GEOMETRY OF TROPICAL MODULI SPACES AND LINKAGE OF GRAPHS

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Abstract. We prove the following "linkage" theorem: two p-regular graphs of the same genus can be obtained from one another by a finite alternating sequence of one-edge-contractions; moreover this preserves 3-edge-connectivity. We use the linkage theorem to prove that various moduli spaces of tropical curves are connected through codimension one.

Contents

1. Introduction

This paper is made of two parts, with the second partially motivating the first. The second part studies the moduli spaces of tropical curves; in order to establish some remarkable connectedness properties, we encounter some questions about graphs which are of interest in their own right. The solution of these graph theoretic problems occupies the first part of this paper.

Let us describe the two parts in some details. The first is concerned with classification of p-regular (every vertex has valency p) connected graphs. It is quite easy to see that there exists a unique 1-regular graph, namely two vertices joined by a unique edge. Similarly, 2-regular graphs are classified by the number of their edges, indeed there exists a unique 2-regular graph with n-edges: the cycle on n vertices, and these are all the 2-regular graphs. As soon as $p \geq 3$ the situation gets complicated; as a matter of fact, as far as we are aware of, the number of 3-regular graphs with fixed first Betti number is not known. And this number would be very interesting for several reasons; for instance, it counts the 0-dimensional combinatorial cycles in the moduli space of Deligne-Mumford stable curves, $\overline{M}_g$. See [1] for more on this issue.

Our main result in the first part of the paper is Theorem 2.4.3. This states, first of all, that any two p-regular connected graphs Γ and Γ', with the same first Betti number, are "linked", i.e. they can be obtained one from the other with a finite sequence of alternating one-edge contractions as follows. There exists a finite sequence

$$\begin{tikzcd}
\Gamma=\Gamma_1 \arrow[dr] & & \Gamma_3 \arrow[dl] \arrow[dr] & \cdots & \cdots & & \Gamma_{2h+1}=\Gamma' \arrow[dl] \\
& \Gamma_2 & & \cdots & & \Gamma_{2h} &
\end{tikzcd} \tag{1.1}$$

where every arrow is the map contracting precisely one edge and leaving everything else unchanged. Also, every odd-indexed graph in the diagram above is p-regular. Secondly, we prove that if Γ and Γ' are 3-edge-connected there exists a diagram as above where the graph Γ_i is 3-edge-connected, for every $i = 1, \ldots, 2h+1$. This second part makes the proof seriously more complicated, but it does play an important role in the application of this result to the second part of the paper. We refer to this property as the "conservation of 3-edge-connectivity".

In case $p = 3$ the result, without the conservation of 3-edge-conectivity, is due to A.Hatcher and W.Thurston [10], by a non combinatorial argument; a combinatorial proof valid for simple graphs is given by Y.Tsukui [15].

Our proof is purely combinatorial. We first reduce it to hamiltonian graphs (in Subsection 2.2), and then show that every hamiltonian graph is linked to a special type of graph called the p-polygon (in Subsection 2.4).

Now we turn to the part concerning moduli of tropical curves, which occupies Section 3. The moduli space, $M_g^{\rm trop}$, of tropical curves of genus g, and the moduli space of n-pointed tropical curves, $M_{g,n}^{\rm trop}$, are here treated simply as topological spaces. The point is, their geometry is so complex that they don't look like tropical varieties (the case $g = 0$ is an exception); in fact the problem to find a "good" category in which they should be placed is under investigation, and still awaits to be resolved. In a similar vein, it is interesting to study which topological properties of tropical varieties are also valid for those moduli spaces.

One of the characterizing properties of tropical varieties (defined by prime ideals) is that they are "connected through codimension one"; see the Structure Theorem in [11, Ch. 3]. The goal of Section 3 is thus to establish that several moduli spaces of tropical curves are connected through codimension one; our motivating observation was that this property is strictly related to the linkage properties of graphs studied in the first part of the paper.

Let us now give more details. In this paper, together with the original notion of tropical curve, here called "pure tropical curve", due to G. Mikhalkin (see [13]), we use the generalization given by S. Brannetti, M. Melo and F. Viviani in [2]; the advantage of the generalized notion is that, with it, the moduli space is closed under specialization, while the moduli space of pure tropical curves is not (see [2] or [3] for details).

We are interested in the spaces $M_{g,n}^{\rm trop}$, and also in the Schottky space, $\mathrm{Sch}_g^{\rm trop}$, defined as the quotient of $M_g^{\rm trop}$ via the Torelli map, studied in [2] and [5]. They are easily seen to be connected, however a stronger property holds, as we are going to explain. Let us focus on $M_g^{\rm trop}$ for simplicity; we have a finite decomposition $M_g^{\rm trop} = \sqcup_{i\in I} M_i$ where each M_i is a connected orbifold, and $M_g^{\rm trop}$ is the closure of the union of those M_i having maximal dimension (equal to $3g-3$). Every M_i has a clear geometric interpretation, for example the above mentioned dense union

$$M_g^{\rm reg} := \bigsqcup_{i\in I:\dim M_i = 3g-3} M_i \subset M_g^{\rm trop} = \bigsqcup_{i\in I} M_i$$

parametrizes genus-g tropical curves whose underlying graph is 3-regular. Moreover $M_g^{\rm reg}$ is open in $M_g^{\rm trop}$. Now, $M_g^{\rm reg}$ is clearly not connected, whereas $M_g^{\rm trop}$ is so, therefore one can ask: if we add to $M_g^{\rm reg}$ all strata M_i of codimension one (i.e. of dimension $3g-4$), do we get a connected space? Equivalently: is $M_g^{\rm trop}$ connected through codimension one?

The answer to the question is yes, and, as we said, this follows from the linkage theorem for graphs. In fact, by Proposition 3.3.3 , connectedness through codimension one holds for all $M_{g,n}^{\rm trop}$. The proof is based on an extension of Theorem 2.4.3 for $p = 3$ to graphs with legs, Proposition 3.3.2.

Next, by the tropical Torelli theorem of [4], and its generalization in [2], the Schottky locus $\mathrm{Sch}_g^{\rm trop}$ is the image via the Torelli map of the locus in $M_g^{\rm trop}$ parametrizing 3-edge-connected tropical curves. This motivates our interest in 3-edge-connected graphs. Indeed, the fact that graph linkage preserves 3-edge-connectivity, enables us to prove that $\mathrm{Sch}_g^{\rm trop}$ is connected through codimension one; see Theorem 3.3.6.

Acknowledgments. I am grateful to Margarida Melo and Filippo Viviani for their precious remarks on this paper and its previous versions, and to the referees for correcting several inaccuracies.

2. The linkage theorem

2.1. **Terminology.** Throughout the paper, p, g and n will be integers with $p \geq 3$ and $g, n \geq 0$.

Γ always denotes a graph (i.e. a one dimensional finite simplicial complex), $V(\Gamma)$ the set of its vertices (or 0-cells) and $E(\Gamma)$ the set of its edges (or 1-cells). Every $e \in E(\Gamma)$ joins two, possibly equal, vertices, called the *endpoints* of e. If the two endpoints of e coincide we say that e is a *loop*.

We assume all graphs to be connected, unless we specify otherwise. The combinatorial definition of graph is in Definition 3.1.2.

The first Betti number, or the genus, of Γ is $b_1(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + c$ where c is the number of connected components of Γ.

Using the standard terminology (see [7]) a graph Γ is called

(1) *p-regular* if every vertex has valency (or degree) equal to p;
(2) a *path* if its first Betti number is equal to 0, and if it contains no vertex of valency ≥ 3. A path Γ satisfies $|V(\Gamma)| = |E(\Gamma)| + 1$; we shall say that $|E(\Gamma)|$ is the length of the path.
(3) a *cycle* if it is 2-regular. A cycle has $b_1(\Gamma) = 1$, and hence an equal number of edges and vertices; this number will be called its length.
(4) *p-edge-connected* if $|V(\Gamma)| \geq 1$ and if $\Gamma \smallsetminus F$ is connected for any $F \subset E(\Gamma)$ with $|F| < p$.

2.1.1. *Contractions and linkage.* Fix Γ and $e \in E(\Gamma)$. Let Γ/e be the graph obtained by contracting e to a point and leaving everything else unchanged ([7, sect I.1.7]). Then there is a natural surjective map $\Gamma \to \Gamma'$, called the *contraction* of e. More generally, if $S \subset E(\Gamma)$ is a set of edges, we denote by Γ/S the contraction of every edge in S and denote by $\sigma : \Gamma \to \Gamma/S$ the associated map. Let $T := E(\Gamma) \smallsetminus S$. Then there is a natural identification between $E(\Gamma/S)$ and T. Moreover σ induces a surjection

$$\sigma_V : V(\Gamma) \longrightarrow V(\Gamma/S); \quad v \mapsto \sigma(v).$$

Notice that every connected component of $\Gamma - T$ (the graph obtained from Γ by removing every edge in T) gets contracted to a vertex of Γ/S; conversely, for every vertex $\overline{v}$ of Γ/S its preimage $\sigma^{-1}(\overline{v}) \subset \Gamma$ is a connected component of $\Gamma - T$. In particular, we obtain the following useful identity:

$$b_1(\Gamma - T) = \sum_{\overline{v} \in V(\Gamma/S)} b_1(\sigma_V^{-1}(\overline{v})). \tag{2.1}$$

Remark 2.1.2. Let $\sigma : \Gamma \to \Gamma/S$ be the contraction of S as above. The following facts are well known and easy to prove.

(1) $b_1(\Gamma) = b_1(\Gamma/S) + b_1(\Gamma \smallsetminus T)$.
(2) If Γ is p-edge-connected so is Γ/S.

Definition 2.1.3. (1) Let Γ_1 and Γ_2 be two graphs. We say that Γ_1 and Γ_2 are *strongly linked* if for $i = 1, 2$ there exists a non-loop edge $e_i \in E(\Gamma_i)$ such that the contraction of e_1 and the contraction of e_2 coincide, i.e.

$$\Gamma_1 \xrightarrow{\sigma_1} \Gamma_1/e_1 = \Gamma_2/e_2 \xleftarrow{\sigma_2} \Gamma_2,$$

and $\sigma_1(e_1) = \sigma_2(e_2)$ (i.e. e_1 and e_2 are mapped to the same vertex).

(2) Let Γ and Γ' be two graphs. We say that Γ and Γ' are *linked* if there exists a finite sequence of graphs

$$\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}, \Gamma_n = \Gamma'$$

such that Γ_i and Γ_{i-1} are strongly linked, for every $i = 2, \ldots, n$.

We are particularly interested in 3-edge-connected graphs, so we need the following variant.

Definition 2.1.4. Let Γ and Γ' be two 3-edge-connected graphs. We say that Γ and Γ' are *3-linked* if there exists a finite sequence of 3-edge-connected graphs

$$\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}, \Gamma_n = \Gamma'$$

such that Γ_i and Γ_{i-1} are strongly linked, for every $i = 2, \ldots, n$.

Remark 2.1.5. Being linked, or 3-linked is an equivalence relation. Linked graphs have the same number of edges and vertices.

Example 2.1.6. The next picture represents two strongly linked 3-regular graphs, with Γ_1/e_1 equal to Γ_2/e_2. Γ_1 is called "Petersen" graph.

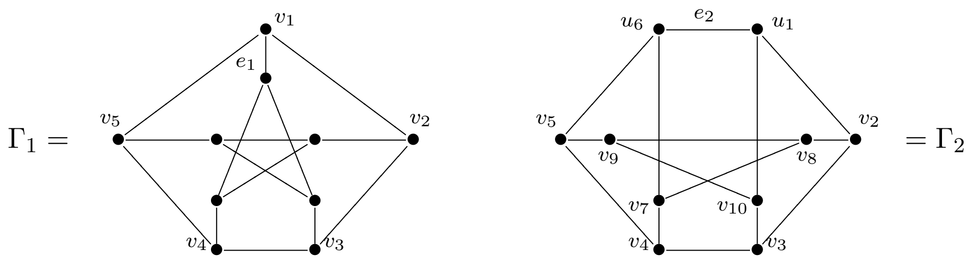

FIGURE 1. Petersen graph strongly linked to a hamiltonian graph.

Remark 2.1.7. Let Γ be a p-regular graph with $p \geq 3$; set $b = b_1(\Gamma)$. We have $|E(\Gamma)| = |V(\Gamma)|p/2$ hence

$$b = 1 + \frac{(p-2)|V(\Gamma)|}{2}, \quad |V(\Gamma)| = \frac{2b-2}{p-2} \text{ and } |E(\Gamma)| = \frac{p(b-1)}{p-2}.$$

If Γ is 3-regular, $|E(\Gamma)| = 3b - 3$ and $|V(\Gamma)| = 2b - 2$.

2.2. p-regular hamiltonian graphs.

Definition 2.2.1. A graph Γ is called *hamiltonian* if $|V(\Gamma)| \geq 2$ and if it contains a *hamiltonian cycle*, i.e. a cycle passing through every vertex (exactly once). A p-regular hamiltonian graph free from loops is called a *p-hamiltonian* graph.

Examples of p-hamiltonian graphs are all the graphs in figures 3, 4 and 5. In Figure 1, the graph Γ_2 (on the right) is hamiltonian, with $b_1(\Gamma_2) = 5$, whereas the graph Γ_1 is not hamiltonian. So, Example 2.1.6 implies that the non hamiltonian graph Γ_1 is linked to the 3-hamiltonian graph Γ_2. This is true in general, by the following Proposition 2.2.2.

In the next two proofs we will use the following terminology. To every edge e of a graph Γ we associate two *half-edges*, h and h', defined as follows. Call $v, v' \in V(\Gamma)$ the two (possibly equal) endpoints of e. Then h and h' are line segments such that $h \subsetneq e$, $h' \subsetneq e$, $e = h \cup h'$ and such that each of them contains precisely one end of e, so $v \in h$ and $v' \in h'$. There are many possible choices for the half-edges of any edge, but we shall assume that such a choice is made; in fact everything we will say does not depend on this choice. For example: the valency of $v \in E(\Gamma)$ is equal to the number of half-edges of Γ touching v (see also 3.1.2).

Proposition 2.2.2. *Every p-regular graph Γ is linked to a p-hamiltonian graph. Every p-regular, 3-edge-connected graph is 3-linked to a 3-edge-connected p-hamiltonian graph.*

Proof. Let Γ be our p-regular graph, and $b = b_1(\Gamma)$. Call $\ell(\Gamma)$ the maximal length of a cycle contained in Γ; by Remark 2.1.7 we have $\ell(\Gamma) \leq |V(\Gamma)| = \frac{2b-2}{p-2}$. We shall use descending induction on $\ell(\Gamma)$. If $\ell(\Gamma) = \frac{2b-2}{p-2}$ there is nothing to prove, so the basis of the induction is done.

Assume $\ell(\Gamma) < \frac{2b-2}{p-2}$; let $\Delta \subset \Gamma$ be a cycle of length $\ell = \ell(\Gamma)$.

For consistency with Definition 2.1.3 we set $\Gamma_1 = \Gamma$. We shall explicitly construct a p-regular graph, Γ_2, strongly linked to Γ and such that $\ell(\Gamma_2) >$

$\ell(\Gamma)$. If Γ is 3-edge-connected so will be Γ_2. Using the induction hypothesis on Γ_2 will suffice to complete the proof. Denote $V(\Delta) = \{v_1, \ldots, v_\ell\} \subsetneq V(\Gamma)$. The forthcoming construction is pictured in Figure 2.

Pick a vertex $v \in V(\Gamma)$ such that $v \notin V(\Delta)$ and such that there is an edge e joining v to one of the vertices of Δ; obviously $e \notin E(\Delta)$. We can assume, with no loss of generality, that the endpoints of e are v_1 and v. Let us call e_1 and e_ℓ the two edges of Δ meeting at v_1.

Since v has valency p, there are $p-1$ half-edges containing v and not contained in e; let us call them $h_1, \ldots, h_{p-1}$. Similarly as v_1 has valency p there are $p-3$ half-edges containing v_1 and not contained in e, e_1 or e_ℓ; we call these $h_p, \ldots, h_{2p-4}$. It is clear that no half-edge h_i lies in Δ. Consider the contraction of e

$$\sigma_1 : \Gamma_1 = \Gamma \longrightarrow \Gamma/e = \Gamma'.$$

Clearly $w := \sigma_1(e)$ is a vertex of valency $2p-2$, indeed the images via σ_1 of $e_1, e_\ell, h_1, \ldots h_{2p-4}$ all touch w, and there is no other edge touching w.

Now we perform a valency reducing extension on w (cf. [4, A.2.2]); namely we introduce an edge contracting map $\sigma_2 : \Gamma_2 \longrightarrow \Gamma'$ from a new graph Γ_2 such that Γ' is obtained from Γ_2 as the contraction to w of a unique edge, which we call $e_{\ell+1}$; hence $\sigma_2(e_{\ell+1}) = w$ and σ_2 leaves everything else unchanged. The two endpoints of $e_{\ell+1}$ are two vertices of valency p, which we call $u_{\ell+1}, u_1 \in V(\Gamma_2)$. In Γ_2 we distribute the $2p-2$ half-edges $e_1, e_\ell, h_1, \ldots, h_{2p-4}$ so that $p-1$ of them touch u_1 and the remaining $p-1$ touch $u_{\ell+1}$. Moreover we have the (old) edge e_1 touching u_1, the old edge e_ℓ touching $u_{\ell+1}$, and the new edge $e_{\ell+1}$ joining u_1 with $u_{\ell+1}$. Therefore the graph Γ_2 is p-regular. Summarizing, we have

$$\Gamma_2 \xrightarrow{\sigma_2} \Gamma_2/e_{\ell+1} = \Gamma/e \xleftarrow{\sigma_1} \Gamma.$$

Therefore Γ and Γ_2 are strongly linked. Now the given cycle $\Delta \subset \Gamma$ is mapped to a cycle of the same length by σ_1, whereas $\sigma_2^{-1}(\sigma_1(\Delta))$ is a cycle of length at least $\ell+1$ (as it contains the vertices $\{u_1, v_2, \ldots, v_\ell, u_{\ell+1}\}$). Therefore $\ell(\Gamma) < \ell(\Gamma_2)$. It is clear that by iterating this construction we arrive at a p-regular graph $\widehat{\Gamma}$ with $\ell(\widehat{\Gamma}) = (2b-2)/(p-2)$, so that $\widehat{\Gamma}$ is hamiltonian graph. It is also clear that $\widehat{\Gamma}$ and Γ are linked.

Now suppose that Γ is 3-edge-connected; then Γ' is also 3-edge-connected by Remark 2.1.2. To prove that $\widehat{\Gamma}$ is 3-edge-connected we need to prove that the extension of w used during the proof may be constructed so as to yield a 3-edge-connected graph Γ_2. This follows from the proof of [4, Prop. A.2.4], with trivial modifications. Finally, by the next lemma 2.2.3, we can take $\widehat{\Gamma}$ free from loops. ■

The next picture illustrates this proof. We represent the relevant portions of Γ, on the left, of Γ' and of Γ_2. The vertices v_2 and v_ℓ belong to the cycle Δ, hence they are joined by a path (not drawn) not intersecting h_1 and h_2.

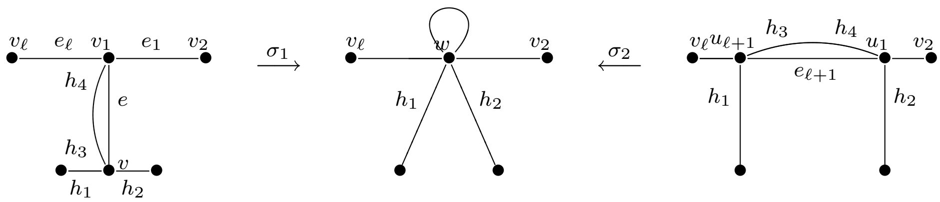

FIGURE 2. Increasing the length of Δ in the proof of 2.2.2.

Lemma 2.2.3. *Every p-regular hamiltonian graph is linked to a p-hamiltonian graph.*

Every p-regular hamiltonian 3-edge-connected graph is 3-linked to a p-hamiltonian 3-edge-connected graph.

Proof. It suffices to exhibit a procedure which decreases the number of loops, preserving the property of being hamiltonian, p-regular and 3-edge-connected.

Let $\Delta \subset \Gamma$ be a fixed hamiltonian cycle; denote by $E(\Delta) = \{e_1, \dots, e_t\}$ and $V(\Gamma) = \{v_1, \dots, v_t\}$ with e_i joining v_i and v_{i+1} as usual. Suppose that Γ contains a loop ℓ, and assume (with no loss of generality) that this loop is based at v_1. Let ℓ_1 and ℓ_2 be the half-edges of the loop (so ℓ_1 and ℓ_2 touch v_1). We know that v_1 is connected to v_2 by the edge e_1. Since v_2 has valency $p \geq 3$ there is a half-edge h touching v_2, not contained in the hamiltonian cycle Δ, and not contained in an edge touching v_1 (for otherwise v_2 would have valency less than that of v_1). Let us consider Γ/e_1, and call w the vertex into which e_1 is contracted. The valency of w is $2p-2$.

Let Γ_2 be the graph obtained from Γ by changing the loop ℓ into an edge, called f_1, joining v_1 with v_2, and by changing the half-edge h into a half-edge touching v_1. This operation does not create any new loop (as the edge of Γ containing h does not touch v_1), and eliminates the loop ℓ. So the number of loops of Γ_2 is less than that of Γ. It is clear that Γ_2 is p-regular (we added and removed a half-edge from v_1 and v_2, and left everything else unchanged). The hamiltonian cycle Δ is clearly contained in Γ_2, so Γ_2 is hamiltonian. Finally, $\Gamma/e_1 = \Gamma_2/e_1$, so Γ and Γ_2 are strongly linked.

It remains to show that if Γ is 3-edge-connected so is Γ_2. This follows from [4, Prop.A.2.4], in fact the extension of w given by $\Gamma_2 \to \Gamma/e_1$ is the same as in Step 1 in the proof of that proposition (with obvious modifications). ■

2.3. p-polygons.

2.3.1. *Fixing a hamiltonian cycle in a p-hamiltonian graph.* We now introduce some useful conventions. Let Γ be a p-hamiltonian graph, with $b := b_1(\Gamma)$. We fix a hamiltonian cycle, Δ, and refer to it as the *distinguished* hamiltonian cycle; let $\gamma = |V(\Gamma)|$ be the length of Δ. The choice of Δ enables us to use the following terminology. The edges of Γ which do not lie in Δ will be called *chords*. The number of chords of Γ is easily computed:

$$\text{(2.2)} \quad \text{Number of chords of } \Gamma = |E(\Gamma)| - \gamma = \frac{p(b-1)}{p-2} - \frac{2(b-1)}{p-2} = b - 1.$$

The vertices of Γ will be labeled according to the cyclic structure of Δ, i.e. $V(\Gamma) = V(\Delta) = \{v_1, v_2, \dots, v_\gamma\}$ so that there exists an edge $e_i \in E(\Delta) \subset E(\Gamma)$ joining v_i with v_{i+1} for every $i = 1, \dots, \gamma$ (with the cyclic convention $v_{\gamma+1} = v_1$); hence $E(\Delta) = \{e_1, \dots, e_\gamma\}$. The starting vertex v_1 can be picked arbitrarily; furthermore, for any choice of v_1, there are two cyclic labelings of the vertices (corresponding to the two cyclic orientations of Δ). Once a distinguished cycle Δ is chosen, we shall always use such a labeling.

Let Γ be a p-hamiltonian graph where a distinguished cycle Δ has been fixed. Every chord has two distinct endpoints (Γ being free from loops). We shall denote by $d_{i,j}$ a chord joining v_i with v_j, and always assume $i < j$. If $p \geq 4$ there may be more than one chord joining v_i with v_j; if we need to distinguish between them we will use superscripts, i.e. we denote $\{d^\alpha_{i,j},\ \alpha = 1, \dots, m\}$ the chords joining v_i and v_j; notice that $m \leq p - 2$.

We also need a notation for a chord of which only one end is known. So, the chord having one end at the vertex v_j and the other end at some other vertex will be denoted $d_{j,*}$.

Let $d_{i,j}$ be a chord as above. Then $d_{i,j}$ determines two paths of the cycle Δ, namely the two paths Λ and Λ' contained in Δ, having extremes v_i and v_j. Hence $\Lambda \cap \Lambda' = \{v_i, v_j\}$ and $\Lambda \cup \Lambda' = \Delta$. We call such two paths the *sides* of $d_{i,j}$. It is obvious that one of them has length $j - i$ and the other has length $\gamma - j + i$. We define the *amplitude*, $\alpha(d_{i,j})$, of $d_{i,j}$ as the minimum between these two lengths:

$$\alpha(d_{i,j}) := \min\{j - i, \gamma - j + i\}. \tag{2.3}$$

It is clear that $\alpha(d_{i,j})$ does not depend on the choice of the labeling.

Lemma - Definition 2.3.2. *Let Γ be a p-hamiltonian graph with a distinguished hamiltonian cycle. Set $\gamma := |V(\Gamma)| = (2b_1(\Gamma) - 2)/(p - 2)$.*

1. *For any chord $d_{i,j}$ we have $1 \leq \alpha(d_{i,j}) \leq \gamma/2$. If $\alpha(d_{i,j}) \leq \gamma/2 - 1$ we say that $d_{i,j}$ is* short*.*
2. *Let $d_{i,j}$ be a short chord. The side of $d_{i,j}$ having length $\alpha(d_{i,j})$ will be called the* short side *of $d_{i,j}$.*
3. *If $\alpha(d_{i,j}) = \lfloor \gamma/2 \rfloor$ for every chord, or equivalently, if Γ has no short chords, then Γ is uniquely determined, it will be denoted by Π^p_γ and will be called the p-*polygon *with γ vertices (see Figures 3 and 4).*

   If γ is even the graph Π^p_γ has $p - 2$ chords between v_i and $v_{i+\gamma/2}$ for every $i = 1, \dots, \gamma/2$, and no other chord.

   If γ is odd then p is even. For every $i = 1, \dots, (\gamma - 1)/2$, the graph Π^p_γ has $(p - 2)/2$ chords between v_i and $v_{i+(\gamma-1)/2}$, $(p - 2)/2$ chords between v_i and $v_{i+(\gamma+1)/2}$, and no other chord.

Proof. Since Γ has no loops we have, for any chord $d_{i,j}$, $1 \leq \alpha(d_{i,j})$. If γ is even (respectively, odd) the maximal amplitude of a chord is obviously $\gamma/2$ (respectively, $(\gamma - 1)/2$).

Now let γ be even. If there are no short chords, every chord is of type $d_{i,i+\gamma/2}$ for $i = 1, \dots, \gamma/2$. Moreover, every pair of vertices v_i, $v_{i+\gamma/2}$ is joined by exactly $p - 2$ chords, because Γ is p-regular. This shows that Γ is uniquely determined.

Now suppose that γ is odd, and that Γ has no short chord. Then every chord is either of type $d_{i,i+(\gamma-1)/2}$ or of type $d_{i,i+(\gamma+1)/2}$. Since $|E(\Gamma)| = p\gamma/2$

we have that p is even; set $r = (p-2)/2$. For every vertex there are $2r$ chords touching it.

We claim that there are exactly r chords of type $d_{i,i+(\gamma-1)/2}$ and r chords of type $d_{i,i+(\gamma+1)/2}$ for every $i = 1, \dots, (\gamma-1)/2$. By contradiction, suppose (with no loss of generality) that there are more than r chords joining v_1 with $v_{(\gamma+1)/2}$; hence there are less than r chords joining v_1 with $v_{(\gamma+3)/2}$. But then there are more than r chords joining v_2 with $v_{(\gamma+3)/2}$ and less than r chords joining v_2 with $v_{(\gamma+5)/2}$. Continuing in this way we get that there are less than r chords joining $v_{(\gamma-1)/2}$ with v_γ. The remaining chords touching v_γ are the ones touching also $v_{(\gamma+1)/2}$; since there are already more than r chords of type $d_{1,(\gamma+1)/2}$, there can only be less than r chords of type $d_{\gamma,(\gamma+1)/2}$. We conclude that there are less than $2r$ chords touching v_γ. A contradiction. This shows that Γ is uniquely determined. ■

Example 2.3.3. If $p = 3$ we have $\gamma = 2b_1(\Gamma) - 2$ and Π^3_γ has no multiple edge.

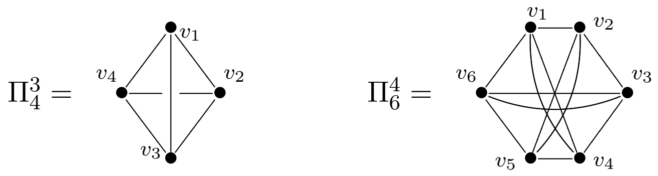

FIGURE 3. Some p-polygons with even number of vertices.

If γ is odd, then Π^p_γ has no multiple edges if and only if $p \leq 4$.

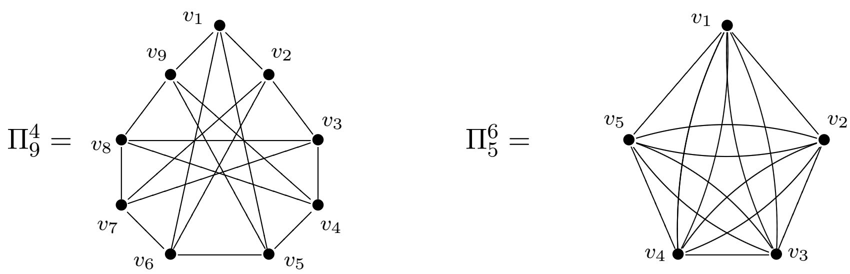

FIGURE 4. Some p-polygons with an odd number of vertices.

We need a criterion for 3-edge-connectivity.

Lemma 2.3.4. *(1) Let Γ be a graph such that for every edge e there exist two distinct cycles Δ_1 and Δ_2 in Γ and such that $E(\Delta_1) \cap E(\Delta_2) = \{e\}$. Then Γ is 3-edge-connected.*

(2) Let Γ_1 be a 3-edge-connected graph and let Γ_2 be a graph strongly linked to Γ_1, so that $\Gamma_1/e_1 = \Gamma_2/e_2$ with $e_i \in E(\Gamma_i)$ (notation in Def. 2.1.3). Then Γ_2 is 3-edge-connected if it contains two cycles $\Delta_1 \neq \Delta_2$ such that $E(\Delta_1) \cap E(\Delta_2) = \{e_2\}$.

(3) The p-polygon Π^p_γ is 3-edge-connected for every $p \geq 3$.

Proof. For part (1), we notice that Γ has no separating edges (a separating edge is not contained in any cycle). Suppose by contradiction that Γ is not 3-edge-connected; let (e, e') be a separating pair of edges of Γ. By [4, Lemma 2.3.2 (iv) and (iii)], (e, e') is a separating pair if and only if e and e' belong to the same cycles of Γ. By our assumption, this is clearly impossible.

Now part (2). The graph $\Gamma_1/e_1 = \Gamma_2/e_2$ is 3-edge-connected as Γ_1 is. Therefore any separating pair of edges of Γ_2 must contain e_2. The proof of part (1) shows that our hypothesis implies that e_2 is not contained in any separating pair of edges, hence we are done.

To prove part (3) we use again part (1). Pick a chord $d_{i,j}$; then there obviously exist two cycles having only $d_{i,j}$ as common edge: just take the two cycles obtained by adding to $d_{i,j}$ one of its two sides (terminology in subsection 2.3.1). To prove that we can apply (1) on the remaining edges we need to distinguish two cases, according to the parity of γ.

Suppose γ even. By Lemma 2.3.2 in Π^p_γ there exists at least one chord $d_{i,i+\gamma/2}$ joining v_i with $v_{i+\gamma/2}$, for every $i = 1, \ldots, \gamma/2$. Pick an edge which is not a chord, $e = e_1$. Now Π^p_γ contains the chords $d_{1,\gamma/2+1}$ and $d_{2,\gamma/2+2}$. Then $\Delta_1 = (e_1, \ldots, e_{\gamma/2}, d_{1,\gamma/2+1})$ and $\Delta_2 = (e_1, e_\gamma, \ldots, e_{\gamma/2+2}, d_{2,\gamma/2+2})$ are two cycles having only e as common edge. Therefore Π^p_γ is 3-edge-connected.

Now suppose that γ is odd; again we use (1). By Lemma 2.3.2 in Π^p_γ there exists at least one chord joining v_i with $v_{i+(\gamma-1)/2}$, and at least one chord joining v_i with $v_{i+(\gamma+1)/2}$. Let $e = e_1$ be an edge which is not a chord. Let $\Delta_1 = (e_1, d_{2,(\gamma+3)/2}, d_{1,(\gamma+3)/2})$ and $\Delta_2 = (e_1, e_2, \ldots, e_{(\gamma-1)/2}, d_{1,(\gamma+1)/2})$; these are two cycles whose only edge in common is e_1. Hence Π^p_γ is 3-edge-connected. ■

We say that two chords $d_{i,j}$ and $d_{k,l}$ do not cross if $i < j < k < l$.

Lemma 2.3.5. *Let Γ be a p-hamiltonian graph with a distinguished hamiltonian cycle(cf. 2.3.1). Let $d_{i,j}$ be a short chord. Then there exists a short chord $d_{k,l}$ with $j < k$ (i.e. $d_{i,j}$ and $d_{k,l}$ do not cross) and such that the short side of $d_{i,j}$ does not intersect the short side of $d_{k,l}$.*

Proof. We denote by Δ the fixed hamiltonian cycle. We may assume that $i = 1$, so that the given chord $d_{1,j}$ has $j \leq \gamma/2$ (i.e. the short side of $d_{1,j}$ has vertices $v_1, v_2, \ldots, v_j$). We must prove that there exists a short chord $d_{k,l}$ such that

(a) $j < k$ ($d_{1,j}$ and $d_{k,l}$ do not cross).

(b) $l - k < \lfloor \gamma/2 \rfloor$ (the short side of $d_{k,l}$ has vertices $v_k, v_{k+1}, \ldots, v_{l-1}, v_l$).

Let us denote by D the set of chords satisfying (a); we begin by bounding $|D|$ from below. Consider the j vertices $v_1, v_2, \ldots, v_j$; there are at most $p-2$ chords touching each of them. Therefore the total number of distinct chords touching these vertices is at most $j(p-2) - 1$ (to explain the "-1" notice that the chord $d_{1,j}$ joins v_1 with v_j, hence it must not be counted twice). Since Γ has $b - 1$ chords, we get

$$|D| \geq b - 1 - j(p-2) + 1 = b - j(p-2). \tag{2.4}$$

In particular, since $b = 1 + (p-2)\gamma/2$ and $j \leq \gamma/2$, we have that $|D| \geq 1$, i.e. D is not empty. To prove that there exists at least one chord in D satisfying (b) we argue by contradiction. Suppose that every chord $d_{k,l} \in D$ satisfies

$l - k \geq \gamma/2$. This is to say that the path $\Lambda \subset \Delta$ from v_{j+1} to v_γ contains a side of length at least $\gamma/2$ for every non multiple chord in D. Let us restrict our attention to the subgraph $\Gamma' = \Gamma \smallsetminus \{v_1, \dots, v_j\}$, obtained by removing the vertices $\{v_1, \dots, v_j\}$ and all edges adjacent to them. So, Γ' is made of Λ together with every chord in D. Now, two vertices of Λ are joined by a chord of D only if they are separated by at least $\gamma/2$ edges. Moreover, every two vertices can be joined by at most $p-2$ chords. Therefore the length of the path Λ satisfies, using (2.4),

$$\text{length}(\Lambda) \geq \frac{\gamma}{2} + \frac{|D|}{p-2} - 1 \geq \frac{b-1}{p-2} + \frac{b}{p-2} - j - 1 = \frac{2b-1}{p-2} - j - 1 > \gamma - j - 1$$

(since $\gamma = \frac{2b-2}{p-2}$). On the other hand Λ is a path from v_{j+1} to v_γ, whose length is easily computed:

$$\text{length }(\Lambda) = \gamma - (j+1) = \gamma - j - 1,$$

which is in contradiction with the above estimate on length(Λ). ∎

2.4. **Proof of the linkage theorem.**

2.4.1. *Twisting pairs of chords in a p-hamiltonian graph.* Let Γ be a p-hamiltonian graph with a distinguished hamiltonian cycle, as in 2.3.1; pick two chords $d_{i,j}$ and $d_{k,l}$. In this subsection we momentarily suspend the general convention $i < j$ and $k < l$ (which would be too restrictive). We introduce the graph Γ' obtained from Γ by swapping two endpoints of the above chords. So, Γ' is obtained from Γ by replacing the chord $d_{i,j}$ with a new chord, $d_{i,k}$, joining v_i and v_k, and by replacing $d_{k,l}$ with a chord $d_{j,l}$. Everything else is left unchanged. We shall say that Γ' is a *twist* of Γ, and that Γ' is obtained from Γ by *twisting* the pair of chords $(d_{i,j}, d_{k,l})$ into the pair $(d_{i,k}, d_{j,l})$. We shall also say that we swapped the end points v_j and v_l.

With no loss of generality we may set $i = 1$; the graph Γ' is obviously a p-regular hamiltonian graph; a distinguished hamiltonian cycle will be naturally induced by the one of Γ. So the vertices of Γ and Γ' will have the same names, and all the edges of Γ other than $d_{1,j}$ and $d_{k,l}$ correspond to edges of Γ' other than $d_{1,k}$ and $d_{j,l}$.

The picture below represents two 3-hamiltonian graphs related by twisting a pair of chords (the dotted chords are the ones that are not changed).

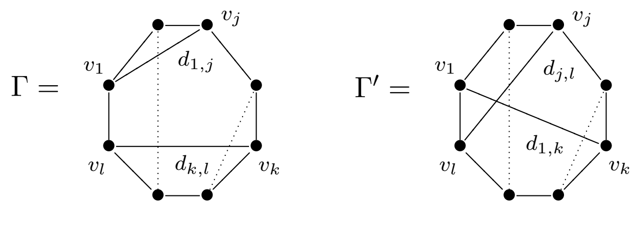

FIGURE 5. Twisting $d_{1,j}$ and $d_{k,l}$ into $d_{1,k}$ and $d_{j,l}$.

The following technical lemma is used in the proof of Theorem 2.4.3.

Lemma 2.4.2. *Let Γ be a p-hamiltonian graph with a distinguished hamiltonian cycle.*

(1) If Γ' is a twist of Γ, then Γ and Γ' are linked.

(2) Let Γ be 3-edge-connected and fix a chord $d_{i,j}$ of Γ, with $i < j$. Let $d_{j+1,*}$ be a chord of Γ starting at the vertex v_{j+1}; suppose that either (a) or (b) below hold.

(a) $d_{j+1,*} = d_{j+1,h}$ with $j+1 < h$ (i.e. $d_{j+1,*}$ does not cross $d_{i,j}$).

(b) $d_{j+1,*} = d_{h,j+1}$ with $i < h < j$ and there exists a third chord $d_{x,y}$ such that $1 \leq i < h < x < j < j+1 < y$.

Then the graph obtained by twisting $(d_{i,j}, d_{j+1,*})$ into $(d_{i,j+1}, d_{j,*})$ is 3-edge-connected and strongly linked to Γ.

Proof. We can assume $i = 1$ so that $d_{i,j} = d_{1,j}$. The edges of the distinguished hamiltonian cycle Δ will be called, as usual, $e_1, e_2, \ldots, e_{2b-2}$ with e_i joining v_i with v_{i+1}. Let Γ' be a twist of Γ. We prove Γ' is linked to Γ by induction on $k - j$ (i.e. on the distance along Δ of the two swapped vertices). If $k = j+1$ let e be the edge of Δ between v_j and v_{j+1}. Then the graph obtained from Γ by contracting e is the same as the graph obtained from Γ' by contracting e; hence Γ and Γ' are strongly linked. Now assume $k - j \geq 2$. Let Γ_1 be the graph obtained from Γ by twisting the chord $d_{1,j}$ with a chord ending at v_{j+1}, denoted $d_{j+1,*}$. So, in Γ_1 we have the chords $d^1_{1,j+1}$ and $d^1_{j,*}$, where the superscript keeps track of the graph to which the chords belong. We already proved that Γ and Γ_1 are linked. Now consider the graph Γ_2 obtained from Γ_1 by twisting $d^1_{1,j+1}$ and $d^1_{k,l}$, replacing them with $d^2_{1,k}$ and $d^2_{j+1,l}$; since $k - (j+1) < k - j$, by induction Γ_1 and Γ_2 are linked. Finally, let Γ_3 be obtained from Γ_2 by twisting $d^2_{j+1,l}$ and $d^2_{j,*}$, replacing them with $d^3_{j,l}$ and $d^3_{j+1,*}$. Again by induction (we are swapping v_j and v_{j+1}) Γ_3 is linked to Γ_2, and therefore Γ_3 is linked to Γ. It is obvious that $\Gamma_3 = \Gamma'$.

Let us prove the second part. Consider e_j, the edge between v_j and v_{j+1} (which, abusing notation as usual, is an edge of both Γ and Γ'). It is easy to check that Γ and Γ' are strongly linked, as the graphs $\overline{\Gamma}$ and $\overline{\Gamma'}$ obtained from Γ and Γ' by contracting e_j are obviously isomorphic. We need to prove that if either (a) or (b) holds, then Γ' is 3-edge-connected if Γ is. By Lemma 2.3.4 (2), it is enough to show that the edge e_j belongs to two distinct cycles Δ_1 and Δ_2 of Γ', such that $E(\Delta_1) \cap E(\Delta_2) = \{e_j\}$.

Suppose (a) holds, so we are twisting $(d_{1,j}, d_{j+1,h})$ into $(d_{1,j+1}, d_{j,h})$, with $h > j+1$. Then in Γ' we have the cycles Δ_1 and Δ_2 whose edge sets are

$$E(\Delta_1) = \{e_j, d_{1,j+1}, e_1, \ldots, e_{j-1}\}$$

and

$$E(\Delta_2) = \{e_j, e_{j+1}, \ldots, e_{h-1}, d_{j,h}\}.$$

It is clear that Δ_1 and Δ_2 are cycles and that $E(\Delta_1) \cap E(\Delta_2) = \{e_j\}$.

Now assume (b). We are twisting $(d_{1,j}, d_{h,j+1})$ into $(d_{1,j+1}, d_{h,j})$. Let $d_{x,y}$ be a chord crossing both $d_{h,j}$ and $d_{1,j}$. We have

$$1 < h < x < j < j+1 < y.$$

Now the edges of the two cycles containing e_j and sharing no other edge are

$$E(\Delta_1) = \{e_j, d_{1,j+1}, e_1, \ldots, e_{h-1}, d_{h,j}\}$$

and

$$E(\Delta_2) = \{e_j, e_{j+1}, \dots, e_{y-1}, d_{x,y}, e_x, e_{x+1}, \dots, e_{j-1}\}.$$

Since $1 < h < x$ we have $E(\Delta_1) \cap E(\Delta_2) = \{e_j\}$. ■

We are ready to prove the linkage theorem.

Theorem 2.4.3. *Let Γ_1 and Γ_2 be p-regular graphs with $b_1(\Gamma_1) = b_1(\Gamma_2)$.*
Then Γ_1 and Γ_2 are linked.
If Γ_1 and Γ_2 are 3-edge-connected, then they are 3-linked.

Proof. By Proposition 2.2.2 we can assume that Γ_1 and Γ_2 are p-hamiltonian.

We shall prove the theorem by showing that every p-hamiltonian graph Γ is linked to the p-polygon Π^p_γ, where $\gamma = \frac{2b-2}{p-2}$ and $b = b_1(\Gamma)$. Moreover, if Γ is 3-edge-connected, we will prove that it is 3-linked to Π^p_γ, which is 3-edge-connected by Lemma 2.3.4.

Let us fix a distinguished hamiltonian cycle Δ of Γ and use the notation of 2.3.1. Now set

$$\epsilon(\Gamma) := \sum \Big(\lfloor \gamma/2 \rfloor - \alpha(d_{i,j}) \Big)$$

where the sum is over all the chords of Γ. By Lemma 2.3.2 we have $\epsilon(\Gamma) \geq 0$, and $\epsilon(\Gamma) = 0$ if and only if Γ has no short chord, if and only if $\Gamma = \Pi^p_\gamma$.

We will prove the theorem by induction on $\epsilon(\Gamma)$. By what we just observed, if $\epsilon(\Gamma) = 0$ there is nothing to prove, so the induction basis is settled.

Assume now that $\epsilon(\Gamma) > 0$ and let us pick a short chord; we may call it $d_{1,j}$ and assume that $j \leq \gamma/2$. By Lemma 2.3.5 we have that there exist chords $d_{k,l}$ satisfying

$$1 < j < k < l \quad \text{and} \quad l - k < \lfloor \gamma/2 \rfloor. \tag{2.5}$$

We can assume (up to changing the labeling of the vertices) that there exists one of them such that the path (in Δ) from v_j to v_k is not longer than the path from v_l to v_1; i.e. we can assume that

$$k - j \leq \gamma + 1 - l. \tag{2.6}$$

We shall pick the pair $(d_{1,j}, d_{k,l})$ such that $k - j$, i.e. the length of the path in Δ from v_j to v_k, is minimal with respect to all pairs satisfying (2.5) and (2.6); we shall refer to this as the "minimality property" of $(d_{1,j}, d_{k,l})$.

Now that we have fixed our two chords, we can assume, up to switching them and changing the labeling on the vertices, that

$$j - 1 = \alpha(d_{1,j}) \leq \alpha(d_{k,l}) = l - k. \tag{2.7}$$

With these settings, we have

$$k \leq \gamma/2 + 1, \quad \text{more exactly} \quad k \leq \lfloor \gamma/2 \rfloor + 1. \tag{2.8}$$

Let us prove (2.8) by contradiction. Suppose $k \geq \lfloor \gamma/2 \rfloor + 2$. Then (2.7) implies

$$\lfloor \gamma/2 \rfloor + 2 \leq k \leq l - j + 1.$$

Now, by (2.6), we have $l - j + 1 \leq \gamma + 2 - k$. Therefore $\lfloor \gamma/2 \rfloor + 2 \leq \gamma + 2 - k$ and hence $k \leq \lfloor \gamma/2 \rfloor + 1$; a contradiction.

Claim 2.4.4. *Let Γ' be the graph obtained from Γ by twisting the pair of chords $(d_{1,j}, d_{k,l})$ into the pair $(d_{1,k}, d_{j,l})$. Then $\epsilon(\Gamma') < \epsilon(\Gamma)$.*

To prove the claim, consider a chord d' of Γ'. For notational clarity, we will denote by $d'_{*,*}$ the chords of Γ'. If d' is not equal to $d'_{1,k}$ or $d'_{j,l}$, then d' corresponds to a unique chord d of Γ such that $\alpha(d) = \alpha(d')$. Therefore we have

$$\epsilon(\Gamma) - \epsilon(\Gamma') = -\alpha(d_{1,j}) - \alpha(d_{k,l}) + \alpha(d'_{1,k}) + \alpha(d'_{j,l}). \tag{2.9}$$

We know that $\alpha(d_{1,j}) = j - 1$ and $\alpha(d_{k,l}) = l - k$, by construction and by (2.5). Furthermore, by (2.8) we have $\alpha(d'_{1,k}) = k - 1$.

To compute the remaining term we need to distinguish two cases.
Case 1: $l - j \leq \gamma/2$. Then $\alpha(d'_{j,l}) = l - j$. Therefore

$$\epsilon(\Gamma) - \epsilon(\Gamma') = 1 - j + k - l + k - 1 + l - j = 2k - 2j \geq 2$$

by (2.5). So the claim is proved in this case.
Case 2: $l - j \geq \gamma/2 + 1$. Now $\alpha(d'_{j,l}) = \gamma + j - l$. Therefore

$$\epsilon(\Gamma) - \epsilon(\Gamma') = 1 - j + k - l + k - 1 + \gamma + j - l = \gamma + 2(k - l) \geq 2$$

as $l - k < \lfloor \gamma/2 \rfloor$ by (2.5). The claim is proved.

Lemma 2.4.2 says that Γ and Γ' are linked. By the claim we may apply induction, getting that Γ' is linked to Π^p_γ; hence the first part of the Theorem is proved.

Before continuing, we analyze the chords having one end at a vertex v_g, with $j + 1 \leq g \leq k - 1$. Let $d_{g,*}$ be one such chord. We claim that with our choice of the pair $(d_{1,j}, d_{k,l})$, we have

$$d_{g,*} = d_{g,m}, \quad m \geq k. \tag{2.10}$$

By contradiction, suppose $m < k$. If $m < g$ we have (as $g \leq k - 1$ and $m \geq 1$)

$$g - m \leq k - 1 - 1 \leq \gamma/2 - 1$$

by (2.8). Therefore $d_{m,g}$ satisfies the properties satisfied by $d_{1,j}$: it is a short chord whose short side does not intersect the short side of $d_{k,l}$, and it verifies (2.6), i.e. the path from v_g to v_k is not shorter than the path from v_l to v_m. Now, the path from v_g to v_k is obviously shorter than the path from v_j to v_k, contradicting the minimality property of $(d_{1,j}, d_{k,l})$.

Suppose now that $g < m < k$. Again, $d_{g,m}$ satisfies (2.6) and v_m is closer to v_k than v_j. Therefore, in order to respect the minimality property of $(d_{1,j}, d_{k,l})$, we must have $m - g \geq \gamma/2$. This implies ($m \leq k - 1 \leq \gamma/2$ by (2.8) and $g \geq j + 1$)

$$\gamma/2 \leq m - g \leq \gamma/2 - j - 1$$

which is obviously impossible. (2.10) is proved.

To finish the proof of the theorem, it is enough to show that, if Γ is 3-edge-connected, then Γ' is 3-edge-connected and 3-linked to Γ. To do that we shall factor the twist of $(d_{1,j}, d_{k,l})$ into $(d_{1,k}, d_{j,l})$ by a series of twists swapping consecutive vertices, each of which preserves 3-edge-connectivity. We do that with two sets of twists. To define the first set, we make a choice of a chord $d_{h+1,*}$ for every $j \leq h \leq k - 1$. This choice will be irrelevant.

(I.1) Twist $(d_{1,j}, d_{j+1,*})$ into $(d_{1,j+1}, d_{j,*})$.

(I.2) Twist $(d_{1,j+1}, d_{j+2,*})$ into $(d_{1,j+2}, d_{j+1,*})$.

....

(I.h+1-j) Twist $(d_{1,h}, d_{h+1,*})$ into $(d_{1,h+1}, d_{h,*})$, with $j \leq h \leq k-1$.

....

(I.k-j) Twist $(d_{1,k-1}, d_{k,l})$ into $(d_{1,k}, d_{k-1,l})$

Observe that in each of the above twists, the two chords getting twisted, $d_{1,h}$ and $d_{h+1,*}$, do not cross, i.e. $d_{h+1,*} = d_{h+1,m}$ with $m > h+1$. This is obvious for the last step, $(I.k-j)$, as $1 < k-1 < k < l$. For the remaining steps, for which $h \leq k-2$, we use (2.10), according to which every $d_{h+1,*}$ is of type $d_{h+1,m}$ with $m \geq k$. Hence $1 < h < h+1 \leq k-1 < m$, as claimed.

Therefore condition (a) of Lemma 2.4.2 holds, and we conclude that the graph Γ'', obtained after the above set of twists, is 3-edge-connected and 3-linked to Γ.

Notice that Γ'' contains the chord $d_{1,k}$ and the chord $d_{k-1,l}$. The second set of twists, starting from Γ'' is the following.

(II.1) Twist $(d_{k-1,l}, d_{k-2,*})$ into $(d_{k-2,l}, d_{k-1,*})$.

(II.2) Twist $(d_{k-2,l}, d_{k-3,*})$ into $(d_{k-3,l}, d_{k-2,*})$.

....

(II.k-h) Twist $(d_{h,l}, d_{h-1,*})$ into $(d_{h-1,l}, d_{h,*})$, where $j+1 \leq h \leq k-1$.

....

(II.k-j-1) Twist $(d_{j+1,l}, d_{j,*})$ into $(d_{j,l}, d_{j+1,*})$

where the chords $d_{h-1,*}$ are those chosen for the first set of twists. Observe that the chord $d_{1,k}$ (which lies in every graph appearing in the above twists) crosses every chord $d_{h,l}$ with $j+1 \leq h \leq k-1$. If $d_{1,k}$ crosses also $d_{h-1,*}$ Lemma 2.4.2 applies to the step (II.k-h) above (condition (b) of Lemma 2.4.2 holds), and hence 3-edge-connectivity is preserved (to fit in precisely with the notation of Lemma 2.4.2, one translates the starting vertex after v_h, sets $h-1 = j$ and $h = j+1$ so that $d_{h-1,*}$ becomes $d_{i,j}$ and $d_{h,l}$ becomes $d_{j+1,*}$.).

What if $d_{1,k}$ does not cross $d_{h-1,*}$? Recall that by (2.10) we have $d_{h-1,*} = d_{h-1,m}$ with $m \geq k$. Therefore $d_{1,k}$ does not cross $d_{h-1,m}$ only if $k = m$. Let us show that twisting $(d_{h,l}, d_{h-1,k})$ into $(d_{h-1,l}, d_{h,k})$ preserves 3-edge-connectivity; let $\widetilde{\Gamma}$ be the graph obtained after this twist. By Lemma 2.3.4(2) it suffices to show that $\widetilde{\Gamma}$ contains two cycles, Δ_1 and Δ_2, whose only edge in common is the edge e_{h-1} (joining the two swapped vertices v_{h-1} and v_h). Here are the two cycles

$$\Delta_1 = (e_{h-1}, d_{h,k}, d_{1,k}, e_1, \ldots, e_{h-2})$$

and

$$\Delta_2 = (e_{h-1}, e_h, \ldots, e_{l-1}, d_{h-1,l}).$$

Therefore the graph Γ''' obtained from Γ by our two sets of twists is 3-edge-connected and 3-linked to Γ. Let us check that Γ''' coincides with the Γ' of Claim 2.4.4. The chords $d_{1,j}$ and $d_{k,l}$ of Γ are twisted into $d_{1,k}$ and $d_{j,l}$ in Γ' and Γ'''. The remaining chords of Γ and Γ' are the same. The chord $d_{h+1,m} \in E(\Gamma)$ with $j \leq h \leq k-2$, in the first set of twists, is changed into the chord $d_{h,m} \in E(\Gamma'')$, which is changed back into $d_{h+1,m} \in E(\Gamma''')$ by the second set of twists. All other chords of Γ are not touched by our twists. So $\Gamma''' = \Gamma'$ and we are done. ■

3. Moduli of Tropical curves

3.1. **Tropical curves and tropical equivalence.** In this subsection we recall several basic facts. The original definition of a tropical curve can be given in terms of metric graphs, by [12] or [14]. In the following, we use a terminology slightly different from the cited references. Recall that our graphs are assumed connected.

- A *pure tropical curve* is a pair (Γ, ℓ) where Γ is a graph and ℓ is a *length* function on the edges
$$\ell : E(\Gamma) \to \mathbb{R}_{>0} \cup \{\infty\}$$
such that $\ell(e) = +\infty$ if and only if e is adjacent to a 1-valent vertex. The genus of (Γ, ℓ) is $g(\Gamma, \ell) = b_1(\Gamma)$.
- More generally, following [2], a *(weighted) tropical curve* is a triple (Γ, w, ℓ) where Γ is a graph, $w : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ a *weight* function on the vertices, and ℓ a length function
$$\ell : E(\Gamma) \to \mathbb{R}_{>0} \cup \{\infty\}$$
such that $\ell(e) = +\infty$ if and only if e is adjacent to a 1-valent vertex of weight 0.

  The genus of (Γ, w, ℓ) is defined as follows:
$$g(\Gamma, w, \ell) = g(\Gamma, w) = b_1(\Gamma) + \sum_{v \in V(\Gamma)} w(v). \tag{3.1}$$

  By "tropical curve" without attribute we shall mean a weighted tropical curve. If $w = \underline{0}$, i.e. $w(v) = 0$ for every $v \in V(\Gamma)$, the weighted tropical curve is pure.
- Two tropical curves are *(tropically) equivalent* if they can be obtained from one another by adding or removing 2-valent vertices of weight 0, or 1-valent vertices of weight 0, together with their adjacent edge.

3.1.1. *Pointed tropical curves.* Before giving more details, we want to extend our discussions to curves with points on them, so-called "pointed tropical curves". First, we introduce a generalized notion of graphs, namely, graphs with legs. Here is the combinatorial definition.

Definition 3.1.2. A graph Γ with n *legs* is the following set of data:

(1) A finite non-empty set $V(\Gamma)$, the set of *vertices*.
(2) A finite set $H(\Gamma)$, the set of *half-edges*.
(3) An involution $\iota : H(\Gamma) \to H(\Gamma)$ with n fixed points called the *legs* of Γ; the set of legs is denoted by $L(\Gamma)$.
   A pair $e = \{h, \iota(h)\}$ of distinct elements in $H(\Gamma)$ is called an *edge*; the set of edges is denoted by $E(\Gamma)$.
(4) A map $\epsilon : H(\Gamma) \to V(\Gamma)$.

If $\epsilon(h) = v$ we say that h is adjacent to v, or that v is its endpoint. The *valency* of $v \in V(\Gamma)$ is the number $|\epsilon^{-1}(v)|$ of half-edges adjacent to v. We say that Γ is p-regular if every $v \in V(\Gamma)$ has valency p.

It is clear how to associate to the above combinatorial object a topological space. Namely let Γ be a graph as defined above, with vertex set V and edge set E. The topological graph associated to it has V as the set of 0-cells; then we add a 1-cell for every $e = \{h, \iota(h)\} \in E$, so that the boundary of this 1-cell is $\{\epsilon(h), \epsilon(\iota(h))\}$. If Γ has a non empty set of legs L, we add an open 1-cell for every $h \in L$ in such a way that one extreme of the 1-cell contains $\epsilon(h)$ in its closure.

Of course, if L is empty we have the same graphs treated in the previous section of the paper. We shall henceforth view graphs with legs also as topological spaces, and we shall freely switch between the combinatorial and the topological viewpoint. As in the previous part of the paper, we shall assume that all our graphs are connected.

Now, a point of a tropical curve can be efficiently represented by a leg of the corresponding graph. Here is a list of basic definitions and properties, the first of which generalize those stated in Subsection 3.1; see [3] for details and examples.

(1) An *n-pointed tropical curve* is a triple (Γ, w, ℓ) where Γ is a graph with n legs, $w : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ a *weight* function on the vertices, and ℓ is a *length* function

$$\ell : E(\Gamma) \cup L(\Gamma) \to \mathbb{R}_{>0} \cup \{\infty\}$$

such that $\ell(x) = +\infty$ if and only if either $x \in L(\Gamma)$ or x is an edge adjacent to a 1-valent vertex of weight 0.

The legs of Γ are the marked points of the curves. The genus of (Γ, w, ℓ) is $g(\Gamma, w, \ell) = g(\Gamma, w)$ as defined in (3.1).

(2) As before, an n-pointed tropical curve is called *pure* if $w = \underline{0}$.

An n-pointed tropical curve is called *regular* if it is pure and if its underlying graph Γ is 3-regular.

(3) The pair (Γ, w) is called a *weighted graph* (with n legs); we say that (Γ, w) is the combinatorial type of the curve (Γ, w, ℓ).

(4) (Γ, w) is called *stable* if every vertex of weight 0 has valency at least 3, and every vertex of weight 1 has valency at least 1 (in other words, an isolated vertex is stable only if it has weight at least 2; see [3, Ex. 2.4.7] for more on this point).

Stable graphs of genus g with n legs exist if and only if $2g-2+n > 0$.

(5) Two n-pointed tropical curves are *(tropically) equivalent* if they can be obtained from one another by adding or removing

(a) 2-valent vertices of weight 0,

or

(b) 1-valent vertices of weight 0, together with their adjacent edge.

Tropical equivalence preserves the genus and the number of legs.

(6) Suppose $2g - 2 + n > 0$. Every tropical equivalence class of n-pointed tropical curves contains a unique representative whose combinatorial type is stable.

(7) Two n-pointed tropical curves (Γ_1, w_1, ℓ_1) and (Γ_2, w_2, ℓ_2) are *isomorphic* if there exists a triple $(\alpha_V, \alpha_E, \alpha_L)$, where $\alpha_V : V(\Gamma_1) \to V(\Gamma_2)$, $\alpha_E : E(\Gamma_1) \to E(\Gamma_2)$ and $\alpha_L : L(\Gamma_1) \to L(\Gamma_2)$ are bijections such that α_V maps the endpoints of $x \in E(\Gamma_1) \cup L(\Gamma_1)$ to the endpoints of

$\alpha_E(x)$ or $\alpha_L(x)$ for every $x \in E(\Gamma_1) \cup L(\Gamma_1)$. Moreover $\forall v \in V(\Gamma_1)$ and $\forall e \in E(\Gamma_1)$ we have $w_1(v) = w_2(\alpha_V(v))$ and $\ell_1(e) = \ell_2(\alpha_E(e))$.

(8) The automorphism group $\mathrm{Aut}(\Gamma, w)$ of a weighted graph (Γ, w) is given by triples $\alpha = (\alpha_V, \alpha_E, \alpha_L)$ as in the previous item, ignoring the condition on the length.

(9) A weighted graph with n legs, and hence an n-pointed tropical curve, has finitely many automorphisms.

Remark 3.1.3. In the definition of n-pointed tropical curve we did not require that the points be distinct, i.e. that the legs have different endpoints. This is because we shall work modulo tropical equivalence, which does not preserve this property. On the other hand, every tropical equivalence class of n-ponted curves contains representatives whose marked points are distinct; see [3, Prop. 2.4.10].

The addition of a weight function to a tropical curve (introduced in [2]) is a way to fix the fact that the set of pure tropical curves of given genus in not closed under specialization. More precisely, families of tropical curves are given by letting the length of the edges vary. Now if some length goes to zero, it may very well happen that some cycle gets contracted, and hence the first Betti number drops. This problem does not arise when considering weighted tropical curves, as we are going to explain.

First, let us formalize the process of edge length going to zero. Fix a weighted graph (Γ, w), and $S \subset E(\Gamma)$. The *weighted contraction* of S is the weighted graph $(\Gamma/S, w/S)$, where Γ/S is defined in subsection 2.1.1 in case $L(\Gamma) = \emptyset$; if $L(\Gamma)$ is not empty, the definition is trivially adjusted so that there is a natural identification between $L(\Gamma)$ and $L(\Gamma/S)$. To define w/S, recall that we have a natural map $\sigma : \Gamma \to \Gamma/S$ and a natural surjection $\sigma_V : V(\Gamma) \to V(\Gamma/S)$. We set for every $\overline{v} \in V(\Gamma/S)$

$$w/S(\overline{v}) = b_1(\sigma^{-1}(\overline{v})) + \sum_{v \in \sigma_V^{-1}(\overline{v})} w(v). \tag{3.2}$$

We write

$$(\Gamma, w) \geq (\Gamma', w') \text{ if } (\Gamma', w') \text{ is a weighted contraction of } (\Gamma, w). \tag{3.3}$$

Remark 3.1.4. Suppose $(\Gamma, w) \geq (\Gamma', w')$. Then one easily checks the following properties

(1) $|L(\Gamma)| = |L(\Gamma')|$.
(2) $g(\Gamma, w) = g(\Gamma', w')$ (by identity (2.1) and remark 2.1.2).
(3) If (Γ, w) is stable, so is (Γ', w').

Therefore, the set of stable genus-g graphs with n legs is closed under weighted contractions.

3.2. **The moduli space of pointed tropical curves.** From now on we shall consider tropical curves up to tropical equivalence. Therefore we will assume that our weighted graphs are stable.

Let us fix the stable graph (Γ, w) with n legs, let $g = g(\Gamma, w)$, and let us consider the space $M(\Gamma, w)$ of isomorphism classes of tropical curves having (Γ, w) as combinatorial type. More precisely, we have a natural identification:

$$M(\Gamma, w) = (\mathbb{R}_{>0})^{E(\Gamma)} / \operatorname{Aut}(\Gamma, w)$$

where an automorphism $(\alpha_V, \alpha_E, \alpha_L) \in \operatorname{Aut}(\Gamma, w)$ acts by permuting the coordinates of $(\mathbb{R}_{>0})^{E(\Gamma)}$ according to α_E; see item (7). In particular, $M(\Gamma, w)$ is an orbifold of dimension $|E(\Gamma)|$, since $\operatorname{Aut}(\Gamma, w)$ is finite. The set $M(\Gamma, w)$ is thus a topological space, with the quotient topology induced by the euclidean topology.

We recall the following well known and easy to prove fact:

Remark 3.2.1. Let (Γ, w) be a genus g stable graph with n legs. Then $|E(\Gamma)| \leq 3g - 3 + n$ and equality holds if and only if Γ is a 3-regular graph with $b_1(\Gamma) = g$. Moreover, in this case we necessarily have $w = \underline{0}$.

We now introduce the moduli space, $M_{g,n}^{\mathrm{trop}}$, of n-pointed tropical curves of genus g:

$$M_{g,n}^{\mathrm{trop}} = \bigsqcup_{\substack{(\Gamma,w) \text{ stable} \\ \text{genus } g,\ n \text{ legs}}} M(\Gamma, w). \tag{3.4}$$

The following statement is a summary of some of the properties of $M_{g,n}^{\mathrm{trop}}$ (see [3] for details; in the case $n = 0$ some of the properties below are proved also in [2]).

Fact 3.2.2. *Assume $2g-2+n > 0$ and let (Γ, w) be a stable graph of genus g with n legs.*

(1) $M_{g,n}^{\mathrm{trop}}$ is endowed with a topology such that the natural injection $M(\Gamma, w) \hookrightarrow M_{g,n}^{\mathrm{trop}}$ is a homeomorphism with its image.

(2) With the notation (3.3)*, we have*

$$M(\Gamma', w') \subset \overline{M(\Gamma, w)} \Leftrightarrow (\Gamma, w) \geq (\Gamma', w').$$

(3) Let $M_{g,n}^{\mathrm{reg}} \subset M_{g,n}^{\mathrm{trop}}$ be the subset parametrizing regular curves, i.e.

$$M_{g,n}^{\mathrm{reg}} = \bigsqcup_{\substack{|L(\Gamma)|=n,\ b_1(\Gamma)=g \\ \Gamma \ \ 3-\textit{regular}}} M(\Gamma, \underline{0}) \subset M_{g,n}^{\mathrm{trop}}.$$

Then $M_{g,n}^{\mathrm{reg}}$ is open and dense in $M_{g,n}^{\mathrm{trop}}$.

(4) Let $M_{g,n}^{\mathrm{pure}}$ be the subset parametrizing pure tropical curves. Then $M_{g,n}^{\mathrm{pure}}$ is open and dense $M_{g,n}^{\mathrm{trop}}$.

(5) $M_{g,n}^{\mathrm{trop}}$ is a connected, Hausdorff topological space of pure dimension $3g - 3 + n$.

Remark 3.2.3. We need to explain the meaning of the last statement. Recall that a topological space X containing a dense open subset U, where U is an orbifold (locally the quotient of a topological manifold by a finite group) of dimension d, is said to have *pure dimension d*.

Now, by part (3), $M_{g,n}^{\mathrm{trop}}$ contains the dense open subset $U = M_{g,n}^{\mathrm{reg}}$, which is an orbifold has dimension $3g - 3 + n$, by Remark 3.2.1. This explains the claim on the dimension. Connectedness of $M_{g,n}^{\mathrm{trop}}$ is trivial, since every (Γ, w) satisfies $(\Gamma, w) \geq (\Gamma^*, w^*)$ where (Γ^*, w^*) is the graph having no edges, only one vertex of weight g, and n legs attached to it. The fact that $M_{g,n}^{\mathrm{trop}}$ is Hausdorff is proved in [3, sect. 3.2].

3.3. **Connectedness properties of tropical moduli spaces.** In this last subsection we apply our Linkage Theorem 2.4.3 to the geometry of some moduli spaces of tropical curves.

To begin with, we have said that $M_{g,n}^{\rm trop}$ is connected; but a stronger form of connectedness holds, namely $M_{g,n}^{\rm trop}$, and likewise $M_{g,n}^{\rm pure}$, is connected through codimension one; see Definition 3.3.1. This property is one that is fundamental for tropical varieties defined by prime ideals (see [11]). Although $M_{g,n}^{\rm trop}$ and $M_{g,n}^{\rm pure}$ are not known to be tropical varieties in general (the case $g = 0$ is a well known exception), their connectedness through codimension one is a sign of their being somewhat close to tropical varieties.

The next definition is adapted from [11, Definition 3.3.2].

Definition 3.3.1. Let X be a topological space of pure dimension d; see 3.2.3. Assume that X is endowed with a decomposition $X = \sqcup_{i\in I} X_i$, where every X_i is a connected orbifold. We say that X is *connected through codimension one* if the subset

$$\bigsqcup_{i\in I:\dim X_i\geq d-1} X_i \subset X$$

is connected.

Notice that if X is pure dimensional and connected through codimension one, then X is connected.

Now, observe that the notion of linked graphs, given in Definition 2.1.3, extends word for word to graphs with legs. We can therefore state the following result, which is a consequence of Theorem 2.4.3.

Proposition 3.3.2. *Let Γ_1 and Γ_2 be two 3-regular graphs with n legs and $b_1(\Gamma_1) = b_1(\Gamma_2)$. Then Γ_1 and Γ_2 are linked.*

Proof. Of course, $|E(\Gamma_1)| = |E(\Gamma_2)|$; we can assume $|E(\Gamma_i)| \geq 2$ for otherwise the result is trivial. We use induction on n; the base case $n = 0$ is a special case of Theorem 2.4.3.

Suppose $n \geq 1$; let us denote by $\mathcal{G}(n)$ the set of 3-regular graphs of genus g with n legs. Let $\Gamma \in \mathcal{G}(n)$, pick a leg $l \in L(\Gamma)$ and let $v \in V(\Gamma)$ be its endpoint. Let Γ' be the closure of the graph obtained by removing l and v from Γ. It is clear that $\Gamma' \in \mathcal{G}(n-1)$. Notice that, of course, every $\Gamma \in \mathcal{G}(n)$ is obtained by adding a leg and its endpoint to some graph in $\mathcal{G}(n-1)$.

Claim. *Fix a graph $\Gamma' \in \mathcal{G}(n-1)$; any two graphs in $\mathcal{G}(n)$ obtained by adding to Γ' a leg and its endpoint are linked.*

The claim implies our Proposition. Indeed, let $\Gamma_1', \Gamma_2' \in \mathcal{G}(n-1)$ be such that for, some $e_i' \in E(\Gamma_i')$ we have

$$\Gamma_1'/e_1' = \Gamma_2'/e_2'. \tag{3.5}$$

Let $\Gamma_1 \in \mathcal{G}(n)$ be obtained by adding to Γ_1' a leg whose endpoint is not in the interior of e_1'. Then, by (3.5), there exists a $\Gamma_2 \in \mathcal{G}(n)$ obtained by adding a leg and its endpoint to Γ_2' such that $\Gamma_1/e_1 = \Gamma_2/e_2$; so Γ_1 is linked to Γ_2. Hence, by the claim, we get that all graphs in $\mathcal{G}(n)$ obtained from Γ_1' are linked to those obtained from Γ_2'. By the induction hypothesis every pair of elements in $\mathcal{G}(n-1)$ is linked, so we are done.

It remains to prove the claim. For $i = 1, 2$, let $\Gamma_i \in \mathcal{G}(n)$ be the graph obtained by adding to Γ' a vertex v_i (in the interior of some edge or leg of Γ') and a leg l_i adjacent to v_i. We must show that Γ_1 and Γ_2 are linked. Pick $w \in V(\Gamma') \subset V(\Gamma_i)$; for $i = 1, 2$ the vertex v_i can be joined to w by some path Π_i of minimal length contained in Γ_i; let h_i be the edge-length of Π_i, where h_i is a positive integer, since $w \neq v_i$; we call h_i the edge-path length from v_i to w. Let $h = h_1 + h_2$; if $h = 2$, i.e. if $h_1 = h_2 = 1$, there exists an edge $e_i \in E(\Gamma_i)$ whose endpoints are w and v_i. It is clear that

$$\Gamma_1/e_1 = \Gamma_2/e_2$$

so we are done. We continue by induction on h.

Suppose $h \geq 3$, and let $h_1 \geq 2$. Let e_1 be the first edge of Π_1 so that v_1 is an endpoint of e_1; Consider the graph Γ_1/e_1. The coming construction is illustrated in the picture below. Now let u be the other endpoint of e_1 and let f be the next edge of Π_1, starting at u; by construction, f is also an edge of Γ'. Let $\Gamma_3 \in \mathcal{G}(n)$ be the graph obtained from Γ' by adding a vertex v_3 in the interior of f and a leg attached to it. Now, Γ_3 has a unique edge e_3 whose endpoints are u and v_3. The edge-path length from v_3 to w is $h_1 - 1$, hence by induction Γ_3 is linked to Γ_2. On the other hand it is immediately clear that

$$\Gamma_1/e_1 = \Gamma_3/e_3,$$

hence Γ_3 and Γ_1 are linked, and so Γ_1 is also linked to Γ_2 . ■

The next picture represents the construction used to prove the claim, with $g = 2$ and $n = 3$..

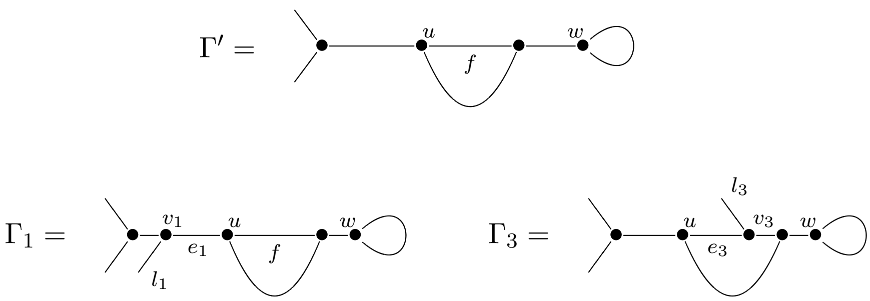

FIGURE 6. Γ_1 and Γ_3 linked, obtained from Γ' (proof of 3.3.2).

From Proposition 3.3.2 we easily get:

Proposition 3.3.3. *The spaces* $M_{g,n}^{\mathrm{trop}}$ *and* $M_{g,n}^{\mathrm{pure}}$ *are connected through codimension one.*

Proof. From Fact 3.2.2 we have that $M_{g,n}^{\mathrm{trop}}$ and $M_{g,n}^{\mathrm{pure}}$ are of pure dimension $3g - 3 + n$. Also, we know that $\dim M(\Gamma, w) = |E(\Gamma)|$. So, by 3.2.2 (2) to prove our statement it suffices to observe that any two 3-regular graphs are linked, as stated in Proposition 3.3.2. ■

Remark 3.3.4. As proved in [2, Prop. 3.2.5], the above result in case $n = 0$ follows from [10, Prop. page 236], which is a remarkable and well known special case of our Theorem 2.4.3.

3.3.5. *The tropical Torelli map and the Schottky locus.* We will now prove that connectedness through codimension one holds for other tropical moduli spaces.

In analogy with the classical situation we have a tropical Torelli map

$$t_g^{\mathrm{trop}} : M_g^{\mathrm{trop}} \to A_g^{\mathrm{trop}}$$

to the moduli space of tropical Abelian varieties, mapping a curve to its tropical Jacobian (see [14], [4] and [2] for details). We denote by $\mathrm{Sch}_g^{\mathrm{trop}}$ the image of t_g^{trop}, and refer to it, as it is customary, as the tropical Schotty locus in A_g^{trop}. A detailed analysis of $\mathrm{Sch}_g^{\mathrm{trop}}$ for small values of g is carried out in [5].

For our purposes $\mathrm{Sch}_g^{\mathrm{trop}}$ can be identified with the topological quotient

$$\mathrm{Sch}_g^{\mathrm{trop}} := M_g^{\mathrm{trop}} / \equiv_{t_g^{\mathrm{trop}}}$$

where $[(\Gamma, \ell, w)] \equiv_{t_g^{\mathrm{trop}}} [(\Gamma, \ell, w)] \Leftrightarrow t_g^{\mathrm{trop}}([(\Gamma, \ell, w)]) = t_g^{\mathrm{trop}}([(\Gamma', \ell', w')])$. For more structure on A_g^{trop} and $\mathrm{Sch}_g^{\mathrm{trop}}$ we refer to [2]. In particular, Theorem 5.2.4 of loc. cit. gives a precise characterization of the tropical Schotty locus $\mathrm{Sch}_g^{\mathrm{trop}}$ in A_g^{trop}, in such a way that the Schottky problem has a satisfactory answer in tropical geometry.

As proved in [4, Thm 4.1.9], and generalized by [2, Thm 5.3.3], the Torelli map identifies curves having the same so-called "3-edge-connected class". More precisely, let us denote by $M_g^{\mathrm{trop}}[3]$ the locus of tropical curves with 3-edge-connected graph:

$$M_g^{\mathrm{trop}} \supset M_g^{\mathrm{trop}}[3] := \{[(\Gamma, \ell, w)] : \ \Gamma \text{ is 3-edge-connected}\}.$$

Then we have

$$t_g^{\mathrm{trop}}(M_g^{\mathrm{trop}}[3]) = t_g^{\mathrm{trop}}(M_g^{\mathrm{trop}}) = \mathrm{Sch}_g^{\mathrm{trop}} \subset A_g^{\mathrm{trop}}.$$

Furthermore, the restriction of t_g^{trop} to $M_g^{\mathrm{trop}}[3]$, denoted by $t_g^{\mathrm{trop}}[3]$, is injective on every subspace $M(\Gamma, w) \subset M_g^{\mathrm{trop}}[3]$, and it identifies two such spaces, $M(\Gamma, w)$ and $M(\Gamma', w')$, only if the graphs Γ and Γ' are cyclically equivalent (i.e. 2-isomorphic in the sense of Whitney, see [4, Def 2.2.3]). In particular, $t_g^{\mathrm{trop}}[3]$ has finite fibers.

The previous results hold in the special case of pure tropical curves (in fact, they were first proved in this case, and then generalized to weighted tropical curves). With self-explanatory notation, the Torelli map for pure tropical curves is a surjection

$$t_g^{\mathrm{pure}} : M_g^{\mathrm{pure}} \longrightarrow \mathrm{Sch}_{\mathrm{g}}^{\mathrm{pure}} := \mathrm{M}_{\mathrm{g}}^{\mathrm{pure}} / \equiv_{\mathrm{t}_{\mathrm{g}}^{\mathrm{pure}}} \subset \mathrm{A}_{\mathrm{g}}^{\mathrm{trop}},$$

and the restriction of t_g^{pure} to the locus of pure tropical curves with 3-edge connected graph, $M_g^{\mathrm{pure}}[3] \subset M_g^{\mathrm{pure}}$, behaves exactly as $t_g^{\mathrm{trop}}[3]$.

Now, the conservation of 3-edge-connectivity under linkage, proved in Theorem 2.4.3, enables us to obtain the following result.

Theorem 3.3.6. *The spaces* $M_g^{\mathrm{trop}}[3]$ *and* $\mathrm{Sch}_g^{\mathrm{trop}}$ *have pure dimension equal to* $3g-3$ *and are connected through codimension one.*

The same holds for the spaces $M_g^{\mathrm{pure}}[3]$ *and* $\mathrm{Sch}_{\mathrm{g}}^{\mathrm{pure}}$.

Proof. We prove the result for tropical curves; the proof for pure tropical curves follows precisely the same lines (and it is actually simpler). We introduce the locus of regular, 3-edge-connected curves

$$M_g^{\mathrm{reg}}[3] \subset M_g^{\mathrm{reg}} \subset M_g^{\mathrm{trop}}.$$

We have that the closure in M_g^{trop} of regular, 3-edge-connected curves is the locus of all 3-edge-connected curves, i.e.

$$\overline{M_g^{\mathrm{reg}}[3]} = M_g^{\mathrm{trop}}[3].$$

This follows from [4, Prop A.2.4], whose proof (stated there only for pure regular curves) works also in our setting (i.e. for weighted tropical curves). It is clear that $M_g^{\mathrm{reg}}[3]$ is an orbifold of pure dimension $3g-3$. We conclude that $M_g^{\mathrm{trop}}[3]$ has pure dimension $3g-3$.

Now, the connectedness through codimension one follows from the part of Theorem 2.4.3 concerning 3-edge connected graphs. It suffices to add that if (Γ', w') is obtained from (Γ, w) by contracting only one edge, then $\dim M(\Gamma, w) = \dim M(\Gamma', w')+1$ (as we also did for Proposition 3.3.6). This proves that $M_g^{\mathrm{trop}}[3]$ is connected through codimension one.

Now we turn to the Schottky locus; by what we said before there is a surjection with finite fibers

$$t_g^{\mathrm{trop}}[3] : M_g^{\mathrm{trop}}[3] \longrightarrow \mathrm{Sch}_g^{\mathrm{trop}}$$

obtained by restricting the Torelli map. This surjection induces a homeomorphism with its image of every subspace $M(\Gamma, \underline{0}) \subset M_g^{\mathrm{trop}}[3]$. This implies that $\mathrm{Sch}_g^{\mathrm{trop}}$ has pure dimension $3g-3$. Furthermore, as $t_g^{\mathrm{trop}}[3]$ is injective on every $M(\Gamma, w)$, it preserves the dimension of these subsets; therefore $\mathrm{Sch}_g^{\mathrm{trop}}$ is connected through codimension one, because so is $M_g^{\mathrm{trop}}[3]$. ∎

Remark 3.3.7. What are the consequences on tropical moduli spaces of the linkage theorem when $p \geq 4$? Consider the subset

$$M_g^{p-\mathrm{reg}} := \bigsqcup_{\substack{\Gamma\ p\text{-regular} \\ b_1(\Gamma)=g}} M(\Gamma, \underline{0}) \subset M_g^{\mathrm{pure}}$$

and assume it is not empty. By a proof similar to that of Theorem 3.3.6 one obtains that the closure of $M_g^{p-\mathrm{reg}}$ is of pure dimension equal to $p(g-1)/(p-2)$, by Remark 2.1.7 (this number is an integer by the non-emptyness assumption), and connected through codimension one.

The same holds if the above disjoint union is restricted to all 3-edge-connected and p-regular graphs with $b_1(\Gamma) = g$. That is, with self-explanatory notation, the closure of $M_g^{p-\mathrm{reg}}[3]$ is of pure dimension $p(g-1)/(p-2)$ and connected through codimension one.

A space closely related to M_g^{trop} is the outer space O_g constructed in [6], and its quotient by the group $\mathrm{Out}(F_g)$ (outer automorphisms of the free group on g generators F_g). This quotient can be interpreted as a moduli space for metric graphs, and its connection with M_g^{trop} or M_g^{pure} is currently under investigation; as it has not yet been completely unraveled, we will not be more specific about this point. We just wish to mention that Theorem 2.4.3 applied to O_g yields analogous connectivity properties of certain subcomplexes of a remarkable deformation retract of O_g, called its "spine" (defined in [6, sect 1.1]).

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