Brill-Noether theory over the moduli space of stable curves.

Lucia Caporaso

Outline and references for 4 lectures given for GAEL
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Please note that the reference list is far from complete.

Contents

1. Lecture 1

1.1. **Review of classical Brill-Noether Theory.**

Picard scheme. Picard variety, $\text{Pic}^d X$, for a smooth curve X.
Brill-Noether varieties $W_d^r(X) \subset \text{Pic}\, X$.
Riemann-Roch theorem (and Riemann Theorem in particular).
Special divisor and line bundles. Clifford Theorem.
Reference: [ACGH, Chapter 1, 3, 5]

1.2. **Stable curves.**

Moduli space $\overline{M}_g$.
Combinatorial description of $\overline{M}_g$.
Dual graph of a nodal curve.
Properties of the strata: dimension, irreducibility.
References: [GAC], [HM98].
Further references: [DM69], [MM64], [GIT], [Gie82].

1.3. Picard scheme of a nodal curve.

Pull-back of line bundles under the normalization map

Exact sequence defining the generalized Jacobian as a semiabelian variety.

Failure of properness for the generalized Jacobian.

Examples of singular curves with a projective generalized Jacobian.

Curves of compact type.

References: [K84], [C94], [BLR], [C10].

2. Lecture 2

2.1. Linear series on semistable curves.

Basic estimate on the dimension of a complete linear series on a nodal curve; [C07, Example 2.6 - Prop. 2.1.1].

The Picard scheme of a reducible curve has infinitelly many, hence too many, components. Failure of Riemann and Cliffor theorem for reducible curves; [C08, Example 2.6 and Section 4.3].

Further references: [EM02], [C08a]

2.2. The Picard functor.

Generalities; [C10], [GIT], [BLR, Chapter 8]

Failure of separation. Twisters. Examples. [C05, Section 3]

Application: characterization of uni-nodal hyperelliptic curves.

Hyperelliptic and weakly hypereliptic curves; [C08, Section 5]

Degree class group (see [C94]) and its various incarnations: as component group of the Néron model (see [N64], [R70]), as complexity group of the dual graph (see [OS79], [C05, appendix], [BMS08]) as Jacobian group of a graph.

3. Lecture 3

3.1. The compactification of the Universal Picard variety over $\overline{M}_g$.

Basic facts: GIT construction, integrality, normality of $\overline{P^d_g}$;

Definition of balanced line bundles.

Geometric versus Good GIT quotient; [C94].

Dichotomy of the stacky picture; [C05], [M07].

Examples.

Further reading about compactified Picard scheme: [I78], [AK80], [AK79], [OS79], [Sim94], [E01], [F04], [M08], [BFV].

3.2. The case $d = g - 1$.

Explicit description of compactified jacobians in case $d = g - 1$; [B77], [Ale04], [C07].

Compactified Brill-Noether varieties $\overline{W^r_{d,X}} \subset \overline{P^d_X}$.

Riemann theorem for balanced line bundles; [C08, Thm 2.3].

Clifford theorem for balanced line bundle; [C08, Thms 4.2, 4.4, 4.11].

Counterexamples to Clifford's inequality for balanced line bundles of degree ≥ 5, [C08, Sect. 4.3].

4. Lecture 4

4.1. Hyperelliptic curves.

Combinatorial description of weakly hyperelliptic curves; [C08, Thm 5.9].

4.2. **Theta divisor.**

Abel map and Theta divisor; [AK79], [C07], [CE07].

Geometry of the Theta divisor: [S94], [E97], [Ale04], [C07].

Compactified Torelli map from $\overline{M}_g$ to the moduli space of stable semi-abelic pairs; [Ale02], [Ale04].

4.3. **Torelli theorem for stable curves.**

Failure of injectivity of the Torelli map; [Nam80, Thm 9.30(vi)] . Fibers of the Torelli map; [V03], [CV09].

Open problems.

4.4. **Riemann Singularity Theorem.** Extension of the Riemann Singularity Theorem to integral nodal curves; [CMK].

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Dipartimento di Matematica, Università Roma Tre, Largo S. Leonardo Murialdo 1, 00146 Roma (Italy)

E-mail address: caporaso@mat.uniroma3.it