GONALITY OF ALGEBRAIC CURVES AND GRAPHS

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Abstract. We define d-gonal weighted graphs using "harmonic indexed" morphisms, and prove that a combinatorial locus of $\overline{M_g}$ contains a d-gonal curve if the corresponding graph is d-gonal and of Hurwitz type. Conversely the dual graph of a d-gonal stable curve is equivalent to a d-gonal graph of Hurwitz type. The hyperelliptic case is studied in details. For $r \geq 1$, we show that the dual graph of a (d, r)-gonal stable is the underlying graph of a tropical curve admitting a degree-d divisor of rank at least r.

Contents

1. Introduction and Preliminaries

1.1. **Introduction.** In this paper we study the interplay between the theory of linear series on algebraic curves, and the theory of linear series on graphs.

A smooth curve C is d-gonal if it admits a linear series of degree d and rank 1; more generally, C is (d, r)-gonal if it admits a linear series of degree d and rank r. A stable, or singular, curve is defined to be (d, r)-gonal, if it is the specialization of a family of smooth (d, r)-gonal curves. This rather

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unwieldy definition is due to the fact that the divisor theory of singular curves is quite complex; for example, every reducible curve admits infinitely many divisors of degree d and rank r, for every d and $r \geq 0$. Moreover characterizing (d, r)-gonal curves is a well known difficult problem.

On the other hand, the moduli space of Deligne-Mumford stable curves, $\overline{M_g}$, has a natural stratification into "combinatorial" loci, parametrizing curves having a certain weighted graph as dual graph. It is thus natural to ask whether the existence in a combinatorial locus of a (d, r)-gonal curve can be detected uniquely from the corresponding graph and its divisor theory.

In fact, in recent times a theory for divisors on graphs has been set-up and developed in a purely combinatorial way, revealing some remarkable analogies with the algebro-geometric case; see [BdlHN97], [BN09], [BN07] for example. One of the goals of this paper is to contribute to this developement; we give a new definition for morphisms between graphs, which we call *indexed morphisms*, and then introduce *harmonic indexed morphisms*. Our definition is inspired by the theory of admissible coverings developed by J. Harris and D. Mumford in [HM82], and generalizes the combinatorial definition of *harmonic morphisms* given by M. Baker, S. Norine and H. Urakawa in [BN09] and [U00] for weightless graphs; this is why we use the word "harmonic". Harmonic indexed morphisms have a well defined degree, and satisfy the Riemann-Hurwitz formula with an effective ramification divisors.

We say that a graph is d-gonal if it admits a non-degenerate harmonic indexed morphism, ϕ, of degree d to a tree; furthermore we say that it is of Hurwitz type if the Hurwitz existence problem naturally associated to ϕ has a positive solution; see Definition 2.1 for details. In particular, if $d \leq 3$ every d-gonal graph is of Hurwitz type. Then we prove the following:

Theorem 1.1. *If (G, w) is a d-gonal weighted stable graph of Hurwitz type, there exists a (stable) d-gonal curve whose dual graph is (G, w). Conversely, the dual graph of a stable d-gonal curve is equivalent to a d-gonal graph of Hurwitz type.*

This Theorem follows immediatly from the more general Theorem 2.11, whose proof combines the theory of admissible coverings with properties of harmonic indexed morphisms.

In the opposite direction, and for all $r \geq 1$, Theorem 3.1 states that the dual graph of a (d, r)-gonal curve always has a refinement admitting a divisor of rank r and degree d. The proof of this theorem uses different methods than the previous one: the theory of stable curves, and a generalization, from [AC11], of Baker's specialization lemma [B08, Lemma 2.8].

Testing whether a graph admits a divisor of given degree and rank involves only a finite number of steps, and can be done by a computer; hence Theorem 3.1 yields a handy necessary condition for a curve to be (d, r)-gonal.

This theorem has also consequences on tropical curves. In fact the moduli space of tropical curves of genus g, M_g^{trop}, has a partition indexed by stable weighted graphs exactly as $\overline{M_g}$. Using our results we obtain that if a combinatorial stratum of $\overline{M_g}$ contains a (d, r)-gonal curve, so does the corresponding stratum of M_g^{trop}; see Subsection 3.1 for more details. The connections between the divisor theories of algebraic and tropical curves have been object of much interest in recent years; in fact some closely related issues are currently being investigated, under a completely different perspective, in a joint project of O. Amini, M. Baker, E. Brugallé and J. Rabinoff. We refer also to [BPR11], [BMV11], [C11], [Ch11] and [LPP] for some recent work on the relation between algebraic and tropical geometry.

The paper is organized in four sections; the first recalls definitions and results from algebraic geometry and graph theory needed in the sequel, mostly from [HM82], [GAC], [BN07] and [AC11]. In Section 2 we study the case $r = 1$ and prove Theorem 2.11 (and Theorem 1.1). The next section studies the case $r \geq 1$ and extends the analysis to tropical curves; the main result here is Theorem 3.1. In Section 4 we concentrate on the hyperelliptic case, and develop the basic theory by extending some of the results of [BN09]. It turns out that for this case the analogies between the algebraic and the combinatorial setting are stronger; see Theorem 4.8.

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1.2. **Graphs and dual graphs of curves.** Details about the forthcoming topics may be found in [GAC] and [C11].

Unless we specify otherwise, by the word "curve" we mean reduced, projective algebraic variety of dimension one over the field of complex numbers; we always assume that our curves have at most nodes as singularities. The genus of a curve is the arithmetic genus.

The graphs we consider, usually denoted by a "G" with some decorations, are connected graphs (no metric) admitting loops and multiple edges, unless differently stated. For the reader's convenience we recall some basic terminology from graph theory. Our conventions are chosen to fit both the combinatorial and algebro-geometric set up. For a graph G we denote by $V(G)$ the set of its vertices, by $E(G)$ the set of its edges and by $H(G)$ the set of its half-edges. The set of half-edges comes with a fixed-point-free involution whose orbits, written $\{h, \overline{h}\}$, bijectively correspond to $E(G)$, and with a surjective *endpoint* map $\epsilon : H(G) \to V(G)$. For $e \in E(G)$ corresponding to the half-edges $h, \overline{h}$ we often write $e = [h, \overline{h}]$.

A *loop-edge* is an edge $e = [h, \overline{h}]$ such that $\epsilon(h) = \epsilon(\overline{h})$.

A *leaf* is a pair, (e, v), of a vertex and an edge, where e is the unique edge adjacent to v. We say that e is a *leaf-edge* and v is a *leaf-vertex*.

A *bridge* is an edge e such that $G \smallsetminus e$ is disconnected.

Let $v \in V(G)$; we denote by $E_v(G) \subset E(G)$, respectively by $H_v(G) \subset H(G)$, the set of edges, resp. of half-edges, adjacent to v.

In some cases we will need to consider graphs endowed with legs, then we will explicitly speak about *graphs with legs*. A leg of a graph G is a one-dimensional open simplex having exactly one endpoint $v \in V(G)$. We denote by $L(G)$ the set of legs of G, and by $L_v(G)$ the set of legs having v as endpoint.

The *valency*, $\mathrm{val}(v)$, of a vertex $v \in V(G)$ is defined as follows

$$\mathrm{val}(v) := |H_v(G)| + |L_v(G)|. \tag{1}$$

Let now X be a curve (having at most nodes as singularities), and let G_X be its so-called dual graph. So, the vertices of G_X correspond to the irreducible components of X, and we write $X = \cup_{v\in V(G_X)}C_v$ with C_v irreducible curve. The edges of G_X correspond to the nodes of X, and we denote the set of nodes of X by $X_{\text{sing}} = \{N_e,\ e \in E(G_X)\}$. The endpoints of the edge e correspond to the components of X glued at the node N_e. Finally, the set of half-edges $H(G_X)$ is identified with the set of points of the normalization of X lying over the nodes, so that a pair $\{h,\overline{h}\} \subset H(G_X)$ corresponding to the edge $e \in E(G_X)$ is identified with a pair of points $p_h, p_{\overline{h}}$ on the normalization of X in such a way that, denoting by $v, \overline{v}$ the endpoints of e, with h adjacent to v and $\overline{h}$ adjacent to $\overline{v}$, we have that p_h lies on the normalization of C_v and $p_{\overline{h}}$ on the normalization of $C_{\overline{v}}$. This yields a handy description of X:

$$X = \frac{\sqcup_{v\in V(G_X)}C_v^\nu}{\{p_h = p_{\overline{h}},\ \forall h \in H(G_X)\}} \tag{2}$$

where C_v^ν denotes the normalization of C_v.

Next, let $(X; x_1, \dots, x_b)$ be a *pointed* curve , i.e. X is a curve and $x_1, \dots, x_b$ are nonsingular points of X. To $(X; x_1, \dots, x_b)$ we associate a graph with legs, written

$$G_{(X;x_1,\dots,x_b)}$$

by adding to the dual graph G_X described above one leg ℓ_i for each marked point x_i, so that the endpoint of ℓ_i is the vertex v such that $x_i \in C_v$.

A *weighted graph* is a pair (G, w) where G is a graph (possibly with legs) and w a weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$. The genus of a weighted graph is

$$g_{(G,w)} := b_1(G) + \sum_{v\in V(G)} w(v).$$

A *tree* is a connected graph of genus zero (hence weights equal zero).

A weighted graph (G, w) with legs is *stable* (respectively *semistable*), if for every vertex v we have

$$w(v) + \mathrm{val}(v) + |L_v(G)| \geq 3 \quad \text{(resp. } \geq 2).$$

Definition 1.2. Let (G, w) be a weighted graph of genus at least 2. Its *stabilization* is the stable graph obtained obtained by removing from (G, w) all leaves (v, e) such that $w(v) = 0$ and all 2-valent vertices of weight zero (see below). We say that two graphs are *(stably) equivalent* if they have the same stabilization.

The stabilization does not change the genus.

As in the previos definition, we shall often speak about graphs obtained by "removing" a 2-valent vertex, v, from a given graph, G. By this we mean that after removing v, the topological space of the so-obtained graph is the same as that of G, but the sets of vertices and edges are different. The operation opposite to removing a 2-valent vertex is that of "inserting" a vertex (necessarily 2-valent) in the interior of an edge.

A *refinement* of a weighted graph (G, w) is a weighted graph obtained by inserting some weight zero vertices in the interior of some edges of G.

Let now X be a curve as before. The *(weighted) dual graph* of X is the weighted graph (G_X, w_X), with G_X as defined above, and for $v \in V(G_X)$ the value $w_X(v)$ is equal to the genus of the normalization of C_v.

It is easy to see that the genus of X is equal to the genus of (G_X, w_X).

The (weighted) dual graph of a pointed curve $(X; x_1, \ldots, x_b)$ is the graph with legs $(G_{(X;x_1,\ldots,x_b)}, w_X)$.

Remark 1.3. A pointed curve $(X; x_1, \ldots, x_b)$ is *stable*, or *semistable*, if and only if so is $(G_{(X;x_1,\ldots,x_b)}, w_X)$.

A curve X is *rational* (i.e. it has genus zero) if and only if (G_X, w_X) is a tree.

Remark 1.4. Let X be a curve of genus ≥ 2 and (G_X, w_X) its dual graph. There exists a unique stable curve X^s of genus g with a surjective map $\sigma : X \to X^s$, such that σ is birational away from some smooth rational components that get contracted to a point. X^s is called the *stabilization* of X. The dual graph of X^s is the stabilization of (G_X, w_X); see Definition 1.2.

For a stable graph (G, w) of genus g, we denote by $M^{\text{alg}}(G, w) \subset \overline{M_g}$ the locus of curves whose dual graph is (G, w), and we refer to it as a *combinatorial locus* of $\overline{M_g}$ (the superscript "alg" stands for algebraic, versus tropical, see Subsection 3.1). Of course, we have

$$\overline{M_g} = \bigsqcup_{(G,w) \text{ stable, genus } g} M^{\text{alg}}(G, w). \tag{3}$$

1.3. **Admissible coverings.** Details about this subsection may be found in [HM82], [HMo] and [GAC]. Let $\overline{M_g}$ be the moduli space of stable curves of genus $g \geq 2$ and $M_g \subset \overline{M_g}$ its open subset parametrizing smooth curves. We denote by $\overline{M^r_{g,d}}$ the closure in $\overline{M_g}$ of the locus, $M^r_{g,d}$, of smooth curves admitting a divisor of degree d and rank r; in symbols:

$$M^r_{g,d} := \{[X] \in M_g : W^r_d(X) \neq \emptyset\} \tag{4}$$

where $W^r_d(X)$ is the set of linear equivalence classes of divisors D on X such that $h^0(X, D) \geq r + 1$.

The case of hyperelliptic curves, $r = 1$ and $d = 2$, has traditionally a simpler notation: one denotes by $H_g \subset M_g$ the locus of hyperelliptic curves and by $\overline{H_g}$ its closure in $\overline{M_g}$. So, $\overline{H_g} = \overline{M^1_{g,2}}$.

Definition 1.5. Let X be a connected curve of genus $g \geq 2$.

If X is stable, then X is *hyperelliptic* if $[X] \in \overline{H_g}$; more generally X is (d, r)*-gonal*, respectively d*-gonal*, if $[X] \in \overline{M^r_{g,d}}$, resp. if $[X] \in \overline{M^1_{d,g}}$.

If X is arbitrary, we say X is hyperelliptic, (d, r)-gonal, or d-gonal if so is its stabilization.

A connected curve of genus $g \leq 1$ is d-gonal for all $d \geq 2$.

We recall the definition of admissible covering, due to J. Harris and D. Mumford [HM82, Sect. 4], and introduce some useful generalizations.

Definition 1.6. Let Y be a connected nodal curve of genus zero, and $y_1, \ldots, y_b$ be nonsingular points of Y; let X be a connected nodal curve.

(A) A *covering* (of Y) is a regular map $\alpha : X \to Y$ such that the following conditions hold:
  (a) $\alpha^{-1}(Y_{\text{sing}}) = X_{\text{sing}}$.
  (b) α is unramified away from X_{sing} and away from $y_1, \ldots, y_b$.
  (c) α has simple ramification (i.e. a single point with ramification index equals 2) over $y_1, \ldots, y_b$.
  (d) For every $N \in X_{\text{sing}}$ the ramification indeces of α at the two branches of N coincide.

(B) A covering is called *semi-admissible* (resp. *admissible*) if the pointed curve $(Y; y_1, \ldots, y_b)$ is semistable (resp. stable), i.e. for every irreducible component D of Y we have

$$|D \cap \overline{Y \smallsetminus D}| + |D \cap \{y_1, \ldots, y_b\}| \geq 2 \quad \text{(resp. } \geq 3). \tag{5}$$

We shall write $\alpha : X \to (Y; y_1, \ldots, y_b)$ for a covering as above, and sometimes just $\alpha : X \to Y$. In fact the definition of a covering (without its being semi-admissible) does not need the points $y_1, \ldots, y_b$, as conditions (Ab) and (Ac) may be replaced by imposing that α has ordinary ramification away from X_{sing}. The following are simple consequences of the definition.

Remark 1.7. Let $\alpha : X \to Y$ be a covering.

(A) There exists an integer d such that for every irreducible component $D \subset Y$ the degree of $\alpha_{|D} : \alpha^{-1}(D) \to D$ is d. We say that d is the degree of α.
(B) Every irreducible component of X is nonsingular.
(C) If α is admissible of degree 2, then X is semistable.

In [HM82] the authors construct the moduli space $\overline{H_{d,b}}$ for admissible coverings, as a projective irreducible variety compactifiying the Hurwitz scheme (parametrizing admissible coverings having smooth range and target), and show that it has a natural morphism

$$\overline{H_{d,b}} \longrightarrow \overline{M_g}; \qquad [\alpha : X \to Y] \mapsto [X^s] \tag{6}$$

where X^s is the stabilization of X and g is its genus, so that $b = 2d + 2g - 2$. For example, if $d = 2$ we have $\overline{H_{2,2g+2}} \longrightarrow \overline{M_g}$.

Moreover, the image of $\overline{H_{2,2g+2}}$ coincides with the locus of hyperelliptic stable curves, $\overline{H_g}$, and more generally the image of (6) is the closure in $\overline{M_g}$ of the locus of d-gonal curves, here denoted by $\overline{M^1_{g,d}}$.

The description of an explicit admissible covering is in Example 2.12.

1.4. **Divisors on graphs.** For any graph G, or any weighted graph (G, w), its divisor group, $\operatorname{Div} G$, or $\operatorname{Div}(G, w)$, is defined as the free abelian group generated by the vertices of G. We use the following notation for a divisor D on (G, w)

$$D = \sum_{v \in V(G)} D(v) v \tag{7}$$

where $D(v) \in \mathbb{Z}$. For loopless and weightless graphs we use the divisor theory developed in [BN07]. If G is a weighted graph with loops, we extend this theory as in [AC11]. We begin with a definition.

Definition 1.8. Let (G, w) be a weighted graph.

We denote by G^0 the loopless graph obtained from G by inserting a vertex in the interior of every loop-edge, and by (G^0, w^0) the weighted graph such that w^0 extends w and is equal to zero on all vertices in $V(G^0) \smallsetminus V(G)$.

We denote by G^w the weightless, loopless graph obtained from G^0 by adding $w(v)$ loops based at v for every $v \in V(G)$ and then inserting a vertex in the interior of every loop-edge.

Notice that (G, w), (G^0, w^0) and G^w have the same genus, and that $(G^0)^{w^0} = G^w$.

For every $D \in \operatorname{Div}(G, w)$ its rank, $r_{(G,w)}(D)$, is set equal to $r_{G^w}(D)$. Linearly equivalent divisors have the same rank. A weighted graph (G, w) of genus g has a canonical divisor $K_{(G,w)} = \sum_{v \in V(G)}(2w(v) - 2 + \operatorname{val}(v))v$ of degree $2g - 2$ such that the following Riemann-Roch formula holds

$$r_{(G,w)}(D) - r_{(G,w)}(K_{(G,w)} - D) = \deg D - g + 1.$$

Remark 1.9. A consequence of the Riemann-Roch formula is the fact that if $g \leq 1$ then for any divisor D of degree $d \geq 0$ we have $r_{(G,w)}(D) = d - g$.

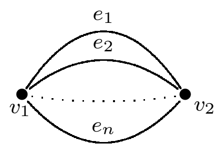
For a weighted graph (G, w) we denote by $\operatorname{Jac}^d(G, w)$ the set of linear equivalence classes of degree-d divisors, and set

$$W^r_d(G, w) := \{[D] \in \operatorname{Jac}^d(G, w) : r_{(G,w)}(D) \geq r\}.$$

Definition 1.10. We say that a graph (G, w) is *divisorially d-gonal* if it admits a divisor of degree d and rank at least1, that is if $W^1_d(G, w) \neq \emptyset$.

A *hyperelliptic* graph is a divisorially 2-gonal graph.

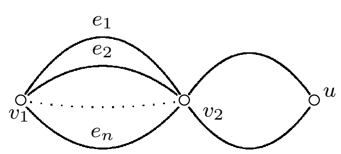
Example 1.11. Consider the following graph G with $n \geq 2$.

$G \quad =$ 

G is obviously hyperelliptic, as $r_G(v_1 + v_2) = 1$. Notice also that

$$r_G(2v_1) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}$$

Now fix on G the weight function given by $w(v_1) = 0$ and $w(v_2) = 1$. Here is the picture of G^w (drawing weight-zero vertices by a "$\circ$")

$G^w \quad =$ 

We have $r_{(G,w)}(v_1+v_2) = r_{(G,w)}(u+v_1) = r_{(G,w)}(u+v_2) = 0$ for every $n \geq 2$. On the other hand

$$r_{(G,w)}(2v_1) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3 \end{cases}$$

and the same holds for $2v_2 \sim 2u$. Therefore (G,w) is hyperelliptic if and only if $n = 2$ (in fact $n \leq 2$). This example is generalized in Corollary 4.5

2. Admissible coverings and harmonic morphisms

2.1. **Harmonic morphisms of graphs.** Let $\phi : G \to G'$ be a morphism; we denote by $\phi_V : V(G) \to V(G')$ the map induced by ϕ on the vertices. ϕ is a *homomorphism* if $\phi(E(G)) \subset E(G')$; in this case we denote by $\phi_E : E(G) \to E(G')$ and by $\phi_H : H(G) \to H(G')$ the induced maps on edges and half-edges. A morphism between weighted graphs (G,w) and (G',w') is defined as a morphism of the underlying graphs, so we write either $G \to G'$ or $(G,w) \to (G',w')$ depending on the situation.

In the next definition, extending the one in [BN09, Subsect. 2.1], we introduce some extra structure on morphisms between graphs;.

Definition 2.1. Let (G,w) and (G',w') be loopless weighted graphs.

(A) An *indexed morphism* is a morphism $\phi : (G,w) \to (G',w')$ enriched by the assignment, for every $e \in E(G)$, of a non-negative integer, the *index* of ϕ at e, written $r_\phi(e)$, such that $r_\phi(e) = 0$ if and only if $\phi(e)$ is a point. An indexed morphism is *simple* if $r_\phi(e) \leq 1$ for every $e \in E(G)$. Let $e = [h, \overline{h}]$ with $h, \overline{h} \in H(G)$; we set $r_\phi(h) = r_\phi(\overline{h}) = r_\phi(e)$.

(B) An indexed morphism is *pseudo-harmonic* if for every $v \in V(G)$ there exists a number, $m_\phi(v)$, such that for every $e' \in E_{\phi_V(v)}(G')$ (and, redundantly for convenience, every $h' \in H_{\phi_V(v)}(G')$) we have

$$m_\phi(v) = \sum_{e \in E_v(G):\phi(e)=e'} r_\phi(e) = \sum_{h \in H_v(G):\phi(h)=h'} r_\phi(h). \tag{8}$$

(C) A pseudo-harmonic indexed morphism is *non-degenerate* if $m_\phi(v) \geq 1$ for every $v \in V(G)$.

(D) A pseudo-harmonic indexed morphism is *harmonic* if for every $v \in V(G)$ we have, writing $v' = \phi(v)$,

$$\sum_{e \in E_v(G)} (r_\phi(e) - 1) \leq 2\Big(m_\phi(v) - 1 + w(v) - m_\phi(v)w'(v')\Big). \tag{9}$$

In the sequel, all graph morphisms will be indexed morphisms, hence we shall usually omit the word "indexed".

For later use, let us observe that if $w' = \underline{0}$ (i.e. G' is weightless) condition (9) simplifies as follows

$$\sum_{e \in E_v(G)} (r_\phi(e) - 1) \leq 2(m_\phi(v) - 1 + w(v)). \tag{10}$$

Remark 2.2. Suppose that ϕ contracts a leaf-edge e whose leaf-vertex v has $w(v) = 0$. Then $r_\phi(e) = m_\phi(v) = 0$ and condition (9) is not satisfied on v. So, loosely speaking, a harmonic morphism contracts no weight-zero leaves.

Remark 2.3. *Relation with harmonic morphisms of* [BN09]*.* For simple morphisms of weightless graphs our definition of harmonic morphism coincides with the one given in [BN09, Sec. 2.1] for morphisms which contract no leaves. Indeed, it is clear that any simple pseudo-harmonic morphism is harmonic in the sense of [BN09]. Conversely, a harmonic morphism in the sense of [BN09] satisfies (10) (with $w(v) = 0$) if and only if ϕ contracts no leaves; see the previous remark.

Lemma - Definition 2.4. *Let $\phi : (G, w) \to (G', w')$ be a pseudo-harmonic morphism. Then for every $e' \in E(G')$ and $v' \in V(G')$ we can define the degree of ϕ as follows*

$$\deg\phi = \sum_{e\in E(G):\phi(e)=e'} r_\phi(e) = \sum_{v\in\phi^{-1}(v')} m_\phi(v) \tag{11}$$

(i.e. the above summations are independent of the choice of e' and v').

Proof. Trivial extension of the proof of [BN09, Lm. 2.2 and Lm. 2.3]. ∎

Let $\phi : (G, w) \to (G', w')$ be a pseudo-harmonic morphism. As in [BN09, Subs. 2.3] we define a pull-back homomorphism $\phi^* : \mathrm{Div}(G', w') \to \mathrm{Div}(G, w)$ as follows: for every $v' \in V(G')$

$$\phi^* v' = \sum_{v\in\phi^{-1}(v')} m_\phi(v)v \tag{12}$$

and we extend this linearly to all of $\mathrm{Div}(G', w')$. By (11) we have

$$\deg D = \deg\phi \deg D'. \tag{13}$$

For a pseudo-harmonic morphism ϕ the *ramification divisor* R_ϕ is defined as follows.

$$R_\phi = \sum_{v\in V(G)} \Big(2\big(m_\phi(v)-1+w(v)-m_\phi(v)w'(v')\big)-\sum_{e\in E_v(G)}(r_\phi(e)-1)\Big)v. \tag{14}$$

The next result, generalizing the analog in [BN09], implies that harmonic morphisms are characterized, among pseudo-harmonic morphisms, by a Riemann-Hurwitz formula with effective ramification divisor.

Proposition 2.5 (Riemann-Hurwitz)**.** *Let $\phi : (G, w) \to (G'w')$ be a pseudo-harmonic morphism of weighted graphs of genus g and g' respectively. Then*

$$K_{(G,w)} = \phi^* K_{(G',w')} + R_\phi. \tag{15}$$

ϕ is harmonic if and only if $R_\phi \geq 0$ (equivalently $2g - 2 \geq \deg\phi(2g' - 2)$).

Proof. We write $K = K_{(G,w)}$ and $K' = K_{(G',w')}$. For every $v \in V(G)$ we have $K(v) = 2w(v) - 2 + \mathrm{val}(v)$ (notation in (7)). Hence, writing $v' = \phi(v)$, by (12) we have

$$K(v) - \phi^*K'(v) = 2w(v) - 2 + \mathrm{val}(v) - m_\phi(v)\Big(2w(v') - 2 + \mathrm{val}(v')\Big) =$$

$$= 2\Big(m_\phi(v) - 1 + w(v) - m_\phi(v)w(v')\Big) + \mathrm{val}(v) - m_\phi(v)\,\mathrm{val}(v').$$

On the other hand by (11)

$$\sum_{e\in E_v(G)}(r_\phi(e)-1)=\sum_{e\in E_v(G)}r_\phi(e)-\mathrm{val}(v)=m_\phi(v)\,\mathrm{val}(v')-\mathrm{val}(v).$$

The two above identities imply $K(v)-\phi^*K'(v)=R_\phi(v)$, so (15) is proved.

By definition, ϕ is harmonic if and only if its ramification R_ϕ divisor is effective. The equivalence in parenthesis follows from (13). ■

Remark 2.6. Other results proved in [BN09] for simple harmonic morphisms extend. In particular, if D' and E' are linearly equivalent divisors on (G',w'), their pull-backs ϕ^*D' and ϕ^*E' under a pseudo-harmonic morphisms ϕ are linearly equivalent.

2.2. **The Hurwitz existence problem.** Our goal is to use harmonic morphisms to characterize graphs that are dual graphs of d-gonal curves. This brings up the "Hurwitz existence problem", about the existence of branched coverings of $\mathbb{P}^1$ with prescribed ramification profiles; to state it precisely we need some terminology.

Let $d\geq 1$ be an integer and let $\underline{P}=\{P_1,\dots,P_b\}$ be a set of partitions of d, so that we write $P_i=\{r_i^1,\dots,r_i^{n_i}\}$ with $r_i^j\in\mathbb{Z}_{\geq 1}$ and $\sum_{j=1}^{n_i}r_i^j=d$.

We say that $\underline{P}$ is a *Hurwitz partition set*, or that $\underline{P}$ is of Hurwitz type, if the following condition holds. There exist b permutations $\sigma_1,\dots,\sigma_b\in S_d$ (S_d the symmetric group) whose product is equal to the identity, such that σ_i is the product of n_i disjoint cycles of lengths given by P_i, and such that the subgroup $<\sigma_1,\dots,\sigma_b>$ is transitive.

Notice that if $\underline{P}$ is of Hurwitz type and we add to it the *trivial* partition $\{1,1,\dots,1\}$, the resulting partition set is again of Hurwitz type.

Remark 2.7. By the Riemann existence theorem, $\underline{P}$ is a Hurwitz partition set if and only if there exists a degree-d connected covering $\alpha: C\to\mathbb{P}^1$ with $q_1,\dots,q_b\in\mathbb{P}^1$ such that α is unramified away from $q_1,\dots,q_b$ and such that for all $i=1,\dots,b$ we have $\alpha^*(q_i)=\sum_{j=1}^{n_i}r_i^jp_i^j$. The genus g of C is determined by the Riemann-Hurwitz formula:

$$2g-2=-2d+\sum_{i=1}^{b}\sum_{j=1}^{n_i}(r_i^j-1), \tag{16}$$

so that we shall also say that $\underline{P}$ is a *Hurwitz partition set of genus g and degree d.*

Remark 2.8. It is a fact that a partition set $\underline{P}$ satisfying (16) is not necessarily of Hurwitz type. Indeed, the Hurwitz existence problem can be stated as follows: characterize Hurwitz partition sets among all $\underline{P}$ satisfying (16). This problem turns out to be very difficult and it is open in general. Easy cases in which every $\underline{P}$ satisfying (16) is of Hurwitz type are $P_i=(2,1,\dots,1)$ for every i, or $d\leq 3$, or $b\leq 2$.

On the other hand if $d=4$ the partition set $\underline{P}=\{(3,1);(2,2);(2,2)\}$ is not of Hurwitz type, but the Riemann-Hurwitz formula (16) holds with $g=0$; see [PP] for this and other results on the Hurwitz existence problem.

Let now $\phi:(G,w)\to T$ be a non-degenerate pseudo-harmonic morphism, where T is a tree; let $v\in V(G)$. For any half-edge $h'\in H(T)$ in the image of some half-edge adjacent to v we define, using (8), a partition of $m_\phi(v)$:

$$P_{h'}(\phi, v) := \{r_\phi(h),\ \forall h \in H_v(G):\ \phi(h) = h'\}.$$

Now we associate to v and ϕ the following partition set:

$$\underline{P}(\phi, v) = \{P_{h'}(\phi, v),\ \forall h' \in \phi_H(H_v(G))\}. \tag{17}$$

In the next definition we use the terminology of Remark 2.7.

Definition 2.9. **(A)** Let (G, w) be a loopless weighted graph. We say that (G, w) is *d-gonal* if it admits a non-degenerate, degree-d harmonic morphism $\phi : (G, w) \to T$ where T is a tree.

If such a ϕ has the property that for every $v \in V(G)$ the partition set $\underline{P}(\phi, v)$ is contained in a Hurwitz partition set of genus $w(v)$, we say that ϕ is a morphism of *Hurwitz type*, and that (G, w) is a d-gonal graph of *Hurwitz type.*

(B) Let (G, w) be any graph. We say that it is d-gonal, or of Hurwitz type, if so is (G^0, w^0), with (G^0, w^0) as in Definition 1.8.

Example 2.10. A harmonic morphism with indeces at most equal to 2 is of Hurwitz type. Hence if $d \leq 3$ a d-gonal graph is always of Hurwitz type.

The following is one of the principal results of this paper, of which Theorem 1.1 is a special case. Recall the terminology introduced in Definition 1.2.

Theorem 2.11. *Let (G, w) be a d-gonal graph of Hurwitz type; then there exists a d-gonal curve whose dual graph is (G, w).*

Conversely, let X be a d-gonal curve; then its dual graph is equivalent to a d-gonal graph of Hurwitz type.

The proof of the first part of the theorem will be given in Subsection 2.4. The converse is easier, and will be proved earlier, in Corollary 2.14.

2.3. **The dual graph-map of a covering.**

To prove Theorem 2.11 we shall associate to any covering $\alpha : X \to Y$ an indexed morphism of graphs, called the *dual graph-map* of α, and denoted by

$$\phi_\alpha : (G_X, w_X) \longrightarrow G_Y.$$

As all components of Y have genus zero, we omit the weight function for Y. We sometimes write just $G_X \to G_Y$ for simplicity.

We use the notation of subsection 1.2; denote by

$$Y = \cup_{u \in V(G_Y)} D_u$$

the irreducible component decomposition of Y. For any $v \in V(G_X)$ we have that $\alpha(C_v)$ is an irreducible component of Y, hence there is a unique $u \in V(G_Y)$ such that $\alpha(C_v) = D_u$; this defines a map $\phi_{\alpha,V} : V(G_X) \to V(G_Y)$ mapping v to u.

Next, $E(G_X)$ and $E(G_Y)$ are identified with the set of nodes of X and Y. To define $\phi_{\alpha,E} : E(G_X) \to E(G_Y)$ let $e \in E(G_X)$; then e corresponds to the node N_e of X. The image $\alpha(N_e)$ is a node of Y, corresponding to a unique edge of G_Y, which we set to be the image of e under $\phi_{\alpha,E}$.

It is trivial to check that the pair $(\phi_{\alpha,V}, \phi_{\alpha,E})$ defines a morphism of graphs, $\phi_\alpha : G_X \to G_Y$.

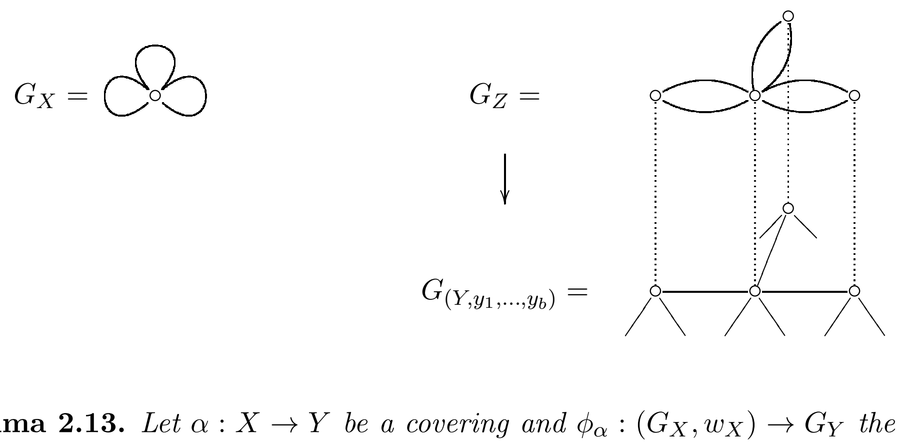
Let us now define the indeces of ϕ_α. For any $e \in E(G)$ let N_e be the corresponding node of X. By Definition 1.6, the restriction of α to each of the two branches of N_e has the form $u = x^r$ and $v = y^r$ where x and y are local coordinate at the branches of N_e, and u, v are local coordinate at the branches of $\alpha(N_e)$ (which is a node of Y). We set $r_{\phi_\alpha}(e) = r$.

If we need to keep track of the branch points of $\alpha : X \to (Y; y_1, \ldots, y_b)$, we endow the dual graph of Y with b legs, in the obvious way, and write $\phi_\alpha : G_X \to G_{(Y;y_1,\ldots,y_b)}$.

Example 2.12. *Dual graph-map for the admissible covering of an irreducible hyperelliptic curve.* Let $X \in \overline{H_g}$ be an irreducible singular hyperelliptic curve. Such curves are completely characterized; we here choose X irreducible with g nodes, so that its normalization is $\mathbb{P}^1$. Let us describe an admissible covering $\alpha : Z \to Y$ which maps to X under the map (6). As we noticed in Remark 1.7, Z cannot be equal to X. In fact, Z is the "blow-up" of X at its g nodes, so that $Z = \cup_{i=0}^{g} C_i$ is the union of $g+1$ copies of $\mathbb{P}^1$, with one copy, C_0, corresponding to the normalization of X, and the remaining copies corresponding to the "exceptional" components. Hence $|C_i \cap C_0| = 2$ and $|C_i \cap C_j| = 0$ for all $i, j \neq 0$. Now, since X is hyperelliptic, its normalization C_0 has a two-to-one map to $\mathbb{P}^1$, written $\alpha_0 : C_0 \to D_0 \cong \mathbb{P}^1$, such that $\alpha_0(p_i) = \alpha_0(q_i) = t_i \in D_0$ for every pair $p_i, q_i \in C_0$ of points lying over the i-th node of X. Let $y_0, y_1 \in D_0$ be the two branch points of α_0.

We assume that in X the component C_0 is glued to C_i along the pair p_i, q_i. For $i \geq 1$ we pick a two-to-one map $\alpha_i : C_i \to D_i \cong \mathbb{P}^1$ such that the two points of C_i glued to X have the same image, s_i, under α_i. Let $y_{2i}, y_{2i+1} \in D_i$ be the two branch points of α_i.

We define Y as the following nodal curve $Y := \sqcup_{i=0}^{g} D_i /_{\{t_i = s_i, \ \forall i = 1,\ldots,g\}}$. Now, $(Y; y_{2i}, y_{2i+1}, \ \forall i = 0, \ldots g)$ is stable, and it is clear that the α_i glue to an admissible covering $\alpha : Z \to Y$. The dual graphs and graph-map are in the following picture, where $g = 3$.



Lemma 2.13. *Let $\alpha : X \to Y$ be a covering and $\phi_\alpha : (G_X, w_X) \to G_Y$ the dual graph-map defined above. Then ϕ_α is a harmonic homomorphism of Hurwitz type.*

If $\deg \alpha = 2$ and X has no separating nodes, then ϕ_α is simple.

Proof. It is clear that G_Y has no loops. By Remark 1.7 (B), every component C_v of X is nonsingular, hence G_X has no loops.

Since α is a covering, we have that $\phi_{\alpha,V}$ and $\phi_{\alpha,E}$ are surjective, and ϕ_α does not contract any edge of G_X; hence ϕ_α is a homomorphism. We shall abuse notation by writing ϕ_α for $\phi_{\alpha,V}$, $\phi_{\alpha,H}$ and $\phi_{\alpha,E}$.

Let now $v \in V(G_X)$ and $h' \in H_{\phi(v)}(G_Y)$, so that h' corresponds to a point in the image of C_v via α, i.e. to a point in $D_{\phi(v)} \subset Y$. Consider the restriction of α to C_v:

$$\alpha_{|C_v} : C_v \longrightarrow D_{\phi(v)}.$$

This is a finite morphism, and it is clear that for every $h' \in H_{\phi(v)}(G_Y)$

$$\sum_{h \in H_v(G_X):\phi(h)=h'} r_{\phi_\alpha}(h) = \deg \alpha_{|C_v}.$$

The right hand side above does not depend on h', hence we may set

$$m_{\phi_\alpha}(v) := \deg \alpha_{|C_v}. \tag{18}$$

Therefore ϕ_α is pseudo-harmonic. To prove that ϕ_α is harmonic we must prove that for every $v \in V(G_X)$ we have

$$\sum_{e \in E_v(G_X)} (r_{\phi_\alpha}(e) - 1) \leq 2(m_{\phi_\alpha}(v) - 1 + w_X(v)). \tag{19}$$

Let $R \in \mathrm{Div}(C_v)$ be the ramification divisor of the map $\alpha_{|C_v}$ above. Then, by the Riemann-Hurwitz formula applied to $\alpha_{|C_v}$ we have,

$$\deg R = 2(m_{\phi_\alpha}(v) - 1 + w_X(v)).$$

On the other hand the map $\alpha_{|C_v}$ has ramification index $r_{\phi_\alpha}(h)$ at all $p_h \in H_v(G_X)$, hence we must have

$$R - \sum_{h \in H_v(G_X)} (r_{\phi_\alpha}(h) - 1)p_h \geq 0$$

from which (19) follows. The fact that ϕ_α is of Hurwitz type follows immediatly from Remark 2.7.

Assume $\deg \alpha = 2$ and X free from separating nodes. We must prove the indeces of ϕ are all equal to one, i.e. that α_{C_v} does not ramify at the points p_h, for every $h \in H(G_X)$. By contradiction, suppose $\alpha_{|C_v}$ is ramified at p_h; hence, as $\deg \alpha = 2$, it is totally ramified at p_h, so that $\alpha^{-1}(\alpha(p_h)) \cap C_v = p_h$. Since α is an admissible covering, we have exactly the same situation at the other branch of N_e, i.e. at $p_{\overline{h}}$. Therefore

$$\alpha^{-1}(\alpha(N_e)) = \{N_e\}.$$

Now $\alpha(N_e)$ is a node of Y, and hence it is a separating node. So, the above identity implies that N_e is a separating node of X; a contradiction. ■

Corollary 2.14. *The second part of Theorem 2.11 holds.*

Proof. Let X be a d-gonal curve; we must prove that the dual graph of X is equivalent to a d-gonal graph of Hurwitz type. By hypothesis there exists an admissible covering $\widehat{X} \to Y$ of degree d such that the stabilization of $\widehat{X}$ is the same as the stabilization of X; see the end of subsection 1.3. Therefore the dual graph of $\widehat{X}$ is equivalent to the dual graph of X. By Lemma 2.13 the dual graph of $\widehat{X}$ is of Hurwitz type, hence we are done. ■

The following is a converse to Lemma 2.13.

Proposition 2.15. *Let (G,w) be a weighted graph of genus ≥ 2 and let T be a tree. Let $\phi : (G,w) \to T$ be a harmonic homomorphism of Hurwitz type. Then there exists a covering $\alpha : X \to Y$ whose dual graph map is ϕ.*

Proof. As ϕ is harmonic, for every $v \in V(G)$ condition (10) holds.

We will abuse notation and write ϕ also for the maps $V(G) \to V(T)$, $H(G) \to H(T)$ and $E(G) \to E(T)$ induced by ϕ . We begin by constructing two curves X and Y whose dual graphs are (G,w) and T.

For every $u \in V(T)$ we pick a pointed curve (D_u, Q_u) with $D_u \cong \mathbb{P}^1$, and such that the (distinct) points in Q_u are indexed by the half-edges adjacent to u:

$$Q_u = \{q_h, \ \forall h \in H_u(T)\}.$$

We have an obvious identification $\cup_{u\in V(T)} Q_u = H(T)$. To glue the curves D_u to a connected nodal curve Y we proceed as in 2.3, getting

$$Y = \frac{\sqcup_{u\in V(T)} D_u}{\{q_h = q_{\overline{h}}, \ \forall h \in H(T)\}}.$$

By construction, T is the dual graph of Y.

Now to construct X we begin by finding its irreducible components C_v with their gluing point sets P_v. Pick $v \in V(G)$ and $u = \phi(v) \in V(T)$. By hypothesis, $m_\phi(v) \geq 1$; we claim that there exists a morphism from a smooth curve C_v of genus $w(v)$ to D_u

$$\alpha_v : C_v \longrightarrow D_u \tag{20}$$

of degree equal to $m_\phi(v)$ such that for every $h' \in H_u(T)$ the pull-back of the divisor $q_{h'}$ has the form

$$\alpha_v^* q_{h'} = \sum_{\phi_H(h)=h'} r_\phi(h) p_h$$

for some points $\{p_h, \ h \in H(G)\} \subset C_v$; we set $P_v = \{p_h, \ h \in H(G)\}$.

Indeed, the degree of the ramification divisor of a degree-m morphism from a curve of genus $w(v)$ to $\mathbb{P}^1$ of is equal to $2(m-1+w(v))$. Therefore assumption (10) guarantees that the ramification conditions we are imposing are compatible; now as ϕ is of Hurwitz type, the Riemann Existence theorem yields that such an α_v exists; see Remark 2.7. Observe that α_v may have other ramification, in which case we can easily impose that any extra ramification and branch point lie $C_v \smallsetminus P_v$, respectively in $D_u \smallsetminus Q_u$, and that they are all simple.

Now that we have the pointed curves (C_v, P_v) for every $v \in V(G)$ such that C_v is a smooth curve of genus $w(v)$ we can define X:

$$X := \frac{\sqcup_{v\in V(G)} C_v}{\{p_h = p_{\overline{h}}, \ \forall h \in H(G)\}},$$

so, (G,w) is the dual graph of X.

Let us prove that the morphisms $\{\alpha_v, \ \forall v \in V(G)\}$ glue to a morphism $\alpha : X \to Y$. It suffices to check that for every pair $(p_h, p_{\overline{h}})$ we have $\alpha_v(p_h) = \alpha_{\overline{v}}(p_{\overline{h}})$, where $p_h \in C_v$ and $p_{\overline{h}} \in C_{\overline{v}}$. We have $\alpha_v(p_h) = q_{\phi(h)}$ and $\alpha_{\overline{v}}(p_{\overline{h}}) =$

$q_{\phi(\overline{h})}$. Now, looking at the involution of $H(T)$ (see subsection 1.2), we have $\phi(\overline{h}) = \overline{(\phi(h))}$, and hence $\alpha : X \to Y$ is well defined.

We now show that α is a covering. It is obvious that $\alpha^{-1}(Y_{\text{sing}}) = X_{\text{sing}}$. Next, for every node N_e of X, the ramification indeces at the two branches, $p_h, p_{\overline{h}}$ where $[h, \overline{h}] = e$, are equal, as they are equal to $r_\phi(h)$ and $r_\phi(\overline{h})$. As we have imposed that α_v has only ordinary ramification points away from the nodes of X, condition (Ac) of Definition 1.6 is satisfied. Therefore the map $\alpha : X \to Y$ is a covering; obviously α has ϕ as dual graph-map. ■

We will need the following:

Lemma 2.16. *Let $\phi : (G, w) \to T$ be a degree-d morphism of Hurwitz type. Then there exists a degree-d homomorphism $\widehat{\phi} : (\widehat{G}, \widehat{w}) \to \widehat{T}$ of Hurwitz type fitting in a commutative diagram whose vertical arrows are edge contractions*

$$\begin{tikzcd} \widehat{G} \arrow[r, "\widehat{\phi}"] \arrow[d] & \widehat{T} \arrow[d] \\ G \arrow[r, "\phi"] & T \end{tikzcd} \tag{21}$$

Moreover, $(\widehat{G}, \widehat{w})$ is equivalent to (G, w).

Proof. The picture after the proof illustrates the forthcoming construction. Since (G^0, w^0) is equivalent to (G, w) we can assume G loopless. Consider the set of "vertical" edges of ϕ:

$$E_\phi^{\text{ver}}(G) := \{e \in E(G) : \ \phi(e) \in V(G')\}$$

and set $E_\phi^{\text{hor}}(G) := E(G) \smallsetminus E_\phi^{\text{ver}}(G)$. Of course, if $E_\phi^{\text{ver}}(G) = \emptyset$ there is nothing to prove. So, let $e \in E_\phi^{\text{ver}}(G)$ and v_1, v_2 be its endpoints. We set $u = \phi(v_1) = \phi(v_2) = \phi(e)$ and write

$$\phi_V^{-1}(u) = \{v_1, v_2, \ldots, v_n\} \tag{22}$$

with $n \geq 2$ and the v_i distinct. Set $m_i := m_\phi(v_i)$ for $i = 1, \ldots, n$.

We begin by constructing $\widehat{G}$. First, we insert a weight zero vertex $\widehat{v}_e$ in the interior of e, and denote by $\widehat{e}_1, \widehat{e}_2$ the two edges adjacent to it. Next, we attach $m_1 - 1$ leaves at v_1, $m_2 - 1$ leaves at v_2, and m_i leaves at v_i for all $i \geq 3$; all these leaf-vertices are given weight zero. We denote the j-th leaf-edge attached to v_i by $l_{e,j^{(i)}}^{(i)}$ and its leaf-vertex by $w_{e,j^{(i)}}^{(i)}$, with $j^{(i)} = 1, \ldots, m_i - 1$ if $i = 1, 2$ and $j^{(i)} = 1, \ldots, m_i$ if $i \geq 3$.

We repeat this construction for every $e \in E_\phi^{\text{ver}}(G)$, and we denote the so obtained graph by $\widehat{G}$. We have identifications

$$E(\widehat{G}) = E_\phi^{\text{hor}}(G) \sqcup \{\widehat{e}_1, \widehat{e}_2, \ \forall e \in E_\phi^{\text{ver}}(G)\} \sqcup \{l_{e,j^{(i)}}^{(i)} \ \ \forall e \in E_\phi^{\text{ver}}(G), \forall i, \forall j^{(i)}\}$$

and

$$V(\widehat{G}) = V(G) \sqcup \{\widehat{v}_e, \ \forall e \in E_\phi^{\text{ver}}(G)\} \sqcup \{w_{e,j^{(i)}}^{(i)} \ \ \forall e \in E^{\text{ver}}({}_\phi G), \forall i, \forall j^{(i)}\}.$$

There is a contraction $\widehat{G} \to G$ given by contracting, for every $e \in E_\phi^{\text{ver}}(G)$, the edge $\widehat{e}_1$ and all leaf edges $l_{e,j^{(i)}}^{(i)}$. It is clear that G and $\widehat{G}$ are equivalent.

Let us now construct $\widehat{T}$; for every $e \in E^{\mathrm{ver}}(G)$ we add to T a leaf based at $u = \phi(e)$; we denote by $\widehat{l}_e$, and $\widehat{w}_e$ the edge and vertex of this leaf. We let $\widehat{T}$ be the tree obtained after repeating this process for every $e \in E_\phi^{\mathrm{ver}}(G)$. There is a contraction $\widehat{T} \to T$ given by contracting all leaf edges $\widehat{l}_e$.

Let $G' := G - E_\phi^{\mathrm{ver}}(G)$, so that G' is also a subgraph of $\widehat{G}$. Denote by $\phi' : G' \to T$ the restriction of ϕ to G'; observe that ϕ' is a harmonic homomorphism. To construct $\widehat{\phi} : \widehat{G} \to \widehat{T}$ we extend ϕ' as follows. For every $e \in E_\phi^{\mathrm{ver}}(G)$ we set, with the above notations,

$$\widehat{\phi}(\widehat{e}_1) = \widehat{\phi}(\widehat{e}_2) = \widehat{\phi}(l_{e,j^{(i)}}^{(i)}) = \widehat{l}_e$$

and

$$\widehat{\phi}(\widehat{v}_e) = \widehat{\phi}(w_{e,j^{(i)}}^{(i)}) = \widehat{w}_e$$

for every i and $j^{(i)}$. Finally, we define the indeces of $\widehat{\phi}$

$$r_{\widehat{\phi}}(\widehat{e}) = \begin{cases} r_\phi(\widehat{e}) & \text{if } \widehat{e} \in E_\phi^{\mathrm{hor}}(G) \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $\widehat{\phi}$ is a homomorphism and that diagram (21) is commutative.

Let us check that $\widehat{\phi}$ is pseudo-harmonic. Pick $e \in E^{\mathrm{ver}}(G)$. Consider a leaf vertex $w_{e,j^{(i)}}^{(i)}$ of $\widehat{G}$. Then it is clear that condition (8) holds with $m_{\widehat{\phi}}(w_{e,j^{(i)}}^{(i)}) = 1$. Next, consider a vertex $\widehat{v}_e$. It is again clear that condition (8) holds with $m_{\widehat{\phi}}(\widehat{v}_e) = 2$. Finally, consider the vertices $v_1, \dots, v_n$ introduced in (22). Recall that $\widehat{\phi}(v_i) = \phi(v_i) = u$ and condition (8) holds for any edge in $E(T) \subset E(\widehat{T})$ adjacent to u with $m_{\widehat{\phi}}(v_i) = m_i$. We need to check that the same holds for the leaf-edges $\widehat{l}_e \in E(\widehat{T})$. For v_1 and any leaf $\widehat{l}_e$ adjacent to $\widehat{\phi}(v_1)$ we have

$$\sum_{\widehat{e} \in E_{v_1}(\widehat{G}) : \widehat{\phi}(\widehat{e}) = \widehat{l}_e} r_{\widehat{\phi}}(\widehat{e}) = \sum_{j^{(1)}=1}^{m_1 - 1} r_{\widehat{\phi}}(l_{e,j^{(1)}}^{(1)}) + r_{\widehat{\phi}}(\widehat{e}_1) = m_1 - 1 + 1 = m_1,$$

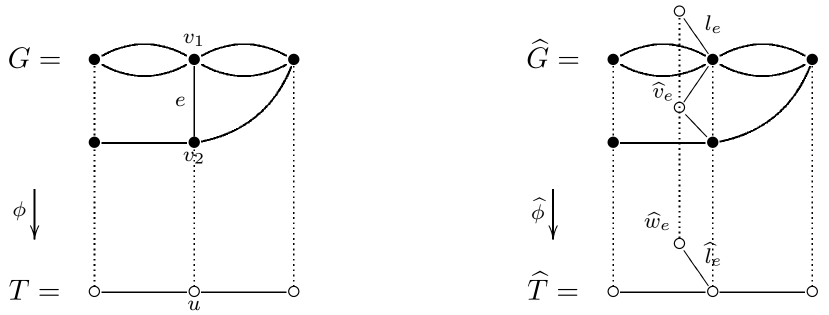
(as $r_{\widehat{\phi}}(l_{e,j^{(1)}}^{(1)}) = r_{\widehat{\phi}}(\widehat{e}_1) = 1$) Similarly for v_2. Next, for v_i with $i = 3, \dots, n$ we have

$$\sum_{\widehat{e} \in E_{v_i}(\widehat{G}) : \widehat{\phi}(e) = \widehat{l}_e} r_{\widehat{\phi}}(\widehat{e}) = \sum_{j^{(i)}=1}^{m_i} r_{\widehat{\phi}}(l_{e,j^{(i)}}^{(i)}) = m_i.$$

Since ϕ' is pseudo-harmonic there is nothing else to check; hence $\widehat{\phi}$ is pseudo-harmonic. Now, to prove that $\widehat{\phi}$ is harmonic we must check that condition (10) holds; since ϕ' is harmonic, this follows immediatly from the fact that the index of $\widehat{\phi}$ at each of the new edges is 1.

Finally, to prove that $\widehat{\phi}$ is of Hurwitz type, pick a vertex of $\widehat{G}$; if this vertex is of type $\widehat{v}_e$ or $w_{e,j^{(i)}}^{(i)}$ then the associated partition set contains only the trivial partition, and hence it is obviously contained in some partition set of Hurwitz type. The remaining case is that of a vertex v of G. Then either $\underline{P}(\phi, v) = \underline{P}(\widehat{\phi}, v)$ (if v is not adjacent to $e \in E^{\mathrm{ver}}$), or $\underline{P}(\widehat{\phi}, v)$ is obtained by adding the trivial partition to $\underline{P}(\phi, v)$; in both cases, since by hypothesis $\underline{P}(\phi, v)$ is contained in a partition set of Hurwitz type, so is $\underline{P}(\widehat{\phi}, v)$. ■

The following picture illustrates $\widehat{\phi}$ for a 3-gonal morphism ϕ. All indeces of ϕ are set equal to 1, with the exception of the vertical edge e for which $r_\phi(e) = 0$.



2.4. **Proof of Theorem 2.11.** By Corollary 2.14 we need only prove the first part of the Theorem. We first assume that G is free from loops.

By hypothesis we have a non-degenerate, degree-d, harmonic morphism $\phi : G \to T$ of Hurwitz type, where T is a tree. We let $\widehat{\phi} : \widehat{G} \to \widehat{T}$ be a degree-d, harmonic homomorphism associated to ϕ by Lemma 2.16, Now $\widehat{\phi} : \widehat{G} \to \widehat{T}$ satisfies all the assumptions of Proposition 2.15, hence there exists a covering $\alpha : \widehat{X} \to \widehat{Y}$ whose dual graph-map is $\widehat{\phi} : \widehat{G} \to \widehat{T}$. We denote by $y_1, \ldots, y_b \in Y$ the smooth branch points of α.

Suppose now that (G, w) is stable; we claim that α is admissible, i.e. that $(Y; y_1, \ldots, y_b)$ is stable. We write $Y = \cup_{\widehat{u} \in V(\widehat{T})} D_{\widehat{u}}$ as usual. For every branch point y_i we attach a leg to $\widehat{T}$, having endpoint $\widehat{u} \in V(\widehat{T})$ such that $y_i \in D_{\widehat{u}}$. We must prove that the graph $\widehat{T}$ with these b legs has no vertex of valency less than 3. Pick a vertex of $\widehat{T}$. There are two cases, either it is a vertex $u \in V(T)$ or it is a leaf vertex $\widehat{w}_e$.

In the first case the preimage of u via $\widehat{\phi}$ is made of vertices of the original graph G. So, pick $v \in V(G)$ with $\phi(v) = u$. The map $\alpha_v : C_v \to D_u$ has degree $m_\phi(v)$. If $m_\phi(v) = 1$, then, of course, $C_v \cong \mathbb{P}^1$ and we have $\mathrm{val}(u) \geq \mathrm{val}(v)$, and $\mathrm{val}(v) \geq 3$ as G is stable; hence we are ok. Notice that this is the only place where we use that (G, w) is stable, the rest of the proof works for any d-gonal graph. If $m_\phi(v) \geq 2$ then the map α_v has at least two branch points, each of which corresponds to a leg adjacent to u. If α_v has more than two branch points, then u has more than two legs adjacent to it, hence we are done; if α_v has exactly two branch points, then, by Riemann-Hurwitz, $C_v \cong \mathbb{P}^1$ and hence $C_v \subsetneq X$ as X has genus ≥ 2. Therefore $C_v \cap \overline{X \smallsetminus C_v} \neq \emptyset$, and hence there is at least one edge of T adjacent to u, hence $\mathrm{val}(u) \geq 3$.

Now consider a vertex of type $\widehat{w}_e$. By construction, its preimage contains the vertex $\widehat{v}_e$, for which $m_{\widehat{\phi}}(\widehat{v}_e) = 2$; hence the corresponding component of $\widehat{X}$ maps two-to-one to the component corresponding to $\widehat{w}_e$, and hence there are at least 2 legs attached to $\widehat{w}_e$ (corresponding to the two branch points). There is also at least one edge because, as before, $\widehat{w}_e$ is not an isolated vertex of $\widehat{T}$. So, $\mathrm{val}(\widehat{w}_e) \geq 3$. This proves that α is an admissible covering.

Now, $\widehat{X}$ is a curve whose dual graph is $(\widehat{G}, \widehat{w})$. Its stabilization is a stable curve, X, whose dual graph is clearly the original (G, w). As we already mentioned, the fact that X is d-gonal follows from [HM82, Sect 4], observing that X is the image of the admissible covering $\alpha : \widehat{X} \to (Y; y_1, \ldots, y_b)$ under the morphism (6). This concludes the proof in case (G, w) is stable and loopless.

Now let us drop the stability assumption on (G, w). If α is admissible, the previous argument yields that the stabilization of $\widehat{X}$ is d-gonal. But the stabilization of $\widehat{X}$ is the same as the stabilization of X, hence we are done.

Suppose α is not admissible; then there are two cases. First case: $\widehat{T}$ has a vertex u of valency 1. By the previous part of the proof this can happen only if every vertex $v \in \phi_V^{-1}(u)$ has valency 1 and α induces an isomorphism $C_v \cong \mathbb{P}^1$; such components of $\widehat{X}$ are called rational tails. We now remove the component D_u from $\widehat{Y}$, and all the rational tails mapping to D_u from $\widehat{X}$. Observe that this operation does not change the stabilization of $\widehat{X}$. This corresponds to removing one leaf from $\widehat{T}$ and all its preimages (all leaves) under ϕ. We repeat this process until there are no 1-valent vertices left.

Second case, $\widehat{T}$ has a vertex u of valency 2. Again by the previous part this happens only if every $v \in \phi_V^{-1}(u)$ has valency 2 and α induces an isomorphism $C_v \cong \mathbb{P}^1$. We collapse the component D_u of $\widehat{Y}$ and all the exceptional components of $\widehat{X}$ mapping to D_u. Again, this operation does not change the stabilization of $\widehat{X}$. We repeat this process until there are no 2-valent vertices left.

In this way we arrive at two curves X' and $(Y'; y_1, \ldots, y_b)$, the latter being stable, endowed with a covering $\alpha' : X' \to Y'$ induced by α, by construction; indeed the process did not touch the branch points $y_1, \ldots, y_b$, which are now the smooth branch points of α'. The covering α' is admissible, hence the stabilization of X' is d-gonal (as before). Since the stabilization of X' is equal to the stabilization of X we are done. The loopless case is now proved.

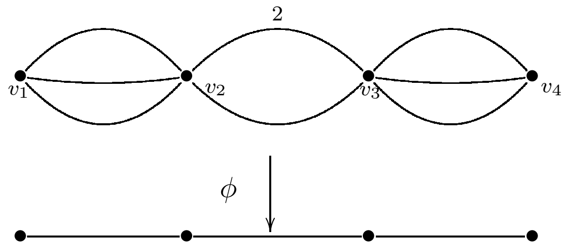
We now suppose that G has some loop; let (G^0, w^0) be its loopless model. By Definition 2.9, (G^0, w^0) is d-gonal. The previous part yields that there exists a curve X^0 whose dual graph is (G^0, w^0) and whose stabilization is d-gonal. Since the stabilization of X is equal to the stabilization of X^0 we are done. Theorem 2.11 is proved. ■

Remark 2.17. *Hyperelliptic and 2-gonal graphs.* It is easy to construct hyperelliptic (i.e. divisorially 2-gonal) graphs that are not 2-gonal; for example the weightless graph G in Example 1.11 for $n \geq 3$.

On the other hand every 2-gonal stable graph is hyperelliptic, by Theorem 2.11 and Proposition 4.6; see also Theorem 4.8. More generally, using Remark 2.6 one can prove directly that if a graph admits a pseudo-harmonic morphism of degree 2 to a tree, then it is hyperelliptic. We omit the details.

Example 2.18. *A* 3*-gonal graph which is not divisorially* 3*-gonal.*

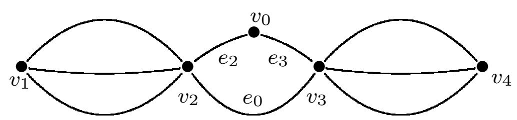
In the following picture we have a pseudo-harmonic morphism ϕ of degree 3 from a weightless graph G of genus 5. There is one edge, joining v_2 and v_3, where the index is 2, and all other edges have index 1. The graph G is easily seen to be 3-gonal, but not divisorially 3-gonal, i.e. $W^1_3(G) = \emptyset$, hence not strongly 3-gonal. We omit the details.



Example 2.19. *A divisorially* 3*-gonal graph which is not* 3*-gonal.* In the graph G below, weightless of genus 5, we have

$$3v_1 \sim 3v_2 \sim -v_2 + 2v_0 + 2v_3 \sim 3v_3 \sim v_0 + v_2 + v_3 \sim 3v_4$$

so the graph is divisorially 3-gonal.



Let us show that G does not admit a non-degenerate pseudo-harmonic morphism of degree 3 to a tree. By contradiction, let $\phi : G \to T$ be such a morphism. Then the edges adjacent to v_1 cannot get contracted (if one of them is contracted, all of them will be contracted, for T has no loops; but if all of them get contracted then $m_\phi(v_1) = 0$, which is not possible). Therefore the three edges adjacent to v_1 are all mapped to the unique edge, e'_1, joining $\phi(v_1)$ with $\phi(v_2)$. Similarly, the edges adjacent to v_4 are all mapped to the unique edge e'_2 joining $\phi(v_4)$ with $\phi(v_3)$. Therefore, as ϕ as degree 3, all edges between v_1 and v_2, and all edges between v_3 and v_4 have index 1, hence $m_\phi(v_1) = m_\phi(v_2) = m_\phi(v_3) = m_\phi(v_4) = 3$.

Now, if $\phi(v_2) = \phi(v_3)$ then one easily checks that e_0 is contracted and e_2, e_3 are mapped to the same edge e'_3 of T, which is different from e'_1 and e'_2. Therefore we have $1 \leq r_\phi(e_i) \leq 2$ for $i = 1, 2$. But then by (8) we have

$$m_\phi(v_2) = \sum_{e \in E_{v_2}(G) : \phi(e) = e'_3} r_\phi(e) = r_\phi(e_2) \leq 2$$

and this is a contradiction.

It remains to consider the case $\phi(v_2) \neq \phi(v_3)$, let $e'_0 = \phi(e_0)$. Then v_0 is either mapped to $\phi(v_2)$ by contracting e_2, or to $\phi(v_3)$ by contracting e_3 (for otherwise T would not be a tree). With no loss of generality, set

$\phi(v_2) = \phi(v_0)$ so that $r_\phi(e_2) = 0$. Now, since $\phi(e_3) = \phi(e_0) = e'_0$ we have $r_\phi(e_0) \leq 2$. Hence

$$m_\phi(v_2) = \sum_{e \in E_{v_2}(G):\phi(e)=e'_0} r_\phi(e) = r_\phi(e_0) \leq 2$$

and this is a contradiction.

3. Higher gonality and applications to tropical curves

3.1. **Basics on tropical curves.** A (weighted) tropical curve is a weighted metric graph $\Gamma = (G, w, \ell)$ where (G, w) is a weighted graph and $\ell : E(G) \to \mathbb{R}_{>0}$. The divisor group $\mathrm{Div}(\Gamma)$ is, as usual, the free abelian group generated by the points of Γ (viewed as a metric space). The weightless case has been carefully studied in [GK08], for example; the general case has been recently treated in [AC11], to which we refer for the definition of the rank $r_\Gamma(D)$ of any $D \in \mathrm{Div}(\Gamma)$ and its basic properties. Here we just need the following facts. Given $\Gamma = (G, w, \ell)$ we introduce the tropical curve $\Gamma^w = (G^w, \underline{0}, \ell^w)$ such that G^w is as in Definition 1.8, the weight function is zero (hence denoted by $\underline{0}$), and ℓ^w is the extension of ℓ such that $\ell^w(e) = 1$ for every $e \in E(G^w) \smallsetminus E(G)$. We have a natural commutative diagram

$$\begin{tikzcd} \mathrm{Div}(G,w) \arrow[r, hook] \arrow[d, hook] & \mathrm{Div}(G^w) \arrow[d, hook] \\ \mathrm{Div}(\Gamma) \arrow[r, hook] & \mathrm{Div}(\Gamma^w) \end{tikzcd} \tag{23}$$

the above injections will be viewed as inclusions in the sequel. Then, for any $D \in \mathrm{Div}(\Gamma)$ we have, by [AC11, Sect. 5]

$$r_\Gamma(D) = r_{\Gamma^w}(D). \tag{24}$$

So, the horizontal arrows of the above diagram preserve the rank. If the length functions on Γ and Γ^w are identically equal to 1, then, by [L11, Thm 1.3], also the vertical arrows of the diagram preserve the rank.

For a tropical curve Γ we denote by $W^r_d(\Gamma)$ the set of equivalence classes of divisors of degree d and rank at least r; we say that Γ is (d, r)-gonal if $W^r_d(\Gamma) \neq \emptyset$.

The moduli space of equivalence classes of tropical curves of genus g is denoted by M^{trop}_g, and the locus in it of curves whose underlying weighted graph is (G, w) is denoted by $M^{\mathrm{trop}}(G, w)$. This gives a partition

$$M^{\mathrm{trop}}_g = \sqcup M^{\mathrm{trop}}(G, w)$$

indexed by all stable graphs (G, w) of genus g.

3.2. **From algebraic gonality to combinatorial and tropical gonality.**

Theorem 3.1. *Let $X \in \overline{M^r_{g,d}}$ and let (G, w) be the dual graph of X. Then*

(A) *there exists a refinement $(\widehat{G}, \widehat{w})$ of (G, w), such that $W^r_d(\widehat{G}, \widehat{w}) \neq \emptyset$;*
(B) *there exists a tropical curve $\Gamma \in M^{\mathrm{trop}}(G, w)$ such that $W^r_d(\Gamma) \neq \emptyset$.*

The proof is similar to the proof of the "Existence theorem" for weightless graphs, in [C11b].

Proof. By hypothesis there exists a family of curves, $f : \mathcal{X} \to B$, with B smooth, connected, of dimension one, such that there is a point $b_0 \in B$ over which the fiber of f is isomorphic to X, and the fiber over any other point of B is a smooth curve whose W^r_d is not empty.

If the total space $\mathcal{X}$ is singular, it can only have (isolated) singular points of type A_n at the nodes of X. Therefore the minimal resolution of the singularities of $\mathcal{X}$, written $\mathcal{Z} \to \mathcal{X}$, yields a new family of curves over B, denoted by $h : \mathcal{Z} \to B$. The fiber of h over b_0 is a semistable curve Z_0 whose stabilization is X; all remaining fibers are isomorphic to the original fibers of f. In particular, for every $b \in B^* = B \smallsetminus \{b_0\}$ we have $W^r_d(Z_b) \neq 0$, (Z_b is the fiber of h over b). Now, write $h^* : \mathcal{Z}^* \to B^*$ for the smooth family obtained by restricting h to $\mathcal{Z} \smallsetminus Z_0$. Recall that as b varies in B^* the $W^r_d(Z_b)$ form a family ([AC81][Sect. 2] or [GAC, Ch. 21]), i.e. there exists a morphism of schemes

$$W^r_{d,h^*} \to B^* \tag{25}$$

whose fiber over b is $W^r_d(X_b)$. Up to replacing $\mathcal{Z} \to B$ with an étale covering (which, of course, does not alter the regularity of the total space), we may assume that the morphism (25) has a section, and that this section corresponds to a line bundle $\mathcal{L}^* \in \operatorname{Pic} \mathcal{Z}^*$; see [BLR]. Since $\mathcal{Z}$ is nonsingular, a line bundle on $\mathcal{Z}^*$ extends to some line bundle on $\mathcal{Z}$; therefore there exists a line bundle $\mathcal{L} \in \operatorname{Pic} \mathcal{Z}$ such that $r(Z_b, \mathcal{L}_{|Z_b}) \geq r$ for every $b \in B$.

Let $(\widehat{G}, \widehat{w})$ be the dual graph of Z_0. We can apply the weighted specialization Lemma [AC11, Thm 4.9] to $\mathcal{Z} \to B$ with respect to the line bundle $\mathcal{L}$. This gives, viewing the multidegree $\underline{\deg}\, \mathcal{L}_{|Z_0}$ as a divisor on $\widehat{G}$,

$$r_{(\widehat{G},\widehat{w})}(\underline{\deg}\, \mathcal{L}_{|Z_0}) \geq r(Z_b, \mathcal{L}_{|Z_b}) \geq r$$

and therefore $W^r_d(\widehat{G}, \widehat{w}) \neq \emptyset$.

Now, by construction $(\widehat{G}, \widehat{w})$ a refinement of (G, w) (the dual graph of X). Hence the first part is proved.

For the next part, consider the tropical curve $\widehat{\Gamma} = (\widehat{G}, \widehat{w}, \widehat{\ell})$ with $\widehat{\ell}(e) = 1$ for every $e \in E(\widehat{G})$. Let $D \in W^r_d(\widehat{G}, \widehat{w})$. Then D is also a divisor on $\widehat{\Gamma}$ (cf. Diagram (23)). We claim that $r_{\widehat{\Gamma}}(D) \geq r$.

We have, by definition,

$$r \leq r_{(\widehat{G},\widehat{w})}(D) = r_{\widehat{G}^{\widehat{w}}}(D).$$

Let $\widehat{\Gamma}^{\widehat{w}} = (\widehat{G}^{\widehat{w}}, \underline{0}, \widehat{\ell^{w}})$ be the tropical curve such that $\widehat{\ell^{w}}(e) = 1$ for every $e \in E(\Gamma^{\widehat{w}})$; so D is also a divisor on $\widehat{\Gamma}^{\widehat{w}}$. By [L11, Thm 1.3], we have

$$r_{\widehat{G}^{\widehat{w}}}(D) = r_{\widehat{\Gamma}^{\widehat{w}}}(D).$$

Now, as we noticed in (24) we have

$$r_{\widehat{\Gamma}^{\widehat{w}}}(D) = r_{\widehat{\Gamma}}(D).$$

The claim is proved; therefore $W^r_d(\widehat{\Gamma}) \neq \emptyset$.

The supporting graph $(\widehat{G}, \widehat{w})$ of $\widehat{\Gamma}$ is not necessarily stable; its stabilization, obtained by removing every 2-valent vertex of weight zero, is the original (G, w), so that $\widehat{\Gamma}$ is tropically equivalent to a curve $\Gamma \in M^{\mathrm{trop}}_g(G, w)$.

Since the underlying metric spaces of Γ and $\widehat{\Gamma}$ coincide, we have

$$W^r_d(\Gamma) = W^r_d(\widehat{\Gamma}) \neq \emptyset.$$

The statement is proved. ■

Corollary 3.2. *Every d-gonal stable weighted graph admits a divisorially d-gonal refinement.*

Proof. Let (G, w) be a d-gonal stable graph. By Theorem 2.11 there exists $X \in \overline{M^1_{g,d}}$ whose dual graph is (G, w). By Theorem 3.1 we are done. ■

The proof of Theorem 3.1 gives a more precise result, to state which we need some further terminology.

Let X be any curve. A *one-parameter smoothing* of X is a morphism $f : \mathcal{X} \to (B, b_0)$, where B is smooth connected with $\dim B = 1$, b_0 is a point of B such that $f^{-1}(b_0) = X$, and all other fibers of f are smooth curves. By definition, $\mathcal{X}$ is a surface having only singularities of type A_n at the nodes of X. To f we associate the following length function ℓ_f on G_X:

$$\ell_f : E(G_X) \longrightarrow \mathbb{R}_{>0}; \qquad e \mapsto n(e)$$

where $n(e)$ is the integer defined by the fact that $\mathcal{X}$ has a singularity of type $A_{n(e)-1}$ at the node of X corresponding to e. In particular, if $\mathcal{X}$ is nonsingular, then ℓ_f is constant equal to one. This defines the following tropical curve associated to f:

$$\Gamma_f = (G_X, w_X, \ell_f).$$

Similarly, we define a refinement of the dual graph of X by inserting $n(e)-1$ vertices of weight zero in e, for every $e \in E(G_X)$; we denote this refinement by (G_f, w_f). Now, if $\mathcal{Z} \to \mathcal{X}$ is the minimal resolution of singularities and $h : \mathcal{Z} \to B$ the composition with f, then (G_f, w_f) is the dual graph of the fiber of h over b_0; we denote by X_f this fiber.

For example, the surface $\mathcal{X}$ is nonsingular if and only if $X = X_f$, if and only if $(G_X, w_X) = (G_f, w_f)$

The following is a consequence the proof of Theorem 3.1, where X_f corresponds to the curve Z_0, while $(G_f, w_f) = (\widehat{G}, \widehat{w})$, and $\Gamma_f = \Gamma$.

Proposition 3.3. *Let $f : \mathcal{X} \to (B, b_0)$ be a one-parameter smoothing of the curve X. If the general fiber of f is (d, r)-gonal (i.e. if $W^r_d(f^{-1}(b)) \neq \emptyset$ for every $b \neq b_0$) then the following facts hold.*

(1) $W^r_d(G_f, w_f) \neq \emptyset$.
(2) $W^r_d(\Gamma_f) \neq \emptyset$.
(3) $W^r_d(X_f) \neq \emptyset$.

Remark 3.4. The tropical curve Γ_f may be interpreted as a Berkovich skeleton of the generic fiber $\mathcal{X}_K$ of $\mathcal{X} \to B$, where K is the function field of B. Then the theorem says that the Berkovich skeleton of a (d, r)-gonal smooth algebraic curve over K is a (d, r)-gonal tropical curve.

4. The hyperelliptic case

4.1. **Hyperelliptic weighted graphs.** Recall that a graph is hyperelliptic if it has a divisor of degree two and rank one. Hyperelliptic graphs free from loops and weights have been thoroughly studied in [BN09]. In this subsection we extend some of their results to weighted graphs admitting loops.

Recall the notation of Definition 1.8. We will use the following terminology. A 2-valent vertex of is said to be *special* if its removal creates a loop. For example, given (G, w), every vertex in $V(G^w) \smallsetminus V(G)$ is special.

Lemma 4.1. *Let (G, w) be a weighted graph of genus g. Then (G, w) is hyperelliptic if and only if so is G^w if and only if so is (G^0, w^0).*

Proof. By Remark 1.9 we can assume $g \geq 2$. By definition, if G is hyperelliptic so is G^w. Conversely, assume G^w hyperelliptic and let $D \in \mathrm{Div}(G^w)$ be an effective divisor of degree 2 and rank 1. If $\mathrm{Supp}\, D \subset V(G)$ we are done, as $r_{(G,w)}(D) = r_{G^w}(D)$. Otherwise, suppose $D = u + u'$ with $u \in V(G^w) \smallsetminus V(G)$. So, u is a special vertex whose removal creates a loop based at a vertex v of G. As $r_{G^w}(u + u') = 1$, it is clear that $u' \neq v$ (e.g. by [AC11, Lm. 2.5(4)]), and a trivial direct checking yields that $u' = u$. Moreover, we have $2u \sim 2v$ and hence $r_{G^w}(2v) = 1$, by [AC11, Lm. 2.5(3)]).

As $(G^0)^{w^0} = G^w$, the second double implication follows the first. ■

Let e be a non-loop edge of a weighted graph (G, w) and let $v_1, v_2 \in V(G)$ be its endpoints. Recall that the *(weighted) contraction* of e is defined as the graph (G_e, w_e) such that e is contracted to a vertex $\overline{v}$ of G_e, and $w_e(\overline{v}) = w(v_1) + w(v_2)$, whereas w_e is equal to w on every remaining vertex of G_e.

We denote by $(\overline{G}, \overline{w})$ the 2-edge-connected weighted graph obtained by contracting every bridge of G as described above.

By [BN09, Cor 5.11] a weightless, loopless graph is hyperelliptic if and only if so is $\overline{G}$. The following Lemma extends this fact to the weighted case.

Lemma 4.2. *Let (G, w) be a loopless weighted graph of genus at least 2. Then (G, w) is hyperelliptic if and only if so is $(\overline{G}, \overline{w})$.*

Proof. By Lemma 4.1, (G, w) is hyperelliptic if and only if so is G^w. Similarly, $(\overline{G}, \overline{w})$ is hyperelliptic if and only if so is $\overline{G}^{\overline{w}}$. Now, $\overline{G}^{\overline{w}}$ is obtained from G^w by contracting all of its bridges (indeed, the bridges of G and G^w are in natural bijection). Therefore, as we said above, G^w is hyperelliptic if and only if so is $\overline{G}^{\overline{w}}$. So we are done. ■

Recall, from [BN09], that a loopless, 2-edge-connected, weightless graph G is hyperelliptic if and only if it has an involution ι such that G/ι is a tree. If G has genus at least 2, this involution is unique and will be called the *hyperelliptic* involution. Furthermore, the quotient map $G \to G/\iota$ is a non-degenerate harmonic morphism, unless $|V(G)| = 2$; see [BN09, Thm 5.12 and Cor 5.15] We are going to generalize this to the weighted case.

Remark 4.3. Let G be a loopless, 2-edge-connected hyperelliptic graph of genus ≥ 2 and ι its hyperelliptic involution. Let $v \in V(G)$ be a special vertex whose removal creates a loop based at the vertex u. Then $\iota(v) = v$, $\iota(u) = u$ and ι swaps the two edges adjacent to v.

Indeed, G/ι is a tree, hence the two edges adjacent to v are mapped to the same edge by $G \to G/\iota$. As v has valency 2 and u has valency at least 3 (G has genus at least 2), ι cannot swap v and u. Hence $\iota(v) = v$ and $\iota(u) = u$.)

Lemma 4.4. *Let (G, w) be a loopless, 2-edge-connected weighted graph of genus at least 2. Then (G, w) is hyperelliptic if and only if G has an involution ι, the* hyperelliptic involution*, fixing every vertex of positive weight and such that G/ι is a tree.*

ι is unique and, if $|V(G)| \geq 3$, then the quotient $G \to G/\iota$ is a non-degenerate harmonic morphism of degree 2.

Proof. Assume that G has an involution as in the statement; then we extend ι to an involution ι^w of G^w by requiring that ι^w fix all the (special) vertices in $V(G^w) \smallsetminus V(G)$ and swap the two edges adjacent to them. It is clear that G^w/ι^w is the tree obtained by adding $w(v)$ leaves to the vertex of G/ι corresponding to every vertex $v \in V(G)$. Hence G^w is hyperelliptic, and hence so is (G, w) by Lemma 4.1.

Conversely, suppose G^w hyperelliptic and let ι^w be its hyperelliptic involution. Let $v \in V(G) \subset V(G^w)$ have positive weight. Then there is a 2-cycle in G^w attached at v; let e^+ and e^- be its two edges, and u its special vertex. By Remark 4.3 we know that ι^w fixes v and u and swaps e^+ and e^-. Notice that the image in G^w/ι^w of every such 2-cycle is a leaf.

We obtain that the restriction of ι^w to G is an involution of G, written ι, fixing all vertices of positive weight. Finally, the quotient G/ι is the tree obtained from G^w/ι^w by removing all the above leaves, so we are done.

As G is 2-edge-connected, by Remark 2.3 we can apply some results from [BN09]. In particular, the uniqueness of ι follows from Corollary 5.14. Next, if $|V(G)| \geq 3$ then $G \to G/\iota$ is harmonic and non-degenerate by Theorem 5.14 and Lemma 5.6. ∎

Corollary 4.5. *Let (G, w) be a loopless, 2-edge-connected graph of genus at least 2, having exactly two vertices, v_1 and v_2. Then (G, w) is hyperelliptic if and ony if either $|E(G)| = 2$, or $|E(G)| \geq 3$ and $w(v_1) = w(v_2) = 0$.*

Proof. Assume (G, w) hyperelliptic. Let $|E(G)| \geq 3$; by contradiction, suppose $w(v_1) \geq 1$. By Lemma 4.4 the hyperelliptic involution fixes v_1, and hence it fixes also v_2; therefore G/ι has two vertices. Since there are at least three edges between v_1 and v_2, such edges fall into at least two orbits under ι, and each such orbit is an edge of the quotient G/ι, which therefore cannot be a tree. This is a contradiction. The other implication is trivial; see Example 1.11. ∎

4.2. **Relating hyperelliptic curves and graphs.**

Proposition 4.6. *Let X be a hyperelliptic stable curve. Then its dual graph (G_X, w_X) is hyperelliptic.*

Proof. We write $(G, w) = (G_X, w_X)$ for simplicity. By Theorem 3.1, there exists a hyperelliptic refinement, $(\widehat{G}, \widehat{w})$, of (G, w). Then the weightless graph $\widehat{G}^{\widehat{w}}$ is hyperelliptic. By Lemma 4.1 it is enough to prove that the weightless graph G^w is hyperelliptic. Now, one easily checks that G^w is obtained from $\widehat{G}^{\widehat{w}}$ by removing every non-special 2-valent vertex of weight zero, and possibly some special vertex of weight zero. On the other hand, by Lemma 4.1, the removal of any special vertex of weight zero does not alter being hyperelliptic. Therefore G^w is hyperelliptic if so is the graph obtained by removing every 2-valent vertex of weight zero from $\widehat{G}^{\widehat{w}}$. This follows from the following Lemma 4.7. ■

Lemma 4.7. *Let $(\widehat{G}, \widehat{w})$ be hyperelliptic of genus at least 2 and let (G, w) be the graph obtained from $\widehat{G}$ by removing every 2-valent vertex of weight zero. Then G is hyperelliptic.*

Proof. By Lemma 4.2, contracting bridges does not alter being hyperelliptic, hence we may assume that $\widehat{G}$ is 2-edge-connected. By Lemma 4.1 up to inserting some special vertices of weight zero we can also assume that $\widehat{G}$ has no loops. Finally, we can assume that $\widehat{G}$ has at least three vertices, for otherwise the result is trivial.

It suffices to prove that the loopless model (G^0, w^0) (see Definition 2.9) of (G, w) admits an involution ι fixing every vertex of positive weight and such that G^0/ι is a tree, by Lemma 4.4. As $(\widehat{G}, \widehat{w})$ is hyperelliptic, it admits such an involution, denoted by $\widehat{\iota}$. Recall that the quotient map $\widehat{G} \to \widehat{G}/\widehat{\iota}$ is a non-degenerate harmonic morphism.

Observe that G^0 is obtained from $\widehat{G}$ by removing all the non-special 2-valent vertices of weight zero. Let $\widehat{v} \in V(\widehat{G})$ be such a vertex and write $\widehat{e}_1, \widehat{e}_2$ for the edges of $\widehat{G}$ adjacent to $\widehat{v}$. To prove our result it suffices to show that if one removes from a hyperelliptic graph either a non-special 2-valent vertex of weight zero fixed by the hyperelliptic involution, or a pair of non-special 2-valent vertices swapped by the hyperelliptic involution, then the resulting graph is hyperelliptic.

First, let $\widehat{\iota}(\widehat{v}) = \widehat{v}$ and let (G', w') be the graph obtained by removing $\widehat{v}$. We have $\widehat{\iota}(\widehat{e}_1) = \widehat{e}_2$ (as $\widehat{G} \to \widehat{G}/\widehat{\iota}$ is non-degenerate), and $\widehat{v}$ is mapped to a leaf of $\widehat{G}/\widehat{\iota}$. Now, $V(G') = V(\widehat{G}) \smallsetminus \{\widehat{v}\}$, and $E(G') = \{e\} \cup E(\widehat{G}) \smallsetminus \{\widehat{e}_1, \widehat{e}_2\}$ where e is the edge created by removing $\widehat{v}$. We define the involution ι' of G' by restricting $\widehat{\iota}$ on $V(G')$ and on $E(\widehat{G}) \smallsetminus \{\widehat{e}_1, \widehat{e}_2\}$, and by setting $\iota'(e) = e$. Since ι' swaps the two endpoints of e (because so does $\widehat{\iota}$), we have that e is contracted to a point by the quotient $G' \to G'/\iota'$. Therefore G'/ι' is the tree obtained from $\widehat{G}/\widehat{\iota}$ by removing the leaf corresponding to $\widehat{v}$. It is clear that ι' fixes all vertices of positive weight, hence (G', w') is hyperelliptic.

Next, let $\widehat{\iota}(\widehat{v}) = \widehat{v}' \neq \widehat{v}$; with $\widehat{v}$ and $\widehat{v}'$ non-special and 2-valent, then the vertex of $\widehat{G}/\widehat{\iota}$ corresponding to $\{\widehat{v}, \widehat{v}'\}$ is 2-valent as well. Moreover, $\widehat{v}$ and $\widehat{v}'$ have weight zero, by Lemma 4.4. Let us show that the graph (G'', w'') obtained by removing $\widehat{v}$ and $\widehat{v}'$ is hyperelliptic. Now $\widehat{\iota}$ maps $\widehat{e}_1, \widehat{e}_2$ to the two edges adjacent to $\widehat{v}'$. We denote by e and e' the new edges of G''. We define ι'' on $V(G'') = V(\widehat{G}) \smallsetminus \{\widehat{v}, \widehat{v}'\}$ by restricting $\widehat{\iota}$; next, we define ι'' on $E(G'')$ so that $\iota''(e) = e'$ and ι'' coincides with $\widehat{\iota}$ on the remaining edges. It is clear that ι'' is an involution fixing positive weight vertices and such that the quotient G''/ι'' is the tree obtained from $\widehat{G}/\widehat{\iota}$ by removing the 2-valent vertex corresponding to $\{\widehat{v}, \widehat{v}'\}$. We have thus proved that (G'', w'') is hyperelliptic. The proof is now complete. ■

Theorem 4.8. *Let (G, w) be a stable graph of genus $g \geq 2$. Then the following are equivalent.*

(A) *$M^{\mathrm{alg}}(G, w)$ contains a hyperelliptic curve.*

(B) *(G, w) is hyperelliptic and for every $v \in V(G)$ the number of bridges of G adjacent to v is at most $2w(v) + 2$.*

(C) *Assume $|V(G)| \neq 2$; the graph (G, w) is 2-gonal.*

Proof. (C) $\Rightarrow$ (A) by Theorem 2.11 and Example 2.10.

(A) $\Rightarrow$ (B). Let X be a hyperelliptic curve such that $(G_X, w_X) = (G, w)$. Then, by Proposition 4.6, (G, w) is hyperelliptic. Let $\alpha : \widehat{X} \to Y$ be an admissible covering corresponding to X; by Remark 1.7 (C), $\widehat{X}$ is semistable. Therefore the dual graph of $\widehat{X}$, written $(\widehat{G}, \widehat{w})$, is a refinement of (G, w) (as X is the stabilization of $\widehat{X}$).

Let $v \in V(G) \subset V(\widehat{G})$ and $C_v \subset \widehat{X}$ be the component corresponding to v, recall that C_v is nonsingular (by Remark 1.7) of genus $w(v)$. Now let $\widehat{e} \in E(\widehat{G})$ be a bridge of $\widehat{G}$ adjacent to v. Then the corresponding node $N_{\widehat{e}}$ of $\widehat{X}$ is a separating node of $\widehat{X}$, and hence $\alpha^{-1}(\alpha(N_{\widehat{e}})) = N_{\widehat{e}}$. This implies that the restriction of α to C_v ramifies at the point corresponding to $N_{\widehat{e}}$. By the Riemann-Hurwitz formula, the number of ramification points of $\alpha_{|C_v}$ is at most $2w(v) + 2$, therefore the number of bridges of $\widehat{G}$ adjacent to v is at most $2w(v) + 2$.

Now, by construction, we have a natural identification $E_v(\widehat{G}) = E_v(G)$ which identifies bridges with bridges. Hence also the number of bridges of G adjacent to v is at most $2w(v) + 2$, and we are done.

(B) $\Rightarrow$ (C) *assuming* $|V(G)| \neq 2$. We can assume $|V(G)| \geq 3$ for the case $|V(G)| = 1$ is clear; see Example 2.12. Let us first assume that G has no loops. By Lemma 4.2, the 2-edge-connected graph $(\overline{G}, \overline{w})$ is hyperelliptic.

Suppose $|V(\overline{G})| > 2$. By Lemma 4.4, $\overline{G}$ has an involution $\overline{\iota}$ such that

$$\overline{\phi} : \overline{G} \longrightarrow \overline{T} := \overline{G}/\overline{\iota}$$

is a non-degenerate harmonic morphism of degree 2, with $\overline{T}$ a tree. Let us show that $\overline{\phi}$ corresponds to a non-degenerate pseudo-harmonic morphism of degree 2, $\phi : G \to T$, with T a tree, such that $r_\phi(e) = 2$ for every bridge e. Suppose that G has a unique bridge e, which is contracted to the vertex $\overline{v}$ of $\overline{G}$; let $\overline{u} = \overline{\phi}(\overline{v}) \in V(\overline{T})$. Let T be the tree obtained from $\overline{T}$ by replacing the vertex $\overline{u}$ by a bridge e' and its two endpoints in such a way that there exists a morphism $\phi : G \to T$ mapping e to e' fitting in a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\phi"'] & \overline{G} \arrow[d, "\overline{\phi}"] \\ T \arrow[r] & \overline{T} \end{tikzcd} \tag{26}$$

where the horizontal arrows are the maps contracting e and e' (it is trivial to check that such a ϕ exists). To make ϕ into an indexed morphism of degree

2 we set $r_\phi(e) = 2$ and we set all other indeces to be equal to 1. Since $\overline{\phi}$ was harmonic and non-degenerate, we have that ϕ is pseudo-harmonic and non-degenerate.

If G has any number of bridges, we iterate this construction one bridge at the time. This clearly yields a pseudo-harmonic, degree 2, non-degenerate morphism $\phi : G \to T$ where T is a tree.

We claim that condition (10) holds. Indeed, we have $r_\phi(e) = 2$ if and only if e is a bridge. Therefore (10) needs only be verified at the vertices of G that are adjacent to some bridge; notice that for any such vertex v we have $m_\phi(v) = 2$. Writing $\mathrm{brdg}(v)$ for the number of bridges adjacent to v, we have, as by hypothesis, $\mathrm{brdg}(v) \leq 2w(v) + 2$,

$$\sum_{e \in E_v(G) \cap E_\phi^{\mathrm{hor}}(G)} (r_\phi(e) - 1) \leq \mathrm{brdg}(v) \leq 2w(v) + 2 = 2(w(v) + m_\phi(v) - 1).$$

This proves that (10) holds, that is, (G, w) is a 2-gonal graph. So we are done.

Suppose $|V(\overline{G})| = 2$, hence the bridges of G are leaf-edges. By Corollary 4.5, if $|E(\overline{G})| \geq 3$, then all the weights are zero, hence, as G is stable, $G = \overline{G}$, which is excluded. If $|E(\overline{G})| = 2$, then the vertices must be fixed by the hyperelliptic involution (for otherwise they would have weight zero by Lemma 4.4, contradicting that the genus be at least 2). But then $\overline{G}$ has clearly an involution $\overline{\iota}$ swapping its two edges and fixing the two vertices, whose quotient is a non-degenerate harmonic morphism of degree 2 to a tree, as in the previous part of the proof, which therefore applies also in the present case.

Suppose $|V(\overline{G})| = 1$. Then G is a tree, hence the identity map $G \to G$ with all indeces equal to 2 is a pseudo-harmonic morphism, ϕ, of degree 2. Arguing as in the previous part we get ϕ is harmonic; so we are done.

Finally, suppose G admits some loops. Let (G^0, w^0) be the loopless model; then $|V(G^0)| \geq 3$. By the previous part we have that (G^0, w^0) is 2-gonal, hence so is (G, w).

(B) $\Rightarrow$ (A) *assuming* $|V(G)| = 2$. If G has loops, then $|V(G^0)| \geq 3$ and we can use the previous implications (B) $\Rightarrow$ (C) $\Rightarrow$ (A). So we assume G loopless. By [HM82], hyperelliptic curves with two components are easy to describe. Let $X = C_1 \cup C_2$ with C_i smooth, hyperelliptic of genus $w(v_i)$ and such that $X \in M^{\mathrm{alg}}(G, w)$. If $|E(G)| = 1$ for X to be hyperelliptic it suffices to glue $p_1 \in C_1$ to $p_2 \in C_2$ with p_i Weierstrass point of C_i for $i = 1, 2$.

If $|E(G)| = 2$ for X to be hyperelliptic it suffices to glue $p_1, q_1 \in C_1$ to $p_2, q_2 \in C_2$ with $h^0(C_i, p_i + q_i) \geq 2$ for $i = 1, 2$.

If $|E(G)| \geq 3$, by Corollary 4.5 all weights are zero. For X to be hyperelliptic it suffices to pick two copies of the same rational curve with $|E(G)|$ marked points, and glue the two copies at the corresponding marked points. The theorem is proved. ∎

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