THETA-CHARACTERISTICS ON TROPICAL CURVES

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ABSTRACT. We give an explicit description of theta-characteristics on tropical curves and characterize the effective ones. We construct the moduli space, T_g^{trop}, for tropical theta-characteristics of genus g as a generalized cone complex. We describe the fibers of the specialization map from the moduli space $\mathcal{S}_g$, of theta-characteristics on algebraic curves, to T_g^{trop}.

1. INTRODUCTION

Let X be a smooth algebraic curve of genus g over an algebraically closed field of characteristic different from 2. A theta-characteristic on X is a line bundle of degree $g - 1$ on X such that $L^{\otimes 2} \cong K_X$, where K_X is the canonical line bundle of X. Theta-characteristics on algebraic curves have been extensively studied over the years because of their remarkable geometry. In particular, there is a notion of parity on theta-characteristics (given by the parity of the dimension of its space of global sections) which is deformation invariant (see [Mum71]). Each algebraic curve admits 2^{2g} theta-characteristics, $(2^g + 1)2^{g-1}$ of which are even and $(2^g - 1)2^{g-1}$ of which are odd. The moduli space $\mathcal{S}_g$ of theta-characteristics on curves of genus g splits in two connected components, according to the parity.

Let now Γ be a tropical curve of genus g and let K_Γ be the canonical divisor on Γ. A tropical theta-characteristic on Γ is a divisor class $[T] \in \mathrm{Pic}^{g-1}(\Gamma)$ such that $[2T] = [K_\Gamma]$. Tropical theta-characteristics on pure tropical curves have been studied by Zharkov in [Zha10] and shown to be associated to orientations given by the choice of a cyclic subcurve of Γ. There are therefore 2^g theta-characteristics on a pure tropical curve of genus g, and among these exactly one is non-effective.

In this paper, we study tropical theta-characteristics on any tropical curve (not necessarily pure). In Proposition 3.6 we generalize Zharkov's results by exhibiting explicit representatives for tropical theta-characteristics obtained from orientations associated to suitable flows on the curve and classify the ones that are effective (see Theorem 3.7). Our choice of representatives is different from Zharkov's, and it is inspired by the geometry of the moduli space of stable spin curves $\overline{\mathcal{S}}_g$ defined by Cornalba in [C89]. Our representatives are well behaved with respect to specialization, and we are able to construct,

in Theorem 4.1, a moduli space, T_g^{trop}, for tropical theta-characteristics as a generalized cone complexes.

Unlike the situation for classical algebraic curves, there is no natural notion of parity on tropical theta-characteristics. In our paper [CMP20], we introduced the notion of tropical spin curve, which is a tropical curve together with a spin structure, i.e., a theta-characteristic together with a suitable sign function, which encodes the notion of parity. Theta-characteristics on tropical curves are obtained from spin structures simply by forgetting the sign function. In loc. cit, we constructed a moduli space S_g^{trop} for tropical spin curves of given genus and we showed that this space is related with Cornalba's moduli space via Berkovich analytification. There is a natural morphism of generalized cone complexes, $S_g^{\mathrm{trop}} \longrightarrow M_g^{\mathrm{trop}}$, to the moduli space of tropical curves, M_g^{trop}. From the results of Section 4 we obtain a natural factorization

$$S_g^{\mathrm{trop}} \longrightarrow T_g^{\mathrm{trop}} \longrightarrow M_g^{\mathrm{trop}}$$

via morphisms of generalized cone complexes.

Let now X_K be a smooth algebraic curve of genus g over a non-Archimedian algebraically closed field K of characteristic different from 2, and let Γ be the genus g tropical curve which is the skeleton of X_K. There is a specialization map $\tau \colon \mathrm{Pic}(X_K) \to \mathrm{Pic}(\Gamma)$, which is well-known to be a homomorphism. Let S_{X_K} be the set of theta-characteristics on X_K and let T_Γ^{trop} be the set of (tropical) theta-characteristics on Γ. Since τ is a homomorphism, it restricts to a map $S_{X_K} \to T_\Gamma^{\mathrm{trop}}$. It is quite interesting to understand the fibers of the specialization map over a given tropical theta-characteristic of Γ. This problem has been considered by various authors: in the case of tropical plane quartics by Baker, Len, Morrison, Pflueger and Ren in [BLMPR16], Chan, Jiradilok in [CJ] and Len, Markwig in [LM20], for hyperelliptic curves by Panizzut in [Pan16] and by Jensen and Len in [JL18] in the case of pure tropical curves. Jensen and Len showed that the specialization map $S_{X_K} \to T_\Gamma^{\mathrm{trop}}$ is surjective. Moreover, there are 2^g even theta-characteristics specializing to the non-effective tropical theta-characteristic, and each effective theta-characteristic is the image of 2^{g-1} even theta-characteristics and 2^{g-1} odd theta-characteristics. Jensen and Len's result follows by carefully analyzing the Weil pairing on the set of 2-torsion points in the Jacobian of smooth curves.

In the second part of this paper we propose a different approach to study the specialization map for theta-characteristics, based on the description of tropical theta-characteristics given in the first part. Our results rely on our approach to tropical theta-characteristics in [CMP20], which allows us to give an interpretation to the specialization map using the combinatorial description of the boundary of Cornalba's moduli space $\overline{\mathcal{S}_g}$. In Theorem 5.4 we use this approach to obtain a generalization of Jensen and Len's results for any tropical curve (not necessarily pure). We point out that by using

the results in [CMP20] the proof of Theorem 5.4 is quite simple compared to the previous approaches.

2. Preliminaries

2.1. **Graphs.** In this paper we denote by $G = (V, E)$ a connected graph. Given a vertex $v \in V$, we write $\deg_G(v)$, sometimes $\deg(v)$, for the number of edges of G incident to v, with loops counting twice. Sometimes we will use the notation $E(G)$ and $V(G)$ for the set of edges and vertices of G.

Given a subset $R \subset E$, we will frequently abuse the notation by writing R to denote the subgraph of G generated by R. For $R \subset E$, the R-subdivision (or simply subdivision) of G, written $\widehat{G}_R$, is the graph obtained from G by inserting exactly one vertex in the interior of each edge $e \in R$.

An orientation O on G is a pair $O = (\sigma, \tau)$ where $\sigma, \tau \colon E \to V$, called source and target map, are such that $\{\sigma(e), \tau(e)\} = \{u_e, v_e\}$, where we write u_e, v_e for the ends of e. A sub-orientation on G is an orientation on some subdivision of G.

Given an orientation O on G, if $\widehat{G}$ is the R-subdivision of G for $R \subset E$, we have a natural orientation induced on $\widehat{G}$ (i.e. a sub-orientation on G), which we call, again, O. More precisely, let e_0, e_1 be the edges of $\widehat{G}$ obtained after inserting one vertex in the interior of $e \in R$. If e_0 is incident to $\sigma(e)$ and e_1 to $\tau(e)$, then we set $\sigma(e_0) = \sigma(e)$ and $\tau(e_1) = \tau(e)$, as in the picture.

A (vertex weighted) graph (G, w) is a (connected) graph G endowed with a function $w \colon V(G) \to \mathbb{Z}_{\geq 0}$. The genus of (G, w) is

$$g = \sum_{v \in V} w(v) + b_1(G) = \sum_{v \in V} w(v) + |E| - |V| + 1.$$

We say that (G, w) is stable if $2w(v) - 2 + \deg_G(v) > 0$, for every $v \in V$.

We denote by $\mathcal{E}_G$, respectively $\mathcal{V}_G$, the vector space over $\mathbb{F}_2$ spanned by E, respectively by V. We consider the linear map $\partial \colon \mathcal{E}_G \to \mathcal{V}_G$ such that for every subset $S \subset E$,

$$\partial \left(\sum_{e \in S} e \right) = \sum_{e \in S} (u_e + v_e).$$

We denote by $\mathcal{C}_G$ the kernel of ∂, and we call an element of $\mathcal{C}_G$ cyclic or a cyclic subgraph of G. It is well known that $\mathcal{C}_G$ is generated by the cycles of G, (i.e. by the connected subgraphs all of whose vertices have degree 2.)

2.2. **Tropical curves.** A (connected) *tropical curve* is a triple $\Gamma = (G, w, \ell)$, where (G, w) is a stable graph and ℓ a length function, $\ell \colon E \to \mathbb{R}_{>0}$. We call (G, w) the combinatorial type of Γ. The *genus* of $\Gamma = (G, w, \ell)$ is the genus of (G, w), usually denoted by g.

We view Γ as a metric space as follows. Every edge e is identified with a closed interval of length $\ell(e)$ if e is not a loop, and with a circle of length $\ell(e)$ if e is a loop. Then every path in Γ has a well defined length. Now given two points $p, q \in \Gamma$ we define the distance between them, $d(p, q)$, as the shortest length of a path in Γ from p to q.

For any edge e of G, we let $p_e \in \Gamma$ be the mid-point of e.

We extend the weight function to all points of Γ as follows

$$w_\Gamma(p) = \begin{cases} w(p) & \text{if } p \in V \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we set

$$\deg_\Gamma(p) = \begin{cases} \deg_G(p) & \text{if } p \in V \\ 2, & \text{otherwise.} \end{cases}$$

We write $w = w_\Gamma$ and $\deg = \deg_\Gamma$ when no confusion is likely.

A *divisor* on Γ is a formal sum $D = \sum_{p \in \Gamma} D(p)p$, with $D(p) \in \mathbb{Z}$, where $D(p)$ is nonzero only for a finite set of points of Γ. We write $D \geq 0$ if $D(p) \geq 0$ for all $p \in \Gamma$, and we say D is *effective*. The degree of D is the integer $\sum_{p \in \Gamma} D(p)$. We let $\mathrm{Div}(\Gamma)$ be the group of divisors on Γ and $\mathrm{Div}^d(\Gamma) \subset \mathrm{Div}(\Gamma)$ the subset of divisors of degree d.

A *rational function* f on Γ is a continuous piecewise linear function $f \colon \Gamma \to \mathbb{R}$ with integral slopes and finitely many pieces. The points of Γ where f is not linear will be called *critical* points of f. Given a rational function f on Γ, for every $p \in \Gamma$, we let $\mathrm{ord}_p(f)$ be the sum of the slopes of f outgoing p (there are $\deg_\Gamma(p)$ of them). If f is linear at p then $\mathrm{ord}_p(f) = 0$, therefore the following

$$\mathrm{div}(f) := \sum_{p \in \Gamma} \mathrm{ord}_p(f)p$$

is a finite sum, hence a divisor, and it is easy to see that $\mathrm{div}(f) \in \mathrm{Div}^0(\Gamma)$. The divisors of the form $\mathrm{div}(f)$ are called *principal divisors*. The set of all principal divisors is a subgroup of $\mathrm{Div}(\Gamma)$ denoted by $\mathrm{Prin}(\Gamma)$. The *Picard group* of Γ is

$$\mathrm{Pic}(\Gamma) := \mathrm{Div}(\Gamma)/\mathrm{Prin}(\Gamma),$$

and $\mathrm{Pic}^d(\Gamma) \subset \mathrm{Pic}(\Gamma)$ is the set of divisor classes of degree d. For $D \in \mathrm{Div}(\Gamma)$ we denote by $[D] \in \mathrm{Pic}(\Gamma)$ its class and we write $D \sim D'$ if $[D] = [D']$.

fe **Example 2.1.** Fix a non-loop edge e of Γ and let u_e, v_e be its ends. Let us identify e with the interval $[-\ell(e)/2, \ell(e)/2]$ so that its mid-point, p_e, gets

identified with 0. Now consider the rational function f_e on Γ defined as follows.

$$f_e(x) = \begin{cases} \ell(e)/2 + x & \text{if } x \in [-\ell(e)/2, 0] \\ \ell(e)/2 - x & \text{if } x \in [0, \ell(e)/2] \\ 0 & \text{otherwise, i.e. if } x \notin e. \end{cases}$$

Then

$$\text{div}(f_e) = u_e + v_e - 2p_e.$$

If e is a loop based at a vertex $v_e \in V$, we identify e with the circle obtained from the interval $[-\ell(e)/2, \ell(e)/2]$ by identifying its ends. Then f_e defines again a rational function, and we have $\text{div}(f_e) = 2v_e - 2p_e$.

From the previous example we highlight the following basic relation:

(1) $$u_e + v_e \sim 2p_e$$

for any edge e of Γ with (possibly equal) ends u_e, v_e.

Let $O = (\sigma, \tau)$ be a sub-orientation on G supported on the subdivision $\widehat{G}$. For $p \in \Gamma$ we denote by $\deg_O^-(p)$ the number of edges directed towards p, i.e.

$$\deg_O^-(p) = \begin{cases} |\{e \in E(\widehat{G}) : \tau(e) = p\}|, & \text{if } p \in V(\widehat{G}); \\ 1, & \text{otherwise.} \end{cases}$$

We define the following divisor on Γ:

$$D_O^- := \sum_{p \in \Gamma} (\deg_O^-(p) - 1 + w(p))p.$$

The divisor D_O^- has degree $g - 1$ and is supported on some subset of $V(\widehat{G})$.

3. Theta-characteristics on tropical curves

3.1. **Tropical square roots and theta-characteristics.** The canonical divisor of a tropical curve $\Gamma = (G, w, \ell)$ is

$$K_\Gamma := \sum_{p \in \Gamma} (2w(p) - 2 + \deg(p))p = \sum_{v \in V} (2w(v) - 2 + \deg(v))v$$

The divisor K_Γ has degree $2g - 2$, with g the genus of Γ.

Definition 3.1. A *(tropical) square root (of zero)* of Γ is a divisor class $[D] \in \text{Pic}(\Gamma)$ such that $[2D] = 0$. The set of square roots of Γ is denoted by R_Γ^{trop}.

A *(tropical) theta-characteristic* of Γ is a divisor class $[D] \in \text{Pic}(\Gamma)$ such that $[2D] = [K_\Gamma]$. The set of theta-characteristics of Γ is denoted by T_Γ^{trop}.

Obviously, $R_\Gamma^{\text{trop}} \subset \text{Pic}^0(\Gamma)$ and $T_\Gamma^{\text{trop}} \subset \text{Pic}^{g-1}(\Gamma)$.

Our next goal is to give an explicit description of all square roots and all theta-characteristics.

Let $P \in \mathcal{C}_G$, then $\deg_P(v)$ is even for every $v \in V(G)$. We can thus define the following divisor on Γ

`F_P` (2)
$$F_P := \sum_{v \in V} \frac{\deg_P(v)}{2} v - \sum_{e \in P} p_e.$$

We have, using Example 2.1,

$$2F_P = \sum_{v \in V} \deg_P(v) v - 2 \sum_{e \in P} p_e = \sum_{e \in P} (v_e + u_e - 2p_e) = \sum_{e \in P} \mathrm{div}(f_e).$$

Therefore $2F_P$ is a principal divisor, in other words, $[F_P]$ is a square root (of zero). Let us show that, as P varies in $\mathcal{C}_G$, the classes of the divisors F_P are distinct and give all the square roots of Γ.

`distinct` **Proposition 3.2.** *Let $\Gamma = (G, w, \ell)$ be a tropical curve. Then the following is a bijection*

$$\mathcal{C}_G \longrightarrow R_\Gamma^{\mathrm{trop}}; \qquad P \mapsto [F_P]$$

Proof. Recall from [MZ08] and [BMV11] that there is a natural group isomorphism,

$$\mathrm{Pic}^0(\Gamma) \cong H_1(G, \mathbb{R})/H_1(G, \mathbb{Z}),$$

and the set of 2-torsion points of $\mathrm{Pic}^0(\Gamma)$ (the square roots of Γ) is identified with $H_1(G, \mathbb{Z}/2\mathbb{Z}) = \mathcal{C}_G$. Hence R_Γ^{trop} has the same cardinality as $\mathcal{C}_G$. There remains to prove that the $[F_P]$ are all distinct. It is easy to check that for every $P, P' \in \mathcal{C}_G$ we have

$$F_P + F_{P'} - F_{P+P'} = \sum_{e \in P \cap P'} \mathrm{div}(f_e).$$

Hence

$$F_P - F_{P'} \sim F_P - F_{P'} + 2F_{P'} = F_P + F_{P'} \sim F_{P+P'}.$$

Therefore it suffices to show that $[F_P] \neq 0$ for every nonzero $P \in \mathcal{C}_G$. By (2) we have $F_P(p) = 0$ unless p is either a vertex of P (in which case $F_P(p) \geq 1$) or the mid-point p_e of an edge of P (in which case $F_P(p_e) = -1$). By contradiction, assume that for some (nonzero) cycle P we have $\mathrm{div}(f) = F_P$, with f a rational function on Γ. Let $\Gamma_{\max}$ be the locus of Γ where f attains its maximum. Since Γ is compact, $\Gamma_{\max}$ is a non-empty closed subcurve, moreover $\Gamma_{\max} \neq \Gamma$ (as otherwise $\mathrm{div}(f) = 0$).

We know that the slopes of f outgoing $\Gamma_{\max}$ are all negative, while the slopes of f internal to $\Gamma_{\max}$ are zero (f is constant on $\Gamma_{\max}$). Therefore $\mathrm{div}(f)(p) < 0$ for every $p \in \Gamma_{\max} \cap \overline{\Gamma \smallsetminus \Gamma_{\max}}$. Since the only points of Γ where F_P is negative are the mid-points p_e of the edges in P, we deduce

`Gammamax` (3)
$$\Gamma_{\max} \cap \overline{\Gamma \smallsetminus \Gamma_{\max}} \subset \{p_e, \ e \in P\}.$$

Let us show that no vertex of P can be contained in $\Gamma_{\max}$. If $v \in V(P)$ were in $\Gamma_{\max}$, it would have to lie in its interior, by (3), and hence $\mathrm{div}(f)(v) = 0$ (as f is constant on $\Gamma_{\max}$), which contradicts $F_P(v) \geq 1$.

We then conclude that $\Gamma_{\max}$ is a union of mid-points p_e, for some edges e of P. Since the slopes outgoing $\Gamma_{\max}$ are negative we obtain $\mathrm{div}(f)(p_e) \leq -2$ for every such p_e, in contradiction with $F_P(p_e) = -1$. ♣

We now turn to theta-characterstics; again, let $P \in \mathcal{C}_G$. Consider the following divisor on Γ,

$$\boxed{\text{TP}} \quad (4) \qquad T_P := \sum_{v \in V} \left(\frac{\deg_P(v)}{2} - 1 + w(v) \right) v + \sum_{e \in E \smallsetminus P} p_e.$$

Let us show that as P varies the $[T_P]$ give all the theta-characteristics of Γ.

$\boxed{\text{distinctt}}$ **Proposition 3.3.** *Let* $\Gamma = (G, w, \ell)$ *be a tropical curve and* $P \in \mathcal{C}_G$. *Then* $[T_P]$ *is a theta-characteristic and the following is a bijection*

$$\mathcal{C}_G \xrightarrow{\ \beta\ } T_\Gamma^{\mathrm{trop}}; \qquad P \mapsto [T_P].$$

Proof. Recalling the definition of F_P we have

$$\boxed{\text{T\_P}} \quad (5) \qquad T_P = \sum_{v \in V} (w(v) - 1)\, v + F_P + \sum_{e \in E} p_e.$$

As $2F_P \sim 0$ we get

$$2T_P \sim \sum_{v \in V} (2w(v) - 2)\, v + 2 \sum_{e \in E} p_e \sim \sum_{v \in V} (2w(v) - 2)\, v + \sum_{e \in E}(u_e + v_e)$$

using (1). Since $\sum_{e \in E}(u_e + v_e) = \sum_{v \in V} \deg_G(v) v$ we have $2T_P \sim K_\Gamma$, i.e. $[T_P]$ is a theta-characteristic.

It remains to show that if $P \neq P'$ then $[T_P] \neq [T_{P'}]$. Using (5) we have

$$T_P - T_{P'} = F_P - F_{P'}$$

and $F_P - F_{P'} \not\sim 0$, by Proposition 3.2. The proof is complete. ♣

Notice that the following

$$K_\Gamma/2 = \frac{\sum_{v \in V}(2w(v) - 2 + \deg(v))v}{2}$$

is a divisor on Γ if and only if G has only vertices of even degree, i.e. if and only if G is cyclic. Now, for any $P \in \mathcal{C}_G$ we denote $\overline{P} = P \cup V$ and

$$\Gamma_{\overline{P}} = (\overline{P}, w_{|\overline{P}}, \ell_{|P}) \subset \Gamma,$$

in other words $\Gamma_{\overline{P}}$ is the subcurve of Γ whose underlying graph is $\overline{P}$, with the same weight function as Γ. Then $K_{\Gamma_{\overline{P}}}/2$ is a divisor (on both Γ and $\Gamma_{\overline{P}}$) and we have

$$T_P = K_{\Gamma_{\overline{P}}}/2 + \sum_{e \in E \smallsetminus P} p_e.$$

3.2. **Effective theta-characteristics via flows.** We say that a theta-characteristic $[T_P]$ on a tropical curve Γ is *effective* if its rank is nonnegative, i.e. if $r_\Gamma(T_P) \neq -1$. Equivalently, $[T_P]$ is effective if there exists an effective divisor E on Γ such that $E \sim T_P$. Our next goal is to characterize effective theta-characteristics, and describe the effective representatives.

Recall the definition $T_P = \sum_{v \in V} \left(\frac{\deg_P(v)}{2} - 1 + w(v) \right) v + \sum_{e \in E \smallsetminus P} p_e$. We write $T_P = T_P^+ - T_P^-$ with T_P^+ and T_P^- effective. We have

$$T_P^- = \sum_{\substack{v \in V \smallsetminus V(P) \\ w(v) = 0}} v.$$

In other words, $T_P \geq 0$ if and only if every vertex of G having weight zero is contained in P.

Definition 3.4. Let $P \in \mathcal{C}_G$ and $W \subset V$. Assume that $P \cup W$ is not empty. The *cyclic subcurve* of Γ associated to P and W is the tropical subcurve $\Gamma_{P,W} \subset \Gamma$ supported on the graph $P \cup W$ (we consider W to be a subgraph of G with vertex set W and no edges).

Consider a cyclic subcurve $\Gamma_{P,W}$ of Γ. Let us define a sub-orientation $O_{P,W}$ on G and a divisor $D_{P,W}$ on Γ. Informally, $O_{P,W}$ will be the *flow* away from $\Gamma_{P,W}$. More precisely, we choose on P a cyclic orientation; there are $2^{b_1(P)}$ ways to do it, the choice of which will be irrelevant. Next, we orient the edges in the complement of P away from $\Gamma_{P,W}$, as follows. Consider the distance function

$$d_{P,W} : \Gamma \longrightarrow \mathbb{R}_{\geq 0}$$
$$p \longmapsto d(p, \Gamma_{P,W}),$$

where $d(p, \Gamma_{P,W})$ denotes the distance from p to $\Gamma_{P,W}$. The function $d_{P,W}$ is easily seen to be continuous and piecewise linear with slope 0 on P and ± 1 away from P. Moreover, for any edge e of G, the function $d_{P,W}$ has at most one critical point lying in the interior of e.

We consider the subdivision $\widehat{G}_{P,W}$ of G given by inserting a vertex at each critical point of $d_{P,W}$. We orient each edge of $\widehat{G}_{P,W} - P$ towards the maximum of $d_{P,W}$, so that we have a sub-orientation $O_{P,W}$ on G. We set

<div style="border:1px solid; display:inline-block; padding:2px;">eq:DPW</div> (6) $$D_{P,W} := D_{O_{P,W}}^- = \sum_{v \in V(\widehat{G}_{P,W})} \left(\deg_{O_{P,W}}^-(v) - 1 + w_\Gamma(v) \right) v.$$

Remark 3.5. If $W = V$ every vertex v not lying on P is a source (i.e. $\deg_{O_{P,V}}^-(v) = 0$) and p_e is a sink (hence $\deg_{O_{P,V}}^-(p_e) = 2$) for every edge e not in P. Therefore by (4) we have

<div style="border:1px solid; display:inline-block; padding:2px;">eq:DP</div> (7) $$D_{P,V} = T_P.$$

<div style="border:1px solid; display:inline-block; padding:2px;">DPS</div> **Proposition 3.6.** *Let $\Gamma_{P,W}$ and $\Gamma_{P',W'}$ be two cyclic subcurves of a tropical curve Γ. Then*

<div style="border:1px solid; display:inline-block; padding:2px;">DPS1</div>   *(1) $D_{P,W} \sim D_{P',W'}$ if and only if $P = P'$.*

| DPS2 | *(2) $[D_{P,W}]$ is a theta-characteristic on Γ, that is, $2D_{P,W} \sim K_\Gamma$.* |

Proof. We first prove $D_{P,W} \sim D_{P,W \cup \{v\}}$ for any vertex $v \in \Gamma$. Consider the following function f on Γ

$$f := \frac{d_{P,W \cup \{v\}} - d_{P,W}}{2}.$$

Of course, f is continuous and piecewise linear with finitely many pieces. We will show that f has integer slopes, i.e. is a rational function, and that $D_{P,W} - D_{P,W \cup \{v\}} = \operatorname{div}(f)$.

Set $O_1 := O_{P,W}$ and $O_2 := O_{P,W \cup \{v\}}$. Assume without loss of generality that O_1 and O_2 coincide over P. For $p \in \Gamma$ we have

eq:difference (8)
$$D_{P,W}(p) - D_{P,W \cup \{v\}}(p) = \deg_{O_1}^-(p) - \deg_{O_2}^-(p).$$

Let $B(p, \epsilon) \subset \Gamma$ be the closed ball with center p and radius $\epsilon \in \mathbb{R}_{>0}$. If ϵ is small enough $B(p, \epsilon)$ is the union of segments h_i of length ϵ and incident to p, for $i = 1, \ldots, \deg_\Gamma(p)$, over which both $d_{P,W}$ and $d_{P,W \cup \{v\}}$ are linear. Fix one such h_i.

If O_1 and O_2 coincide along h_i, then $d_{P,W}$ and $d_{P,W \cup \{v\}}$ have the same slope on it. Therefore, their difference is a constant function on h_i, and the slope of f at p along h_i is 0.

If O_1 and O_2 do not coincide along h_i, then h_i is not contained in P, the slopes of $d_{P,W}$ and $d_{P,W \cup \{v\}}$ have absolute value 1 and opposite signs. Therefore their sum is even and f has slope ± 1 along h_i. So f is a rational function. Suppose that p is the target of h_i according to O_1, and the source according to O_2, Then the contribution to $\deg_{O_1}^-(p)$ along h_i is 1, and the contribution to $\deg_{O_2}^-(p)$ is zero, so their difference is 1. Moreover, the slope of $d_{P,W}$ on h_i is equal to -1 whereas the slope of $d_{P,W \cup \{v\}}$ is $+1$, hence the contribution to $\operatorname{div}(f)(p)$ along h_i is $(1 - (-1))/2 = 1$.

Repeating this for all segments h_i we get $\operatorname{div}(f)(p) = \deg_{O_1}^-(p) - \deg_{O_2}^-(p)$, hence by (8) we have $\operatorname{div}(f)(p) = D_{P,W}(p) - D_{P,W \cup \{v\}}(p)$, as wanted.

Therefore $D_{P,W} \sim D_{P,W \cup \{v\}}$, hence $D_{P,W} \sim D_{P,W'}$ for any W, W'.

In particular $D_{P,W} \sim D_{P,V}$, hence, as $D_{P,V} = T_V$, by Proposition 3.3 $D_{P,W}$ is a theta-characteristic. Part (2) is proved.

It remains to prove that $D_{P,W}$ and $D_{P',W'}$ are not equivalent if $P \neq P'$. Now it is enough to show that $D_{P,V}$ and $D_{P',V}$ are not equivalent, i.e. that T_P and $T_{P'}$ are not equivalent, which follows again by Proposition 3.3. ♣

We write $V_+(\Gamma) = \{v \in V : w(v) > 0\}$, sometime just $V_+ = V_+(\Gamma)$. We say that Γ is a *pure* tropical curve if $V_+(\Gamma) = \emptyset$, equivalently, if $w = 0$.

We now prove that theta-characteristics are always effective with the only exception of the theta-characteristic $[T_0]$ on a pure tropical curve.

thm:rank **Theorem 3.7.** *Let $\Gamma = (G, w, \ell)$ be a tropical curve and $P \in \mathcal{C}_G$. Then $r_\Gamma(T_P) = -1$ if and only if $P = 0$ and Γ is pure.*

The case of a pure tropical curve is known; see [Zha10]. We include the full proof for completeness and better clarity. First we note the following simple fact.

eff

Lemma 3.8. *Let $P \in \mathcal{C}_G$ and $W \subset V$ with $P \cup W$ non empty. Then*

eff1   *(1) $D_{P,W} \geq 0$ if and only if $w(v) > 0$ for every $v \in W \smallsetminus V(P)$.*

effz   *(2) $D_{0,W} \geq 0$ if and only if $W \subset V_+$.*

effe   *(3) $D_{P,\emptyset} \geq 0$.*

Proof. Notice that (2) and (3) follow trivially from (1).

Let us prove (1); we have

$$\deg^-_{O_{P,W}}(v) \begin{cases} = 0 & \text{if } v \in W \smallsetminus V(P) \\ \geq 1 & \text{otherwise.} \end{cases}$$

Therefore, writing $D_{P,W} = D^+_{P,W} - D^-_{P,W}$ as the difference of two effective divisors, we have

$$D^-_{P,W} = \sum_{\substack{v \in W \smallsetminus V(P) \\ w(v) = 0}} v$$

which is zero if and only if $w(v) > 0$ for every $v \in W \smallsetminus V(P)$.   ♣

Now we prove Theorem 3.7.

Proof. By Lemma 3.8(3), if $P \neq 0$ then $D_{P,\emptyset} \geq 0$. By Proposition 3.6 we have $D_{P,\emptyset} \sim D_{P,V}$ and since $D_{P,V} = T_P$ we conclude that $[T_P]$ is effective.

We are left with the case $P = 0$.

If Γ is not pure then $V_+ \neq \emptyset$, hence we can consider D_{0,V_+}, which is a representative for $[T_0]$. By Lemma 3.8 we have $D_{0,V_+} \geq 0$, hence $[T_0]$ is effective. This concludes the proof in case Γ is not pure.

Now suppose Γ is pure. To finish the proof we need to show

$$r_\Gamma(T_0) = -1$$

It suffices to prove that $r_\Gamma(D_{0,v}) = -1$ for any vertex v of Γ. By definition, $D_{0,v}$ is associated to the sub-orientation $O_{0,v}$, which is clearly acyclic (having v as a source). Therefore, $D_{0,v}$ is a reduced divisor with respect to v in the sense of [BN07], so the fact that $D_{0,v}(v) < 0$ implies that it has rank equal -1 (see e.g., Lemma 3.8(a) in [CLM15]). In fact, $D_{0,v}$ is a moderator in the sense of [MZ08, Def. 7.8], so the fact that it has rank -1 follows also from [MZ08, Lm. 7.10].   ♣

Remark 3.9. If Γ is a pure tropical curve, then the theta-characteristics described by Zharkov in [Zha10] are the classes of the divisors $D_{P,W}$ where $W = \emptyset$ for $P \neq 0$, and $|W| = 1$ for $P = 0$.

4. The moduli space of tropical theta-characteristics

The moduli space, M_g^{trop}, of tropical curves of genus g, first constructed in [BMV11], has the structure of a generalized cone complex (see [ACP15]). The points in M_g^{trop} are in bijective correspondence with (equivalence classes of) tropical curves of genus g.

Similarly, the moduli space S_g^{trop}, of spin tropical curves of genus g, constructed in [CMP20, sec. 2.5] is a generalized cone complex, and there is a natural morphism of generalized cone complexes

$$\pi^{\mathrm{trop}} : S_g^{\mathrm{trop}} \longrightarrow M_g^{\mathrm{trop}}.$$

This morphism has an explicit connection with the analogous moduli spaces of Deligne-Mumford stable algebraic curves, $\overline{\mathcal{M}}_g$, and stable spin curves, $\overline{\mathcal{S}}_g$, for which we refer to loc. cit. Theorem C.

We shall now construct the moduli space of tropical theta-characteristics on tropical curves of genus g as a generalized cone complex, and relate it to M_g^{trop} and S_g^{trop}. Since the procedure is very similar to the one used in [ACP15] and [CMP20] we will skip many details.

By Proposition 3.3, for any tropical curve $\Gamma = (G, w, \ell)$ of combinatorial type (G, w), we have the isomorphism $\mathcal{C}_G \cong T_\Gamma^{\mathrm{trop}}$ associating to each cyclic subgraph P of G the theta-characteristic $[T_P]$. Therefore, for each tropical curve of combinatorial type (G, w), there is exactly one theta-characteristic associated to the choice of each cyclic subgraph P of G.

Notice that a non-trivial automorphism of Γ may fix a cyclic subgraph of G, that is, $\mathrm{Aut}(\Gamma)$ may act non-trivially on T_Γ^{trop}.

For any $P \in \mathcal{C}_G$ we denote by $T_{(G,P)}^{\mathrm{trop}}$ the set of isomorphism classes of all theta-characteristics of type $[T_P]$ on all tropical curves $\Gamma = (G, w, \ell)$.

We consider the poset of cyclic graphs of genus g, i.e.:

$$\mathcal{C}_g := \bigsqcup_{G \in \mathcal{G}_g} \mathcal{C}_G$$

where $\mathcal{G}_g$ is the poset of stable graphs of genus g, partially ordered by edge contraction. Elements in $\mathcal{C}_g$ are written as pairs (G, P) with G a stable graph and $P \in \mathcal{C}_G$. The poset structure on $\mathcal{C}_g$ is given by edge contraction, as follows. For (G, P) and (G', P') in $\mathcal{C}_g$ we say that $(G, P) \geq (G', P')$ if there exists a contraction $\gamma \colon G \to G'$ such that $\gamma_* P = P'$ (in this case we have $G \geq G'$ by definition). By [CMP20, Prop. 2.3.1] the poset $\mathcal{C}_g$ is connected and the forgetful map $\mathcal{C}_g \to \mathcal{G}_g$ is a quotient of posets. We also consider the poset $[\mathcal{C}_g] := \sqcup_{G \in \mathcal{G}_g} \mathcal{C}_G / \mathrm{Aut}(G)$.

We introduce the category, CYC_g, whose objects are isomorphism classes of pairs (G, P) with $G \in \mathcal{G}_g$ and $P \in \mathcal{C}_G$, and whose arrows are generated by contractions and automorphisms of pairs (that is, the elements of the subgroup $\mathrm{Aut}(G, P)$ of $\mathrm{Aut}(G)$ fixing P).

We now define a contravariant functor from the category CYC_g to the category of rational polyhedral cones. To the isomorphism class of (G, P)

we associate the cone
$$\sigma_{(G,P)} = \mathbb{R}_{\geq 0}^{E(G)},$$
with the integral structure determined by the sub-lattice parametrizing tropical curves having integral edge-lengths. As usual, we write $\sigma^o_{(G,P)} = \mathbb{R}_{>0}^{E(G)}$.

To a contraction $\gamma \colon (G,P) \to (G',P')$ we associate the injection of cones $\iota_\gamma \colon \sigma_{(G',P')} \hookrightarrow \sigma_{(G,P)}$ whose image is the face of $\sigma_{(G,P)}$ where the coordinates corresponding to $E(G) \smallsetminus E(G')$ vanish. If $\gamma \in \mathrm{Aut}(G,P)$ then ι_γ is the corresponding automorphism of $\mathbb{R}_{\geq 0}^{E(G)}$. By the results in [CMP20, sect. 2], this is indeed a contravariant functor. We can therefore consider the colimit of the diagram of cones $\sigma_{(G,P)}$ using the inclusions ι_γ, for all arrows, γ, in CYC_g, which is the following generalized cone complex
$$T_g^{\mathrm{trop}} := \varinjlim \left(\sigma_{(G,P)}, \iota_\gamma \right).$$

Theorem 4.1. *The following properties hold.*

    (1) The space T_g^{trop} is the moduli space of tropical theta-characteristics and we have a stratification
$$T_g^{\mathrm{trop}} = \bigsqcup_{[G,P] \in [\mathcal{C}_g]} T_{(G,P)}^{\mathrm{trop}}.$$

    (2) We have $T_{(G,P)}^{\mathrm{trop}} \cong \sigma^o_{(G,P)}/\mathrm{Aut}(G,P)$.

    (3) T_g^{trop} is connected and has pure dimension $3g-3$.

    (4) $T_{(G',P')}^{\mathrm{trop}} \subset \overline{T_{(G,P)}^{\mathrm{trop}}}$ if and only if $(G,P) \geq (G',P')$.

Proof. A point $\underline{l} = (l_e, \ e \in E) \in \sigma_{(G,P)}$ corresponds to a pair $(\Gamma_{\underline{l}}, P)$ where $\Gamma_{\underline{l}} = (G, w, \ell,)$ is the tropical curve whose length function is $\ell(e) = l_e$ for all $e \in E$. Hence $\underline{l}$ corresponds to the theta-characteristic $T_P \in T_{\Gamma_{\underline{l}}}^{\mathrm{trop}}$. By extending this reasoning to all tropical curves of genus g we have a surjection
$$\bigcup_{\Gamma \in M_g^{\mathrm{trop}}} T_\Gamma^{\mathrm{trop}} \longrightarrow T_g^{\mathrm{trop}}$$
which identifies (Γ, T_P) with $(\Gamma', T_{P'})$ if and only if $\Gamma = \Gamma'$ and there is an automorphism of Γ mapping T_P to $T_{P'}$. From this (1) follows.

The rest is similar to the proofs of [CMP20, Props 2.5.1 and 2.5.2].   ♣

We have a canonical morphism of generalized cone complexes
$$\psi^{\mathrm{trop}} \colon T_g^{\mathrm{trop}} \longrightarrow M_g^{\mathrm{trop}}$$
sending a tropical theta-characteristic $[T_P]$ in $\Gamma = (G, w, \ell)$ (associated to the cyclic subgraph P of G) to the tropical curve $(G, w, \psi^{\mathrm{trop}}(\ell))$, where $\psi^{\mathrm{trop}}(\ell)(e) = 2\ell(e)$ if $e \in E \smallsetminus P$, and $\psi^{\mathrm{trop}}(\ell)(e) = \ell(e)$ if $e \in P$.

By the same argument as in [CMP20, Prop. 2.5.3] we have, for every tropical curve $[\Gamma] \in M_g^{\mathrm{trop}}$,
$$(\psi^{\mathrm{trop}})^{-1}(\Gamma) \cong T_\Gamma^{\mathrm{trop}}/\mathrm{Aut}(\Gamma).$$

Remark 4.2. The canonical map $\pi^{\mathrm{trop}} : S_g^{\mathrm{trop}} \longrightarrow M_g^{\mathrm{trop}}$ factors through ψ^{trop}.

5. Lifting theta-characteristics on tropical curves

From now on, we fix an algebraically closed field k of characteristic different from 2.

Let X be a stable curve over k. We let (G, w) be the (stable) dual graph of X, with $G = (V, E)$. Recall that V corresponds to the set of irreducible components of X and E to the set of its nodes. We often identify edges with nodes. The weight of a vertex is the genus of the desingularization of the corresponding component, therefore (G, w) is a stable graph and has the same genus as X.

Recall that a *spin curve* over X is a pair $(\widehat{X}, \widehat{L})$, where $\widehat{X} = X_R^\nu \cup Z$ is a quasistable curve, with X_R^ν the normalization of X at the subset of nodes corresponding to some $R \subset E$ and Z the disjoint union of smooth connected rational components meeting X_R^ν at 2 points (called *exceptional components*), and $\widehat{L}$ is a line bundle on $\widehat{X}$ such that

(1) the restriction, L_R, of $\widehat{L}$ to X_R^ν satisfies $L_R^2 \cong \omega_{X_R^\nu}$;
(2) the restriction of $\widehat{L}$ to each exceptional component E has degree 1.

We denote by $\mathcal{S}_X$ the scheme of spin curves over X. We will use the terminology *theta-characteristic* for spin curves such that $\widehat{X} = X$ (in this case, $\widehat{L}^2 \cong \omega_X$). The *parity* of a spin curve refers to the parity of $h^0(\widehat{X}, \widehat{L})$.

The *dual spin graph* of a spin curve $(\widehat{X}, \widehat{L})$ on X is the tern (G, P, s), where:

(1) G is the dual graph of X;
(2) $P = E \smallsetminus R$ (it is a cyclic subgraph of G);
(3) $s : V(G/P) \to \mathbb{Z}/2\mathbb{Z}$ is a function taking $v \in V(G/P)$ to $s(v)$, the parity of $h^0(Z_v, \widehat{L}_{|Y_v})$, where Y_v is the connected component of X_R^ν corresponding to v (the set of connected components of X_R^ν is in bijection with the set of vertices $V(G/P)$).

Recall that (G, P, s) is a spin graph in the sense of [CMP20, Def. 2.1.1] and its parity is the parity of $\sum_{v \in V(G/P)} s(v)$.

In what follows, we fix an algebraically closed non-Archimedean field K whose valuation ring is R and residue field is k.

We let X_K be a genus-g smooth curve over K, we assume that X_K extends to a stable curve $\mathcal{X}$ over $\mathrm{Spec}\, R$, whose special fiber we denote by X.

We let $\Gamma = (G, w, \ell)$ be the tropical curve given as the skeleton of the Berkovich analitification X_K^{an}. Recall that the length ℓ is defined by setting $\ell(e) = \mathrm{val}_K(f_e)$ where $xy = f_e$ is an étale local equation of $\mathcal{X}$ at the node corresponding to the edge e.

The natural retraction map $\tau\colon X_K^{\mathrm{an}} \to \Gamma$ induces by linearity a specialization homomorphism:

$$\tau_{\mathrm{Div}}\colon \mathrm{Div}(X_K) \longrightarrow \mathrm{Div}(\Gamma)$$

where $\mathrm{Div}(X_K)$ is the group of divisors on X_K.

Let us describe τ_{Div} explicitly. Let D_K be a divisor on X_K. Then there is a semistable curve $\widehat{\mathcal{X}}$ over $\mathrm{Spec}\,R$ with $\mathcal{X}$ as stable model, such that D_K extends to a Cartier divisor $\mathcal{D}$ over $\widehat{\mathcal{X}}$; we let $\widehat{X}$ be the special fiber of $\widehat{\mathcal{X}}$. The dual graph $\widehat{G}$ of $\widehat{X}$ is obtained by inserting n_e vertices of weight zero in the interior of any edge e of G, where n_e is the number of rational components of $\widehat{X}$ lying over the node of X corresponding to e. Hence we can view $V(\widehat{G})$ as a subset of Γ (see also Remark 5.1). The multidegree, $\underline{\deg}\mathcal{D}|_{\widehat{X}}$, of the restriction of $\mathcal{D}$ to $\widehat{X}$ is thus a divisor of Γ supported at $V(\widehat{G})$, and we have

<div style="text-align:right">(9)</div>

$$\tau_{\mathrm{Div}}(D_K) = \underline{\deg}\mathcal{D}|_{\widehat{X}}.$$

`eq:tau*`

`addpt`

Remark 5.1. In passing from $\mathcal{X}$ to $\widehat{\mathcal{X}}$, the tropical equivalence class of the skeleton Γ does not change. Indeed, this amounts to inserting weight zero vertices in the interior of the edges of Γ, getting a tropical curve, $\widehat{\Gamma} = (\widehat{G}, \widehat{w}, \widehat{\ell})$, having the same metric structure, and hence equivalence class, of Γ. More precisely (following [ACP15] Section 8) let v be the vertex of $\widehat{G}$ corresponding to the exceptional component, C_v, of $\widehat{X}$. Let e_0 and e_1 be the edges of $\widehat{\Gamma}$ corresponding to the two nodes in $C_v \cap \overline{\widehat{X} \smallsetminus C_v}$, and, for $i = 0, 1$, let $xy = f_{e_i}$ be an étale local equation of $\widehat{\mathcal{X}}$ at e_i. Then an étale local equation of $\mathcal{X}$ at the node e to which C_v gets contracted is $xy = f_{e_0} f_{e_1}$, therefore

$$\ell(e) = \mathrm{val}_K(f_{e_0} f_{e_1}) = \mathrm{val}_K(f_{e_0}) + \mathrm{val}_K(f_{e_1}) = \hat{\ell}(e_0) + \hat{\ell}(e_1).$$

The map τ_{Div} takes principal divisors on X_K to principal divisors on Γ (see [BR15]), hence we get a homomorphism

$$\tau_{\mathrm{Pic}}\colon \mathrm{Pic}(X_K) \longrightarrow \mathrm{Pic}(\Gamma).$$

Let $\mathcal{S}_{X_K} \subset \mathrm{Pic}(X_K)$ and $T_\Gamma^{\mathrm{trop}} \subset \mathrm{Pic}(\Gamma)$ be, respectively, the sets of theta-characteristics of X_K and of Γ. Since τ_{Pic} is a homomorphism taking class of the canonical divisor of X_K to the class of the canonical divisor of Γ, it restricts to a map:

$$\tau_{\mathcal{S}}\colon \mathcal{S}_{X_K} \longrightarrow T_\Gamma^{\mathrm{trop}}.$$

In Proposition 3.3 we introduced the bijection $\beta : \mathcal{C}_G \longrightarrow T_\Gamma^{\mathrm{trop}}$ mapping P to $[T_P]$. We now use it to construct a useful factorization of the map $\tau_{\mathcal{S}}$. First of all, we define a map

$$\alpha\colon \mathcal{S}_{X_K} \longrightarrow \mathcal{C}_G$$

as follows. Given a theta-characteristic L_K on X_K, there is a unique pair $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ with the following properties. $\widehat{\mathcal{X}}$ is a semistable curve over $\mathrm{Spec}\,R$ having $\mathcal{X}$ as stable model, $\widehat{\mathcal{L}}$ is a line bundle on $\widehat{\mathcal{X}}$ extending L_K; we let $\widehat{X}$

be the special fiber of $\widehat{\mathcal{X}}$ and $\widehat{L} = \widehat{\mathcal{L}}_{|\widehat{X}}$. Finally (which ensures uniqueness) $(\widehat{X}, \widehat{L})$ is a spin curve on X. We denote by (G, P, s) the dual spin graph of $(\widehat{X}, \widehat{L})$. By definition, $P \in \mathcal{C}_G$, and we define

$$\alpha(L_K) = P.$$

lem:factor **Lemma 5.2.** *With the above notation, we have the following factorization*

$$\tau_{\mathcal{S}} \colon \mathcal{S}_{X_K} \xrightarrow{\alpha} \mathcal{C}_G \xrightarrow{\beta} T_\Gamma^{\mathrm{trop}}.$$

Proof. We continue to use the notation before the statement. We need to show that $\tau_{\mathcal{S}}(L_K)$ is equal to $[T_P]$. By (9)

$$\tau_{\mathcal{S}}(L_K) = \underline{\deg}\widehat{\mathcal{L}}_{|\widehat{X}} = \underline{\deg}\widehat{L}.$$

To compute the multidegree of $\widehat{L}$, observe that the dual graph, $\widehat{G}$, of $\widehat{X}$ is the R-subdivision of G, where $R = E \setminus P$. For a vertex $v \in V(\widehat{G})$, we let $\widehat{C}_v$ be the irreducible component of $\widehat{X}$ corresponding to v. Recall that $\widehat{L}$ restricts to a theta-characteristic on the complement of the exceptional components of $\widehat{X}$, i.e. on the subcurve of $\widehat{X}$ whose dual graph is $(V(G), P)$.

Therefore $\forall v \in V(G) \subset V(\widehat{G})$ we have

$$\deg_{\widehat{C}_v} \widehat{L} = w(v) - 1 + \frac{\deg_P(v)}{2}.$$

Comparing this with (4) we conclude that the coefficient of v in $\underline{\deg}\widehat{L}$ and T_P coincide, as wanted.

Consider now $v \in V(\widehat{G}) \smallsetminus V(G)$, so that $\widehat{C}_v$ is an exceptional component lying over a node of X corresponding to an edge $e \in E \smallsetminus P$. We know that $\deg_{\widehat{C}_v} \widehat{L} = 1$. To conclude the proof, in view of (4), we need to show that in the tropical curve Γ the vertex v coincides with p_e, the mid-point of e.

We use Remark 5.1; let e_0 and e_1 be the nodes of $\widehat{X}$ lying on $\widehat{C}_v$. Recall that there are étale neighborhoods of $\widehat{\mathcal{X}}$ around e_0 and e_1 in which the local equations of $\widehat{\mathcal{X}}$ are, respectively, $x_0 y_0 = h_e$ and $x_1 y_1 = h_e$, for the same $h_e \in R$ (see [CCC07, Eq. (4), Section 3.2]). This implies that the lengths of e_0 and e_1 are equal, and therefore v is the mid point of e. ♣

We let $\mathcal{S}_{X_K}^+$ and $\mathcal{S}_{X_K}^-$, respectively, be the loci in $\mathcal{S}_{X_K}$ corresponding to even and odd theta-characteristics of X_K. Thus $\tau_{\mathcal{S}}$ restricts to maps

$$\tau_{\mathcal{S}}^+ \colon \mathcal{S}_{X_K}^+ \longrightarrow T_\Gamma^{\mathrm{trop}} \quad \text{and} \quad \tau_{\mathcal{S}}^- \colon \mathcal{S}_{X_K}^- \longrightarrow T_\Gamma^{\mathrm{trop}}.$$

Similarly, we restrict α

$$\alpha^+ \colon \mathcal{S}_{X_K}^+ \longrightarrow \mathcal{C}_G \quad \text{and} \quad \alpha^- \colon \mathcal{S}_{X_K}^- \longrightarrow \mathcal{C}_G$$

Next, let $\mathcal{S}_X$ be the scheme of spin curves over X. For a spin structure (P, s) on G we define

$$\mathcal{S}_{(X,P,s)} = \{(\widehat{X}, \widehat{L}) \in \mathcal{S}_X : \text{the dual spin graph of } (\widehat{X}, \widehat{L}) \text{ is } (G, P, s)\}$$

Denote by $\mathcal{S}_X^+$ and $\mathcal{S}_X^-$, respectively, the subschemes of $\mathcal{S}_X$ corresponding to even and odd spin curves.

Recall that SP_G^+ and SP_G^- denote, respectively, the set of pairs (P, s) such that (G, P, s) is an even and odd spin graph. For every $P \in \mathcal{C}_G$, we define

$$\mathcal{S}_{(X,P)}^+ = \cup_{(P,s)\in SP_G^+}\mathcal{S}_{(X,P,s)} \quad \text{and} \quad \mathcal{S}_{(X,P)}^- = \cup_{(P,s)\in SP_G^-}\mathcal{S}_{(X,P,s)}.$$

Lemma 5.3. *With the above notation, for every $P \in \mathcal{C}_G$ we have*

$$|(\alpha^{\pm})^{-1}(P)| = 2^{b_1(G)-b_1(P)}|\mathcal{S}_{(X,P)}^{\pm}|.$$

lem:ramif

Proof. As we saw above, for every theta-characteristic L_K on X_K there is a unique pair $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ over $\operatorname{Spec} R$ extending (X_K, L_K). By construction, the special fiber of $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ is contained in $\mathcal{S}_X$. For every $(\widehat{X}, \widehat{L}) \in \mathcal{S}_X$, we let $N(\widehat{X}, \widehat{L})$ be the number of theta-characteristics L_K on X_K whose extension $(\widehat{\mathcal{X}}, \widehat{\mathcal{L}})$ has $(\widehat{X}, \widehat{L})$ as a special fiber. Clearly, we have

$$|(\alpha^{\pm})^{-1}(P)| = \sum_{(\widehat{X},\widehat{L})\in\mathcal{S}_{(X,P)}^{\pm}} N(\widehat{X}, \widehat{L}).$$

Let $\pi\colon S_{\mathcal{X}} \to \operatorname{Spec} R$ be the moduli space of spin curves of the family $\mathcal{X} \to \operatorname{Spec} R$. The map π is finite and flat and the multiplicity of the special fiber of π at a point $(\widehat{X}, \widehat{L}) \in \mathcal{S}_{(X,P)}^{\pm}$ is $2^{b_1(G)-b_1(P)}$ (see [C89] and [CC03]). Thus $\widehat{L}$ lifts to $2^{b_1(G)-b_1(P)}$ theta-characteristics on X_K, that is, $N(\widehat{X}, \widehat{L}) = 2^{b_1(G)-b_1(P)}$, and we are done.   ♣

Given an integer m, we set

$$N_m^+ := 2^{m-1}(2^m + 1) \quad \text{and} \quad N_m^- := 2^{m-1}(2^m - 1).$$

Theorem 5.4. *For every $P \in \mathcal{C}_G$, the following properties hold.*

thm:main2

thm:main2a

(1) If $P \neq 0$, then

$$|(\tau_{\mathcal{S}}^+)^{-1}(T_P)| = |(\tau_{\mathcal{S}}^-)^{-1}(T_P)| = 2^{2g-b_1(G)-1}.$$

thm:main2b

(2) If $P = 0$, then

$$|(\tau_{\mathcal{S}}^+)^{-1}(T_0)| = 2^{b_1(G)} \sum_{\substack{U \subset V \\ |U| \equiv 0 \mod (2)}} \left(\prod_{v \in U} N_{w(v)}^- \prod_{v \in V \setminus U} N_{w(v)}^+ \right).$$

and

$$|(\tau_{\mathcal{S}}^-)^{-1}(T_0)| = 2^{b_1(G)} \sum_{\substack{U \subset V \\ |U| \equiv 1 \mod (2)}} \left(\prod_{v \in U} N_{w(v)}^- \prod_{v \in V \setminus U} N_{w(v)}^+ \right).$$

Proof. Consider $(\widehat{X}, \widehat{L}) \in S_{(X,P)}$. The dual graph of $\widehat{X}$ is the R-subdivision of G, where $R = E \smallsetminus P$. We let $Z \subset \widehat{X}$ be the union of the exceptional components, and $X_1, \ldots, X_n$ the connected components of $\overline{\widehat{X} \setminus Z}$. It is well

known (as the restriction of $\widehat{L}$ to every exceptional component is $\mathcal{O}(1)$) that we have the following identity

`eq:sum` (10)
$$h^0(\widehat{X}, \widehat{L}) = \sum_{1 \leq i \leq n} h^0(X_i, \widehat{L}|_{X_i}).$$

Assume $P \neq 0$. We can assume that the dual graph P_1 of X_1 has $b_1(P_1) \neq 0$. By [Har82, Cor. 2.13], we have the same number, M, of odd and even theta-characteristics on X_1. We argue that $|\mathcal{S}_{(X,P)}^+| = |\mathcal{S}_{(X,P)}^-|$. By (10), this is clear if $n = 1$. Otherwise, let Y be the disjoint union of $X_2, \ldots, X_n$, and denote by A and B the number of even and odd theta-characteristics on Y, respectively. We have $h^0(\widehat{X}, \widehat{L}) = h^0(X_1, \widehat{L}|_{X_1}) + h^0(Y, \widehat{L}|_Y)$, therefore
$$|\mathcal{S}_{(X,P)}^+| = |\mathcal{S}_{(X,P)}^-| = AM + BM.$$
Therefore, by [CC03, Sect. 1.3],
$$|\mathcal{S}_{(X,P)}^+| = |\mathcal{S}_{(X,P)}^-| = 2^{2\sum_{v \in V} w(v)} 2^{b_1(P)-1} = 2^{2g-2b_1(G)} 2^{b_1(P)-1}.$$
Then by Lemmas 5.2 and 5.3 we have
$$|(\tau_{\mathcal{S}}^\pm)^{-1}(T_P)| = 2^{b_1(G)-b_1(P)} |\mathcal{S}_{(X,P)}^\pm| = 2^{2g-b_1(G)-1}.$$
This concludes the proof of (1).

Assume now $P = 0$. A spin curve in $\mathcal{S}_{(X,0)}$ is given by the datum of a theta-characteristic on each component of the normalization of X. This spin curve is even if and only if we have an even number of components for which the theta-characteristic is odd. Hence
$$|\mathcal{S}_{(X,0)}^+| = \sum_{\substack{U \subset V \\ |U| \equiv 0 \mod (2)}} \left(\prod_{v \in U} N_{w(v)}^- \prod_{v \in V \setminus U} N_{w(v)}^+ \right)$$
which, by Lemmas 5.2 and 5.3, gives the stated number for $|(\tau_{\mathcal{S}}^+)^{-1}(T_0)|$. A similar reasoning gives the stated number for $|(\tau_{\mathcal{S}}^-)^{-1}(T_0)|$. ♣

Remark 5.5. Notice that Theorem 5.4 recovers [JL18, Theorem 1.1, (2) and (3)]. Indeed, assume that Γ is pure, so that $b_1(\Gamma) = g$. For every $P \neq 0$, by Theorem 5.4 we get
$$|(\tau_{\mathcal{S}}^+)^{-1}(T_P)| = |(\tau_{\mathcal{S}}^-)^{-1}(T_P)| = 2^{g-1}.$$
Moreover, for $\epsilon \in \{0, 1\}$, using that $N_0^+ = 1$ and $N_0^- = 0$, we have
$$\sum_{\substack{U \subset V \\ |U| \equiv \epsilon \mod (2)}} \left(\prod_{v \in U} N_0^- \prod_{v \in V \setminus U} N_0^+ \right) = \begin{cases} 1, & \text{if } \epsilon = 0, \\ 0, & \text{if } \epsilon = 1, \end{cases}$$
(the unique non-zero summand in the left hand side is the one corresponding to $\epsilon = 0$ and $U = \emptyset$). By Theorem 5.4,
$$|(\tau_{\mathcal{S}}^+)^{-1}(T_0)| = 2^g \quad \text{and} \quad |(\tau_{\mathcal{S}}^-)^{-1}(T_0)| = 0.$$

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