ON CERTAIN UNIFORMITY PROPERTIES OF CURVES OVER FUNCTION FIELDS

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Abstract. A uniform version of the Shafarevich Conjecture for function fields (Theorem of Parshin-Arakelov) is proved, together with a uniform version of the geometric Mordell conjecture (Theorem of Manin). Such results are generalized to base varieties of arbitrary dimension (i.e. function fields of arbitrary transcendence degree).

1. Introduction

The problem. Let B be a non-singular, projective, complex curve and let S be a finite subset of points of B. The theorem below was conjectured by Shafarevich and proved by Parshin (assuming $S = \emptyset$) and Arakelov (general case) in [P] and [A].

Theorem P-A. *Fix $g \geq 2$. There exist only finitely many non-isotrivial families of smooth curves of genus g over $B - S$.*

Some motivation for the Shafarevich conjecture was that, as Parshin shows in [P], the Mordell conjecture for function fields (Theorem M below, which had already been proved by Manin [Ma]) can be obtained as a corollary of the above Theorem P-A. Moreover, the same implication holds true for the number field analog of the conjectures of Shafarevich and Mordell.

Theorem M. *Let K be a function field and let X be a non-isotrivial curve of genus at least 2 defined over K. Then X has finitely many K-rational points.*

Denote by $X(K)$ the set of K-rational points of X. Let V be a smooth variety over $\mathbb{C}$ and let T be a closed subset of V. Define $F_g(V,T)$ as the set of equivalence classes of non-isotrivial families of smooth curves of genus $g \geq 2$ over $V - T$ (see below).

The problem studied in this paper is how $|F_g(V,T)|$ and $|X(K)|$ vary as V, T, X and K vary. The case where $V = B$ is a curve and $K = \mathbb{C}(B)$ is its function field is treated first, to obtain uniform versions of Theorem P-A and Theorem M. More precisely it will be proved that $|F_g(B,S)|$ and $|X(K)|$ are uniformly bounded by a function of g, the genus of B and the cardinality of S (Theorems 3.1 and 4.2).

The generalizations to higher dimensional bases are obtained by specializing to suitable curves in V. We obtain a uniformity result where the varieties V are quantified by their degree in some fixed projective space (3.5 and 4.4). From this a more intrinsic uniformity statement is obtained for so-called "canonically polarized varieties", that is, smooth varieties V with ample canonical bundle K_V. A natural invariant quantifying canonically polarized varieties is the Hilbert polynomial $h(x)$ of the canonical polarization (i.e. $h(n) = \chi(K_V^{\otimes n})$). In such a case the cardinalities of $F_g(V,T)$

and $X(K)$ are bounded by a function of g, the Hilbert polynomial $h(x)$, and the canonical degree of T (see Theorems 3.4 and 4.3).

The proof of the Shafarevich conjecture is made of two parts, usually called "boundedness" and "rigidity". We proceed like in [P], giving essentially a relative version of Parshin's proof of the boundedness part. To obtain our uniform version, we combine this with techniques from the modern theory of stable curves and their moduli. This paper is about families of curves; for the analog of the Shafarevich conjecture for families of higher dimensional varieties, see [F] and the very recent preprint [BV].

We prove the uniform version of Manin's Theorem as a corollary of such a uniform version of the Theorem of Parshin-Arakelov. Another perspective on this issue was taken by Miyaoka in [Mi], where he indicated how an effective version of the Mordell conjecture could be obtained from the Sakai inequality. Our approach here is completely different.

It is interesting to compare our results with analogous problems in the arithmetic setting. The uniformity conjecture for rational points of curves over number fields remains open, despite much work on the subject, in recent years.

Uniformity Conjecture. *Fix $g \geq 2$. There exists a number $N(g)$ such that for any number field F there exist at most finitely many curves of genus g, defined over F, and having more than $N(g)$ F-rational points.*

This is often called the "strong" Uniformity conjecture, to distinguish it from its weaker variant, where the number field is fixed. At the moment, they are both open (nor was anyone able to give evidence for their being false). Interest in such types of problems was revived after it was proved in [CHM] that the Uniformity conjectures are a consequence of the so-called Lang diophantine conjectures (also open) about the distribution of rational points on varieties of general type.

Preliminaries. We work over $\mathbb{C}$. Throughout the paper g, q and s will be non-negative integers such that $g \geq 2$; B will be a smooth, projective complex curve of genus q and $S \subset B$ a subset of cardinality s (unless otherwise specified).

By a "stable" (respectively "semistable") curve we mean a curve which is stable (respectively semistable) in the sense of Deligne and Mumford [DM]. If C is a semistable curve, we denote by C^{st} the stable model of C. M_g (respectively $\overline{M}_g$) denotes the coarse moduli space of smooth (resp. stable) curves of genus g. The divisor of $\overline{M}_g$ parametrizing singular curves is denoted by Δ. A nodal curve is said to have genus g if its arithmetic genus is g.

If $f : X \longrightarrow V$ is a morphism, we denote by X_v the fiber of X over $v \in V$; if $\phi : X \longrightarrow Y$ is a V-map, we denote by $\phi_v : X_v \longrightarrow Y_v$ the restriction of ϕ to the fiber over $v \in V$. By a "family of curves" we mean a projective, flat morphism $X \longrightarrow V$ of integral varieties such that X_v is a curve for every $v \in V$, smooth for v in some dense, open subset of V. A family of curves is called "isotrivial" if all of its smooth fibers are isomorphic. A "family of stable curves of genus g" is a family of curves $X \longrightarrow V$ all of whose fibers are stable curves of genus g.

Given two families of curves $f : X \longrightarrow U$ and $h : Y \longrightarrow U$ we say that they are "equivalent" if there is an automorphism α of U and an isomorphism $\epsilon : X \longrightarrow Y$ such that $h \circ \epsilon = \alpha \circ f$. This notion is really necessary only in the case $\dim U > 1$, where U would typically be $V - T$ (see the introduction).

A fibration is a proper morphism $f : X \longrightarrow V$ such that X and V are smooth integral varieties and the fiber X_v is smooth for v in some open dense subset of V. A semistable fibration is a fibration all of whose fibers are semistable curves. Given a semistable fibration $f : X \longrightarrow V$ its stable model is the family $Y \longrightarrow V$ of stable curves such that for every $v \in V$ $Y_v = X_v^{st}$. Y is uniquely determined by the mapping $X \longrightarrow \mathbb{P}(f_* \omega_f^{\otimes n}) \longrightarrow B$ for $n \geq 3$ (where ω_f is the relative dualizing sheaf).

Given a polynomial $p(x)$ we denote by $\mathrm{Hilb}^{p(x)}[\mathbb{P}^r]$ the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^r$ having Hilbert polynomial equal to $p(x)$. Given a scheme Z and a scheme $Y \longrightarrow Z$ quasi-projective and flat over Z, we denote by $\mathrm{Hilb}_Z[Y]$ the Hilbert scheme parametrizing closed, flat Z-subschemes of Y. Given $X \longrightarrow Z$ projective and flat over Z, the functor $\mathcal{H}om_Z[X,Y]$ of Z-morphisms is represented by a scheme $\mathrm{Hom}_Z[X,Y]$ which is described as an open subscheme of $\mathrm{Hilb}_Z[X \times_Z Y]$.

Let X be a curve of genus g, B a curve of genus q and $S \subset B$ a set of cardinality s; let $\mathrm{f}(X,B,S)$ be a numerical function of X,B and S. It will be convenient to use the following terminology.

Definition. f is "uniformly bounded" if there exists a constant $\mathtt{c}(g,q,s)$ such that for all X, B, S as above we have $\mathrm{f}(X,B,S) \leq \mathtt{c}(g,q,s)$. We say that a (finite) set is uniformly bounded if its cardinality is uniformly bounded.

An example of a uniformly bounded set which is relevant to this paper is the following. Let $e(g)$ be some positive integer depending on g, let $C(B,S,e(g))$ be the set of finite coverings of B having degree at most equal to $e(g)$, and ramified only over S. Then $C(B,S,e(g))$ is a uniformly bounded set (this would fail in positive characteristic). The claim, of course, follows from Riemann's theorem and from the fact that an element in $C(B,S,e(g))$ is given by a group homomorphism from the fundamental group of $B \setminus S$ to the symmetric group on $e(g)$ elements.

Let K be a field containing $\mathbb{C}$, let C be a smooth, irreducible curve defined over K. Define

$$\mathrm{Q}_K(C) := \{(\phi, D) : D \text{ smooth curve of genus } \geq 2 \text{ over } K,\ \phi : C \to D \text{ finite map}\}.$$

So that if $g(C) \leq 1$, then $\mathrm{Q}_K(C)$ is empty. If instead $g(C) = 2$ then $\mathrm{Q}_K(C) = \mathrm{Aut}_K(C)$. We recall a famous theorem about $\mathrm{Q}_{\mathbb{C}}(C)$:

Theorem D. *$\mathrm{Q}_{\mathbb{C}}(C)$ is a finite set. More precisely, there exists a number $\mathtt{D}_{\mathbb{C}}(g)$ such that for every curve $C \in M_g$ we have $|\mathrm{Q}_{\mathbb{C}}(C)| \leq \mathtt{D}_{\mathbb{C}}(g)$.*

The first assertion is the classical theorem of De Franchis ([D]). The second is a modern refinement, which actually comes with an explicit bound for $\mathtt{D}_{\mathbb{C}}(g)$. See [AP] and [K] for current records and related references. We observe that a non-effective proof of the existence of a uniform bound $\mathtt{D}_{\mathbb{C}}(g)$ (which is all we need here), given the theorem of De Franchis, is a simple application of the existence of Hilbert schemes. We need a uniform version over function fields.

Proposition 1.1. *Fix $g \geq 2$. There exists a number $\mathtt{D}(g)$ such that for any function field K and for any smooth curve X of genus g defined over K we have $|\mathrm{Q}_K(X)| \leq \mathtt{D}(g)$.*

Remark. We shall see that $\mathtt{D}(g) = \mathtt{D}_{\mathbb{C}}(g)$.
Proof. Let $(\phi^i, Y^i) \in \mathrm{Q}_K(X)$ be distinct elements, for $i = 1, ..., N$. By contradiction, suppose that $N > \mathtt{D}_{\mathbb{C}}(g)$. We need the following
Claim. *Let B be an integral scheme, let X, Y^1 and Y^2 be families of stable curves of genus at least 2 over B. For $i = 1,2$, let $\phi^i : X \longrightarrow Y^i$ be finite B-morphisms. Then there exists an open subset U of B such that $\forall b \in U$, $\phi^1_b \neq \phi^2_b$.*
If $\phi^1 = \phi^2$ we take $U = \emptyset$, hence we shall assume that the two morphisms are distinct. The Lemma is also clear if Y^1 and Y^2 are not fiberwise isomorphic. Suppose then that $\forall b \in B$ we have $Y^1_b \cong Y^2_b$. This does not imply that the total spaces are isomorphic. Suppose first that they are, so that $Y^1 \cong Y^2 \cong Y$ then ϕ^i determines a map $\Phi^i : B \longrightarrow \mathrm{Hom}_B[X,Y]$ defined by $\Phi^i(b) = \phi^i_b$. Clearly $\phi^1_b = \phi^2_b$ if and only if $\Phi^1(b) = \Phi^2(b)$, since $\Phi^1(B) \cap \Phi^2(B)$ is closed we are done.

Now the general case. Given that Y^1 and Y^2 are fiberwise isomorphic, we can construct a natural covering of B where they become isomorphic. This will be defined by the scheme over

B $\mathrm{Isom}_B[Y^1, Y^2] \longrightarrow B$, representing the functor of B-isomorphisms of the two families. It is well known that $\mathrm{Isom}_B[Y^1, Y^2] \longrightarrow B$ is a finite and unramified map (cf [DM]). Let B' be a connected component dominating B and let X', Z^1, Z^2 be the base changed families of X Y^1, and Y^2 respectively. Now $Z^1 \cong Z^2$ over B'. Let $\chi^i : X' \longrightarrow Z^i$ for $i = 1, 2$ be the pull-back of ϕ^i. By the previous discussion, there exists an open subset $U' \subset B'$ where χ^1 and χ^2 have distinct restrictions on the fibers. Since B' is finite over B, the image of U' in B is open, and we are done.

Let then $U_{i,j}$ be the open subset of B such that $\forall b \in U_{i,j}$ we have $\phi^i_b \neq \phi^j_b$. For $i \neq j$ we have that $U_{i,j}$ is not empty. Let $U := \bigcap_{1 \leq i < j \leq N} U_{i,j}$, of course U is open and non empty. Let $b \in U$, then by construction we have N distinct elements (ϕ^i_b, Y^i_b) in $Q_{\mathbb{C}}(X_b)$. But $N > \mathtt{D}_{\mathbb{C}}(g)$, while by the refinement of the theorem of De Franchis over the complex numbers we know that $|Q_{\mathbb{C}}(X_b)| \leq \mathtt{D}_{\mathbb{C}}(g)$. We reached a contradiction. $\square$

2. Boundedness results for stable families

The following set up will be fixed throughout the section. Let $f : X \longrightarrow B$ be a non-isotrivial family of stable curves of genus g. The surface X needs not be smooth: it will be normal with isolated singularities of type $xy = t^n$ at the nodes of the fibers. We shall consider the relatively minimal desingularization $r : Y \longrightarrow X$ of X, with its natural map $h = f \circ r : Y \longrightarrow B$. Then $h : Y \longrightarrow B$ is a semistable fibration. We shall denote by $S \subset B$ the set parametrizing singular fibers (of both f and h), and we denote $s := |S|$.

A large part of the material of this section is adapted from [P] and [S]. Recall that, by Arakelov's Theorem, ω_f is an ample line bundle (since the family is not isotrivial). It easily follows that ω_h is a big and nef line bundle on Y.

All the results of this section hold uniformly for any smooth complex curve B of genus q and for any stable, non-isotrivial family $f : X \longrightarrow B$ of curves of genus g having at most s singular fibers. We shall not repeat this in every statement.

Lemma 2.1. *Let $g \geq 2, q, e$ be fixed integers. There exists a number $\mathtt{n}(g, q, e)$ such that for any line bundle $E \in \mathrm{Pic}B$ with $\deg E = e$ and for any $n \geq \mathtt{n}(g, q, e)$:*

(1) $\omega_f^{\otimes n} \otimes f^*E$ *is ample*;
(2) $\omega_h^{\otimes n} \otimes h^*E$ *is nef and big.*

Remark. The bound above does not depend on the number of singular fibers.
Proof. Clearly (2) follows from (1). To prove (1) we use the Nakai Moishezon criterion (which holds for a normal surface). Denote $L_n = \omega_f^{\otimes n} \otimes f^*E$; it is easy to see that if $e \geq 0$ then L_n is ample for all $n \geq 1$, hence we can assume that $e < 0$. We have

$$L_n^2 = n^2\omega_f^2 + 2ne(2g - 2)$$

and, since $\omega_f^2 \geq 1$, if $n > -2e(2g - 2)$ we have that $L_n^2 > 0$.

Let now C be an integral curve in X, then $\omega_f \cdot C > 0$ because ω_f is ample. If C is contained in a fiber, $L_n \cdot C = \omega_f^{\otimes n} \cdot C > 0$.

Suppose now that $f_{|C} : C \longrightarrow B$ is a finite map. We need the fact that there exists $\mathtt{n}'(g, q, e)$ such that if $m \geq \mathtt{n}'(g, q, e)$, then L_m is effective.

Before proving this, let us see how it implies the Lemma. Let $L_m = \mathcal{O}(\sum r_i C_i) \otimes \mathcal{O}(V)$ such that V is an effective divisor supported on the fibers of f, $r_i > 0$ and C_i are irreducible curves that cover B via f.

If $C \neq C_i$, then $L_m \cdot C \geq 0$, as sum of intersections of effective, irreducible divisors with distinct supports; therefore $L_{m+h} \cdot C > 0$ for every $h \geq 1$.

If $C = C_i$, then $\deg\{f_{|C} : C \longrightarrow B\} \leq m(2g-2)$, hence

$$L_m \cdot C \geq m(\omega_f \cdot C) + em(2g-2) > em(2g-2).$$

Therefore if $h > -em(2g-2)$ we have that $L_{m+h} \cdot C > 0$. It remains to prove that there is $\mathtt{n}'(g,q,e)$ such that L_m is effective for $m \geq \mathtt{n}'(g,q,e)$.

For $n \geq 2$, $R^1 f_* L_n = 0$ hence

$$\chi(L_n) = \chi(f_* L_n) = \binom{n}{2}(12\lambda - \delta) + \lambda - (2n-1)(-e+q-1)(g-1)$$

where we used the formula $\deg f_* \omega_f^n = \binom{n}{2}(12\lambda - \delta) + \lambda$ for $n \geq 2$ proved in [Mu] (see [Mu] for the definition of λ and δ). Then, if $q \neq 0$,

$$\chi(L_n) \geq \frac{n^2}{2}(12\lambda - \delta) - n\big(\frac{12\lambda - \delta}{2} + 2(q-e+1)(g-1)\big)$$

(recall that we are assuming $e < 0$). We conclude that if $n > 1 + 4(q-e+1)(g-1)$ and $q \geq 1$, we have $\chi(L_n) > 0$; we leave it to the reader to find the bound if $q = 0$.

It is easy to see that $h^2(X, L_n) = 0$ (see the proof of 2.3(c) below) and hence $H^0(X, L_n) \neq 0$ for such a choice of n. $\square$

Let ω_X be the dualizing sheaf of X. We need a basic

Lemma 2.2. *Let $f : X \longrightarrow B$ be a family of stable curves. Let $A \in \mathrm{Pic}X$ be a relatively very ample line bundle. Then A is very ample on X if, for any line bundle $M \in \mathrm{Pic}B$ such that $\deg M = -1, -2$, we have that $A \otimes \omega_X^{-1} \otimes f^* M$ is nef and big.*

Proof. It is well known that A is very ample if and only if for every pair of (not necessarily distinct) points p and q in X, we have that $H^1(X, A \otimes \mathcal{I}_p) = H^1(X, A \otimes \mathcal{I}_p \otimes \mathcal{I}_q) = 0$. Let F_p and F_q be the fibers of f passing through p and q respectively. We have exact sequences:

$$0 \longrightarrow A \otimes \mathcal{I}_{F_p} \longrightarrow A \otimes \mathcal{I}_p \longrightarrow \mathcal{I}_p \otimes A_{|F_p} \longrightarrow 0$$

and

$$0 \longrightarrow A \otimes \mathcal{I}_{F_p}\mathcal{I}_{F_q} \longrightarrow A \otimes \mathcal{I}_p \otimes \mathcal{I}_q \longrightarrow \mathcal{I}_q\mathcal{I}_p \otimes A_{|F_p \cup F_q} \longrightarrow 0.$$

Since A is relatively very ample, we have $H^1(F_p, \mathcal{I}_p \otimes A_{|F_p}) = H^1(F_p \cup F_q, \mathcal{I}_q\mathcal{I}_p \otimes A_{|F_p \cup F_q}) = 0$ for every p and q in X. Therefore, to show that A is very ample, it is enough to prove that

$$H^1(X, A \otimes \mathcal{I}_{F_p}) = H^1(X, A \otimes \mathcal{I}_{F_p} \otimes \mathcal{I}_{F_q}) = 0,$$

that is, it is enough to prove that for every line bundle M on B of degree equal to -1 or -2 we have $H^1(X, A \otimes f^* M) = 0$.

By Serre's duality, $H^1(X, A \otimes f^* M) \cong H^1(X, \omega_X \otimes (A \otimes f^* M)^{-1})^{-1}$. We conclude by the Kawamata-Viehweg vanishing theorem, which can be applied to our surface X (see [V] 2.35). According to this theorem, if L is nef and big, then $H^1(X, L^{-1}) = 0$; we apply it to $L = \omega_X^{-1} \otimes A \otimes f^* M$, and conclude the proof. $\square$

Corollary 2.3. *Fix $g \geq 2$ and q. There exists an integer $\mathrm{n}_0(g,q)$ such that for $n \geq \mathrm{n}_0(g,q)$,*

(a) *ω_f^n is very ample on X.*
(b) *$H^1(X, \omega_f^n) = H^1(Y, \omega_h^n) = 0$.*
(c) *$H^2(X, \omega_f^n) = H^2(Y, \omega_h^n) = 0$.*

Remark. (a) is in [S] (Lemme 9). We include a proof as it takes two lines after what we already proved. Observe also that from Matsusaka's big theorem one derives that there exists a constant k_0 depending on the Hilbert polinomial $\chi(\omega_f^n)$ such that ω_f^n is very ample for $n \geq k_0$. This is a weaker version of (a) below, as k_0 depends on g, q and s.
Proof. By Lemma 2.2, ω_f^n is very ample if $\omega_f^{n-1} \otimes f^*(M \otimes K_B^{-1})$ is nef and big for every M of degree -1 or -2 on B. Then we apply Lemma 2.1 and obtain (a).

(b) follows by Lemma 2.1and Kawamata-Viehweg vanishing theorem (see above) applied to the line bundle $\omega_f^{n-1} \otimes f^* K_B^{-1}$. Similarly for ω_h.

It is enough to prove (c) for ω_h. By Serre duality

$$H^2(X, \omega_h^n) = H^0(X, \omega_h^{\otimes(1-n)} \otimes h^* K_B)$$

now $\omega_h^{\otimes(1-n)} \otimes h^* K_B \cdot X_b < 0$ for $n \geq 2$. Since $X_b^2 = 0$ we obtain that $\omega_h^{\otimes(1-n)} \otimes h^* K_B$ cannot possibly be effective. $\square$

Lemma 2.4. *ω_h^2 and $\chi(\mathcal{O}_Y)$ are uniformly bounded.*

Proof. The first part of the statement is in [S], Théorème 3, where more precisely it is proven that $\omega_h^2 \leq 8g(g-1)(q-1+\frac{s}{2})$. The uniform boundedness of $\chi(\mathcal{O}_Y)$ follows from [P] Proposition 1. $\square$

Corollary 2.5. *Fix n so that 2.3 holds. Then there exists a number $r_0 = \mathrm{r}_0(g,q,s)$ such that $h^0(X, \omega_f^n) \leq r_0$.*

Proof. It is sufficient to show the statement for ω_h^n. We have by Riemann-Roch and by 2.3

$$h^0(Y, \omega_h^n) = \chi(\omega_h^n) = \chi(\mathcal{O}_Y) + \frac{n^2\omega_h^2 - n(\omega_h \cdot K_Y)}{2}$$

and $(\omega_h \cdot K_Y) = \omega_h^2 + \omega_h \cdot h^* K_B = \omega_h^2 + (2g-2)(2q-2) > (2g-2)(2q-2)$. We conclude by the previous result. $\square$

The above analysis can be now applied to perform a useful construction. Fix $n \geq \mathrm{n}_0(g,q)$ so that 2.3 holds, and let $d = n(2g-2)$. Given $f : X \longrightarrow B$, the line bundle ω_f^n determines an embedding

$$\phi : X \hookrightarrow \mathbb{P}^r = \mathbb{P}(H^0(X, \omega_f^n)^*)$$

where $r \leq \mathrm{r}_0(g,q,s)$ as in Corollary 2.5 above. For every $b \in B$ the fiber X_b is mapped isomorphically by ϕ to a curve spanning a linear subspace $\mathbb{P}^k \subset \mathbb{P}^r$, where $k = d - g$. In fact $H^1(X, \omega_f^n \otimes \mathcal{I}_{X_b}) = 0$, because we have chosen n so that $\omega_f^n \otimes \mathcal{I}_{X_b}$ is ample (see the proof of 2.3).

Let $p(x) = dx - g + 1$, fix $\mathbb{P}^r$ as above and consider the Hilbert scheme $\mathrm{Hilb}^{p(x)}[\mathbb{P}^r]$ with its universal curve $\mathcal{Y} \subset \mathrm{Hilb}^{p(x)}[\mathbb{P}^r] \times \mathbb{P}^r$; we now introduce the following subscheme of $\mathrm{Hilb}^{p(x)}[\mathbb{P}^r]$

$$J^r := \{h \in \mathrm{Hilb}^{p(x)}[\mathbb{P}^r] : Y_h \text{ is stable and spans a } \mathbb{P}^k,\ \omega_{Y_h}^{\otimes n} \cong \mathcal{O}_{\mathbb{P}^k}(1) \otimes \mathcal{O}_{Y_h}\}.$$

where Y_h denotes the fiber over $h \in \mathrm{Hilb}^{p(x)}[\mathbb{P}^r]$ of the universal curve. We denote by $\Delta^r \subset J^r$ the locus of singular curves.

Lemma 2.6. *J^r is a smooth, irreducible, quasi-projective variety of dimension $r(k+1)+k+3g-3$. Δ^r is a reduced divisor with normal crossings singularities.*

Proof. If $k = r$ we denote $J^r = H_g$; this is the subset of the Hilbert scheme parametrizing stable curves embedded by the n-canonical linear series; since $r = d - g$ these curves are not contained in any hyperplane. The geometry of H_g is well understood, in fact the geometric invariant theory quotient of H_g by the natural action of $SL(r+1)$ is identified with $\overline{M}_g$. Results of [G] give our statement for $r = k$. If $k < r$ there is a natural surjective morphism

$$\lambda : J^r \longrightarrow \mathbb{G}(k, r)$$

where $\mathbb{G}(k, r)$ is the Grassmannian of k-dimensional linear subspaces in $\mathbb{P}^r$. The map λ maps the point h to the linear span of the fiber Y_h in $\mathbb{P}^r$. Its fibers are all isomorphic to H_g. To conclude just notice that $\mathbb{G}(k, r)$ is a smooth and irreducible projective variety of dimension $(k+1)(r-k)$. $\square$

The embedding $\phi : X \hookrightarrow \mathbb{P}^r$ defined above defines a morphism $\Phi : B \longrightarrow J^r$. Fix a projective model $B \times J^r \subset \mathbb{P}^s$ and let Γ be the graph of Φ, so that $\Gamma \subset B \times J^r \subset \mathbb{P}^s$.

Lemma 2.7. *There exists $\mathsf{e}_0(g, q, s)$ such that* $\deg \Gamma \leq \mathsf{e}_0(g, q, s)$.

This is essentially Lemma 5 in [P]. To adjust Parshin's proof to our situation one needs to consider heights with respect to the polarization given by ω_f^n rather than K_X^6. Effective versions of such a result are given in [EV], for example, or in the more recent [T] where other relevant references are given.

3. Uniform versions of the theorem of Parshin-Arakelov

One-dimensional base. Fix B a curve and $S \subset B$ as usual.

Definition. Let $F_g(B, S)$ to be the set of equivalence classes of non-isotrivial fibrations $f : X \to B$, with X a smooth, relatively minimal surface and, for all $b \notin B$, X_b is a smooth curve of genus g.

By the existence and unicity of relatively minimal models over B for fibrations of genus g over an open subset of B, this definition is consistent with the one given in the introduction for $F_g(V, T)$. The goal of this section is the following result

Theorem 3.1. *Fix $g \geq 2$, q and s. There exists a number $\mathsf{P}(g, q, s)$ such that for any curve B of genus q and for any subset $S \subset B$ of cardinality s, we have $|F_g(B, S)| \leq \mathsf{P}(g, q, s)$.*

There are few cases where the theorem is already known (see [B]): if $q = 0$ and $s \leq 2$ then $F_g(\mathbb{P}^1, S) = \emptyset$ for every g; if $q = 1$ and $S = \emptyset$ again for every g, $F_g(B, \emptyset) = \emptyset$. This fact allows us to ignore the equivalence relation for families over B (all automorphism groups being uniformly bounded).

The theorem is proved in three steps. First we show that for every pair (B, S) as above, the set of non-constant morphisms $\psi : B \longrightarrow \overline{M}_g$ such that $\psi^{-1}(\Delta) \subset S$ and such that ψ is the moduli map (see below) for some family of stable curves over B, is uniformly bounded. Then we show that for any fixed ψ the set of families having ψ as moduli map is uniformly bounded. Finally, we reduce the general case to the stable case.

The first part is a consequence of the construction of a moduli space for moduli maps of curve of genus q to $\overline{M}_g$. Let N be a coarse moduli space (for example, $N = M_g$ or $N = \overline{M}_g$)

Definition. A morphism $\psi : B \longrightarrow N$ is a "moduli map" if ψ is non-constant and if there exists a family $f : X \longrightarrow B$ of objects parametrized by N such that $\psi : b \mapsto [X_b]$ is the moduli map of f.

Clearly, if N is a fine moduli space, every non-constant map is a moduli map. We denote $M[B, g]$ (respectively $\overline{M}[B, g]$) the set of moduli maps from B to $N = M_g$ (respectively $N = \overline{M}_g$). Let S be a finite subset of B, denote by $\overline{M}[(B, S), g]$ the set of all moduli maps ψ of B to $\overline{M}_g$ such that $\psi^{-1}(\Delta) \subset S$. That is, $\overline{M}[(B, S), g]$ is the set of moduli maps with degeneracy locus contained in S; it is a finite set, by the theorem of Parshin-Arakelov.

Proposition 3.2. *Fix $g \geq 2$, q, and s. There exists a number $\mathtt{P}^0(g, q, s)$ such that for any curve B of genus q and any finite subset $S \subset B$ of cardinality s we have $|M\big[(B, S), g\big]| \leq \mathtt{P}^0(g, q, s)$.*

Proof. In the relative setting, let $h : \mathcal{B} \longrightarrow Z$ be a projective morphism of integral varieties (we can assume it to be smooth). We shall say that a $Z-$morphism $\Psi : \mathcal{B} \longrightarrow N \times Z$ is a moduli map over Z if there exists a family $\mathcal{X} \longrightarrow \mathcal{B}$ such that $\forall z \in Z$ the restriction $\psi_z : B_z \longrightarrow N \times \{z\}$ is the moduli map for the family $X_z \longrightarrow B_z$.

We are mainly interested in the following variant. Given $h : \mathcal{B} \longrightarrow Z$ as above, consider the special case in which h is a family of smooth curves, assume in addition that h has s non-intersecting sections $\sigma_i : Z \longrightarrow \mathcal{B}$ with $h \circ \sigma_i = \mathrm{id}_Z$ for every $i = 1, ..., s$. Denote $\underline{\sigma} = \{\sigma_1, ..., \sigma_s\}$, $\mathcal{S} = \cup \sigma_i(Z) \subset \mathcal{B}$, for $z \in Z$ denote by S_z the finite subset of B_z given by $\{\sigma_1(z), ..., \sigma_s(z)\}$. Consider the Z-set $M_Z[(\mathcal{B}, \underline{\sigma}), g] \longrightarrow Z$ whose fiber over $z \in Z$ is the set $M[(B_z, S_z), g]$.

As a set, the above space of moduli maps with fiberwise fixed degeneracies is naturally a subset

$$M_Z[(\mathcal{B}, \underline{\sigma}), g] \subset \mathrm{Hom}_Z\big[\mathcal{B}, \overline{M}_g \times_Z Z\big].$$

We are going to show that $M_Z[(\mathcal{B}, \underline{\sigma}), g]$ is a finite union of irreducible, quasi-projective varieties; to do that we construct an auxiliary parameter scheme A which has a natural morphism to $\mathrm{Hom}_Z\big[\mathcal{B}, \overline{M}_g \times_Z Z\big]$, whose image we shall identify with $M_Z[(\mathcal{B}, \underline{\sigma}), g]$.

Let J^r be as defined at the end of the previous section and let $\mathcal{J}^r := J^r \times Z$. Define $\mathrm{Hom}_Z\big[(\mathcal{B}, \underline{\sigma}), \mathcal{J}^r\big]$ to be the subscheme of $\mathrm{Hom}_Z[\mathcal{B}, \mathcal{J}^r]$ parametrizing morphisms Φ such that $\Phi^{-1}(\Delta^r \times Z) \subset \mathcal{S}$ and such that $\forall z \in Z$ the restriction $\phi_z : B_z \longrightarrow J^r \times \{z\}$ is not constant. There is an inclusion of schemes

$$\mathrm{Hom}_Z\big[(\mathcal{B}, \underline{\sigma}), \mathcal{J}^r\big] \hookrightarrow \mathrm{Hilb}_Z\big[\mathcal{B} \times_Z \mathcal{J}^r\big]$$

associating to a morphism its graph. Since $\mathcal{B}$ is projective over Z, for a fixed integer e we can consider the quasi-projective scheme over Z parametrizing morphisms of degree e, that is we consider

$$\mathrm{Hom}^e_Z\big[(\mathcal{B}, \underline{\sigma}), \mathcal{J}^r\big] \hookrightarrow \mathrm{Hilb}^{ex-q+1}_Z\big[\mathcal{B} \times_Z \mathcal{J}^r\big].$$

Let

$$A := \bigcup_{r \leq r_0} \bigcup_{e \leq e_0} \mathrm{Hom}^e_Z\big[(\mathcal{B}, \underline{\sigma}), \mathcal{J}^r\big]$$

where $e_0 = \mathtt{e}_0(g, q, r)$ and $r_0 = \mathtt{r}_0(g, q, r)$ are as in 2.7 and 2.5 respectively. A is thus a scheme of finite type over Z. We now show that A carries a natural family of moduli maps to $\overline{M}_g$, with fixed degeneracies. This follows by the functorial properties of the various moduli spaces involved; we have a diagram of objects an morphisms over Z:

$$\begin{array}{ccccccc}
 & & A & \overset{i}{\hookrightarrow} & \mathrm{Hilb}_Z\big[\mathcal{B}\times_Z\mathcal{J}\big] & & \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{G} & \overset{j}{\longrightarrow} & \mathcal{C} & \hookrightarrow & \mathcal{B}\times\mathcal{J}\times\mathrm{Hilb}_Z\big[\mathcal{B}\times_Z\mathcal{J}\big] \\
 & & \uparrow & & \uparrow & & \uparrow \\
A\times\mathcal{B} & \longleftarrow & \mathcal{W} & \longrightarrow & \mathcal{D} & \hookrightarrow & \mathcal{B}\times\mathcal{Y}\times\mathrm{Hilb}_Z\big[\mathcal{B}\times_Z\mathcal{J}\big]
\end{array}$$

where $\mathcal{J}$ is the disjoint union of $\mathcal{J}^r$, for $r\leq r_0$, $\mathcal{Y}\longrightarrow\mathcal{J}$ is the universal family, and the Hilbert scheme is defined correspondingly. We denoted $\mathcal{G}=i^*\mathcal{C}$ and $\mathcal{W}=j^*\mathcal{D}$. The above diagram yields

$$\begin{array}{ccccc}
\mathcal{W} & \longrightarrow & A\times_Z\mathcal{B} & & \\
\downarrow & & \downarrow & & \\
A & \longrightarrow & Z & \longleftarrow & \mathcal{B}
\end{array}$$

so that for every $a\in A$, if z is the image of a in Z and B_z is the fiber of $\mathcal{B}$ over z, the restriction $W_a\longrightarrow\{a\}\times B_z$ is a family of stable curves of genus g with singular fibers over $(a,\sigma_i(z))$ for $i=1,....,s$. Therefore the above diagram can be viewed as an element of $\mathcal{H}om_Z\big[\mathcal{B},\overline{M}_g\times_Z Z\big](A)$ and as such it determines a unique morphism

$$\beta: A\longrightarrow \mathrm{Hom}_Z\big[\mathcal{B},\overline{M}_g\times_Z Z\big].$$

Now $\mathrm{Im}\beta$ is a finite union of quasi-projective schemes over Z, and, by construction, it is a subset of the set of moduli maps with fixed degeneracies, that is

$$\mathrm{Im}\beta\subset M_Z[(\mathcal{B},\underline{\sigma}),g].$$

To show that the above inclusion is in fact an equality of sets we apply the results of the previous section. Given a moduli map $\psi: B\longrightarrow\overline{M}_g$, let $f: X\longrightarrow B$ be the corresponding family of stable curves. By the construction at the end of section 2 we can associate to such data a morphism from B to J^r (canonically up to an action of the $PGL(r+1)$); this determines a unique point in $A\subset\mathrm{Hom}_Z\big[(\mathcal{B},\underline{\sigma}),\mathcal{J}^r\big]$ whose image via β is the given $[\psi]$.

To conclude the proof of the proposition, we shall apply the above construction to a suitable $h:\mathcal{B}\longrightarrow Z$ with s sections. Suitable means that Z must be some fine moduli space for curves of genus q with s marked points. There are various choices for this, for example, $Z=M^N_{q,s}$, the moduli space of smooth curves of genus q, with s marked points, and level $N\geq 3$-structure. For every pair (B,S) as above there exists a (not necessarily unique) point $z\in Z$ corresponding to such a pair (that is, $B_z=B$ and $\{\sigma_1(z),....,\sigma_n(z)\}=S$). Now $M_Z[(\mathcal{B},\underline{\sigma}),g]$ is a finite union of irreducible quasi-projective varieties over Z and its fiber over the point z is $M\big[(B,S),g\big]$ (for every $z\in Z$). By the theorem of Parshin-Arakelov, $M_Z[(\mathcal{B},\underline{\sigma}),g]$ has finite fibers over Z; therefore there exists an upper bound $\mathtt{P}^0(g,q,s)$ on the cardinality of the fibers. $\square$

This concludes the first part of the program to prove Theorem 3.1. The discussion that follows is needed in order to handle curves with automorphisms. We start by a general fact about moduli maps over a general base scheme, which will be useful again later. Let V be a fixed quasi-projective variety of any dimension, assume that V is irreducible and smooth, let $\psi: V\longrightarrow\overline{M}_g$ be a morphism such that $\mathrm{Im}\psi\cap M_g\neq\emptyset$.

Definition. Let F_ψ be the set of all equivalence classes of stable familes $f: X\longrightarrow V$ such that for every $v\in V$, X_v is stable and $\psi(v)=[X_v]$.

Lemma 3.3. *For every V and ψ as above, $|F_\psi| \leq \mathtt{D}(g)$.*

Proof. Let $f^i : X^i \longrightarrow V$, $i = 1, ..., n$, be distinct elements of F_ψ, and assume, by contradiction, that $n > \mathtt{D}(g)$ (see 1.1). The families f^i are fiberwise isomorphic. We construct a covering $V' \longrightarrow V$ over which they are all isomorphic to a certain family $X \longrightarrow V'$. Let $V_i := \mathrm{Isom}_V(X^i, X^n)$ so that V_i has a natural, finite morphism to V (Theorem 1.11 of [DM]). Replacing V_i with a connected component dominating V will not alter the argument. Let $X_1^i := X^i \times_V V_1$ for $i = 1, ..., n$. Then $X_1^1 \cong X_1^n$ over V_1. Moreover, for each i there is a natural commutative diagram

$$\begin{array}{ccc} X_1^i & \longrightarrow & X^i \\ {\scriptstyle f_1^i}\downarrow & & \downarrow{\scriptstyle f^i} \\ V_1 & \longrightarrow & V \end{array}$$

and over V_1 we have $n-1$ different familes f_1^i. Iterate this construction to get $V' \longrightarrow V$. We have a commutative diagram for every $i = 1, ..., n$

$$\begin{array}{ccccc} X & \xrightarrow{\phi^i} & Y^i & \longrightarrow & X^i \\ \downarrow & & \downarrow & & \downarrow \\ V' & = & V' & \longrightarrow & V \end{array}$$

where $Y^i = X^i \times_V V'$ and ϕ^i is an isomorphism. By assumption, $\phi^i \neq \phi^j$, we thus obtain $n > \mathtt{D}(g)$ distinct elements in $\mathrm{Q}_K(X)$ (K being the function field of V'), which is not possible, by 1.1. $\square$

Proof of Theorem 3.1. Denote by $F_g(B,S)^s$ the subset of $F_g(B,S)$ consisting of semistable fibrations. We use curves with level structure to show that there is a map

$$\lambda : F_g(B,S) \longrightarrow \cup F_g(B',S')^s$$

where the union is for (B',S') varying in a uniformly bounded set, and the fibers are uniformly bounded sets.

Fix $N \geq 3$ and let $\overline{M}_g^N$ be the moduli space of stable curves of genus g with level N structure. This is a fine moduli space, endowed with a universal family $\mathcal{C}_g^N \longrightarrow \overline{M}_g^N$ and with a finite morphism to $\overline{M}_g$. Let $[f : X \longrightarrow B] \in F_g(B,S)$, we have a commutative diagram

$$\begin{array}{ccccc} & & Z & \longrightarrow & \mathcal{C}_g^N \\ & & \downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & B' & \xrightarrow{\psi'} & \overline{M}_g^N \\ \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow \\ X & \xrightarrow{f} & B & \xrightarrow{\psi} & \overline{M}_g \end{array}$$

where f' is the rigidification of f by level N structures, so that ρ is a finite morphism which is étale with group $Sp(g, \mathbb{Z}/n\mathbb{Z})$ away from S. Therefore ρ, which depends on f, varies in a uniformly bounded set as f varies. Let $S' = \rho^{-1}(S)$.

Away from singular fibers, ψ' is the moduli map of f', so that the pull back Z of $\mathcal{C}_g^N$ to B' is naturally birational to X' over B' (in fact Z is isomorphic to X' away from S'). The family $Z \longrightarrow B'$ is a stable reduction of f.

Now we define λ by sending the fibration $f : X \longrightarrow B$ to the relatively minimal semistable fibration obtained by resolving the singularities of the stable family $Z \longrightarrow B'$. The previous discussion shows that the union defining the range of λ is over (B', S') varying in a uniformly bunded set; in addition, the fibers of λ are uniformly bounded, by 1.1.

It remains to show that $F_g(B, S)^s$ is uniformly bounded. There is a surjection

$$\begin{array}{ccc} F_g(B,S)^s & \longrightarrow & M\big[(B,S),g\big] \\ f & \mapsto & \psi_f \end{array}$$

where $\psi_f(b) = [X_b^{st}]$, that is to say, ψ_f is the moduli map of the stable model of $X \longrightarrow B$. By 3.2 $M\big[(B,S),g\big]$ is uniformly bounded. Since the smooth, relatively minimal model is unique, the fiber of the above map over $\psi_f = \psi$ is naturally in bijective correspondence with F_ψ. But F_ψ is uniformly bounded, by 3.3 hence the Theorem is proved. $\square$

Higher dimensional base. We apply the results of Theorem 3.1 to families of curves parametrized by smooth, projective, integral varieties V of arbitrary dimension. The first issue is how to quantify the base spaces, so as to obtain analog uniformity statements. We proceed as follows: fix a polynomial $h \in \mathbb{Q}[x]$ such that $h(\mathbb{Z}) \subset \mathbb{Z}$. Let $M_{h(x)}$ be the moduli space of smooth projective varieties V having ample canonical line bundle K_V and such that $\chi(K_V^{\otimes n}) = h(n)$ It is well known (see [V]) that $M_{h(x)}$ is a quasi-projective scheme, called the moduli space of "canonically polarized" varieties. For example, if $\deg h = 1$, in order for $M_{h(x)}$ not to be empty one needs that there exist an integer $q \geq 2$ such that $h(x) = (2q-2)x - q + 1$ and $M_{h(x)}$ is equal to the moduli space of curves of genus q. We have the following generalization of Theorem 3.1:

Theorem 3.4. *Fix non-negative integers $g \geq 2$ and s and fix a polynomial h as above. There exists a number $\mathtt{P}(g, h, s)$ such that for any $V \in M_{h(x)}$, for any closed $T \subset V$ such that $K_V^{\dim V - 1} \cdot T \leq s$ there exist at most $\mathtt{P}(g, h, s)$ equivalence classes of non-isotrivial families of smooth curves of genus g over $V - T$.*

Proof. The proof follows from a more general uniformity statement for polarized varieties (3.5 below). First, we use Matsusaka's big Theorem and fix an integer $\nu = \nu(h)$ depending on the Hilbert polynomial h such that for every $n \geq \nu$ and for every $V \in M_{h(x)}$, the line bundle $K_V^{\otimes n}$ is very ample. By the vanishing Theorem of Kodaira, it follows that $h^0(V, K_V^{\otimes n}) = h(n)$ for $n > \nu$ and for every $V \in M_{h(x)}$. Fix then $n > \nu$ and let $r = h(n) - 1$; we fix a projective space $\mathbb{P}^r$ such that for every $V \in M_{h(x)}$ the line bundle $K_V^{\otimes n}$ determines a projective model of V in $\mathbb{P}^r$ having degree d determined by h (in fact, d equals the leading coefficient of h, multiplied by $\dim V!$). The Theorem follows from the following

Lemma 3.5. *Fix g, d, r and s integers; there exists a number $\mathtt{H}(g, d, r, s)$ satisfying the following property. For any smooth projective variety $V \subset \mathbb{P}^r$ of degree d and for any subvariety $T \subset V$ of degree s, there exist at most $\mathtt{H}(g, d, r, s)$ equivalence classes of non-isotrivial families of smooth curves of genus g over $V - T$.*

Proof. There exists a number $q = \mathtt{q}(d, r)$ such that V is covered by smooth curves B of genus at most q passing through any of its points. In fact such B are just one-dimensional linear sections of V, that is, they are obtained by intersecting V with the correct number of generic hyperplanes in $\mathbb{P}^r$. By a theorem of Castelnuovo (see [ACGH] III.2), the genus of a smooth curve of degree d in $\mathbb{P}^r$ is bounded above by a function $\mathtt{q}(d, r)$ of d and r. In addition, any such B intersects T in at most s points. Let

$$P = \max_{q' \leq q, s' \leq s} \{\mathtt{P}(g, q', s')\}$$

(see 3.1 for the definition of $\mathtt{P}(g,q,s)$). Let $U = V - T$. We show that U has at most P moduli maps to M_g. By contradiction, suppose that for $n > P$ there exist $\psi^1, ..., \psi^n$ different such moduli maps. Thus for every i there exists a non-isotrivial family of smooth curves $X^i \longrightarrow U$ having $\psi^i : U \longrightarrow M_g$ as moduli map. There exists a dense open subset $U' \subset U$ such that for every u in U' we have $\psi^i(u) \neq \psi^j(u)$. By the previous discussion, there exists a smooth curve B of genus at most q such that the restrictions of X^i to B are distinct, non-isotrivial families of smooth curves over $B - S$, where $S := (T \cap B)_{red}$. Let $f_i : Y^i \longrightarrow B$ be the smooth relatively minimal model of $X^i_{|B}$ over B. Then $[f_i] \in F_g(B,S)$ and $[f_i] \neq [f_j]$. Since S has cardinality at most s, $F_g(B,S)$ has at most P elements; we reached a contradiction.

To conclude the proof of the Lemma, fix $\psi : U \longrightarrow M_g$ any such a moduli map; Lemma 3.3 shows that there exist at most $\mathtt{D}(g)$ families of curves over U having ψ as moduli map. We can conclude that the number of equivalence classes of non-isotrivial families of smooth curves of genus g over U is bounded above by $\mathtt{H}(g,d,r,s) = \mathtt{D}(g) \cdot P$. $\square$

Remark. In particular we obtained, of course, that for every pair (V,T), the set $F_g(V,T)$ is finite.

4. Uniformity of rational points of curves over function fields

Parshin's construction revisited. We deal here with a well known construction of Parshin, the goal of which was to show that the Shafarevich conjecture implies the Mordell conjecture. Fix a curve B, a finite subset $S \subset B$ and let K be the field of rational functions of B; let $f : X \longrightarrow B$ be a non-isotrivial fibration having smooth fiber away from S.

Remark. By the existence and unicity of minimal models for (non-ruled) smooth surfaces, giving $f : X \longrightarrow B$ is the same as giving the generic fiber of it. Therefore we shall abuse notation slightly and denote by $X(K)$ the set of sections of f, identifying it with the set of rational points.

The goal of Parshin's method is to obtain a map: $\alpha_X : X(K) \longrightarrow \cup F_{g'}(B', S') =: P_X$ such that the union on the right is over a finite set. The construction is such that α_X has finite fibers. The Shafarevich conjecture (i.e. the Parshin-Arakelov Theorem) gives that $F_g(B,S)$ is always a finite set. Therefore $X(K)$ is finite.

We will revise Parshin's method to show that, as X and B vary among curves with fixed genus g and q, and S varies among sets of fixed cardinality s:

(1) the fibers of α_X are uniformly bounded,

(2) the union defining P_X is over a uniformly bounded set.

By Theorem 3.1 the sets $F_g(B,S)$ are uniformly bounded. We shall hence conclude that rational points are uniformly bounded.

Let $\sigma \in X(K)$ be a K-rational point of X, so that σ can be viewed as a section $\sigma : B \longrightarrow X$ of $f : X \longrightarrow B$. Let $\Sigma = \sigma(B)$ be the image curve. We are going to construct a finite covering $\rho : B' \longrightarrow B$, ramified only over S, a fibration $Y' \longrightarrow B'$ with a commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{\rho} & B \end{array}$$

such that every fiber $Y'_{b'}$ is a finite covering of X_b ($b = \rho(b')$) which ramifies only over $\sigma(b)$. The goal is to carry this out so that all numerical invariants of the new family are bounded with q, g and s. We here provide a brief account, see [S] or [P] for the missing details.

The section σ gives a map $u : X \longrightarrow \mathrm{Pic}^0 X/B$ defined by $u(x) = x - \sigma(f(x))$. Multiplication by 2 in $\mathrm{Pic}^0 X/B$ yields a covering of X over B, which, away from singular fibers, is étale of degree 2^{2g}. Let Y be a connected component of such covering, we get a new family of curves of genus $g(Y) \leq 1 + 2^{2g}(g-1)$; for $b \notin S$, each fiber Y'_b is an étale covering of the corresponding fiber X_b. Let D be the preimage of Σ in Y. Then D is étale of degree at most 2^{2g} over $B \setminus S$.

There exists a covering $B_1 \longrightarrow B$ of degree at most $2^{2g}(2^{2g}-1)$, ramified only over S such that on the relatively minimal desingularization Y_1 of $B_1 \times_B Y$ there are two disjoint sections σ_1 and σ_2 with $\Sigma_i = \mathrm{Im}\sigma_i$ mapping to D via the natural morphism $Y_1 \longrightarrow Y$.

One can construct a further covering $B_2 \longrightarrow B_1$, ramified only over the preimage of S, such that on $Y_2 = Y_1 \times_{B_1} B_2$ the line bundle $\mathcal{O}_{Y_2}(\Gamma_1 + \Gamma_2)$, given by the the pull-back of $\Sigma_1 + \Sigma_2$ admits a square root. Such a B_2 is obtained by first mapping B_1 to $\mathrm{Pic}^0 Y_1/B_1$ via $b \mapsto \sigma_1(b) - \sigma_2(b)$, and then by considering the multiplication by 2 map $\mathrm{Pic}^0 Y_1/B_1 \longrightarrow \mathrm{Pic}^0 Y_1/B_1$ and defining $B_2 = B_1 \times_{\mathrm{Pic}^0 Y_1/B_1} \mathrm{Pic}^0 Y_1/B_1$ (replacing B_2 by a connected component dominating B_1 if necessary).

This ensures (after a further degree-2 base change ramified over singular fibers) that there exists a double covering $Y_3 \longrightarrow Y_2$ having branch locus $\Gamma_1 + \Gamma_2$. Finally, let Y' be the relatively minimal resolution of Y_3 over B_2, let $B' = B_2$, let $\rho : B' \longrightarrow B$ be the covering map and let $S' = \rho^{-1}(S)$. Then $f' : Y' \longrightarrow B'$ is the family we wanted; by construction the fibers of Y' away from S' are coverings of the fibers of f of degree dividing 2^{2g+1}, having two simple ramification points lying over Σ. Hence $g(Y') \leq 2 + 2^{2g+1}(g-1)$, let $\mathtt{g}(g) = 2 + 2^{2g+1}(g-1)$. ρ is a covering of degree bounded above by a function of only g and s, and ramified only over S; let $S' = \rho^{-1}(S)$, then the pair B', S' belongs to a uniformly bounded set I of cardinality at most $\mathtt{c}(g,q,s)$ (some constant depending only on g, q and s). Moreover, the genus of B' is uniformly bounded by a constant $\mathtt{q}(g,q,s)$. We shall define $\alpha_X(\sigma) = [f' : Y' \longrightarrow B']$.

Lemma 4.1. *There exists a number $\mathtt{A}(g,q,s)$ such that for every curve B of genus q, for every finite subset $S \subset B$ of cardinality s and for every non-isotrivial fibration $X \longrightarrow B$ of genus g, having smooth fiber outside of S, the fibers of the map α_X have cardinality at most $\mathtt{A}(g,q,s)$.*

Proof. Let $[f' : Y' \longrightarrow B']$ be an element in $\mathrm{Im}\alpha_X$ which lies in $F_{g'}(B', S')$. As we have seen, a K-rational point $\sigma \in \alpha_X^{-1}([f'])$ determines a commutative diagram of objects and morphisms

$$\begin{array}{ccccccc} Y' & \longrightarrow & Y \times_B B' & \longrightarrow & Y & \longrightarrow & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ B' & = & B' & \xrightarrow{\rho} & B & = & B \end{array}$$

let $\rho' : Y' \longrightarrow X$ be the composition map above, then σ is uniquely determined as the branch locus of ρ'. Now, ρ varies in a uniformly bounded set, as we have just seen. By 1.1 so does ρ'. $\square$

Theorem 4.2. *Fix $g \geq 2$ and q. There exists an integer $\mathtt{M}(g,q,s)$ such that for every curve B of genus q, for every $S \subset B$ with $|S| \leq s$, and for every non-isotrivial curve X of genus g, defined over $K = \mathbb{C}(B)$ and having smooth fiber outside of S, we have $|X(K)| \leq \mathtt{M}(g,q,s)$*

Proof. Maintaining the same notation as before, by the Theorem of Parshin-Arakelov, P_X is a finite set; by the previous Lemma we get: $|X(K)| \leq \mathtt{A}(g,q,s) \cdot |P_X|$. It remains to show that P_X is unformly bounded. Recall that $P_X = \cup F_{g'}(B', S')$ where the union is over a uniformly bounded set. More precisely, the union is taken over
(a) $g' \leq \mathtt{g}(g)$
(b) (B', S') varies in a set of cardinality at most $\mathtt{c}(g,q,s)$
(c) $q' = g(B') \leq \mathtt{q}(g,q,s)$

(d) $s' = |S'| \leq \mathtt{s}(g,q,s)$

By Theorem 3.1, there exists $\mathtt{P}(g,q,s)$ such that for every curve B of genus q and every S of cardinality s we have $|F_g(B,S)| \leq \mathtt{P}(g,q,s)$.

Hence letting $P^M(g,q,s) := \max\{\mathtt{P}(g',q',s')\}$ for g', q' and s' satisfying (a) (c) and (d) above, we conclude $|P_X| \leq P^M(g,q,s) \cdot \mathtt{g}(g) \cdot \mathtt{c}(g,q,s)$ $\square$

Fields of higher transcendence degree. We obtain as a corollary a uniformity result about rational points of curves over function fields of any dimension. In [Ma] it is shown that working by induction on the transcendence degree, the Mordell conjecture for curves over function fields of any dimension follows from the one-dimensional case. Here we show that, by specializing to certain curves, from the uniform version given by 4.2 one can obtain uniformity statements for function fields of any dimension. We use the same notation introduced at the end of the previous section, to state and prove Theorem 3.4.

Theorem 4.3. *Fix non-negative integers $g \geq 2$ and s and fix a polynomial $h \in \mathbb{Q}[x]$. There exists a number $\mathtt{M}(g,h,s)$ such that for any $V \in M_{h(x)}$, for any closed $T \subset V$ such that $K_V^{\dim V-1} \cdot T \leq s$ and for any non-isotrivial curve X of genus g, defined over K, having smooth fiber away from T we have $|X(K)| \leq \mathtt{M}(g,h,s)$, where K is the field of functions of V.*

Proof. Exactly as in the proof of 3.4, one reduces the proof to the statement below.

Lemma 4.4. *Fix g, d, r and s integers; there exists a number $\mathtt{N}(g,d,r,s)$ such that for any smooth irreducible projective variety $V \subset \mathbb{P}^r$ of degree d, for any subvariety $T \subset V$ of degree s and for any non-isotrivial family of curves $f : X \longrightarrow V$ such that for every $v \notin T$ the fiber X_v is a smooth curve of genus g , there exist at most $\mathtt{N}(g,d,r,s)$ rational sections of f.*

Proof. As in the first part of the proof of 3.5, let $q = \mathtt{q}(d,r)$ be the maximum genus of a curve of degree d in $\mathbb{P}^r$. Then V is covered by smooth curves of genus at most q through any of its points. Any such a curve, call it B, intersects T in at most s points.

Let $\mathtt{N}(g,d,r,s) := \max_{q' \leq q, s' \leq s}\{\mathtt{M}(g,q',s')\}$ (see 4.2) and let us show that f has at most $\mathtt{N}(g,d,r,s)$ sections. By contradiction, let $n > \mathtt{N}(g,d,r,s)$ and suppose that there exist $\sigma_1, ...\sigma_n$ distinct sections of f. Observe that there exists an open, dense subset U of V such that σ_i is regular on it and for every $u \in U$, we have $\sigma_i(u) \neq \sigma_j(u)$ if $i \neq j$.

There exists a curve B of genus at most q as above, such that $B \cap U \neq \emptyset$ and such that the restriction $f_B : X_B \longrightarrow B$ is not isotrivial. f_B is a family of curves of genus g having smooth fiber away from the set $(T \cap B)_{red}$ of cardinality at most s. By construction, the restrictions of $\sigma_1, ..., \sigma_n$ to B give n distinct sections of f_B. This is a contradiction to 4.2. $\square$

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References

[AP] A.Alzati, P.Pirola: *Some remarks on the de Franchis Theorem* Ann.Univ.Ferrara -Sez. VII - Sc.Mat. Vol.XXXVI, 45-52 (1990).

[A] S.Ju. Arakelov: *Families of algebraic curves with fixed degeneracies* Izv. Akad. Nauk. SSSR Ser. Mat.35 (1971) no. 6, 1277-1302.

[ACGH] E.Arbarello, M.Cornalba, P.Griffiths, J.Harris: *Geometry of algebraic curves* Grundlehren der mathematischen Wissenschaften 267 Springer.
[B] A.Beauville: *Exposé n. 6* in *Séminaire sur les pinceaux des courbes de genre au moins deux* Astérisque 86 (1981).
[BV] E.Bedulev, E.Viehweg: *On the Shafarevich conjecture for surfaces of general type over function fields* Preprint (1999) AG 9904124.
[CHM] L.Caporaso, J.Harris, B.Mazur: *Uniformity of rational points* Journal of American Mathematics Society, Vol. 7, N. 3, January 97 p. 1-33.
[DF] M.De Franchis: *Un teorema sulle involuzioni irrazionali* Rend.Circ.Mat Palermo 36 (1913), 368.
[DM] P.Deligne, D.Mumford: *The irreducibility of the space of curves of given genus* Publ Math. IHES 36 (1969) 75-120.
[EV] H.Esnault, E.Viehweg: *Effecive bounds for semipositive sheaves and the height of points on curves over complex function fields* Compositio math. 76 (1990) 69-85.
[F] G.Faltings: *Arakelov's Theorem for abelian varieties* Invent.Math. 73 (1983) 337-348.
[G] D.Gieseker: *Lectures on moduli of curves* TIFR Lecture Notes 69 (1982) Springer.
[HM] J.Harris, I.Morrison *Moduli of curves* Graduate texts in Math. 187 (1998) Springer
[K] E.Kani: *Bounds on the number of non rational subfields of a function field* Invent. Math. 85 (1986) pp. 185-198.
[Ma] Y.Manin: *Rational points of algebraic curves over function fields* Izv. Akad. Nauk. 27 (1963), 1395-1440.
[Mi] Y.Miyaoka: *Themes and variations on inequalities of Chern classes* Sûgaku 41 (1989), no. 3, 193-207.
[Mu] D.Mumford: *Stability of projective varieties* L'Enseignement Mathématique 23 (1977) p. 39-110.
[P] A.N.Parshin: *Algebraic curves over function fields* Izv. Akad. Nauk. SSSR Ser. Mat.32 (1968) no. 5, 1145-1170.
[S] L.Szpiro: *Exposé n. 3* in *Séminaire sur les pinceaux des courbes de genre au moins deux* Astérisque 86 (1981).
[T] S. Tan:*Height inequalities of algebraic points on curves over functional fields* J.reine angew. Math. 481 (1995) 123-135.
[V] E. Viehweg: *Quasi-projective moduli for polarized manifolds* Ergebnisse der Mathematik (30) Springer.