INTRODUCTION to MODULI of CURVES

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Sketchy, unfinished course notes full of gaps and inaccuracies. Please do not distribute. Corrections and suggestions will be appreciated by the author.

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1 The set up in scheme theoretic and functorial language

1.1 Generalities

In classical terms, posing a moduli problem in the algebro-geometric category amounts to consider

1. A fixed set $\mathcal{C}$ of algebro-geometric objects defined over an algebraically closed field, together with an equivalence relation.

As a leading example, take $\mathcal{C}$ the set of all connected, smooth, complex projective curves of given genus with the relation of isomorphism.

2. A notion of *families* of objects in $\mathcal{C}$; that is, suitable morphisms of schemes $f : X \longrightarrow B$ such that for every $b \in B$, the fiber $X_b := f^{-1}(b)$ is an object in $\mathcal{C}$.

For our example, a good notion of family will be a flat morphism.

By taking B the spectrum of an algebraically closed field, we obtain 1. as a special case of 2.; however for our purposes the distinction must be kept clear.

Let M be the set of all equivalence classes on $\mathcal{C}$; we want to endow M with a scheme structure satisfying the following natural properties:

A. The (closed) points of M are in bijective correspondence with the equivalence classes of objects in $\mathcal{C}$.

B. If $f : X \longrightarrow B$ is a family (as stated above), then the natural set theoretic map

$$\phi_f : B \longrightarrow M$$

sending a point b to the equivalence class of its fiber X_b, is a morphism of schemes.

A further requirement is the following;

C. There exists a "universal" family $U \longrightarrow M$ (of objects in $\mathcal{C}$) such that every family $f : X \longrightarrow B$ as before is obtained by pulling back to B the family U via ϕ_f. In other words, we ask that $X \cong \phi_f^* U := U \times_M B$.

If a moduli problem admits a uniquely determined scheme M satisfing all three requirements above, M is called a *fine moduli scheme*. If M only satisfies A and B it is called a *coarse moduli scheme*.

All of this is more rigorously stated using the language of functors. Fix a base scheme S and consider the category of all schemes (always assumed of finite type) over S, denoted by SCH_S. We often consider the case $S = \operatorname{Spec} k$, with k a field (in which case we shorten the notation by SCH_k), or the general case of all schemes, that is, $S = \operatorname{Spec} \mathbb{Z}$ (in which case we write simply $\mathrm{SCH} = \mathrm{SCH}_{\operatorname{Spec} \mathbb{Z}}$).

Then the moduli data described above determine a contravariant functor $\mathcal{M}$ from the category of schemes over S to the category of sets, which we denote by SET. The functor $\mathcal{M}$ is defined by assigning to every S-scheme $B \in \mathrm{SCH}_S$[1] the set $\mathcal{M}(B)$ of all equivalence classes of families of objects in $\mathcal{C}$. This functor is well defined only if the properties defining the families are stable under base change, that is to say, if $f : X \longrightarrow B$ is a family and $h : B' \longrightarrow B$ is a morphism of schemes, the base change $h^*f := X \times_B B' \longrightarrow B'$ is also a family. Then we can define $\mathcal{M}(h) : \mathcal{M}(B) \longrightarrow \mathcal{M}(B')$ as the map sending the family $f : X \longrightarrow B$ to the family $h^*f : X \times_B \times B' \longrightarrow B'$.

Recall now

Definition 1.1.1. Let $\mathcal{F}$ be a contravariant functor from the category of schemes over S to that of sets

$$\mathcal{F} : \mathrm{SCH}_S \longrightarrow \mathrm{SET}.$$

A scheme $F \in \mathrm{SCH}_S$ and an element $U_{\mathcal{F}} \in \mathcal{F}(F)$ *represent* $\mathcal{F}$ if for every scheme $B \in \mathrm{SCH}_S$ the natural map of sets

$$\mathrm{Hom}_S(B, F)^2 \longrightarrow \mathcal{F}(B)$$

determined by the pair $(M, U_{\mathcal{F}})$, sending a morphism $\psi : B \longrightarrow F$ to the image of $U_{\mathcal{F}}$ under the map

$$\mathcal{F}(\psi) : \mathcal{F}(F) \longrightarrow \mathcal{F}(B),$$

is a bijection.

Example 1.1.2. The prototypical example of a representable functor is the following. Let X be a scheme; define the functor $\mathcal{H}om(..., X) : \mathrm{SCH} \longrightarrow \mathrm{SET}$ which assigns to a scheme B the set $\mathrm{Hom}(B, X)$. It is easy to check that the pair (X, id_X) represents this functor.

Now, as we shall see, many interesting moduli functors are not representable. This motivates the following weakening of Definition 1.1.1.

Definition 1.1.3. Let $\mathcal{F} : \mathrm{SCH} \longrightarrow \mathrm{SET}$ be a functor as above. A scheme M is said to *coarsely represent* $\mathcal{F}$ if there exists a functor transformation

$$\Phi : \mathcal{F}(...) \longrightarrow \mathcal{H}om(..., M)$$

such that for any algebraically closed field K

A. $\Phi(\operatorname{Spec} K) : \mathcal{F}(\operatorname{Spec} K) \longrightarrow \mathcal{H}om(\operatorname{Spec} K, M)$ is a bijection.

B. (Unicity) If N is a scheme and $\Psi : \mathcal{F}(...) \longrightarrow \mathcal{H}om(..., N)$ a functor transformation, then there exists a unique morphism $\pi : M \longrightarrow N$ such that the corresponding functor transformation $\Pi : \mathcal{H}om(..., M) \longrightarrow \mathcal{H}om(..., N)$ satisfies $\Psi = \Pi \circ \Phi$.

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[1] We shall abuse notation: given a category CAT and an object O in it, we will write $O \in CAT$.

[2] $\mathrm{Hom}_S(B, F)$ denotes the set of S-morphisms, i.e. S-regular maps, from B to F.

Exercise 1.1.4. Suppose that M coarsely represents $\mathcal{F}$ and that the transformation Φ above is an isomorphism of functors (that is, $\Phi(B)$ is a bijection for every $B \in$SCH). Show that M represents $\mathcal{F}$ (in the sense of Definition 1.1.1) by finding $U_{\mathcal{F}}$.

Definition 1.1.5. If $\mathcal{M}$ is a representable moduli functor and M the scheme that represents it, then M is a called a *fine moduli scheme* and $U_{\mathcal{M}}$ the *universal family.*

If a moduli functor is coarsely represented by a scheme M, such M is a called a *coarse moduli scheme.*

1.2 Rational points of schemes and moduli spaces

Let X be a scheme over a field k and let p be a point in X. We denote by $R(p)$ the residue field of p, that is, $R(p) = \mathcal{O}_{X,p}/m_p$. Since X is defined over k there is always a natural injection

$$k \hookrightarrow R(p).$$

If the above injection is an isomorphism, the point p is said to be *rational over* k or called a k-*rational point* of X. Denote

$$X(k) := \{p \in X : \ p \text{ is rational over } k\}$$

The notation $X(k)$ suggests a functorial perspective, in fact we have the following simple basic fact (compare with 1.1.2)

Lemma 1.2.1. *For every scheme X defined over a field k there is a natural identification:*

$$X(k) \cong \operatorname{Hom}(\operatorname{Spec} k, X)$$

Proof. Given a $\psi : \operatorname{Spec} k \longrightarrow X$, denote by p the image point of ψ. Then there is an induced natural homomorphism $\psi^* : \mathcal{O}_{X,p} \longrightarrow k$ which (after moding out by the maximal ideal of the left hand side) descends to a natural homomorphism of fields $R(p) \longrightarrow k$. As we observed above, $R(p)$ always contains k, hence the above morphism in an isomorphism.

Conversely, for every $p \in X(p)$ there is a natural isomorphism $R(p) \cong k$ (by definition). Then the quotient map $\mathcal{O}_{X,p} \longrightarrow R(p) = k$ determines a morphism $\operatorname{Spec} k \longrightarrow X$. □

Example 1.2.2. Let V be a nonsingular irreducible scheme over $\mathbb{C}$, let $k = \mathbb{C}(V)$ be the field of rational functions on V and let X be a scheme projective over k. Then we can choose a projective scheme Y over $\mathbb{C}$ and a dominant morphism $f : Y \longrightarrow V$ whose generic fiber is isomorphic to X (that is, the restriction of f to the generic point of Y is isomorphic to the structure map $X \longrightarrow \operatorname{Spec} k$).

The k-rational points of X can be described geometrically as the rational sections of f that is

$$X(k) = \{\sigma : V \dashrightarrow Y \text{ s.t. } \exists \text{ a dense, open } U \subset V \text{ with } f\sigma(u) = u \ \forall u \in U\}.$$

In fact if $p \in X$ has residue field $k = \mathbb{C}(V)$ then, over $\mathbb{C}$, p is birational to V, and the birational map is natural, that is, it is compatible with the structure morphism and hence with f. The converse is also clear, for the image p of a section $\sigma : V \dashrightarrow Y$ is a subvariety of Y birational to V; hence the residue field of p, which is equal to $\mathbb{C}(p)$, must be isomorphic to k.

A fibration f associated to X, as above, is called a *model* for X over V; notice that is not unique. A canonical choice will always be possible if $\dim V = 1$ (see 2.1.6 and 2.1.8). Moreover, in this case, the sections representing the rational points of X can be taken to be regular morphisms (simply because V is a non singular curve, hence a rational map $\sigma : V \longrightarrow Y$ is regular if X is projective).

Remark 1.2.3. As we explained, the k-rational points of a scheme X are the elements of $\mathrm{Hom}(\mathrm{Spec}\, k, X)$. More generally, if B is a scheme, the elements of $\mathrm{Hom}(B, X)$ are called the *B-rational points of X*.

With this in mind, the functor $\mathcal{H}om(..., X)$ (defined in 1.1.2) is often called *the functor of points of X*.

Thus, to say that a moduli problem admits a fine moduli space M, is to say that, for any scheme B, the families over B are identified with the B-rational points of M. In particular the objects correspond to the k-rational points of M, where k is an algebraically closed field.

Instead, for a moduli problem admitting only a coarse, non-fine, moduli scheme M, the objects (over a fixed algebraically closed field k) are still in bijection with the k-rational points of M, but M fails to carry every information about the families. More precisely, two types of problem can occur: first, for some scheme B the map $\Phi(B) : \mathcal{M}(B) \longrightarrow \mathrm{Hom}(B, M)$ is not injective, that is, there may be B-rational points of M which correspond to more than one family (a typical instance is the existence of so-called *isotrivial* families, see 2.1.2 and Example 2.1.3). Second, the map $\Phi(B)$ might be non-surjective, that is there may be B-rational points of M which do not correspond to any family at all (such rational points, or maps $B \longrightarrow M$ will be called *non-modular*, see 2.1.5).

Exercise 1.2.4. Fix an algebraically closed field k and work with the category of schemes over k. Let $\mathcal{M}$ be a moduli functor, $B \in \mathrm{SCH}_k$ and $f \in \mathcal{M}(B)$. Suppose that there exists an object $C \in \mathcal{M}(\mathrm{Spec}\, k)$ such that every fiber of f is isomorphic to C (families with this property are called *isotrivial*, see 2.1.2). Assume now that there exists a scheme N endowed with a functor transformation $\Psi : \mathcal{M}(...) \longrightarrow \mathcal{H}om(..., N)$. Prove that the morphism from B to N corresponding to the given isotrivial family is constant; in symbols, prove that

$$\Psi(B)(f) : B \longrightarrow N$$

maps B to a point.

1.3 Fine moduli for projective schemes: the Hilbert scheme

A fundamental moduli problem is that of closed subschemes of a fixed projective space $\mathbb{P}^r$ over some fixed algebraically closed field k. To fix some numerical invariants (dimension and degree), let $h(t) \in \mathbb{Q}[t]$ be a polynomial, and let $\mathcal{C}$ be the set of all closed subschemes of $\mathbb{P}^r$ whose Hilbert polynomial is $h(t)$. The equivalence relation is simply the equality. Let us consider as families all proper and flat morphisms $f : X \longrightarrow B$, where B is a scheme, X is a closed subcheme of $\mathbb{P}^r \times B$, f is the projection onto B and all fibers have Hilbert polynomial equal to $h(t)$. The corresponding functor is denoted by

$$\mathcal{H}ilb_{\mathbb{P}^r}^{h(t)} : \mathrm{SCH}_k \longrightarrow SET$$

Thus $\mathcal{H}ilb^{h(t)}_{\mathbb{P}^r}(B)$ is the set of all closed subchemes $X \subset \mathbb{P}^r \times B$ such that the projection to B restricts to a proper, flat morphism $X \longrightarrow B$ all of whose fibers have $h(t)$ as Hilbert polynomial.

The following theorem is in [Gro62]

Theorem 1.3.1 (Grothendieck)**.** *The functor $\mathcal{H}ilb^{h(t)}_{\mathbb{P}^r}$ is represented by a projective scheme $Hilb^{h(t)}_{\mathbb{P}^r}$.*

In other words, there exists a fine moduli scheme $Hilb^{h(t)}_{\mathbb{P}^r}$ for this moduli problem, called the Hilbert scheme; it is endowed with a universal family $U \subset \mathbb{P}^r \times Hilb^{h(t)}_{\mathbb{P}^r}$, from which every other family is obtained by base change. The above theorem holds in greater generality: $\mathbb{P}^r$ may be replaced by a scheme projective over a fixed base scheme S.

Example 1.3.2. It is not hard to work out the special case of hypersurfaces of degree d in $\mathbb{P}^r$. The Hilbert polynomial then is

$$h(t) = \binom{r+t}{r} + \binom{r+t-d}{r} = \frac{d}{(r-1)!} t^{r-1} + \dots.$$

Let V be the k-vector space of homogeneous polynomials of degree d in $r+1$ variables, and let $U \subset \mathbb{P}(V) \times \mathbb{P}^r$ be the incidence correspondence

$$U := \{(F, p) : F(p) = 0\}$$

Then we have $Hilb^{h(t)}_{\mathbb{P}^r} = \mathbb{P}(V)$ and U is the universal family over it (the family morphism $U \longrightarrow Hilb^{h(t)}_{\mathbb{P}^r}$ being, of course, the restriction of the projection).

We shall discuss the proof of the above Theorem in the sequel.

2 Moduli of nonsingular curves

2.1 Existence and coarseness

Consider the class of all nonsingular, irreducible curves projective over S, equipped with the relation of isomorphism (families are defined below). The case $S = \operatorname{Spec}\mathbb{C}$ was considered by Riemann who, in [Rie857] proved that the space of non isomorphic complex structures (which he himself named "moduli") definable on a compact, connected, topological surface of genus $g \geq 2$ has complex dimension $3g - 3$.

Fix now the base scheme $S = \operatorname{Spec}\mathbb{Z}$ and let SCH be the category of all schemes. The moduli functor $\mathcal{M}_g : SCH \longrightarrow SET$ is defined by letting $\mathcal{M}_g(B)$ to be the set of all smooth proper morphisms $f : X \longrightarrow B$ all of whose fibers are nonsingular irreducible projective curves of genus g, modulo the following equivalence relation: the family $f : X \longrightarrow B$ is equivalent to the family $f' : X' \longrightarrow B'$ if there exist isomorphisms $h : B \longrightarrow B'$ and $j : X \longrightarrow X'$ such that $h \circ f = f' \circ j$.

We shall show that $\mathcal{M}_g$ cannot possibly be represented, that is to say, there does not exist a fine moduli scheme for this moduli problem. On the other hand, we shall prove

Theorem 2.1.1. *Let $g \geq 2$. The functor $\mathcal{M}_g$ is coarsely represented by a quasi-projective scheme M_g. M_g is irreducible and* $\dim M_g = 3g - 3$.

The first part of the statement is due to D.Mumford ([GIT65]); the irreducibility part (classically known over $\mathbb{C}$) is proved in the seminal paper by P.Deligne and D.Mumford [DM69], which we will discuss later.

Now, there are two different reasons preventing M_g from being a fine moduli space (see remark 1.2.3), which we want to illustrate. First some terminology:

Definition 2.1.2. $X \longrightarrow B$ is called *isotrivial* if there exists a scheme C and a dense open subset U of B such that for every $u \in U$ the fiber X_u is isomorphic to C. $X \longrightarrow B$ is called *trivial* if X is birational to $C \times B$.

Example 2.1.3. *Existence of (non trivial) isotrivial families of nonsingular curves .* To construct one such, pick a hyperelliptic curve C and let $\iota \in Aut(C)$ be the hyperelliptic involution. Let $B = \mathbb{C}^*$ and act on $C \times B$ by the automorphism α defined by

$$\alpha(p, z) := (\iota(p), -z) \quad \forall p \in C,\ \forall z \in B.$$

Let then X be the quotient of $C \times B$ by the group (of order 2) generated by α. We have that X is an isotrivial family over B, all of whose fibers are isomorphic to C.

To show that such a family $f : X \longrightarrow B$ is not trivial, it suffices to prove that f has only finitely many sections.

In fact, it is not hard to show that the sections of f are exactly the ones obtained by descending the $2g + 2$ (constant) sections of $C \times B \longrightarrow B$ that correspond to the fixed points of ι (the Weierstrass points of C).

The general lesson is that, in the scheme theoretic and functorial framework, moduli spaces disregard all information concerning isotriviality phenomena of the given moduli problem.

Isotriviality becomes an issue when there exist special objects in $\mathcal{C}$ with nontrivial automorphisms. For the case of (nonsingular irreducible projective) curves, we have

Lemma 2.1.4.

(a) A curve of genus $g \geq 2$ has finitely many automorphisms.

(b) If $g \geq 3$ the locus in M_g of curves with nontrivial automorphisms has codimension $g-2$.

For (a), see [ACGH] or [Ha77]. (b) is a good exercise (hint: to estimate the dimension of the locus of curves with an automorphism of order p prime, notice that every such a curve C is a degree p cyclic cover $C \longrightarrow C'$ of a curve C' of lower genus, with branch locus B of bounded order. Conversely, any pair (C', B) determines finitely many such curves C).

We now give a second reason why M_g is not a fine moduli space.

Example 2.1.5. *Existence of non-modular rational points in M_g.* The goal is to construct a morphism $\psi : B \longrightarrow M_g$ which is not a *modular map*, that is, it does not come from a family of curves. To do that, we shall carry out an explicit "stable reduction", (see Theorem 3.3.4).

We use an example from section 3.C of [HM98]. Take $B \subset \mathbb{A}^1_t$ a neighborhood of 0 and consider the family $X \subset \mathbb{P}^2 \times B$ given by the pencil of plane quartics generated by a general smooth quartic $Z(F)$ given by a polynomial F (homogeneous of degree 4) and by a double conic $Z(Q^2)$ corresponding to an irreducible polynomial Q (homogeneous of degree 2); assume also that the conic $Z(Q)$ intersects the quartic $Z(F)$ in 8 distinct points. More precisely, the equation of X is

$$X := Z(tF + Q^2) \subset \mathbb{P}^2 \times B$$

and we let $f : X \longrightarrow B$ to be the projection. If B is a suitable (Zariski open) neighborhood of $0 \in \mathbb{A}^1_t$, we can assume that if $t \neq 0$ the fiber X_t is a nonsingular curve (of genus 3); the only singular curve X_0 is of course nonreduced, being in fact a double rational curve.

From the above equation, it is clear that the total space of X is nonsingular away from the 8 points of the central fiber that correspond to the 8 points where the $Z(F)$ intersects $Z(Q)$ in the plane. On the other hand, locally at one of such 8 points, the equation defining X is formally equivalent to the equation $tx + y^2$ where x and y are local parameters on the initial quartic and conic respectively. Therefore the surface X has 8 singular points of so-called type A_1, to resolve each of which one has to blow up once, replacing the point with a nonsingular rational curve of self intersection -2. After having resolved these 8 singular points, we obtain a nonsingular surface X^r with a birational morphism $\sigma : X^r \longrightarrow X$; it is clear that the composition $f^r := f \circ \sigma$ gives a family of curves $f^r : X^r \longrightarrow B$ with smooth fibers over $t \neq 0$ and with only singular fiber

$$X_0^r = 2E_0 \cup (\cup_{i=1}^8 E_i)$$

where $E_j \cong \mathbb{P}^1$ for $j = 0, ..., 8$.

Consider now the degree 2 covering $\delta : B' \longrightarrow B$ induced by restricting to B the covering (again called δ)

$$\begin{array}{cccc} \delta : & \mathbb{A}^1_u & \longrightarrow & \mathbb{A}^1_t \\ & u & \mapsto & u^2 \end{array}$$

A direct computation shows that the normalization Y of the pull-back $\delta^*(X^r)$ of X^r over B' is a nonsingular surface with a natural morphism to B'. The fibers of $Y \longrightarrow B'$ are all nonsingular with the exception of the the fiber over $0 \in B'$ (in fact Y is isomorphic to $\delta^*(X^r)$ away from the fibers over 0). The special fiber Y_0 is now a reduced curve

$$Y_0 = C \cup (\cup_{i=1}^8 E_i')$$

such that E_i' is the preimage of E_i and it is an exceptional curve of the first kind (i.e. $(E_i')^2 = -1$), and C is a nonsingular hyperelliptic curve of genus 3. In fact C has a degree-2 morphism onto E_0, with ramification locus given by the eight points $C \cap E_i'$.

The final step is the contraction of the 8 exceptional divisors E_i' of Y; this gives a smooth surface Z with a morphism h to B' such that $h : Z \longrightarrow B'$ is a family of nonsingular curves of genus 3 (the central fiber being hyperelliptic). This family $Z \longrightarrow B'$ is a *stable reduction* of the original family $X \longrightarrow B$ (see section 3.3 for the precise definition).

Therefore B' admits a moduli morphism to M_3, determined by $Z \longrightarrow B'$:

$$\phi_h : B' \longrightarrow M_3$$

Since B is a smooth curve, such a morphism ϕ_h factors through a morphism $\psi : B \longrightarrow M_3$ such that $\phi_h = \psi \circ \delta$ and such that the restriction of ψ to $B \setminus \{0\}$ is the moduli map of the family $X \setminus X_0 \longrightarrow B \setminus \{0\}$.

Suppose now that there exists a family of nonsingular curves $W \longrightarrow B$ over B having ψ as moduli map; then W is a nonsingular, relatively minimal surface. By our genericity choice (on the quartics) we can assume that the smooth fibers of $X \longrightarrow B$ have no automorphisms, which implies that, away from $0 \in B$, W and X are isomorphic over B, and hence $W \setminus W_0$ and $X^r \setminus X_0^r$ are isomorphic over B. Thus $X^r \longrightarrow B$ and $W \longrightarrow B$ are non isomorphic (their special fibers are different), relatively minimal families, which are isomorphic over an open subset of B. This contradicts the uniqueness of minimal models for families of smooth curves, whose precise statement we recall (see [DM69] and references therein):

Theorem 2.1.6. *Let R be a discrete valuation ring with algebraically closed residue field; let k be its quotient field, and let Y_k be a nonsingular, geometrically irreducible curve projective over k and of genus at least 2. Then there exists a unique (up to k-isomorphism) curve Y proper over R, such that Y is nonsingular, relatively minimal, and has Y_k as generic fiber.*

Remark 2.1.7. The theorem says nothing about the local geometry of the closed fiber of Y over $\operatorname{Spec} R$, which might be nonreduced, reducible and singular.

In purely geometric terms, the above theorem implies the following fundamental fact: let B be a nonsingular irreducible curve over an algebraically closed field, let $S \subset B$ be a finite set of its points, let $U := B \smallsetminus S$ and let $f_U : X_U \longrightarrow U$ be a family of smooth, irreducible projective curves of genus at least 2. Then there exists a unique (up to isomorphism) completion $f : X \longrightarrow B$ of such a family over the whole of B (so that $f_{|X_U} = f_U$) such that f is flat and proper, and the total space X is a nonsingular surface, relatively minimal over B.

Definition 2.1.8. The curve Y in Theorem 2.1.6 is called the *minimal model* of Y_k over R. Likewise, given B, U and X as above, we shall call the uniquely determined $f : X \longrightarrow B$ the *minimal model* of X_U over B.

Exercise 2.1.9. Construct examples similar to Example 2.1.5 by studying (for instance) pencils of nonsingular plane curves of degree d specializing to a d-fold line. Under suitable "genericity" assumptions on the pencil, the final outcome is the same: the stable reduction (i.e. the analogue of $Z \longrightarrow B'$ in example 2.1.5, see also 3.3) will be a family of non-singular curves and one gets a non-modular morphism to M_g.

2.2 Families of smooth curves

Let B be an integral (for simplicity) scheme. Recall that $\mathcal{M}_g(B)$ is the set of equivalence classes of families of smooth curves of genus g over B. Denote by $\mathcal{M}^*_g(B)$ the subset of $\mathcal{M}_g(B)$ consisting of non-isotrivial families.

Then we have

Theorem 2.2.1 (Shafarevich conjecture)**.** *Fix $g \geq 2$ and an integral scheme B. Then the set $\mathcal{M}^*_g(B)$ is finite.*

Equivalently

There exists a finite number of non-isotrivial families of smooth curves of genus g over B.

The statement was conjectured by I. Shafarevich in 1962, assuming $\dim B = 1$; the proof is due to A. Parshin and Y. Arakelov ([P68], [A71]). The higher dimensional case was proved in [C02] as a consequence of a uniform strengthening of the one-dimensinal case.

We shall focus here on explicit examples. The case when B is a complete curve was the first to be studied. An example (or, rather, series of examples) was exhibited by Kodaira ([K67]); we shall now describe it. Our knowledge of such examples is remarkably scarce, and almost all explicitely known families of smooth curves over a complete base are a variation on the Kodaira construction (see also [BDS01]).

Example 2.2.2. Work over $\mathbb{C}$. We shall construct families $f : X \longrightarrow B$ having the following properties: B is a complete smooth curve of genus q. There exists a smooth curve C_0 of genus at least 2 such that every fiber of f is a smooth curve having a finite map to C_0, with fixed profile (i.e. degree and ramification type). The non-isotriviality will derive from the theorem of De Franchis - Severi, which says that *the set of dominant maps from a fixed smooth curve to curves of genus at least* 2 *is finite*[3].

Let $\epsilon : C_0 \longrightarrow B_0$ be an étale covering of degree e and denote by $g_0 \geq 2$ the genus of C_0. Then we shall construct $f : X \longrightarrow B$ whose fibers will be degree-e coverings of C_0, totally ramified over the fibers of ϵ. The graph $\Gamma_\epsilon \subset C_0 \times B_0$ represents a family over B_0 of effective, reduced divisor of C_0. Notice that if D is a degree-e effective, reduced divisor of C_0, there exist e^{2g_0} cyclic coverings of degree e of C_0 having D as branch locus. To resolve this ambiguity we perform an étale covering δ of B_0, namely we have a fiber product diagram

$$\begin{array}{ccc} B = B_0 \times_{\mathrm{Pic}^0 C_0} \mathrm{Pic}^0 C_0 & \stackrel{\delta}{\longrightarrow} & B_0 \\ \downarrow & & \downarrow \\ \mathrm{Pic}^0 C_0 & \stackrel{\mu_e}{\longrightarrow} & \mathrm{Pic}^0 C_0 \end{array}$$

[3]Such a theorem has been generalized in many ways; in particular it holds uniformly over every field.

where the map $B_0 \longrightarrow \operatorname{Pic}^0 C_0$ is given by sending b to the divisor $\epsilon^{-1}(b) - ep_0$ (for some fixed point $p_0 \in C_0$) and μ_e is the multiplication by e in the Jacobian of C_0. Thus δ is étale of degree e^{2g_0}; notice that the fiber product B may fail to be connected and that we are free to replace it by any of its connected component.

It is now easy to check that over B there is a family X as desired. In fact X is the degree-e cyclic covering of $C_0 \times B$ branched over $(id_{C_0} \times \delta)^{-1}(\Gamma_\epsilon)$ (the existence of such a covering is a direct consequence of the existence of a Poincaré line bundle for C_0, see section 6.1).

Now as b varies in B, the covering $X_b \longrightarrow C_0$ varies in an infinite set (because so does its branch divisor, by construction). Therefore the De Franchis-Severi theorem ensures that $X \longrightarrow B$ is not isotrivial.

Observe finally that the fibers of this family have non-trivial automorphisms, hence they vary in a closed subscheme of M_g, by Lemma 5.1.2.

A wonderful variation on the Kodaira construction is the so-called "Parshin trick", by which Parshin (in [P68]) proved that the Shafarevich conjecture implies the Mordell Conjecture[4] on finiteness of rational points for non-isotrivial curves of genus at least 2.

Lemma 2.2.3 (Parshin). *Assume that for every $g \geq 2$ and for every quasiprojective curve B over $\mathbb{C}$ the set $\mathcal{M}_g^*(B)$ is finite. Then $X(k)$ is finite for every non-isotrivial curve X of genus at least 2 defined over a one-dimensional complex function field k.*

Remark 2.2.4. The above Lemma was proved before the full Shafarevich conjecture (on a one-dimensional base). In fact [P68] treats the case when B is complete. The general case was achieved a few years later in [A71].

Proof. The final outcome of Parshin's construction is a map:

$$\alpha_X : X(k) \longrightarrow \bigcup_{g', B'} \mathcal{M}_{g'}^*(B')$$

such that the union on the right is over a finite set of pairs (g', B') with $g' \geq 2$ an integer and B' a connected curve (non necessarily complete). The construction is such that α_X has finite fibers. The Shafarevich conjecture says that $\mathcal{M}_g^*(B)$ is always a finite set, hence the image of α_X is in a finite set. Therefore $X(k)$ is finite.

Now some details (a careful description can be found in [Sp81]). We can express $k = \mathbb{C}(B)$ with B a smooth projective curve; fix the smooth, relatively minimal model of our given curve X over B (using Theorem 2.1.6 and 2.1.8), which we will continue to denote by X abusing notation (recall however that the smooth, relatively minimal model is uniquely determined if $\dim B = 1$). So that we have fixed a family

$$f : X \longrightarrow B$$

and f might have some singular fibers. Then set

$$S := \{b \in B \text{ such that } X_b \text{ is singular } \}.$$

Let $\sigma \in X(k)$ be a k-rational point of X, viewed as a section of f (see 1.2.1); let $\Sigma \subset X$ be the image curve of σ.

[4] Parshin's argument directly extends to the case of number fields.

We denote by Pic_f the relative Picard scheme of f and by Pic_f^0 the subscheme parametrizing line bundles of degree 0 (see section 6.1 for details). In what follows we are free to ignore what happens over the singular fibers.

The section σ gives a rational map $u : X \longrightarrow \mathrm{Pic}_f^0$ defined by $u(x) = x - \sigma(f(x))$. Multiplication by 2 in Pic_f^0 yields a covering of X over B, which, away from singular fibers, is étale of degree 2^{2g}. Let Y be a connected component of such covering, it forms a new family of curves over B, of genus at most $1 + 2^{2g}(g - 1)$; for $b \notin S$, each fiber Y_b is an étale covering of the corresponding fiber X_b. Let D be the preimage of Σ in Y. Then D is étale of degree at most 2^{2g} over $B \setminus S$.

There exists a covering $B_1 \longrightarrow B$ of degree at most $2^{2g}(2^{2g} - 1)$, ramified only over S, such that on the relatively minimal desingularization Y_1 of $B_1 \times_B Y$ there are two disjoint sections Σ_1 and Σ_2 mapping to D via the natural morphism $Y_1 \longrightarrow Y$. Denote by $f_1 : Y_1 \longrightarrow B_1$ the natural map, (thus $\Sigma_i = Im\sigma_i$ with σ_i a section of f_1).

One can construct a further covering $B_2 \longrightarrow B_1$, ramified only over the preimage of S in B_1, such that on $Y_2 = Y_1 \times_{B_1} B_2$ the line bundle $\mathcal{O}_{Y_2}(\Gamma_1 + \Gamma_2)$, defined as the the pull-back of $\Sigma_1 + \Sigma_2$ admits a square root. Such a B_2 is obtained by first mapping B_1 to $\mathrm{Pic}_{f_1}^0$ via $b \mapsto \sigma_1(b) - \sigma_2(b)$, and by then considering the multiplication by 2 map $\mathrm{Pic}_{f_1}^0 \longrightarrow \mathrm{Pic}_{f_1}^0$ and defining

$$B_2 = B_1 \times_{\mathrm{Pic}_{f_1}^0} \mathrm{Pic}_{f_1}^0$$

(replacing B_2 by a connected component dominating B_1 if necessary).

This ensures (after a further degree-2 base change ramified over singular fibers) that there exists a double covering $Y_3 \longrightarrow Y_2$ having branch locus $\Gamma_1 + \Gamma_2$. Finally, let Y_4 be the relatively minimal resolution of Y_3 over B_2, let $\rho : B_2 \longrightarrow B$ be the covering map, let $S' = \rho^{-1}(S)$ so that S' is the set of points of B_2 over which the fiber of Y_4 is singular; let $B' = B_2 \setminus S'$. Note that ρ is a covering of B of degree bounded above (by a function of g), and ramified only over S; therefore B' varies in a finite set.

Now denote by f' the restricted family $f' : Y' \longrightarrow B'$, where Y' is what remains of Y_4 after removing the (singular) fibers over S': it is the family of smooth curves that we wanted; by construction the fibers of Y' are coverings of the (smooth) fibers of f of degree dividing 2^{2g+1}, having two simple ramification points lying over Σ. Hence $g' = g(Y') \leq 2 + 2^{2g+1}(g-1)$. The so obtained family f' is non-isotrivial (by the theorem of De Franchis-Severi) and it gives the element $\alpha_X(\sigma) \in \mathcal{M}_{g'}^*(B')$ that we wanted. The finiteness of the fibers of α_X follows again from the theorem of De Franchis-Severi, and we are done. □

2.3 The Torelli morphism

Let A_g be the coarse moduli scheme of principally polarized abelian varieties of dimension g. Some details about its functorial properties and its construction are in [GIT65], chapter 7. The literature on the subject is very rich, varying between analysis, geometry and arithmetic.

There is a natural, functorial map, the so-called *Torelli morphism*

$$\tau : M_g \longrightarrow A_g$$

mapping a curve to its Jacobian, polarized by the Theta divisor.

For every algebraically closed base field, the Torelli map is an injective morphism.

Let us work over $\mathbb{C}$ for the rest of this section. Then the analytic theory of abelian varieties provides a description of A_g as the quotient of the Siegel upper half-space H_g modulo the group $Sp(2g, \mathbb{Z})$:

$$H_g \longrightarrow H_g/Sp(2g, \mathbb{Z}) = A_g$$

In particular the (topological) universal covering space of A_g^0 (notation in 5.1.2) is an open subset of H_g. As a consequence we can prove the following "hyperbolicity property" of $\mathcal{M}_g$ (Beauville [B81])

Proposition 2.3.1. *Let $g \geq 1$, then $\mathcal{M}_g^*(B)$ is empty if B belongs to the following list:*

1. *$B = \mathbb{P}^1_{\mathbb{C}}$*
2. *(more generally) $B = \mathbb{P}^1_{\mathbb{C}} \smallsetminus S$ with S a finite subset of cardinality at most* 2
3. *B a complex projective curve of genus* 1

Proof. (From [B81]) If there exists a family $f \in \mathcal{M}_g^*(B)$ then the corresponding moduli map ϕ_f to M_g yields a non constant map to A_g:

$$B \xrightarrow{\phi_f} M_g \stackrel{\tau}{\hookrightarrow} A_g.$$

In turn, the above map $\tau \circ \phi_f$ lifts to a (necessarily non costant) map from the universal covering space $\tilde{B}$ of B to H_g. This is immediate if the image of $\tau \circ \phi_f$ is in A_g^0 (that is, if the fibers of f have trivial automorphism group). More generally, the lifting is obtained by noticing that the local system $R^1 f_* \mathbb{Z}$ becomes trivial on $\tilde{B}$, thus one can simultaneously choose a symplectic basis for the first homology group of all fibers (of the pull-back of f to $\tilde{B}$), thereby obtaining a morphism to H_g.

Now, if B is as one of the listed cases, then $\tilde{B}$ contains (dense) copies of $\mathbb{C}$ and hence we get a non costant map $\mathbb{C} \longrightarrow H_g$. This is impossible, as H_g is isomorphic to a bounded domain. □

Remark 2.3.2. The Proposition states that there does not exist a modular morphism from B to M_g. The assumption that the morphism is modular cannot be removed; see [BD02] for an example of a non constant (and not modular) morphism from $\mathbb{P}^1$ to M_g.

Remark 2.3.3. Examples of families of smooth curves over $\mathbb{P}^1 - S$ where $\#S = 3$ are known for all $g \geq 1$([B81]): $y^2 = x^d - dtx + (d-1)t$.

From the Torelli morphism $\tau : M_g \longrightarrow A_g$ and the basic moduli theory of abelian varieties we see that M_g is not complete. It is in fact well known that the closure of the image of τ contains products of principally polarized abelian varieties. Now, the Jacobian of a smooth curve is never a product, because its theta-divisor is always irreducible.

Remark 2.3.4. A modular completion of M_g will be discussed at length in the sequel. For the moment, we recall (without details, see [HM98] 2.C for a better description and references) the existence of an important completion of A_g: the so-called Satake compactification. The Satake compactification induces a projective compactification of M_g which has a very special feature: the complement of M_g in it (the so-called "boundary") has codimension 2 if $g \geq 3$. This implies that M_g contains plenty of complete curves, namely *every point in M_g lies on a complete curve entirely contained in M_g. Moreover it is easy to see that, for $g \geq 4$, such curves can be chosen to be modular* (because the locus of curves with automorphisms has codimension greater than 1, by Lemma 2.1.4).

3 Stable curves

3.1 Completing M_g

As we have seen, M_g is quasiprojective and not complete (if $g \geq 1$).

We shall describe a compactification of M_g for all $g \geq 2$, which is itself a coarse moduli scheme for a moduli problem "containing" smooth curves: the compactification $\overline{M_g}$ by means of stable curves in the sense of P.Deligne and D.Mumford.

Definition 3.1.1. A *semistable curve* C of genus $g \geq 2$ over k is a reduced, connected projective curve having at most nodes as singularities and such that if $E \subset C$ is a smooth rational component, then $|E \cap \overline{C \smallsetminus E}| \geq 2$.

The genus of C is its arithmetic genus $g = h^1(\mathcal{O}_C)$.

A *stable curve* is a semistable curve such that if $E \subset C$ is a smooth rational component, then $|E \cap \overline{C \smallsetminus E}| \geq 3$.

A *family* of stable (respectively semistable) curves is a proper flat morphism $X \longrightarrow B$ (B a scheme) such that the fibers of X over the points of B are stable (respectively semistable) curves. If B contains an open dense subset over which the fibers are nonsingular, the family is said to be *generically smooth*.

A simple useful observation: if C is a semistable curve and $E \subset C$ is such that $E \cong \mathbb{P}^1$ and $|E \cap \overline{C \smallsetminus E}| = 2$, E is called an *exceptional* component. It is clear that a semistable curve is stable if and only if it contains no exceptional components.

The new functor is thus

$$\overline{\mathcal{M}}_g : SCH \longrightarrow SET$$

that associates to a scheme B the set of all families of stable curves over B (defined in Definition 3.1.1) modulo the same equivalence relation that was used to define $\mathcal{M}_g$.

The main point is that the above functor is coarsely representable over $\operatorname{Spec} \mathbb{Z}$, that is to say, there exists a coarse moduli scheme $\overline{M_g}$ of stable curves, which is projective over $\operatorname{Spec} \mathbb{Z}$ and contains M_g as an open dense subset (see Theorem 3.4.1 below).

The theory of stable curves (first introduced by A. Mayer and D. Mumford) began to be developed in [DM69] with the goal of proving the irreducibility of M_g in any characteristic. In that paper, the authors do not construct the moduli scheme $\overline{M_g}$ (they do define the corresponding stack), but they prove the main properties of stable curves, used later by D. Gieseker in [Gie82] to establish the existence of $\overline{M_g}$ (see Theorem 3.4.1).

3.2 Features of stable curves

Let C be a nodal projective curve; we list a number of basic properties, for which we refer to [HM98], [Ha77] or [ACG03]. The dualizing sheaf ω_C of C exists, is unique and is invertible; on the nonsingular locus of C, ω_C coincides with the sheaf of regular differential forms. Its characteristic property is that of making Serre's duality valid; in particular, for any line bundle L on C we have

$$H^1(C, L) \cong H^0(C, \omega_C \otimes L^{-1})^*.$$

The theorem of Riemann Roch holds: if g is the arithmetic genus of C and L a line bundle, then

$$h^0(C, L) - h^1(C, L) = \deg L - g + 1.$$

In particular, $\deg \omega_C = 2g - 2$ and $g = h^0(C, \omega_C)$. If

$$\nu : C^\nu \longrightarrow C$$

is the normalization, then we have

$$\nu^* \omega_C = K_{C^\nu}(\sum_{i=1}^{\delta}(p_i + q_i))$$

where δ is the number of nodes of C and p_i, q_i are the two points of C^ν lying over the i-th node.

Exercise 3.2.1. As an application, prove the following simple formula for the arithmetic genus g of C: if C has γ irreducible components $C_1, ..., C_\gamma$ and C_i has geometric genus g_i, then

$$g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + c$$

where c is the number of connected components of C.

Lemma 3.2.2. *Let C be a nodal connected curve of genus at least* 2. *Then*

a) C is semistable if and only if for every complete subcurve D of C the degree of the restriction of ω_C to D is nonnegative.

b) C is stable if and only if for every complete subcurve D of C the degree of the restriction of ω_C to D is positive.

Remark 3.2.3. In particular D is such that $\deg_D \omega_C = 0$ if and only if D is a union of exceptional components.

Proof. The proof is straightforward: it clearly suffices to assume that D is connected, in which case if $k_D = |D \cap \overline{C \smallsetminus D}|$, then

$$\deg_D \omega_C = 2p_a(D) - 2 + k_D$$

the rest is left to the reader. □

Throughout the paper we shall use the following notation:

Notation 3.2.4. Let C be a nodal connected curve, let $C_1, ..., C_\gamma$ be its irreducible components, and let D be a complete subcurve of C. Then

i) $k_D := |D \cap \overline{C \smallsetminus D}|$

ii) $w_D := \deg_D \omega_C = \deg \omega_D + k_D = 2p_a(D) - 2 + k_D$

iii) In case $D = C_i$, we denote $k_i := k_{C_i}$ and $w_i := w_{C_i}(= 2p_a(C_i) - 2 + k_i)$.

A particularly useful result is the following

Proposition 3.2.5. *Let C be a nodal conected curve of genus at least* 2

a) C is stable if and only if $Aut(C)$ is finite.

b) C is semistable if and only if $H^1(C, \omega_C^n) = 0$ for every $n \geq 2$.

c) C is stable if and only ω_C^n is very ample for every $n \geq 3$,.

Proof. The proof of a) is simple and it is left to the reader. The last two statements are essentially Theorem 1.2 in [DM69]. □

In fact one often finds in the literature the following equivalent definitions:

• A stable curve is a connected nodal curve of arithmetic genus at least 2 whose dualizing sheaf is ample.

• A stable curve is a connected nodal curve of arithmetic genus at least 2 having finitely many automorphism.

More generally, if $f : X \longrightarrow B$ is a family of connected nodal curves, the relative dualizing sheaf ω_f is a uniquely determined invertible sheaf satisfying the relative analogue of the properties listed above. Naturally, the restriction of ω_f to every fiber X_b is the dualizing sheaf of X_b.

A useful formula, holding if X and B are nonsingular, is the product-like expression

$$K_X = \omega_f \otimes f^* K_B. \tag{1}$$

Consider now a family of stable curves $f : X \longrightarrow B$. An important consequence of 3.2.5 part b), is that for any $n \geq 3$ the sheaf $f_* \omega_f^n$ is locally free of rank $(2n-1)(g+1)$. Therefore the relatively very ample line bundle ω_f^n (3.2.5 part c)) determines an embedding

$$X \hookrightarrow \mathbb{P}_B(f_* \omega_f^n).$$

Notice also that if, more generally, $f : X \longrightarrow B$ is a family of semistable curves, again we have that, if $n \geq 3$ then $H^1(C, \omega_C^n) = 0$ (by 3.2.5), hence $f_* \omega_f^n$ is locally free. By Lemma 3.2.2, ω_f^n determines a morphism $X \longrightarrow \mathbb{P}_B(f_* \omega_f^n)$ which contracts all the exceptional components of the fibers of X and whose image is a family $f^{st} : X^{st} \longrightarrow B$ of stable curves. Such a family f^{st} is called *the stable model of $X \longrightarrow B$.*

3.3 Stable reduction and separation

Throughout this section, we shall assume that B is a nonsingular irreducible curve and $U \subset B$ an open dense subset. Let $f_U : X_U \longrightarrow U$ be a family of smooth curves and let b_0 be a point of B. Let X be the minimal model of X_U over B (see Definition 2.1.8).

Definition 3.3.1. One says that X_U (or X) has *semistable reduction in b_0* if the fiber of X over b_0 is a semistable curve.

One says that X_U admits *semistable reduction (or semistable model) over B* if all fibers of X over B are semistable curves.

If there exists a family of stable curves $f^{st} : X^{st} \longrightarrow B$ such that $X_U \subset X^{st}$ and $f_{|X_U} = f_U$, one says that X_U admits *stable reduction (or stable model) over B*

We now prove that X_U admits semistable reduction over B if and only if it admits stable reduction over B.

Fix $f : X \longrightarrow B$ a family of semistable curves with smooth fibers over some dense open subset of B. Then the exceptional components of the semistable fibers can all be contracted to points: just consider the birational morphism determined by the n-th power of the relative dualizing sheaf ω_f, for n very high. After doing that, call X^{st} the resulting surface, then X^{st} might be singular (it will in fact have only singularities of type A_n) but X^{st} has a natural flat proper morphism to B which makes it into a family of stable curves (the stable model of f, introduced before).

It is clear that $X^{st} \longrightarrow B$ is uniquely determined by $X \longrightarrow B$.

Conversely, let $Z \longrightarrow B$ be a generically smooth family of stable curves. Then a simple local analysis shows that if $N \in Z$ is a singular point, then N must be a node of some fiber and, moreover, N will be a singularity of type A_n of Z. Therefore its resolution consists in replacing it by a chain of $n-1$ exceptional components, so that the "blown-up" fiber will be a semistable curve. Recapitulating, the resolution of the singularities of Z yields a nonsingular surface Z^{res} with a birational morphism $Z^{res} \longrightarrow Z$, which is naturally a family of semistable curves, relatively minimal over B (the family map is of course the composition $Z^{res} \longrightarrow Z \longrightarrow B$).

Summarizing, we proved:

Lemma 3.3.2. *There is a natural bijective correspondence between generically smooth families of stable curves over B and semistable families $X \longrightarrow B$ with X nonsingular (and relatively minimal).*

This correspondence associates to a stable family $Z \longrightarrow B$ the minimal resolution of singularities of Z, $Z^{res} \longrightarrow B$; its inverse associates to a semistable family $X \longrightarrow B$ its stable model $X^{st} \longrightarrow B$.

Remark 3.3.3. The reason why "relatively minimal" is in parenthesis is that it is not hard to show that if $f : X \longrightarrow B$ is a family of semistable curves with smooth total space, then it necessarily is relatively minimal. Suppose in fact that $E \subset X_b \subset X$ is an exceptional divisor of the first type (such that $E^2 = -1$), contained in a fiber X_b of f. Then adjunction together with formula (1) gives

$$-1 = \deg_E K_X = \deg_E \omega_f = \deg_E \omega_{X_b}$$

Now, X_b being semistable, this is in contradiction with Lemma 3.2.2.

As we have seen, not all families admit stable reduction over the completion of their base space (example 2.1.5 and exercise 2.1.9). However, this is always true after a finite base change:

Theorem 3.3.4 (Stable reduction over one dimensional base)**.** *Let B be a nonsingular irreducible curve and let $U \subset B$ be a dense open subset. Let $X_U \longrightarrow U$ be a family of smooth curves. Then there exists a finite morphism $\delta : B' \longrightarrow B$ such that $\delta^* X_U$ admits stable reduction over B'.*

Addendum. One can in fact say more about δ. There exists a point $b_0 \in B$ (possibly not in U) such that δ is not ramified over $U \smallsetminus \{b_0\}$. If the base field has characteristic 0, δ can be taken to be a cyclic covering of B.

For a proof of this result (with the addendum) over $\mathbb{C}$ we refer to the third chapter of [BPV84]. The first step of the proof invokes the theorem on resolution of singularities. This ensures that there exists a nonsingular surface X' with a birational morphism $\sigma : X' \longrightarrow X$ such that if F is a singular fiber of the so-obtained family $X' \longrightarrow B$, then F is a "divisor with normal crossings", that is, the reduced scheme underlying F is a curve having at most nodes as singularities. The rest of the proof starts with a family like $X' \longrightarrow B$ above, and performs (quite explicitely) a series of ramified coverings of the base (always normalizing the base changed families) based upon the local geometry of $X' \longrightarrow B$.

For positive characteristic references are given in the first two pages of [DM69], we suggest also to look at the paper [AW71].

A few nice examples of explicit construction of semistable (and thus stable) reductions are in [HM98].

Exercise 3.3.5 (J. de Jong). Let $B = \mathbb{A}^1_t$ and let $a_1, ..., a_n, b_1, ..., b_l$ be distinct points in B. Let $U = \mathbb{A}^1 \smallsetminus \{a_1, ..., a_n, b_1, ..., b_l\}$ and consider the family $X_U \longrightarrow U$ of nonsingular projective curves given by the family of plane curves of affine equation

$$y^2 = (x - a_1) \cdot .... \cdot (x - a_n) \cdot (x - t)(t - b_1) \cdot .... \cdot (t - b_l)$$

where x, y are affine coordinates in the plane and t the coordinate for U.

1) Prove that X_U has semistable reduction at $t = a_i$ for every i.

2) Prove that X_U does not have semistable reduction at $t = b_i$ for every i.

3) Study the stable reduction of the family locally at b_i and prove that the curve replacing X_{b_i} is nonsingular.

A crucial fact is that the moduli functor of stable curves is is separated (the following is Lemma 1.12 in [DM69]):

Proposition 3.3.6. *Let R be a discrete valuation ring with algebraically closed residue field, and set $B = \operatorname{Spec} R$. Let $X \longrightarrow B$ and $Y \longrightarrow B$ be two generically smooth families of stable curves. Then any isomorphism of their generic fibers extends uniquely to an isomorphism between X and Y.*

Remark 3.3.7. It is easy to see that this result is false if the assumption that the special fiber is stable is weakened by assuming it just nodal or semistable.

Proof. As in [DM69] pp. 84-85. Denote by U the generic point of B and identify the generic fibers $X_U \longrightarrow U$ and $Y_U \longrightarrow U$ by the given isomorphism. The result says that, if it exists, the stable reduction over B is unique.

By Theorem 2.1.6 there exists a unique minimal model $Z \longrightarrow B$ for $X_U = Y_U$. Thus Z is the minimal resolution of the singularities of both X and Y. Now X and Y are generically smooth families of stable curves, therefore, as we have already seen in the proof of Lemma 3.3.2, the morphism $Z \longrightarrow X$ is precisely the contraction of all the -2-curves of Z; since the same holds for $Z \longrightarrow Y$ we conclude that $X = Y$. □

3.4 The moduli scheme of stable curves

The construction (in [Gie82]) of $\overline{M_g}$ is by means of Geometric Invariant Theory, and works over $\operatorname{Spec} \mathbb{Z}$:

Theorem 3.4.1 (Gieseker)**.** *Let $g \geq 2$. There exists a coarse moduli space $\overline{M_g}$ for stable curves of genus g. $\overline{M_g}$ is a reduced projective, normal scheme of dimension $3g-3$, containing M_g as dense open subset.*

Remark 3.4.2. More precisely, as we shall see, $\overline{M_g} = H_g/PGL(r+1)$, where H_g is smooth over $\mathbb{Z}$. In fact H_g is a (non closed) subscheme of a certain Hilbert scheme.

This results is a consequence of Theorem 4.4.5, and will be discussed later.

Corollary 3.4.3. *M_g is irreducible over any field.*

Proof. The result is well known in characteristic 0. The main idea is to obtain it in positive characteristic as specialization from characteristic 0. Consider $\overline{M_g}$ over $S = \operatorname{Spec}\mathbb{Z}$. Denote by $(\overline{M_g})_0$ the generic point (defined of course in characteristic 0) and by $(\overline{M_g})_p$ a chosen closed point, our special point, defined in characteristic p; extend this notation to the fibers of M_g over S and of H_g over S. The generic point is known to be irreducible (because its open subset $(M_g)_0$ is), and hence connected. By the connectedness principle of Enriques-Zariski, the special fibers $(\overline{M_g})_p$ are also connected. Notice that the connectedness principle applies to a projective fibration, this is precisely where the compactification $\overline{M_g}$ of M_g is needed.

Now the fact that $(\overline{M_g})_p$ is a quotient, $(\overline{M_g})_p = (H_g)_p/PGL(r+1)$, and that the group is connected, implies that $(H_g)_p$ is connected. On the other hand, $(H_g)_p$ is smooth and hence irreducible; it follows that $(\overline{M_g})_p$ is irreducible, and we are done.

□

This proof of the irreduciblity of M_g is at the end of [Gie82] and it is considerably simpler than original proof of [DM69].

4 Moduli of canonically polarized varieties

Definition 4.0.4. Let V be a projective variety with ample and locally free canonical line bundle, fix an integer n such that ω_V^n is very ample. Then any image in projective space of X given by ω_V^n is called a *n-canonical model* of V.

4.1 Canonically embedded stable curves

As we have seen in 3.2, if C is any stable curve of genus $g \geq 2$ and $n \geq 3$ is an integer, ω_C^n is a very ample line bundle of degree $d = n(2g-2)$ and $h^0(C, \omega_C^n) = (2n-1)(g-1)$. Consider the projective space $\mathbb{P}^r$ with $r = (2n-1)(g-1)-1$; any choice of a basis for the vector space $H^0(C, \omega_C^n)$ gives an embedding

$$\gamma : C \longrightarrow \mathbb{P}(H^0(C, \omega_C^n)^*)$$

whose image is a curve of degree d which we denote $C_n := \gamma(C)$. We then say that C_n is a *n-canonical model of C*. Here is a simple but crucial

Remark 4.1.1. If C and D are stable curves of genus g and C_n and D_n are n-canonical models of C and D respectively, we have that

C is isomorphic to D if and only if C_n and D_n are projectively equivalent.

The non obvious implication (only if) is an immediate consequence of the uniqueness of the dualizing sheaf: if C and D are isomorphic as abstract curves, any isomorphism between them induces an isomorphism between ω_C and ω_D, hence an isomorphism between their spaces of global sections, hence an automorphism of $\mathbb{P}^r$.

Therefore the set of isomorphism classes of stable curves of genus g is naturally bijective to the set of orbits with respect to the action of $Aut(\mathbb{P}^r)$ on the space of all n-canonical models of curves of genus g. To transfer this set-theoretic fact into algebraic geometry, let us consider the Hilbert polynomial of n-canonical models of curves of genus g:

$$h(t) = dt - g + 1.$$

Then in the Hilbert scheme $Hilb_{\mathbb{P}^r}^{h(t)}$ we can look at the following locus:

$$H_g := \{h \in Hilb_{\mathbb{P}^r}^{h(t)} : \ Z_h \text{ is a non degenerate}^5 \text{ stable curve (of genus } g), \ \mathcal{O}_{Z_h}(1) \cong \omega_{Z_h}^n \ \}$$

where Z_h denotes the fiber of the universal family over the point h. There is a natural action of G on $Hilb_{\mathbb{P}^r}^{h(t)}$, that obviously restricts to an action on H_g. From what we have said, the quotient space H_g/G is a reasonable candidate for a moduli space of stable curves. To place on H_g/G an algebro-geometric structure, we shall invoke Geometric Invariant Theory, which perfectly applies in this set up.

4.2 Geometric Invariant Theory

From what we have seen, we are led to study the action of a linear algebraic group G on a projective scheme H. We shall now give a brief survey of Geometric Invariant Theory, referring to [GIT65] for the many missing details and proofs, and for some nice examples.

For all applications, it suffices to work over an algebraically closed field k, which we shall do throughout the section.

[5] That is, not contained in a hyperplane.

Definition 4.2.1. Fix an embedding of H in some projective space $\mathbb{P}(W) = \mathbb{P}^N$, so that $H = ProjR$ where R is a finitely generated graded algebra over k (R is thus a quotient of the polynomial ring in $N+1$ variables). Then one says that G *acts linearly on* H if the action of G can be lifted to a linear action on W and, compatibly, on R. One also says that the above data are a *linearization* of the action of G on H. Thus a linearization yields a group homomorphism

$$\sigma : G \longrightarrow GL(W)$$

A foundational theorem of Geometric Invariant Theory states that, if G is a reductive group[6], the ring of invariants $R^G := \{f \in R : \ f^g = f \ \forall g \in G\}$ is a graded algebra finitely generated over k. This is the starting point to form an algebro-geometric quotient.

To better illustrate the situation, we study the affine and the projective case in parallel. Thus, consider the natural map

$$\pi : W \smallsetminus \{0\} \longrightarrow \mathbb{P}(W)$$

and the induced map (denoted again π):

$$\pi : SpecR \smallsetminus \{0\} \longrightarrow ProjR = H$$

Let $Y := SpecR$; if $h \in H$ denote by $v_h \in Y$ any point $v_h \in \pi^{-1}(h)$.

Now, since R^G is a finitely generated graded k-algebra, we obtain an affine scheme: $SpecR^G$, and a projective scheme: $ProjR^G$. They both come with natural maps (the duals to the inclusion $R^G \subset R$); namely, a regular map of affine schemes

$$\phi : Y \longrightarrow Y/G := SpecR^G$$

and a rational map of projective schemes

$$q : H \dashrightarrow H/G := ProjR^G.$$

Theorem 4.2.2 (Affine quotients)**.** *The regular morphism $\phi : Y \longrightarrow Y/G$ is G-invariant, affine and surjective.*

(1) $\phi(v) = \phi(v')$ if and only if

$$\overline{O_G(v)} \cap \overline{O_G(v')} \neq \emptyset$$

(2) (maximality) If $\chi : Y \longrightarrow Z$ is a G-invariant morphism, there exists a unique $\pi : Y/G \longrightarrow Z$ such that $\chi = \pi\phi$.

(3) If $Z \subset Y$ is closed and G-invariant, then $\phi(Z)$ is closed

Remark 4.2.3. Notice that G is as close to a set theoretic quotient (i.e. an orbit space) as it can. In fact the fibers of ϕ are exactly the G-orbits if and only if the action of G on Y is closed (in the sense that the orbits are closed).

Exercise 4.2.4.

(a) Prove that if Y is normal, then Y/G is normal.

[6] The groups $GL(n)$, $SL(n)$, $PGL(n)$ are reductive, and so are finite groups, tori and semisimple groups

(b) Prove that if G acts on Y with trivial stabilizers and Y is non singular, then Y/G is nonsingular.

The geometry of the projective quotient is more subtle: to start with, denote H^{SS} the open subscheme of H where q is regular. Then H^{SS} is exactly the locus of *semistable points* defined as follows

Definition 4.2.5. A point $h \in H$ is called *semistable* (with respect to the given linearization) if there exists a homogeneous, noncostant, G-invariant polynomial $f \in R^G$ such that $f(h) \neq 0$. The locus of semistable points of H is denoted by H^{SS}.

A point $h \in H$ is called *stable* (with respect to the given linearization) if it is semistable, if its orbit is closed in the semistable locus and if its stabilizer is finite. The locus of stable points of H is denoted by H^S.

Our definition of stable points is not the most general, but it is the one that works in our applications. In greater generality, that is, for an action where there may be no finite stabilizers, stable points are usually defined as having stabilizers of minimal dimension.

One should stress that semistability and stability depend on the choice of the linearization (i.e. of the embedding of H in projective space) and a more faithful notation would have to include such a dependence. For example, one finds in the literature the notations H^{SS}_R, $H^{SS}(\sigma)$ and $H^{SS}(L)$ (where $L \cong \mathcal{O}_H(1)$) for the locus of semistable points. We will use the less accurate H^{SS} and H^S.

It is not hard to show that

Lemma 4.2.6. *(a) H^{SS} and H^S are open subsets of H.*

For any $h \in H$ and for any $v_h \in \pi^{-1}(h)$, we have:

(b) $h \in H^{SS}$ if and only if $0 \notin \overline{O_G(v_h)}$

(c) $h \in H^S$ if and only if $O_G(v_h)$ is closed and $Stab_G(v_h)$ is finite.

Proof. That H^{SS} is open is obvious, and H^S is open by upper semicontinuity of the fiber dimension.

Now suppose that h is semistable. Then there exists a homogeneous, non costant $f \in R^G$ such that $f(h) \neq 0$ and hence, for all $g \in G$, $f(v_h^g) = \alpha \neq 0$. Then, for every w lying in the closure of the orbit of v_h, we have $f(w) = \alpha$. On the other hand $f(0) = 0$ (because f is homogeneous) and hence 0 is not in the orbit closure of v_h.

Conversely, if h is not semistable, then every non-costant G-invariant homogeneous polynomial vanishes on h and hence on v_h; therefore

$$\phi(v_h) = \phi(0).$$

Now, the orbit of 0 is closed and in fact $O_G(0) = \{0\}$. By part (1) of Theorem 4.2.2 we obtain

$$\emptyset \neq \overline{O_G(v_h)} \cap \overline{O_G(0)} = \overline{O_G(v_h)} \cap \{0\}$$

and hence $0 \in \overline{O_G(v_h)}$.

We leave part (c) as an exercise. □

Theorem 4.2.2 above has the following analogue:

Theorem 4.2.7 (Projective quotients)**.** *The regular morphism $q : H^{SS} \longrightarrow H/G$ is G-invariant, affine and submersive*[7]*.*

(1) $q(h) = q(h')$ if and only if

$$\overline{O_G(h)} \cap \overline{O_G(h')} \cap H^{SS} \neq \emptyset$$

(2) (maximality as in 4.2.2)

(3) The restriction of q to H^S is such that for every h and h' stable points, we have $q(h) = q(h')$ if and only if h and h' are in the same G-orbit.

The practical problems that one often needs to solve when studying quotients in the projective set up, is that of describing semistable and stable points. For example, when H is a parameter scheme for certain geometrical objects (as in our case where $H = Hilb^{h(t)}_{\mathbb{P}^r}$), one hopes to use the geometric features of the objects parametrised by H to characterise semistability and stability.

There is an important "effective" criterion, which is based on the following crucial special case

Example 4.2.8. *Actions of the multiplicative group G_m.* Keeping the above notation, let $G = G_m$ and name λ the linearization morphism (as usual)

$$\lambda : G_m \longrightarrow GL(W)$$

which we assume nonconstant. Such a λ is called a *one parameter subgroup* of $GL(n)$, denoted briefly by "1 PS". Recall that there exist a basis $v_0, ..., v_N$ for W, made of common eigenvectors for λ, and integers $r_0, ..., r_N$ such that, for every $t \in G_m$ the matrix of $\lambda(t)$ with respect to such a basis is diagonal, of type

$$\lambda(t) = diag(t^{r_0}, ..., t^{r_N}).$$

If $v \in W$ is any element and $v = \sum \alpha_i v_i$, then we have

$$\lambda(t)v = \sum \alpha_i t^{r_i} v_i$$

The λ-*weights of* v are defined to be the integers r_i such that the i-th coordinate α_i of v is not equal to 0.

It is not hard to characterize semistable and stable points: using the terminology λ-semistable (and λ-stable) to stress the fact that we are studying the linearization given by λ, we state

Lemma 4.2.9. *Let $h \in H$ and let $v_h \in W$ be any point lying in $\pi^{-1}(h)$. Then*

(a) h is λ-semistable
if and only if $\lim_{t\to 0} \lambda(t)v_h \neq 0$
if and only if the λ-weights of v_h are not all positive

(b) h is λ-stable
if and only if $\lim_{t\to 0} \lambda(t)v_h$ *does not exist*
if and only if v_h admits a negative λ-weight.

[7]That is, $U \subset H/G$ is open if and only if $q^{-1}(U)$ is open.

Proof. Use Lemma 4.2.6. For both (a) and (b), the equivalence of the last two statements is obvious. □

The study of semistability for any group G can be reduced to the analysis of the semistability with respect to all of its one-parameter subgroups: a one-parameter subgroup λ of G is a (non costant) homomorphism

$$\lambda : G_m \longrightarrow G$$

Theorem 4.2.10 (Hilbert-Mumford numerical criterion)**.** *Let G be a reductive group acting on a projective scheme H, let a linearization for such an action be fixed. Then for every $h \in H$*

(1) h is semistable if and only if h is λ-semistable for every one-parameter subgroup λ of G.

(2) h is stable if and only if h is λ-stable for every one-parameter subgroup λ of G.

The proof of this important fact can be found in [GIT65], 2.2.1.

Remark 4.2.11. It is useful to spell out the following equivalent form of (1) above, combining it with Lemma 4.2.9:

(1') h is unstable if and only if there exists a one-parameter subgroup λ of G such that all the λ-weights of h are positive.

4.3 GIT on the Hilbert scheme

Now we consider a Hilbert Scheme $Hilb_{\mathbb{P}^r}^{h(t)}$ and a subgroup $G < GL(r+1)$ which obviously acts on $\mathbb{P}^r$ (we are interested in the cases $G = SL(r+1), GL(r+1)$).

To study the action of G on $Hilb_{\mathbb{P}^r}^{h(t)}$, let us briefly recall how the Hilbert scheme is constructed (more details can be found in [Vie91] and [ACG03], for instance).

Let $Z \subset \mathbb{P}^r$ be a closed subscheme having Hilbert polynomial $h(t)$. Denote by $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^r}$ the sheaf of ideals of Z. For any integer m consider the exact sequence

$$0 \longrightarrow \mathcal{I}_Z(m) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_Z(m) \longrightarrow 0$$

and the associated long cohomology sequence.

The construction of $Hilb_{\mathbb{P}^r}^{h(t)}$ is based on results of Serre saying

Fact. *There exists an integer m' that only depends on the Hilbert polynomial $h(t)$ such that, for every $m \geq m'$, for every scheme Z as above*

(a) $H^1(Z, \mathcal{I}_Z(m)) = 0$

(b) Z is uniquely determined by $\mathcal{I}_Z(m)$.

Fix an $m \geq m'$ as above, denote $S_m = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ the vector space of homogeneous polynomials of degree m in $r+1$ variables. Then, by the first of the two facts above, the long cohomology sequence of the above exact sequence of coherent sheaves yields

$$0 \longrightarrow H^0(\mathbb{P}^r, \mathcal{I}_Z(m)) \longrightarrow S_m \xrightarrow{\rho_m} H^0(Z, \mathcal{O}_Z(m)) \longrightarrow 0.$$

Therefore to our scheme Z we can associate the $h(m)$-dimensional quotient of S_m

$$\rho_m : S_m \longrightarrow H^0(Z, \mathcal{O}_Z(m)) \longrightarrow 0$$

which expresses the fact that hypersurfaces of degree m cut on Z a complete linear series of (projective) dimension $h(m) - 1$.

Of course ρ_m gives a point in the grassmannian $\mathcal{G}$ of $h(m)$-dimensional quotients of S_m.

The second of the two results of Serre above implies that, conversely, the point of $\mathcal{G}$ given by ρ_m uniquely determines Z.

Now introduce the Plücker embedding of $\mathcal{G}$: $\mathcal{G} \hookrightarrow \mathbb{P}(\wedge^{h(m)} S_m)$. We obtain an embedding ϵ_m of our Hilbert scheme in projective space, for every integer $m \geq m'$:

$$\epsilon_m : Hilb_{\mathbb{P}^r}^{h(t)} \hookrightarrow \mathcal{G} \hookrightarrow \mathbb{P}(\wedge^{h(m)} S_m).$$

Now it is clear that this gives, for every $m \geq m'$, a linearized action of G on $Hilb_{\mathbb{P}^r}^{h(t)}$, in fact

(1) G acts linearly on $\mathbb{P}^r$, hence

(2) G acts linearly on S_m, hence

(3) G acts linearly on $\mathbb{P}(\wedge^{h(m)} S_m)$

The fact that $Hilb_{\mathbb{P}^r}^{h(t)}$ is invariant for such an action is a consequence of its algebro-geometric construction (which we left out) as a well defined closed subscheme of $\mathcal{G}$ and hence of $\mathbb{P}(\wedge^{h(m)} S_m)$.

It is important to notice that the linearization of the action of G on m depends on the choice of the integer m.

To understand how G acts, pick a one-parameter subgroup λ of G.

(1) Let $\{x_0, ..., x_r\}$ be a basis for $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ which diagonalizes λ with weights $w_0, ..., w_r$:

$$\lambda(t) x_i = t^{w_i} x_i, \quad \forall i$$

Then

(2) let $\underline{d} = \{d_0, ..., d_r\}$ be a r-uple of nonnegative integers such that $\sum_0^r d_i = m$; let $\{M_{\underline{d}} = x_0^{d_0} \cdot .... \cdot x_r^{d_0} \ \ \forall \underline{d}\}$ be the corresponding (monomial) basis for S_m, which diagonalizes the action of λ on S_m with weights $\sum d_i w_i$. Finally,

(3) let $I = (\underline{d}_1, ..., \underline{d}_{h(t)})$ be a multi-index and let $M_I := M_{\underline{d}_1} \wedge .... \wedge M_{\underline{d}_{h(m)}}$ be vectors of the corresponding basis for $\wedge^{h(m)} S_m$. Such a basis diagonalizes λ with weights

$$w_\lambda(M_I) = \sum_{j=1}^{h(t)} w_\lambda(M_{\underline{d}_j})$$

where by $w_\lambda(x)$ we denote the λ-weight of an eigenvector x of λ.

Consider now the Hilbert point of Z, given by the quotient ρ_m. Denote it by $hilb(Z)$

$$hilb(Z) \in \epsilon_m(Hilb^{h(t)}_{\mathbb{P}^r}) \subset \mathbb{P}(\wedge^{h(m)} S_m)$$

Then the λ-weights of $hilb(Z)$ will be the λ-weights of M_I for every $I = (\underline{d}_1, ..., \underline{d}_{h(t)})$ such that the corresponding $h(m)$ monomials $M_{\underline{d}_1}, ..., M_{\underline{d}_{h(m)}}$ are mapped by

$$\rho_m : S_m \longrightarrow H^0(Z, \mathcal{O}_Z(m))$$

to linearly independent vectors (i.e. to a basis) of $H^0(Z, \mathcal{O}_Z(m))$.

We then have:

Lemma 4.3.1. *The point $hilb(Z)$ is semistable for the action of G on $Hilb^{h(t)}_{\mathbb{P}^r}$ (linearized as above) if and only if for every one-parameter subgroup λ of G there exist $h(m)$ monomials, $M_1, ..., M_{h(m)}$, in the basis diagonalizing λ, for which the two following conditions hold:*

(a) $\{\rho_m(M_1), ..., \rho_m(M_{h(m)})\}$ is a basis for $H^0(Z, \mathcal{O}_Z(m))$.

(b) $\sum w_\lambda(M_i) \leq 0$

The point $hilb(Z)$ is stable if strict inequality holds in (b).

Proof. Follows easily from Theorem 4.2.10 and Lemma 4.2.9.

The first condition ensures that the coordinate of $hilb(Z)$ with respect to the diagonalizing basis vector $M_I := M_1 \wedge ... \wedge M_{h(m)}$ is not zero. The second says that the weight with respect to M_I is non positive, as required by Lemma 4.2.9 for $hilb(Z)$ to be λ-semistable. □

4.4 Moduli of polarized curves

The goal is now to geometrically characterize GIT-semistable points in $Hilb^{h(t)}_{\mathbb{P}^r}$, where now $h(t) = dt - g + 1$, with the same notation of section 4.1. We need a new piece of terminology concerning line bundles on reducible curves. Let C be a nodal connected curve of genus $g \geq 2$. Let $L \in \operatorname{Pic}^d C$ be a line bundle of degree d on C

Definition 4.4.1. We say that L *satisfies the Basic Inequality* if and only if for every complete subcurve $D \subset C$ the degree of the restriction of L to D, $\deg_D L$, satisfies

$$|\deg_D L - \frac{d}{2g-2} w_D| \leq \frac{k_D}{2} \tag{2}$$

where (consistently with the notation in 3.2.4) $k_D := |D \cap \overline{C \smallsetminus D}|$ and $w_D := \deg_D \omega_C = 2p_a(D) - 2 + k_D$.

L is said to satisfy the *strict Basic Inequality* if either the curve C is irreducible[8] or if for every subcurve D of C the strict inequality holds in (2).

Example 4.4.2.

(1) For every n and for every C, ω_C^n satisfies the strict Basic Inequality.

[8]It is clear that on an irreducible curve, every line bundle satisfies the Basic Inequality.

(2) Let C be a semistable curve and E a rational connected chain of exceptional components, that is $E = \cup_1^n E_i$, with $E_i \cong \mathbb{P}^1$, and $\#E_i \cap E_j \leq 1$ with equality if and only if $|i-j| = 1$. Then if L satisfies the Basic Inequality and L is very ample, the degree of L restricted to E is equal to 1. In particular, the chain must have length 1; in other words, if C is a semistable curve containing two exceptional components meeting each other, there are no very ample line bundles on C satisfying the Basic Inequality.

The following elementary result will be used.

Lemma 4.4.3. *Let $f : X \longrightarrow \operatorname{Spec} R$ be a family of nodal curves, let $\mathcal{L}$ be a line bundle on X satisfying the strict Basic Inequality on the special fiber of f. Let $\mathcal{M} \in \operatorname{Pic} X$ be a line bundle which coincides with $\mathcal{L}$ on the generic fiber of f. Then $\mathcal{M}$ satisfies the Basic Inequality on the special fiber if and only if it $\mathcal{M} \cong \mathcal{L}$*

Proof. Denote by X_0 the special fiber of f. If $\mathcal{L}$ and $\mathcal{M}$ coincide on the generic fiber of f, there exists a divisor T supported on X_0 such that, away from the singular points of X we have

$$\mathcal{M} \cong \mathcal{L} \otimes \mathcal{O}_X(T).$$

We can write $T = \sum n_i D_i$ in such a way that the integers n_i are all different, the D_i are complete subcurves of X_0 and two different D_is have no irreducible component in common. We can also assume that $n_1 < n_i$ for every $i > 1$.

Suppose that T is not a multiple of X_0, then there is more than one D_i, and we easily see that

$$\deg_{D_1} \mathcal{O}_X(T) \geq k_{D_1}$$

therefore

$$\deg_{D_1}(\mathcal{L} \otimes \mathcal{O}_X(T)) = \deg_{D_1} \mathcal{L} + \deg_{D_1} \mathcal{O}_X(T) \geq \deg_{D_1} \mathcal{L} + k_{D_1}.$$

Now, since $\deg_{D_1} \mathcal{L}$ satisfies the strict Basic Inequality, $\deg_{D_1}(\mathcal{L} \otimes \mathcal{O}_X(T))$ does not satisfy it: the interval of degrees on D satisfying the Basic Inequality has width k_D for every subcurve D. We conclude that T is a multiple of X_0. Then $\mathcal{O}_X(T)$ is the pull-back via f of a line bundle on $\operatorname{Spec} R$, which, in turn, must be trivial, because $\operatorname{Spec} R$ is affine. We thus obtain that $\mathcal{M} \cong \mathcal{L}$ and we are done. □

Notation 4.4.4. Let $L \in \operatorname{Pic} C$ be a very ample line bundle. We shall denote by $C_L \subset \mathbb{P}(H^0(C, L)^*)$ a projective model of C determined by L. It is clear that C_L is determined by a choice of a basis of $H^0(C, L)$.

We shall denote by $hilb(C_L) \in Hilb_{\mathbb{P}^r}^{h(t)}$ the Hilbert point of the projective scheme C_L.

If $h \in Hilb_{\mathbb{P}^r}^{h(t)}$, we denote by $Z_h \subset \mathbb{P}^r$ the projective scheme parametrized by h (that is, the fiber over h of the universal family over $Hilb_{\mathbb{P}^r}^{h(t)}$) and by $\mathcal{O}_{Z_h}(1) := \mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathcal{O}_{Z_h}$

Hence, if $h = hilb(C_L) \in Hilb_{\mathbb{P}^r}^{h(t)}$ we tautologically have $L = \mathcal{O}_{Z_h}(1)$.

We shall now state the most general theorem about semistability of Hilbert points for curves. Since the terminology "stable-semistable" has been introduced in two different and unrelated contexts (moduli theory for curves and Geometric Invariant Theory) we shall stress their diversity by using the terminology "GIT-stable" and "GIT-semistable" whenever referring to the Geometric Invariant Theory definition.

Theorem 4.4.5. *Fix $g \geq 2$, $d \geq 20(g-1)$, $r = d - g$ and $h(t) = dt - g + 1$. There exist infinitely many linearizations for the action of $SL(r+1)$ on $Hilb_{\mathbb{P}^r}^{h(t)}$ such that the following holds.*

(1) For every semistable curve C of genus g and for every very ample line bundle L on C, having degree d and satisfying the Basic Inequality, $hilb(C_L)$ is GIT-semistable.

Conversely

(2) Let $h \in Hilb_{\mathbb{P}^r}^{h(t)}$ be a GIT-semistable point parametrizing a connected subscheme Z_h of $\mathbb{P}^r$. Then Z_h is a semistable curve of genus g which is not contained in a hyperplane and such that $\mathcal{O}_{Z_h}(1)$ satisfies the Basic Inequality.

Addendum. More precisely, the proof shows that for any stable curve C and any very ample $L \in \operatorname{Pic}^d C$, the point $hilb(C_L)$ is GIT-stable if and only if L satisfies the *strict* Basic Inequality (in particular, if C is irreducible, then $hilb(C_L)$ is always GIT-stable).

The same is false for a semistable curve: there exist (plenty of) GIT-stable points $h \in Hilb_{\mathbb{P}^r}^{h(t)}$ such that the curve Z_h is semistable but not stable, and $\mathcal{O}_{Z_h}(1)$ satisfies the Basic Inequality but not the strict Basic Inequality.

The statement of Theorem 4.4.5 summarizes a series of results that have been proved separately during a rather long interval of time. Chronologically, the main steps are ordered as follows:

• Step A. Part (1) of the Theorem, in the special case that C is a nonsingular curve, is in [Gie82] and contains techniques that had been used by Mumford to prove the GIT-stability of n-canonical models of nonsingular curves (see [Mu77]).

• Step B. Part (2) of the Theorem is the main result of Gieseker in [Gie82]. It is proved independently of Step A.

• Step C. $H_g \cap H^{SS}$ *is closed in* H^{SS}.

Where recall that

$$H_g := \{h \in Hilb_{\mathbb{P}^r}^{h(t)} : \ Z_h \text{ is a non-degenerate stable curve (of genus } g\text{)}, \ \mathcal{O}_{Z_h}(1) \cong \omega_{Z_h}^n \ \}$$

Notice parenthetically that Step A implies that H_g does intersect the locus of GIT-semistable points: just consider the Hilbert points of n-canonical models of nonsingular curves. We shall see at the end that H_g is entirely contained in the locus of GIT-stable points.

Proof. We must prove that n-canonical models of stable curves specialize to n-canonical models within the GIT-semistable locus of the Hilbert scheme.

Consider a family of subschemes of $\mathbb{P}^r$, $f : Z \longrightarrow \operatorname{Spec} R$ with $Z \subset \mathbb{P}^r \times \operatorname{Spec} R$ such that for every point of $\operatorname{Spec} R$, the Hilbert point of its fiber is GIT-semistable and such that the generic fiber Z_U is n-canonically embedded (thus $\mathcal{O}_{Z_U}(1) \cong \omega_{Z_U}^n$). Denote by Z_0 the special fiber. To show that H_g is closed in H^{SS} it suffices to show that

$$\mathcal{O}_{Z_0}(1) \cong \omega_{Z_0}^n$$

Let $\mathcal{M} := \mathcal{O}_Z(1)$ the given polarization. By construction, $\mathcal{M}$ and ω_f^n coincide on the generic fiber. Now, since all fibers have GIT-semistable Hilbert point, $\mathcal{M}$ satisfies the Basic Inequality on every fiber.

We know that, $\omega_{Z_0}^n$ satisfies the strict Basic Inequality; therefore Lemma 4.4.3 applies with $\mathcal{L} = \omega_f^n$ yielding that $\mathcal{O}_Z(1) \cong \omega_f^n$, and we are done. □

• Step D. The proof of part (1) of the Theorem in the case that $L = \omega_C^n$ (and hence C is necessarily a stable curve, for ω_C is ample). That is: *If C is a stable curve and C_n is an n-canonical model of C, then $hilb(C_n)$ is GIT-semistable.*

Proof. We argue (following [Gie82]) by degeneration, using the previous parts. To simplify the notation, let us identify C with C_n, so that $C \subset \mathbb{P}^r$ is an n-canonical curve. Pick a projective deformation of C to nonsingular n-canonical curves, that is, a family $f : X \longrightarrow \operatorname{Spec} R$ such that the special fiber X_0 is C, the generic fiber X_U is non singular, and $X \subset \operatorname{Spec} R \times \mathbb{P}^r$ with $\mathcal{O}_X(1) \cong \omega_f^n$. Then the moduli map ψ_f to the Hilbert scheme is such that the image of the generic point lies in $H_g \cap H^{SS}$, in fact Step A ensures that the Hilbert point of the generic fiber is GIT-semistable.

Consider the quotient $(H_g \cap H^{SS})/G$, which is projective, because of part C. There is a natural map $\phi : \operatorname{Spec} R \longrightarrow H_g \cap H^{SS}/G$, completing the composition of the (rational) moduli map $\psi_f : \operatorname{Spec} R \dashrightarrow H_g \cap H^{SS}$ with the quotient map. Therefore we can, up to replacing $\operatorname{Spec} R$ with a finite, ramified covering, introduce a new family of projective, n-canonical curves, $f^S : Z \longrightarrow \operatorname{Spec} R$ with $Z \subset \mathbb{P}^r \times \operatorname{Spec} R$, such that the moduli map has image in the GIT-semistable locus

$$\psi_{f^S} : \operatorname{Spec} R \longrightarrow H_g \cap H^{SS}$$

and its restriction to the generic point of $\operatorname{Spec} R$ is conjugate to ψ_f by the action of G[9]. Z is the pull-back via ψ_{f^S} of the universal family over $Hilb_{\mathbb{P}^r}^{h(t)}$.

Now $f^S : Z \longrightarrow \operatorname{Spec} R$ is a family of stable curves, because by Step C, the special fiber of f^S is the n canonical model of a stable curve. By construction, the generic fibers Z_U and X_U are isomorphic. Since f and f^S are both families of stable curves, the isomorphism between their generic fibers extends everywhere, by 3.3.6; hence we are done. □

• Step E. The remaining part of the Theorem (that is, the GIT-semistability of Hilbert points of projective curves satisfying the Basic Inequality) was proved much later, in 1993 [C94], by a degeneration technique combined with a combinatorial analysis.

4.5 Construction of $\overline{M_g}$

We shall now prove Theorem 3.4.1. As Gieseker did in [Gie82], to prove the existence of the coarse moduli scheme for stable curves one only needs only the first three steps (A), (B) and (C). The fourth will be used later to construct other moduli spaces.

Proof of Theorem 3.4.1: existence of $\overline{M_g}$.

[9] This is just an application of the so called *GIT-semistable replacement property.*

From now on, fix a linearization of the action of $G = SL(r+1)$ on $Hilb_{\mathbb{P}^r}^{h(t)}$ such that the Theorem holds.

It is clear that the closed subset H_g of H^{SS} is G-invariant. Furthermore, since H_g parametrizes n-canonical models, we have that, for every point $h \in H_g$,

$$\mathrm{Stab}_G(h) = \mathrm{Aut}(Z_h).$$

hence all points of H_g have finite stabilizers (By 3.2.5, part a)). We conclude that all orbits in H_g are closed in the semistable locus and hence all points of H_g are GIT-stable.

Now the crux is that the quotient H_g/G is a projective scheme. This follows from Theorem 4.2.7, which says that if $q : H^{SS} \longrightarrow H/G$ is the quotient map, where $H := Hilb_{\mathbb{P}^r}^{h(t)}$, then H/G is projective and $q(H_g)$ a closed subscheme of it, and hence itself projective. Because H_g is made of GIT-stable points, by Theorem 4.2.7 the restriction of the quotient morphism (denoted again by q)

$$q : H_g \longrightarrow H_g/G$$

is an orbit space, that is to say, the closed points of H_g/G are in bijective correspondence with isomorphism classes of stable curves of genus g.

We then set

$$\overline{M_g} = H_g/G$$

and, from what we said in section 4.1, it is clear that $\overline{\mathcal{M}_g}(\mathrm{Spec}\, k)$ is in bijective correpondence with $\mathrm{Hom}(\mathrm{Spec}\, k, H_g/G)$.

It remains to show that there is a natural transformation of functors

$$\Phi : \overline{\mathcal{M}_g}(...) \longrightarrow \mathcal{H}om(..., H_g/G)$$

satisfying property B. in Definition 1.1.3.

Constructing Φ amounts to the following: pick $f : X \longrightarrow B$ be a family of DM-stable curves over a scheme B; then construct the moduli morphism $\phi_f : B \longrightarrow H_g/G$.

To do that, consider the vector bundle E on B defined as $E := f_*\omega_f^n$ and let $B = \cup U_i$ be an open covering of B such that the restriction of E to every

U_i is trivial. Denote by $f_i : X_i \longrightarrow U_i$ the restriction of f to $X_i = f^{-1}(U_i)$ and notice that $\omega_{f_i}^n$ determines an embedding of X_i in $\mathbb{P}^r \times U_i$ which restricts to the n-canonical embedding on the fibers of f. Therefore, by the functorial properties of the Hilbert scheme, we obtain, for every i, a natural morphism

$$\psi_i : U_i \longrightarrow H_g \subset Hilb_{\mathbb{P}^r}^{h(t)}$$

Denote by ϕ_i the composition of ψ_i with the quotient morphism q

$$\phi_i : U_i \longrightarrow H_g/G$$

Now, on $U_i \cap U_j$ the images of ψ_i and ψ_j are conjugate by the action of G, therefore ϕ_i and ϕ_j coincide on $U_i \cap U_j$, therefore we obtain the seeked for morphism $\phi : B \longrightarrow H_g/G$ by gluing the ϕ_i.

Now that we have the functor transformation Φ, we must prove its maximality property. Let then N be a scheme also endowed with a functor transformation

$$\Psi : \overline{\mathcal{M}_g}(...) \longrightarrow \mathcal{H}om(..., N)$$

To produce the morphism $\pi : H_g/G \longrightarrow N$ consider $\Psi(H_g) : \overline{\mathcal{M}_g}(H_g) \longrightarrow \mathrm{Hom}(H_g, N)$ and look at the image of the universal family $u_g : Z_g \longrightarrow H_g$ (obtained by restricting the universal family over $Hilb_{\mathbb{P}^r}^{h(t)}$). Set then

$$\Psi(H_g)(u_g) = \chi : H_g \longrightarrow N$$

and observe that χ is G-invariant (use Exercise 1.2.4). We can then apply the maximality property of quotients, which gives a morphism $\pi : H_g/G \longrightarrow N$ such that

$$\pi \circ q = \chi$$

Then it is easy to see that π satisfies the condition required by part B of 1.1.3.

5 Geometry of $\overline{M_g}$

5.1 Local properties

As a consequence of the GIT construction of $\overline{M_g}$, we have

Proposition 5.1.1. *$\overline{M_g}$ is an integral, normal, projective scheme.*

Proof. We have already proved that $\overline{M_g}$ is projective (Theorem 4.4.5) and irreducible (in 3.4.3). The fact that it is a reduced normal scheme follows from it being a quotient of a nonsingular scheme H_g (see Exercise 4.2.4). □

Notation 5.1.2. Let Z be a scheme whose points correspond to a given set of geometric objects (e.g. Z is a subscheme of some moduli space, like $Hilb^{h(t)}_{\mathbb{P}^r}$ or of $\overline{M_g}$). If $z \in Z$ denote by V_z the geometric object parametrized by z. We denote by $Z^0 \subset Z$ the subset of Z parametrizing objects with trivial automorphism group:

$$Z^0 := \{z \in Z, \quad \mathrm{Aut}(V_z) \cong \{1\}\}$$

Similarly, if $\mathcal{M}$ is a functor, we denote by $\mathcal{M}^0$ the restriction of $\mathcal{M}$ to objects having no automorphisms other then the identity.

Then we state the following

Lemma 5.1.3. *Let $g \geq 3$. Then*

(a) $\overline{M_g}^0$ is an open subset of $\overline{M_g}$ whose complement has codimension $g - 2$.

(b) $\overline{M_g}^0$ is contained in the non-singular locus of $\overline{M_g}$.

(c) If $g \geq 3$, $\overline{M_g}^0$ is a fine moduli scheme for stable curves whose automorphism group reduces to the identity.

Remark 5.1.4. $\overline{M_g}^0$ does not coincide with the nonsingular locus of $\overline{M_g}$. See example 5.2.2 below.

Proof. (a) follows from 2.1.4. For (b), notice that $\overline{M_g}^0 = H_g^0/G$ and G acts on H_g^0 with trivial stabilzers, by definition. Therefore $\overline{M_g}^0$ is nonsingular. To prove (c) is to prove that $\overline{M_g}^0$ has a universal family

$$u : \mathcal{C}_g \longrightarrow \overline{M_g}^0$$

which is an element of $\overline{\mathcal{M}_g}^0(\overline{M_g}^0)$ and such that the pair $(\mathcal{C}_g, \overline{M_g}^0)$ represents the functor $\overline{\mathcal{M}_g}^0$. The universal family $u : \mathcal{C}_g \longrightarrow \overline{M_g}^0$ is obtained by quotienting the universal family over H_g^0. We leave it as an exercise to check that the moduli property is satisfied. □

5.2 Boundary strata

The "boundary" of $\overline{M_g}$ is the closed subscheme $\overline{M_g} \setminus M_g$. It is a union of $[\frac{g}{2}]+1$ irreducible divisors, denoted $\Delta_0, \Delta_1, \dots, \Delta_{[\frac{g}{2}]}$ which are easily described:

- Δ_0 is the closure of the locus parametrizing irreducible curves with one node.

- For $[\frac{g}{2}] \geq i \geq 1$, Δ_i is the closure of the locus parametrizing reducible curves with one node and containing a component of genus i.

It is easy to show that these are irreducible and distinct.

Definition 5.2.1. Let C be a curve and let N be a node of C. Denote by $C_N \longrightarrow C$ the normalization of C at N. We say that N is *separating* if the number of connected components of C_N is greater than the number of connected components of C.

We say that a nodal connected curve is of *compact type* if all of its nodes are separating.

Example 5.2.2. Δ_1 contains a dense open subset parametrizing curves of type $C = C_1 \cup C_2$ with C_i nonsingular, $\#C_1 \cap C_2 = 1$ and C_1 of genus 1. For all such curves, $\mathrm{Aut}(C)$ is nontrivial: it certainly contains an automorphism (of order 2) which acts nontrivially on C_1, by fixing the point $C_1 \cap C_2$.

Notice also that $\overline{M_g}$ is certainly not singular along Δ_1, in fact Δ_1 has codimension 1 and $\overline{M_g}$ is normal, hence its singular locus has codimension at least 2.

Notation 5.2.3. For a nodal curve X, denote by $\delta(X)$ the number of nodes of X and by $\gamma(X)$ the number of irreducible components of X.

Exercise 5.2.4. Prove that

1. if $\gamma(C) = 1$, then $\delta(C) \leq g$;

2. if C is of compact type then $\delta(C) = \gamma(C) - 1 \leq 2g - 3$.

It is interesting to ask about the "smallest" boundary loci of $\overline{M_g}$. They are described in the next elementary

Lemma 5.2.5. *Let C be a stable curve of genus $g \geq 2$. Then*

(a) C has at most $3g-3$ nodes and at most $2g-2$ irreducible components.

(b) Assume that C has $3g-3$ nodes. Then C has $2g-2$ irreducible components $C_1, \dots, C_{2g-2}$ and, if $\nu_i\colon C_i^\nu \to C_i$ is the normalization, then $C_i^\nu \simeq \mathbb{P}^1$ and $\#\nu_i^{-1}(C_i \cap C_{sing}) = 3$ for all i.

Proof. □

See 5.3.4 for a proof using graphs.

Notice that if C is a curve having $3g-3$ nodes, as in the previous lemma, the degree of the dualizing sheaf of C on every irreducible component is equal to 1. This is why such curves are often called "polygonal" curves.

5.3 Combinatorics of stable curves

The combinatorial aspects of the theory of stable curves are better expressed using graph theory. To a nodal curve C one attaches a graph Γ_C defined as follows.

Definition 5.3.1. The *dual graph* Γ_C of a nodal curve C is a symplicial complex of dimension at most 1, defined to have one *vertex* (i.e. zero-dimensional symplex) for every irreducible component of C, and one *edge* (i.e. one-dimensional symplex) connecting two vertices for every node in which the two corresponding components intersect. Thus Γ_C has $\gamma(C)$ vertices, $\delta(C)$ edges and among the edges there is a loop for every node lying on a single irreducible component of C.

The first Betti number $b_1(\Gamma_C)$ (sometimes called the *cyclomatic number*) is then, if c is the number of connected components of C (and of Γ_C)

$$b_1(\Gamma_C) := \dim_{\mathbb{Z}} H_1(\Gamma_C, \mathbb{Z}) = \delta(C) - \gamma(C) + c$$

Of course, we need an orientation on Γ_C to introduce a chain complex and its homology (see below), but the first Betti number does not depend on the choice of the orientation. Recall also that *the first Betti number of a connected graph is the maximal number of one-dimensional open symplices that can be removed from the graph without disconnecting it.*

Example 5.3.2. Let C be connected.

(1) C is of compact type if and only if Γ_C is a tree, if and only if $b_1(\Gamma_C) = 0$.

(2) By the genus formula $g = \sum g_i + b_1(\Gamma_C)$ we get that $b_1(\Gamma_C) \leq g$. Moreover, $b_1(\Gamma_C) = g$ if and only if all irreducible components of C have geometric genus 0.

Another important invariant of a graph is its *complexity*

Definition 5.3.3. Let Γ be a connected graph. A *spanning tree of* Γ_C is a subgraph $\Gamma' \subset \Gamma$ which is a (connected) tree and such that Γ and Γ' have the same vertices. The *complexity* of Γ, $\tau(\Gamma)$, is defined to be the number of spanning trees that it contains.

For example, if Γ is a tree, then $\tau(\Gamma) = 1$.

The complexity can be computed cohomologically. Fix an orientation on Γ and consider the standard homology operators

$$\partial : C_1(\Gamma, \mathbb{Z}) \longrightarrow C_0(\Gamma, \mathbb{Z})$$

such that for any edge e of Γ, starting in the vertex v and ending in the vertex w, we have $\partial(e) = v - w$. And

$$\delta : C_0(\Gamma, \mathbb{Z}) \longrightarrow C_1(\Gamma, \mathbb{Z})$$

such that for a vertex v, $\delta(v) = \sum e_v^+ - \sum e_v^-$ where e_v^+ are the edges starting at v and e_v^- are those ending in v. Then introduce the group

$$\frac{\partial C_1(\Gamma, \mathbb{Z})}{\partial \delta C_0(\Gamma, \mathbb{Z})}$$

The theorem of Kirchoff-Trent ([?]) states that *such a group is finite and its cardinality is equal to the complexity of* Γ. We shall use this result of graph theory later, to give a moduli-theoretical interpretation of the complexity of the dual graph of a stable curve.

Exercise 5.3.4. Give a proof of Lemma 5.2.5 using the language of graphs.

6 Moduli of line bundles

6.1 The Picard functor and the Picard scheme

Fix a flat projective morphism

$$f : X \longrightarrow S$$

and consider the category SCH_S of schemes over S. We shall use the following notation: if B is an object in SCH_S, we shall denote $X_B := X \times_S B$ and by

$$f_B : X_B \longrightarrow B$$

the projection. The Picard functor $\mathcal{P}ic_f$ associated to the above $f : X \longrightarrow S$ goes from SCH_S to the category of sets, and associates to any object $B \in \mathrm{SCH}_S$ the set

$$\mathcal{P}ic_f(B) = \{\text{equivalence classes of line bundles on } X_B\}$$

where we say that two line bundles L and L' on X_B are equivalent if there exists a line bundle M on B such that

$$L \cong L' \otimes f_B^* M$$

Notice that $\mathcal{P}ic_f(B)$ is a group under tensor product of line bundles.

The representability of the Picard functor was studied by Grothendieck, the following statement summarizes the main facts

Theorem 6.1.1 (Grothendieck)**.** *Let $f : X \longrightarrow S$ be a flat projective morphism with integral geometric fibers.*

1. *There exists a unique group scheme* Pic_f *over S which coarsely represents $\mathcal{P}ic_f$. Moreover for every scheme B over S the natural map $\mathcal{P}ic_f(B) \longrightarrow \mathrm{Hom}_S(B, \mathrm{Pic}_f)$ is injective.*

2. *If f admits a section, then* Pic_f *is a fine moduli scheme for $\mathcal{P}ic_f$.*

Remark 6.1.2. The injectivity of the map $\mathcal{P}ic_f(B) \longrightarrow \mathrm{Hom}_S(B, \mathrm{Pic}_f)$ says that if L and L' are two line bundles on X_B which agree on every fiber of f_B, then L and L' are equivalent, that is, they differ by the pull-back of some line bundle of B. In particular, isotriviality is not an issue for the Picard moduli problem. In this respect, $\mathcal{P}ic_f$ is closer to be representable than $\mathcal{M}_g$ (for which isotriviality plays a subtle role, see 2.1.3).

What prevents $\mathcal{P}ic_f$ from having a fine moduli space is the existence of non-modular maps $B \longrightarrow \mathrm{Pic}_f$. In other words, a continuously varying family of line bundles on the fibers of f_B does not necessarily "glue together" to a line bundle on the total space X_B. The second part of the previous theorem says exactly that such a "gluing" exists if f has a section.

If $S = \operatorname{Spec} k$ where k is an algebraically closed field, and X is a smooth irreducible projective variety over k, we find the classical Picard group of X, usually denoted

$$\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$$

where $\mathcal{O}_X^*$ is the subsheaf of units in the structure sheaf of X. $\operatorname{Pic} X$ is a fine moduli scheme for line bundles on X. The representabilty of $\mathcal{P}ic_X$ amounts to the existence of a *Poincaré* *line bundle* P on $X \times \operatorname{Pic} X$, which is the universal object $P = U_{\mathcal{P}ic_X} \in \mathcal{P}ic_X(\operatorname{Pic} X)$ defined in 1.1.1.

Consider the special case of a projective curve C. Then for $d \in \mathbb{Z}$ denote

$$\operatorname{Pic}^d C := \{L \in \operatorname{Pic} C : \ \deg L = d\}$$

so that $\operatorname{Pic} C$ can be broken up

$$\operatorname{Pic} C = \coprod_{d \in \mathbb{Z}} \operatorname{Pic}^d C$$

If C is nonsingular and irreducible, then $\operatorname{Pic}^d C$ is a projective irreducible variety and there are non-canonical isomorphisms $\operatorname{Pic}^d C \cong \operatorname{Pic}^0 C$. The "identity" component $\operatorname{Pic}^0 C$ of $\operatorname{Pic} C$ is a projective subgroup and therefore an abelian variety, called the Jacobian variety of C and often denoted by $J_C = \operatorname{Pic}^0 C$.

6.2 Line bundles on nodal curves

Let now C be a connected nodal curve having δ nodes and γ irreducible components; consider its normalization

$$\nu : C^\nu \longrightarrow C$$

There is a natural exact sequence

$$1 \longrightarrow \mathcal{O}_C^* \longrightarrow \nu_* \mathcal{O}_{C^\nu}^* \longrightarrow \mathcal{G} \longrightarrow 1$$

where $\mathcal{G}$ is a skyscraper sheaf supported on the singular points of C. The associated long cohomology sequence is

$$1 \longrightarrow H^0(C, \mathcal{O}_C^*) \longrightarrow H^0(C, \nu_*\mathcal{O}_{C^\nu}^*) \longrightarrow H^0(C, \mathcal{G}) \longrightarrow H^1(C, \mathcal{O}_C^*) \longrightarrow H^1(C, \nu_*\mathcal{O}_{C^\nu}^*) \longrightarrow 1$$

that is

$$1 \longrightarrow k^* \longrightarrow (k^*)^\gamma \longrightarrow (k^*)^\delta \longrightarrow \operatorname{Pic} C \longrightarrow \operatorname{Pic} C^\nu \longrightarrow 0.$$

From this, we extract the comparison sequence for the Picard groups

$$1 \longrightarrow (k^*)^{b_1(\Gamma_C)} \longrightarrow \operatorname{Pic} C \xrightarrow{\nu^*} \operatorname{Pic} C^\nu \longrightarrow 0. \qquad (3)$$

where the number $b_1(\Gamma_C)$ is defined in section 5.3 and the map ν^* is, of course, the pull-back.

Having obtained this important sequence cohomologically, we shall now directly explain its geometric meaning. The surjectivity of ν^* expresses the well known fact that one can move the support of a divisor away from any finite set of points (the critical points being the preimages via ν of the nodes of C). Then the sequence says that for any line bundle L^ν on C^ν, the set of line bundles on C which pull back to L^ν is a torus of dimension $b_1(\Gamma_C)$. This means that such a torus parametrizes the truly different gluing data on $L^\nu \in \operatorname{Pic} C^\nu$. Now, for every node N of C, the possible gluings of the two corresponding fibers of L^ν form, of course, a k^*. Suppose first that C has only one node, N. Then either C^ν is connected, and then these gluings give non-isomorphic line bundles on C, or C^ν has two connected components C_1 and C_2, and then L^ν is a pair of line bundles (L_1, L_2), with

$L_i \in \operatorname{Pic} C_i$. In this case, different gluing data give always isomorphic line bundles on C, simply because the endomorphisms of L_i (again a k^*) act transitively on the gluing data.

Iterating this process, we see that if C is of compact type, then ν^* is injective (as it follows easily, see example 5.3.2).

Now, in the general case, we can order the nodes of C so that gluing the first $\gamma(C)-1$ of them from C^ν gives a connected curve C', of compact type, with two natural morphisms factoring the normalization map

$$\nu : C^\nu \longrightarrow C' \longrightarrow C$$

By what we said, any line bundle L^ν on C^ν determines a unique line bundle L' on C' (C' being of compact type). To obtain a line bundle L on C we work on L', gluing pairwise corresponding fibers over the remaining $\delta(C)-\gamma(C)+1 = b_1(\Gamma_C)$ nodes (C' is obviously the normalization of C at such nodes). Since C' is connected, every gluing datum gives a different line bundle, so we obtain a $(k^*)^{b_1(\Gamma_C)}$ of line bundles on C which pull-back to L^ν.

It is particularly interesting to restrict the above sequence to the connected component of $\operatorname{Pic} C$ containing the identity: the generalized jacobian of C, denoted J_C.

Definition 6.2.1. The generalized Jacobian of a nodal curve C is defined as the set J_C of isomorphism classes of line bundles on C having degree 0 on every irreducible component of C. J_C is a commutative algebraic group with respect to $\otimes$, and it fits into an exact sequence of groups (obtained from the exact sequence 3)

$$1 \longrightarrow (k^*)^{b_1(\Gamma_C)} \longrightarrow J_C \xrightarrow{\nu^*} J_{C^\nu} \cong \prod_{i=1}^{\gamma} J_{C_i^\nu} \longrightarrow 0$$

where $C = \cup_1^\gamma C_i$ is the decomposition of C into irreducible components.

Remark 6.2.2. We have that a connected nodal curve is of compact type if and only if its generalized Jacobian is compact.

Write $C = \cup_1^\gamma C_i$ with C_i irreducible component of C. Let $\underline{d} = \{d_1, \ldots, d_\gamma\} \in \mathbb{Z}^\gamma$ and define

$$\operatorname{Pic}^{\underline{d}} C := \{L \in \operatorname{Pic} C : \deg_{C_i} L = d_i\}$$

so that we can decompose

$$\operatorname{Pic}^d C = \coprod_{|\underline{d}|=d} \operatorname{Pic}^{\underline{d}} C$$

where $|\underline{d}| := \sum_1^\gamma d_i$. Now $\operatorname{Pic}^{\underline{d}} C$ is an irreducible quasi-projective variety, in particular

$$\operatorname{Pic}^{\underline{0}} C = J_C$$

and there are non canonical isomorphisms $\operatorname{Pic}^{\underline{0}} C \cong \operatorname{Pic}^{\underline{d}} C$ for every $\underline{d}$.

Consider now a morphism $f : X \longrightarrow S$ whose fibers are projective curves; assume that the general fiber is non-singular and that there is some nodal reducible fiber. The Picard scheme $p : \operatorname{Pic}_f \longrightarrow S$ is such that for every point $s \in S$ the fiber $p^{-1}(s)$ is identified with $\operatorname{Pic} f^{-1}(s)$ and there is a natural decomposition

$$\operatorname{Pic}_f = \coprod_{d \in Z} \operatorname{Pic}_f^d .$$

We have seen that Pic^d_f is not complete (unless all fibers of f are of compact type).

We shall now study the problem of completing it. To do that, there is one basic difficulty: the Picard functor is not separated, as long as reducible fibers occur.

To explain this, let $S = \operatorname{Spec} R$ for some discrete valuation ring R, assume X to be smooth and denote by $C = \cup C_i$ the special fiber of $f : X \longrightarrow S$ which is assumed to be nodal (mantaining the previous notation). Let $L \in \operatorname{Pic} X$ and let $M = L \otimes \mathcal{O}_X(D)$ where $D = \sum n_i C_i$ is a Cartier divisor supported on the special fiber of f and such that the integers n_i are not all equal (to avoid trivial cases, i.e. the case when D equals a multiple of C). Then, of course, L and M are equal on the generic fiber of f, but they are different on the special fiber: notice in fact that even their multidegrees are different:

$$\underline{\deg}_C L \neq \underline{\deg}_C M$$

The distinguished line bundles on C of the form $\mathcal{O}_X(\sum n_i C_i) \otimes \mathcal{O}_C$ (for some X as above and $n_i \in \mathbb{Z}$), prevent the Picard functor from being separated and are usually called *twisters*. The above discussion shows that any proper completion of the Picard scheme must identify line bundles which differ by a twister.

6.3 The Degree class group of a nodal curve

Observe that twisters on a fixed curve C depend on two types of data:

(1) discrete data, given by the choice of the coefficients n_i

(2) continuous data, namely the choice of the one-parameter smoothing $f : X \longrightarrow \operatorname{Spec} R$ of C.

The following analysis (taken from [C94]) shows how to control the discrete data.

The guiding idea is that while the twisters $\mathcal{O}_X(\sum n_i C_i) \otimes \mathcal{O}_C$ depend on X, their multidegree does not. In fact, for every component C_i of C denote, if $j \neq i$

$$k_{i,j} := \#(C_i \cap C_j)$$

and

$$k_{i,i} = -\#(C_i \cap \overline{C \setminus C_i})$$

then it is clear that for every pair i, j and for every non-singular X

$$\deg_{C_j} \mathcal{O}_X(C_i) = k_{i,j}$$

Obviously we have that $k_{i,j} = k_{j,i}$ and that $\sum_{j=1}^{\gamma} k_{i,j} = 0$ for every fixed i .

Now, for every $i = 1, \ldots, \gamma$ set

$$\underline{c}_i := (k_{1,i}, \ldots, k_{\gamma,i}) \in \mathbb{Z}^\gamma$$

and

$$\mathbf{Z} := \{\underline{d} \in \mathbb{Z}^\gamma : \ |\underline{d}| = 0\}$$

so that $\underline{c}_i \in \mathbf{Z}$ and we can consider the sublattice Λ_C of $\mathbf{Z}$ spanned by them

$$\Lambda_C :=< \underline{c}_1, \ldots, \underline{c}_\gamma > .$$

Thus, Λ_C is the lattice formed by the multidegrees of all twisters, inside the abelian group $\mathbf{Z}$ of multidegrees of degree 0.

It is easy to see that Λ_C has rank $\gamma - 1$, in fact any $\gamma - 1$ among the $\underline{c}_1, \dots \underline{c}_\gamma$ are independent over $\mathbb{Z}$, whereas the following natural relation occur

$$\sum_1^\gamma \underline{c}_i = \underline{0}$$

(since $\sum_1^\gamma \underline{c}_i = \underline{\deg}_C \mathcal{O}_X(C) = \underline{0}$).

Definition 6.3.1. The *degree class group* of C is the finite group $\Delta_C := \mathbf{Z}/\Lambda_C$.

Remark 6.3.2. Such a group Δ_C parametrizes classes of multidegrees in the following sense. Let $\underline{d}$ and $\underline{d}'$ be in $\mathbf{Z}$; we say that they are equivalent if their difference is the multidegree of a twister, or equivalently, if $\underline{d} - \underline{d}' \in \Lambda_C$

Δ_C is a purely combinatorial invariant of the curve: using the notation introduced in Section 5.3 we have

Proposition 6.3.3. *For a nodal connected curve C with dual graph Γ_C we have*

$$\Delta_C \cong \frac{\partial C_1(\Gamma_C, \mathbb{Z})}{\partial\delta C_0(\Gamma_C, \mathbb{Z})}.$$

In particular $\#\Delta_C$ is equal to the complexity of the dual graph of C.

Proof. The proof is straightforward using the definitions of ∂ and δ given in 5.3. First we identify $\mathbf{Z} \cong \partial C_1(\Gamma_C, \mathbb{Z})$ (because Γ_C is connected). Then we have $\Lambda_C \cong \partial\delta C_0(\Gamma_C, \mathbb{Z})$, in fact if v_i is the vertex of Γ_C corresponding to the component C_i of C, then, by definition, one obtains $\partial\delta(v_i) = -\underline{c}_i$, for every $i = 1, \dots, \gamma$.

The final sentence follows from the theorem of Kirchoff-Trent, quoted at the end of 5.3. □

Remark 6.3.4. The name "degree class group" was given to Δ_C in [C94] where such a group was introduced and used to compactify the Picard scheme of the universal curve over $\overline{M_g}$. With the above result 6.3.3 in mind, a better name (stressing its intrinsic combinatorial nature) might be the *complexity group of C*.

Example 6.3.5. If C is a curve made of two smooth components meeting at δ points, or if C is a "cycle" of δ smooth components, then Δ_C is the cyclic group of order δ.

6.4 Completing the Picard scheme for curves

We shall apply the results of section 4.4 to construct a modular completion of the Picard scheme for a family of stable curves.

7 Appendix - Problems for Pragmatic

7.1 Modular curves in M_g: estimates on their numerical invariants

Notation and background in 2.2. Work over a fixed algebraically closed field of characteristic 0 (e.g. work over $\mathbb{C}$). Denote by $\mathrm{q}(g)$ the minimum genus for the base of a complete non-isotrivial family of smooth curves of genus g, more precisely

$$\mathrm{q}(g) := \min\{q : \exists\, B \in M_q \text{ such that } \mathcal{M}_g^*(B) \neq \emptyset\}$$

It is known (see 2.3.1) that $\mathrm{q}(g) \geq 2$ for all $g \geq 1$, moreover the cases $g = 1$ and $g = 2$ are not interesting as M_1 and M_2 are affine (this is classically known, see [Ha77], Chapter IV), hence there are no complete curves contained in them. Quite the opposite holds for every $g \geq 3$, as M_g is covered by complete curves (see 2.3.4). Then we can ask:

Problem 1. Fix $g \geq 3$.

(a) Find the best upper bound on $\mathrm{q}(g)$ that can be obtained using the families of Kodaira-type (or any other families, if you can find any!).

(b) Is $\mathrm{q}(g)$ uniformly bounded, as g varies? In other words, does there exist a number Q such that for every $g \geq 2$ we have $\mathrm{q}(g) \leq Q$[10]?

(c) What if we only consider families possessing a rational point (i.e. a section)?

7.2 Hyperbolicity of the moduli functor $\mathcal{M}_g$

Over $\mathbb{C}$. Generalize 2.3.1 in two ways: first by allowing certain (simple) singular fibers and second by replacing $\mathbb{P}^1$ with suitable higher dimensional bases. That is, complete, however you can, the following statement and prove it.

Problem 2. Fix $g \geq 1$. Prove that there does not exist a complete family of curves of compact type over a base which is (!replace $\mathbb{P}^1$!) rational? rationally connected?

Notice that a smooth rationally connected variety is simply connected.

7.3 Modular rational points of M_g: asymptotics.

Fix a smooth projective curve B of genus q, then we know that for every finite set of points $S \subset B$ the set $\mathcal{M}_g^*(B \smallsetminus S)$ is finite (see 2.2.1) and uniformly bounded by a function that depends on g, on q and on the cardinality of the degeneracy locus $s = \#S$ (Theorem 3.1 in [C02]). Expand 3.3.5 to obtain that, for fixed $g \geq 2$ and B, the number $\#\mathcal{M}_g^*(B \smallsetminus S)$ is not uniformly bounded as S gets larger and larger. More precisely

Problem 3. Study the examples of de Jong and Beauville (see [B81]) to measure the asymptotics of such a bound and prove that as g, q or $s = \#S$ go to infinity (one at the time, fixing the remaining two), the set of non-isotrivial families of smooth curves of genus g over B, having singular fibers over S[11], has unbounded cardinality.

[10] The answer to this question is known to be "yes", and for certain special values of g (for example, $g = 25$ and $g = 49$) it is known that $\mathrm{q}(g) = 2$ (due to Bryan and Donagi, see [BDS01]).

[11] Warning: this set is a proper subset of $\mathcal{M}_g^*(B \smallsetminus S)$, as we are requiring all fibers over S to be singular

Consider now the analogous question for modular maps to M_g. Recall first that M_g is not a fine moduli scheme, so there exist non-modular maps to M_g; for any scheme V denote by $\mathrm{Hom}^*(V, M_g)$ the set of non constant modular maps from V to M_g. Then there is a natural surjective map

$$\mathcal{M}_g^*(V) \longrightarrow \mathrm{Hom}^*(V, M_g)$$

which associates to a family f over V its moduli morphism $\phi_f : V \longrightarrow M_g$. The fibers of the above map are finite and bounded above by a function of g (exercise); thus $\mathrm{Hom}^*(V, M_g)$ is a finite and uniformly bounded set if and only if so is $\mathcal{M}_g^*(V)$.

Consider now the moduli space of stable curves $\overline{M_g}$ and let Δ be its boundary (i.e. the complement of M_g in $\overline{M_g}$). Define the set $\mathrm{Hom}^*[(B,S),(\overline{M_g},\Delta)] \subset \mathrm{Hom}^*(B-S, M_g)$ as the subset of modular maps from B to $\overline{M_g}$ corresponding to families over B having a singular fiber over every point of S

$$\mathrm{Hom}^*[(B,S),(\overline{M_g},\Delta)] := \{\phi \in \mathrm{Hom}^*(B,\overline{M_g}) : \ \phi(b) \in M_g \text{ if and only if } b \notin S\}$$

By the above discussion, the cardinality of the above set is bounded above by a function of g, q, and s. We ask again whether such a bound really depends on s:

Problem 4. Having fixed g and B, is the cardinality $\mathrm{Hom}^*[(B,S),(\overline{M_g},\Delta)]$ bounded as S varies among subsets of B of increasing cardinality?

By 3.3.5, the example of de Jong does not prove that the answer to the above question is "no"!

7.4 Weak uniformity of rational points of curves over number fields

Weak uniform versions of the geometric Shafarevich and Mordell conjecture are known (see [C02]). The same facts should hold over number fileds:

Problem 5. Fix a number field k and integers s, g with $g \geq 2$.

1. There exists a number $E_g(k,s)$ such that for every finite set of places S of $\mathcal{O}_k$ such that $\#S \leq s$ there exist at most $E_g(k,s)$ curves of genus g defined over k and having good reduction away from S.

2. There exists a number $F_g(k,s)$ such that for every finite set of places S of $\mathcal{O}_k$ such that $\#S \leq s$ and for every curve C defined over k and having good reduction away from S we have that $\#C(k) \leq F_g(k,s)$

7.5 Uniform Mordell for curves over suitable function fields.

Uniform versions of the Mordell conjecture for curves over suitable function fields. Improve on the results obtained in [C03] (for example Lemma 5) using the moduli theory of curves.

Problem 6. An analog of the Lemma below for families of hyperelliptic curves or other "special" families.

Similarly for families with maximal variation of moduli over a large dimensional base.

We recall Lemma 5 from [C03] below:

Lemma 7.5.1. *Let V be a variety of dimension $3g - 3$ with function field L and let X be a smooth curve of genus g over L having maximal variation of moduli. Then either $X(L) = \emptyset$ or the modular degree of X is a multiple of $2g - 2$.*

Where the modular degree is the degree of the moduli map $V \dashrightarrow M_g$ associated to any model of X over V. Something else that I believe to be true is the following strengthening of the above Lemma:

Problem 7 (Conjecture)**.** With the hypotheses of the Lemma, prove that if $X(L) \neq \emptyset$ then $\#X(L) \leq n$ where the modular degree of X is equal to $n(2g - 2)$.

7.6 Combinatorics of stable curves

Detailed study of the Degree class group (defined in 6.3.1): almost no examples are known, for instance: which stable curves have cyclic degree class group (see 6.3.5)?

Problem 8.

1. Compute the degree class groups of stable curves and its behaviour under standard operations: (a) blowing-up a node, (b) normalize a node.
2. Use the degree class group to classify stable curves and stratify $\overline{M_g}$.

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