the Weak Diophantine conjecture implies that there is a proper closed subset Z of X containing $X(K)$; clearly then $F_K \subset Z$. Thus, the restriction $f_{|Z} : Z \longrightarrow B$ having finite degree (since $\dim X = 2$ and hence $\dim Z = 1$), the fibers of F_K over B have bounded cardinality.

Of course, assuming that X be of general type was crucial. Now, let us weaken such a ridiculously restrictive assumption by the following hypothesis: given a family $f : X \longrightarrow B$ of curves of genus at least 2, there exists a number n such that the n-th fibered power $X \times_B .... \times_B X$ has a dominant map to a variety of general type. Then a subtle elaboration ([CHM97], proof of Theorem 1.1) of the quick reasoning above shows that, if this hypothesis holds, then the Weak Diophantine conjecture implies that the sets of rational points of the fibers of f have bounded cardinality.

At this point the question becomes: when is the above "fibered power hypothesis" satisfied?

The answer is: always! That is, for every family of generically smooth curves of genus at least 2.

The proof of this algebro-geometric fact, called the "Correlation Theorem", is far from being trivial and will be dealt with in the next section.

Here we shall focus on its arithmetic consequences. They are obtained by applying the above machine to a special family of curves: a so-called "global family", that is a morphism $f : X \longrightarrow B$ such that every curve of genus g is isomorphic to some fiber of f. It is well known from moduli theory that there are many choices for such a global family, and that one can assume B to be nonsingular.

The first corollary of the Correlation Theorem is that if the Weak Diophantine conjecture holds, then there exists a uniform bound $B_g(K) < \infty$ such that any curve of genus $g \geq 2$ defined over K has at most $B_g(K)$ K-rational points ([CHM97]). This result has been strenghtened by Abramovich and Pacelli, who showed that such a bound only depends on the degree d of K over $\mathbb{Q}$.

Theorem 13. *The Weak Diophantine Conjecture implies that for any $d \geq 1$ and for any $g \geq 2$ there is a number $B_g(d)$ such that for every K with $[K : \mathbb{Q}] \leq d$ and for every curve V of genus g, defined over K we have $|V(K)| \leq B_g(d)$.*

(This is Theorem 1.1 in [CHM97] with strengthening from [A95] and [Pa97].)
In a similar vein the Strong Diophantine conjecture implies a universal uniformity statement (Theorem 1.2 in [CHM97]):

Theorem 14. *The Strong Diophantine Conjecture implies that for any $g \geq 2$ there is a number N_g such that for every K there exist only finitely many curves having more than N_g points defined over K.*

Of course the (finite) set of curves having more that N_g K-rational points has to depend on K.

At present, it is not known whether either of the two above statements, following from the diophantine conjectures, hold. A true consequence was obtained by Mazur (in [Ma96]):

Corollary 15. *The Strong Diophantine Conjecture implies that for any K there are only finitely many j-invariants of elliptic curves defined over K and having more than 20 K-rational isogenies.*

For $K = \mathbb{Q}$ the above finiteness result is known to be true independently of the Diophantine Conjecture: a theorem of Kenku [Ke82] states that any elliptic curve over $\mathbb{Q}$ has at most 8 rational isogenies.

3.2 The role of algebraic geometry

As already mentioned, the proofs of the two Theorems 13 and 14 rely on an algebro-geometric result, the Correlation theorem, which says that high fibered powers of a family of curves of genus at least 2 dominate a variety of general type (see below). The name Correlation theorem was given to it because of the application involving the Diophantine conjecture: if $X \to B$ is a family of curves, then by the Correlation theorem there exists a variety of general type V such that the n-th fiber power $X \times_B .... \times_B X$ has a (rational) dominant map h to V. Now the Weak Diophantine conjecture says that the rational points of V are contained in a proper, closed subset V' of V; the preimage of V' via h will be a closed subset Z of the fibered product, containing all of its rational points. So the algebraic functions defining Z can be viewed as co-relating functions for n-tuples of rational points of the fibers of $X \to B$.

In [CHM97] the Correlation theorem was proved for curves, using their moduli theory; it was conjectured to hold for families of varieties of any dimension. From a purely algebro-geometric point of view, the issue was interesting in its own right (regardless of its applications to arithmetic), and fit very well with a lively area of research, at the time. In fact it did not take long for its full generalization to be obtained by D. Abramovich. Here it is, with a more appropriate name ([A97]):

Theorem 16 (Fibered Power Theorem). *Let $f : X \to B$ be a proper morphism of integral varieties such that the general fiber is smooth of general type. Then there exists an integer n such that X^n_B has a dominant rational map to a variety of general type.*

Where X^n_B is the irreducible component of the n-th fiber product of X over B which dominates B; we shall denote

$$f_n : X^n_B \longrightarrow B$$

the natural map (so that $X = X^1_B$ and $f = f_1$).

Remark. The proof of the theorem explicitely constructs a variety of general type dominated by X^n_B. Call it V, then if $f : X \longrightarrow B$ is defined over a number field K, so is V. In fact V is tightly related to the given family, for example, if f has maximal variation of moduli (see section 1.4), then V is equal to X^n_B (theorem 18). In general, if the fibers of f have dimension N, then

$$\dim V = Var(f) + n + N$$

($Var(f)$ being the variation of moduli of f, defined in 1.4)

B. Hassett, in [Has96], proved it for families of surfaces, by applying the moduli theory of surfaces of general type; Abramovich obtained the general case combining an argument suggested by Viehweg (using results of Kollár) together with some techniques of de Jong. The cases of curves and surfaces were dealt with by making use of semistable reduction and of existing compactifications for their moduli spaces. Such tools were not available for higher dimensional varieties.

We give now an outline of the proof of this theorem, comparing the original argument of [CHM97] with that in [A97]. Without further mention we will assume that all maps are rational maps, unless we specify otherwise; we will use the notation $X \approx Y$ to mean X birational to Y.

Definition 17. A morphism $f : X \longrightarrow B$ is a *family of relative general type* if f is a proper morphism of integral varieties such that the generic fiber of f is smooth and of general type.

The key result is the special case of "truly varying" families (see section 1.4)

Theorem 18. *Suppose that $X \longrightarrow B$ is a family of relative general type having maximal variation of moduli. Then for n large, X^n_B is of general type.*

Then the Fibered Power Theorem is proved by reducing it to the case of a family with maximal variation of moduli. This is done by means of the following diagram, whose existence follows from standard moduli theory

$$\begin{array}{ccccccccc}
X_1/G & \approx & X & \longleftarrow & X_1 & \approx & T\times_S B_1 & \longrightarrow & T \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & B & \longleftarrow & B_1 & = & B_1 & \longrightarrow & S
\end{array}$$

Where the starting point is a family $X \longrightarrow B$ of relative general type, as in the statement of Theorem 16; $B_1 \longrightarrow B$ is a generically finite surjective Galois morphism with group G, and

$$X_1 := X \times_B B_1$$

G acts as a subgroup of $\mathrm{Aut}_{B_1}(X_1)$ in a way that X_1/G is birational to X. The map $B_1 \longrightarrow S$ is a classifying map (identifying points in B_1 with isomorphic fibers) and $T \longrightarrow S$ a family with maximal variation of moduli, which should be thought of as a tautological, or universal, family (in particular, for every point $s \in S$ the fiber of T over s is isomorphic to the fiber of X_1 over any point in B_1 that maps to s). Finally, X_1 is birational to $T \times_S B_1$ over B_1.

The diagram is such that for every $n \geq 1$ there is a rational dominant map $X^n_B \longrightarrow T^n_S/G$ (G acting diagonally). By Theorem 18, T^n_S is of general type for large n, therefore, by 1.3.4, T^m_S/G is also of general type for m large, hence the fibered power theorem is proved.

The simplest nontrivial example of how this works, is that of an isotrivial family: assume that for every b in some open subset of B we have that $X_b \cong X_0$, where X_0 is a fixed nonsingular variety of general type. Then, since X_0 has finitely many automorphisms (by Theorem 9) one can construct the finite covering $B_1 \longrightarrow B$ so that the base changed family is trivial: $X_1 \approx X_0 \times B_1$. The space S is just one point and $T \cong X_0$ so that X is birational to $(X_0 \times B_1)/G$; G is a subgroup of $\mathrm{Aut}\, F$ such that X dominates X_0/G and therefore, for every n, X^n_B dominates X^n_0/G.

3.3 Semistable reduction and its variants

We are then left with the proof of theorem 18. The information that the fibers are of general type is contained in the relative dualizing sheaf ω_f of the family, which is relatively big, that is, $\omega_f \otimes \mathcal{O}_{X_b} = \omega_{X_b}$ is big for general $b \in B$. The question is: how does ω_f behave as a line bundle on the total space X? The answer is provided by the

Proposition 19. *Let $f : X \longrightarrow B$ be a family of relative general type, with X and B smooth. Assume that the family has maximal variation of moduli. Then*

1. *ω_f is big.*
2. *For n large, $\omega_{X^n_B}$ is big.*

Proof. The first part is the difficult one; an ad hoc proof for families of curves is outlined in [CHM97] (Lemma 3.2). The general case follows from deep results of Kollár and Viehweg ([Vi83] and [Kol87]).

The second part is not hard, and follows from the first: notice to start that ω_{f_n} is also big. To prove that $\omega_{X^n_B}$ is big; we have the relation

$$\omega_{X^n_B} = \omega_{f_n} \otimes f_n^*\omega_B = p_1^*\omega_f \otimes ....p_n^*\omega_f \otimes f_n^*\omega_B$$

where $p_i : X_B^n \longrightarrow X$ is the i-th factor projection. Therefore, what could prevent $\omega_{X_B^n}$ from being big is ω_B. On the other hand the above formula suggests that such an obstruction does not depend on n. In fact one proves that, for n large enough, the positivity of ω_f overcomes the negativity of ω_B so that $\omega_{X_B^n}$ is big. (Lemma 3.1 in [CHM97]). ∎

Our goal is to prove that X_B^n is of general type and we must now use caution, because X_B^n may be singular: as we saw in 1.3.1, the fact that ω_V is big for a singular variety V does not imply that V is of general type.

Let $r : V^r \longrightarrow V$ be a resolution of singularities, we shall say that V has *canonical* singularities if ω_V^m is a line bundle for some integer m, and if, given a regular section σ of ω_V^m, the pull-back $r^*\sigma$ extends to a regular section of $K_{V^r}^m$ (see [KM98] 2.11).

In particular, if V has canonical singularities and ω_V is big, then K_{V^r} is also big and V is of general type.

What is needed now is an analysis of the singularities of X_B^n: if they were canonical, the second part of the above Proposition would, of course, imply that X_B^n is of general type.

We are here facing the most delicate part of the proof, where the argument in [A97] must be substantially different from that in [CHM97], as we will explain.

The problem is that we have no control over the singularities of X_B^n. The solution is to relate the given $X \longrightarrow B$ to a new family whose fibered powers have canonical singularities. The crucial tool that will be used is:

Proposition 20. *Let $f : X \longrightarrow B$ be a proper morphism of generically smooth integral varieties such that locus in B where the fiber is singular is a divisor with normal crossings. Assume that one of the two following conditions hold*

1. *All fibers of f have normal crossings singularities.*
2. *The family $X \longrightarrow B$ is plurinodal*

Then for every n the singularities of X_B^n are canonical.

Where a family $f : X \longrightarrow B$ is called plurinodal if f factors through finitely many morhisms

$$f : X = X_h \longrightarrow X_{h-1} \longrightarrow .... \longrightarrow X_1 \longrightarrow B = X_0$$

where each $X_i \longrightarrow X_{i-1}$ is a family of irreducible, nodal curves.

The proof of part 1 for curves is Lemma 3.3 in [CHM97], for the general case and part 2 see Lemma 3.6 in [Vi83] and section 4.2 in [A97].

Remark. The normal crossings assumption on the degenerate locus of B is always attainable after birational modifications, which will not alter our final conclusions.

The first condition is used if semistable reduction is known, for example, for curves. More precisely, it is well known that, if $f : X \longrightarrow B$ is a family of generically smooth curves, there is a diagram:

$$\begin{array}{ccccc} Z/G \approx & X & \longleftarrow & X_1 & \approx & Z \\ & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow \\ & B & \longleftarrow & B_1 & = & B_1 \end{array}$$

where $X_1 = X \times_B B_1$. The new object $Z \longrightarrow B_1$ is the so called "semistable reduction" of $X \longrightarrow B$, that is, a family of curves with normal crossings singularities (that is, ordinary double points) which coincides with $f_1 : X_1 \longrightarrow B_1$ away from the singular fibers of f_1. Moreover $B_1 \longrightarrow B$ is a generically finite, surjective Galois covering with group G and $G \subset \mathrm{Aut}_{B_1}(Z)$ so that X is birational to Z/G. Thus, if $X \longrightarrow B$ has maximal variation of moduli, so does $Z \longrightarrow B_1$. By Proposition 20 the fibered powers $Z^n_{B_1}$ have canonical singularities; therefore, by Proposition 19, for large n, $Z^n_{B_1}$ is of general type. Finally we get that, for large m, X^m_B is of general type, being birational to the quotient $Z^m_{B_1}/G$ (using 1.3.4).

The question remains as how to treat the higher dimensional case, where a diagram like the above does not exist. Abramovich's method uses techniques developed in [dJ96] and replaces semistable reduction by a sort of plurinodal reduction. Here is the diagram that he constructs. (Lemma 4.1 in [A97]):

$$\begin{array}{ccccc} X & \longleftarrow & X_1 \approx Y/G & \longleftarrow & Y \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow & & \Big\downarrow{\scriptstyle h} \\ B & \longleftarrow & B_1 & = & B_1 \end{array}$$

where the new family $Y \longrightarrow B_1$ is plurinodal. Furthermore, $B_1 \longrightarrow B$ is a generically finite, surjective morphism, $X_1 = X \times_B B_1$ and $G \subset \mathrm{Aut}_{B_1}(Y)$ is a finite group of automorphisms of Y such that $Y \longrightarrow X_1$ is the quotient map (that is, X_1 is birational to Y/G).

Now, condition 2 of Proposition 20 is obviously satisfied by $Y \longrightarrow B_1$, so that for every n the singularities of $Y^n_{B_1}$ are canonical.

The trouble is that Y and the original family $X \longrightarrow B$ are not as naturally related as in the semistable reduction diagram. The only connection is that Y/G (and similarly $Y^n_{B_1}/G$ for G acting diagonally) is birational to the base change of $X \longrightarrow B$ (or of $X^n_B \longrightarrow B$) by a generically finite map.

This last diagram is in fact harder to handle than the previous one: there are a few technical steps, which we omit, needed to descend the positivity properties of ω_h and ω_Y down to X and show that, for n large, X^n_B is of general type.

3.4 Integral points.

For integral points on affine varieties, an argument similar to the proof of theorem 13 yields that the analog of the Diophantine conjectures implies certain uniformity phenomena.

In 1929 Siegel proved that if U is an affine curve whose projective closure has genus at least 1, then U has finitely many integral points over any number field.

Such a theorem of Siegel has a conjectural generalization, due to Lang and Vojta.

Conjecture (Lang-Vojta). *Let X be a variety of logarithmic general type and let $\mathcal{X}$ be a model for X over* $\operatorname{Spec}\mathcal{O}_{K,S}$. *Then the set of S-integral points, $\mathcal{X}(\mathcal{O}_{K,S})$, is not Zariski-dense.*

Where S is a finite set of places of K and

Definition 21. A quasiprojective variety U is of *logarithmic general type* if given a smooth compactification V of a desingularization U' of U, such that $D := V \setminus U'$ is a a divisor with normal crossings, then, for n large, $n(K_V + D)$ gives a birational map of V in projective space.

For our purposes it suffices to know that

1. A projective variety is of logarithmic general type if and only if it is of general type
2. A curve of positive genus with one or more points removed is of logarithmic general type
3. $\mathbb{P}^1$ with n points removed is of logarithmic general type iff $n \geq 3$.

Abramovich extended the techniques of [CHM97] to the case of integral points on elliptic curves. He proved that the Lang-Vojta conjecture implies uniformity of integral points on certain special models of elliptic curves. One of his results is the following (see [Ab97] Corollary 1):

Theorem 22. *The Lang-Vojta conjecture implies that the number of integral points on semistable elliptic curves defined over $\mathbb{Q}$ is uniformly bounded.*

In fact, he proves a more general result, for any number field K and any set of places S.

Finally, just as in the case of rational points, Pacelli ([Pa99]) showed that there exist a uniform bound for S integral points on semistable elliptic curves which only depends on the degree of K over $\mathbb{Q}$ and on S.

The uniformity consequences of the diophantine conjectures that we listed in this survey are open, with the exception of corollary 15 and of a remark of Abramovich (unpublished) who showed that a true consequence of Lang-Vojta conjecture is the already mentioned uniform boundedness of torsion points on elliptic curves, proved in [Me97] (see 2.1).

4 Boundedness results

4.1 Counting rational points on curves.

This section is devoted to uniformity questions about rational points on curves, forgetting from now on the Diophantine conjectures.

We ask: what are the best known bounds on $B_g(K)$ and N_g (see Theorems 13 and 14 for the definitions of $B_g(K)$ and N_g)? In other words:

1. (for $B_g(K)$) How many rational points can a curve have?
2. (for N_g) How many rational points can an infinite collection of curves (all of genus g and defined over the same number field) have?

It would obviously be optimal to have upper bounds for them, but this stands completely open as of today; in particular, $B_g(K)$ and N_g might very well be infinite (which would of course imply that the diophantine conjectures are false!).

We then investigate lower bounds, that is, we look for examples of curves having as many rational points as possible. Until '93, before the results described earlier revived interest in this subject, the records for small genus, over $\mathbb{Q}$, were held by Brumer who had examples showing that $B_2(\mathbb{Q}) \geq 144$ and $B_3(\mathbb{Q}) \geq 72$. Since then, these records have been broken, and as of today one knows that

$$B_2(\mathbb{Q}) \geq 588$$

with the following curve discovered by Kulesz:

$$y^2 = 278271081x^2(x^2 - 9) - 229833600(x^2 - 1)^2$$

which has 12 automorphisms. For genus 3,

$$B_3(\mathbb{Q}) \geq 112$$

with the hyperelliptic curve

$$y^2 = 48397950000(x^2 + 1)^4 - 939127350499(x^3 - x)^2$$

found by Keller and Kulesz. This last curve has at least 16 automorphisms; there are also examples of curves having many rational points, but no extra-automorphisms; for instance the curve below was found by Stahlke, and has 306 points over $\mathbb{Q}$

$$y^2 = 9703225x^6 - 9394700x^5 + 152200x^4 + 1124745x^3 + 119526x^2 - 42957x + 2061.$$

For lower bounds on N_g one has to exhibit infinitely many non-isomorphic curves over a fixed number field K, having a given number of K-rational points. There are a few different methods that have been used for this purpose. One basic technique is to find a surface (in $\mathbb{P}^3$ say) containing as many lines as possible, and then construct a family of curves by taking plane sections of that surface. The points of intersection with the lines are the seeked for rational points.

This method has been exploited in [CHM95] and by Noam Elkies, to give the (partly unpublished) results listed below.

$N_2 > 128$
$N_3 > 100$

$N_4 > 126$
$N_5 > 132$
$N_6 > 146$
$N_9 > 180$
$N_{10} > 192$
$N_{45} > 781$

Different techniques yield results for all genera: the bound below was obtained independently by Brumer and Mestre

$$N_g \geq 16(g+1)$$

A description of Mestre's method can be found in [CHMjr], while that of Brumer is in [C1]. Although the methods of Brumer and Mestre look very different from each other, Brumer discovered a precise relation between them. Notice also that the above bound is not sharp, in fact Elkies got $N_{45} \geq 781$ and, of course, $781 > 16(45+1)$.

4.2 A function field analog.

The uniformity questions that we so far discussed only over number fields, remain interesting for curves over function fields. The rest of the paper is devoted to this case, and we will from now on consider curves defined over a complex function field $L = \mathbb{C}(B)$ where B is a nonsingular projective variety. The main conjecture to keep in mind is the following (discussed for number fields in the previous sections):

Conjecture. *For any complex function field L and any integer $g \geq 2$ there exists a number $B_g(L)$ such that for every $C \in \mathcal{C}_g(L)$ we have $|C(L)| \leq B_g(L)$*

A revision of Parshin's Lemma 8, using the moduli theory of curves, yields that it actually holds uniformly; in other words, if a uniform Shafarevich conjecture holds, then a uniform Mordell conjecture holds.

Therefore a natural attempt is to focus on sets of curves over function fields, looking for their uniformity properties.

The Shafarevich problem can be approached within the framework of moduli theory. Given a curve C of genus g defined over L (always considered modulo L-isomorphism), pick a family $f : X \longrightarrow B$ whose generic fiber is L-isomorphic to C. We can associate to such a family its classifying map (see 1.4)

$$\gamma_f : B \dashrightarrow M_g;$$

since such a map is uniquely determined by the curve C (if $f' : X' \longrightarrow B'$ is another family having C as generic fiber, then $\gamma_{f'}$ coincides with γ_f on some open subset of B), we shall also use the notation $\gamma_C := \gamma_f$.

In this situation sets of maps are easier to handle than sets of curves, we shall therefore mimick Definition 7 and set

Definition 23. Let $L = \mathbb{C}(B)$ be a complex function field.

1. Denote by $\mathcal{M}_g(L)$ the set of rational maps $\gamma : B \dashrightarrow M_g$ such that there exists a curve $C \in \mathcal{C}_g(L)$ having γ as classifying map.

2. Let S be a closed subset of B. Denote by $\mathcal{M}_g(L, S)$ the subset of $\mathcal{M}_g(L)$ parametrizing classifying maps of curves in $\mathcal{C}_g(L)$ having good reduction outside of S

We view $\mathcal{M}_g(L)$ as the set of L-rational points of the moduli functor (or moduli stack) of smooth curves of genus g.

There is a canonical surjection

$$\begin{array}{rccc} \mu : & \mathcal{C}_g(L) & \longrightarrow & \mathcal{M}_g(L) \\ & C & \mapsto & \gamma_C \end{array}$$

and $\mathcal{M}_g(L, S)$ is the image via μ of the locus of curves in $\mathcal{C}_g(L)$ for which there exists a model $f : X \longrightarrow B$ having nonsingular fiber over every point of B which does not lie in S.

The crucial observation is that the map μ has finite fibers; more precisely, there exists a universal constant $D(g)$ (independent of L) such that the fibers of μ have cardinality at most $D(g)$. This follows from foundational moduli theory: given n different curves $C_1, \ldots, C_n$ all defined over L and having the same classifying map

$$\gamma_{C_1} = \ldots = \gamma_{C_n} = \gamma : B \dashrightarrow M_g$$

there exists a finite extension L' of L over which the n curves above become all isomorphic ([DM69]). Since the curves are distinct (by assumption) we find n different L' automorphisms for a curve defined over L'. If n is too large, this gives a contradiction with the fact that the number of automorphism of a curve of genus g is bounded by a constant $D(g)$ that depends only on g (and not on the field of definition).

The conclusion is that uniform boundedness of sets of curves is equivalent to uniform boundedness of rational points of the moduli functor. Combining this with lemma 8 (which, as we said, holds uniformly) we obtain that uniform boundedness of rational points on curves would follow from uniform boundedness of rational points of the moduli functor.

The general result that one obtains in this framework is the following ([C03]):

Theorem 24. *Fix integers $g \geq 2$, d and s. There exist numbers $M_g(d, s)$ and $B_g(d, s)$ such that, for every projective variety $B \subset \mathbb{P}^r$ of degree at most d, and for every closed subset S of B having degree at most s, we have*

1. $|\mathcal{M}_g(L, S)| \leq M_g(d, s)$ *($L = \mathbb{C}(B)$ as always).*

2. *For every $C \in \mathcal{C}_g(L, S)$ we have $|C(L)| \leq B_g(d, s)$.*

The two bounds above do not depend on the dimension of B or of the ambient projective space.

Proof. We outline the argument. The first step is the case $\dim B = 1$, which is the difficult part. One proves that there exists a bound $P_g(q, s)$ such that for every curve B of genus at most q, and for every subset S of B having at most s points, we have $|\mathcal{M}_g(\mathbb{C}(B), S)| \leq P_g(q, s)$.

This is the crucial point, achieved by constructing a moduli space over M_g for the maps parametrized by all sets $\mathcal{M}_g(L, S)$ (see [C02]).

As we explained before, this gives (is equivalent to) an identical statement for $\mathcal{C}_g(L, S)$. Then one applies the uniform version of the Kodaira-Parshin trick to obtain the proof of part 2 of the theorem for $\dim B = 1$.

The higher dimensional statements are both obtained by "slicing" B by hyperplanes to obtain one-dimensional sections. To such curves (whose genus is obviously bounded above by a function of d only) one applies the one-dimensional result to conclude the proof. ■

Remark. In a very different framework, Miyaoka (in [Mi89]) used the theory of vector bundles on surfaces to obtain effective results of the same type as part 2, for rational points of curves defined over one-dimensional function fields and satisfying some other geometric assumptions.

An approach to the Shafarevich conjecture, using Chow varieties rather than moduli spaces, has been recently applied by Heier ([He]) in his Ph.D. thesis, to produce an effective expression for the bound $P_g(q, s)$ above (on the cardinality of $\mathcal{C}_g(\mathbb{C}(B), S)$ when B has dimension 1 and genus q).

It is at this point natural (at least to my opinion) to expect analogous boundedness phenomena to hold in the arithmetic case; for instance, a statement like the following is not known: *there exists a number $M_g(d, s)$ such that for every number field K with $[K : \mathbb{Q}] \leq d$, for every set S of places of K, with $|S| \leq s$, for every curve $C \in \mathcal{C}_g(K, S)$, we have $|C(K)| \leq M_g(d, s)$.*

We conclude with a few very special cases in which the Conjecture holds. Let $L = \mathbb{C}(B)$ be our complex function field.

1. Denote by $\mathcal{C}_g^2(L)$ the set of curves in $\mathcal{C}_g(L)$ having good reduction in codimension 1, that is
$$\mathcal{C}_g^2(L) := \bigcup_{codim_B S \geq 2} \mathcal{C}_g(L, S)$$

2. Denote by $\mathcal{C}_g^{max}(L)$ the set of curves C in $\mathcal{C}_g(L)$ having maximal variation of moduli, that is, the associated moduli map γ_C is generically finite.

Then we have

Proposition 25. *1. There exists a number $B_g^2(L)$ such that for every curve $C \in \mathcal{C}_g^2(L)$ we have $|C(L)| \leq B_g^2(L)$.*

2. Fix $g \geq 24$ and let $trdeg_{\mathbb{C}} L = 3g - 3$. There exists a number $B_g^{max}(L)$ such that for every curve $C \in \mathcal{C}_g^{max}(L)$ we have $|C(L)| \leq B_g^{max}(L)$.

Both statements follow trivially from the fact that the sets of curves in question are finite. The finiteness of $\mathcal{C}_g^2(L)$ can be proved in the same spirit as Theorem 24 (see [C03]).

The finiteness of $\mathcal{C}_g^{max}(L)$ is obtained using the well known theorem of Harris-Mumford and Eisenbud-Harris that $\overline{M_g}$ is of general type for $g \geq 23$ (see [HM98] for proof and references). Combining such a fact with theorem 9 we obtain that the set of rational dominant maps from B to M_g is finite. Now, $\mathcal{C}_g^{max}(L)$ has a natural surjection (the restriction of μ) onto such a set, hence it is finite.

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