Naturality of Abel maps

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Abstract. We give a combinatorial characterization of nodal curves admitting a natural (i.e. compatible with and independent of specialization) d-th Abel map for any $d \geq 1$.

Let X be a smooth projective curve and d a positive integer; the classical d-th Abel map of X, $\alpha_X^d : X^d \longrightarrow \mathrm{Pic}^d X$, associates to $(p_1, \ldots, p_d) \in X^d$ the class of the line bundle $\mathcal{O}_X(p_1 + \ldots + p_d)$ in $\mathrm{Pic}^d X$. This morphism has good functorial properties, it is compatible with specialization and base change.

Now let X be a singular nodal curve occurring as the limit of a family of smooth curves. We ask whether there is a notion of d-th Abel map for X which is limit of the Abel maps of the smooth curves of the family, and which is *natural*, i.e. independent of the choice of the family.

It is known that, although a nodal curve X is endowed with a generalized Jacobian and a Picard scheme which are both natural (i.e. they are the limit of the Jacobians and, respectively, of the Picard schemes of the fibers of every family of curves specializing to X), there are interesting degeneration problems about line bundles and linear series where naturality is a subtle issue. Here we investigate the case of Abel maps.

The main result of this paper, Theorem 1.5, characterizes in purely combinatorial terms nodal curves that possess a natural d-th Abel map.

A consequence of our result is that, if we consider stable curves of genus $g \geq 2$, then the locus in $\overline{M}_g$ of curves that fail to admit a natural d-th Abel map, for a fixed $d \geq 2$, has codimension 2. So, naturality of Abel maps is not to be taken for granted, unless X is irreducible or of compact type, in which case it is not difficult to see that natural Abel maps exist for all d.

What is a good notion of Abel maps for singular curves? The same definition as for the smooth case behaves badly under specialization; moreover, it obviously does not make sense if some of the p_i are singular points of X. This last problem will not be an issue here: we shall only study noncomplete Abel maps, which is enough for our scopes.

So, for us a d-th Abel map is a rational map $\beta : X^d \dashrightarrow \mathrm{Pic}^d X$ arising as the limit of Abel maps of smooth curves specializing to X, for some family. A further requirement is added to ensure separation; see 1.2.

Thus, the target space of our d-th Abel map is the Picard scheme, not any particular compactification of it. Our definition and results should be sufficently general to apply to various compactified Picard schemes existing in the literature (see section 5).

The construction of complete Abel maps for singular curves was carried out by A. Altman and S. Kleiman for irreducible and reduced curves in [AK]; see also [EGK00] for further results. Not much is known for reducible curves. Recently, in [CE06], degree-1 Abel maps of stable curves have been defined, compactified, and shown to be natural. For higher d the completion problem is open in general, see [Co06] for some progress in case $d = 2$. The main result of the present paper indicates that a safe way to approach it is to work with a fixed one-parameter smoothing of the given curve X (as in [CE06] and [Co06]), or to restrict to natural Abel maps.

Among our techniques, the main one is the use of Néron models of Jacobians (as constructed by M. Raynaud in [R70]); this allows us to obtain a concrete description of our axiomatically defined Abel maps. Then we combine a result of E. Esteves and

N. Medeiros about deformation of line bundles and enriched structures (in [EM02]) with a detailed combinatorial analysis.

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1. Statement of the main result

In this section we state the main theorem (1.5) after a few preliminaries.

1.0.1. *Conventions.* We work over an algebraically closed field k. X always denotes a connected, reduced, projective curve defined over k having at most nodes as singularities, and $C_1, \ldots, C_\gamma$ its irreducible components.

By a (one-parameter) *regular smoothing of* X we mean a proper morphism $f : \mathcal{X} \to B = \operatorname{Spec} R$, with R a discrete valuation ring having residue field k and quotient field K, such that X is the closed fiber of f, the total space $\mathcal{X}$ is non-singular, and such that the generic fiber of f, denoted $\mathcal{X}_K$, is a smooth projective curve over K.

Let $X = \cup_1^\gamma C_i$ be a curve as above and $L \in \operatorname{Pic} X$ a line bundle on it. Then L has a *degree*, $d = \deg L \in \mathbb{Z}$, and a *multidegree*, $\underline{d} = \underline{\deg} L \in \mathbb{Z}^\gamma$ defined as $\underline{\deg} L := (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L)$. Denoting $\operatorname{Pic}^d X$, respectively $\operatorname{Pic}^{\underline{d}} X$, the locus in $\operatorname{Pic} X$ of line bundles of degree d, respectively of multidegree $\underline{d}$, we have

$$\operatorname{Pic} X = \coprod_{d \in \mathbb{Z}} \operatorname{Pic}^d X \qquad\qquad \operatorname{Pic}^d X = \coprod_{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = d} \operatorname{Pic}^{\underline{d}} X,$$

where $|\underline{d}| := \sum_1^\gamma d_i$ for every $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$. $\operatorname{Pic}^{\underline{d}} X$ is (non canonically) isomorphic to the generalized Jacobian of X.

If $Z \subset X$ is a (complete) subcurve, we denote by $\underline{d}_Z$ the multidegree of $L_{|Z}$ (L restricted to Z) and by $|\underline{d}_Z|$ the total degree of $L_{|Z}$.

Set $\mathcal{D}(X) := \{D = \sum n_i C_i,\ n_i \in \mathbb{Z}\}$ the free abelian group generated by $C_1, \ldots, C_\gamma$. If $Z \subseteq X$ is a subcurve of X, so that $Z = \cup_{C_i \subset Z} C_i$, we shall, as usual, abuse notation by denoting again $Z = \sum_{C_i \subset Z} C_i \in \mathcal{D}(X)$.

For any $f : \mathcal{X} \longrightarrow \operatorname{Spec} R$ be a regular smoothing of X we have a symmetric bilinear product $(\ \cdot\ ) : \mathcal{D}(X) \times \mathcal{D}(X) \to \mathbb{Z}$, often called the "intersection pairing", which is the same for every f (as long as the total space $\mathcal{X}$ is regular, which we always assume). Recall that $(X \cdot D) = 0$ for all $D \in \mathcal{D}(X)$.

For any subcurve $Z \subset X$ we set $Z' := \overline{X \smallsetminus Z}$ and $k_Z := (Z \cdot Z') = -Z^2$.

Consider the quotient $\overline{\mathcal{D}(X)}$ of $\mathcal{D}(X)$ by the subgroup generated by X: $\overline{\mathcal{D}(X)} := \frac{\mathcal{D}(X)}{\mathbb{Z} \cdot X}$. The intersection pairing descends to $\overline{\mathcal{D}(X)}$.

1.0.2. *Twisters.* For a fixed regular smoothing $f : \mathcal{X} \to B$ of X, set

$$\operatorname{Tw}_f X := \frac{\{\mathcal{O}_{\mathcal{X}}(D)_{|X},\ \forall D \in \mathcal{D}(X)\}}{\cong} \subset \operatorname{Pic}^0 X.$$

The union as f varies among the regular smoothings of X is denoted by

$$\operatorname{Tw} X := \bigcup_{f \text{ reg. sm.}} \operatorname{Tw}_f X \subset \operatorname{Pic}^0 X \tag{1}$$

Elements of $\operatorname{Tw} X$ are special line bundles called *twisters*. Any $T \in \operatorname{Tw}_f X$ determines a $D \in \mathcal{D}(X)$ such that $T \cong \mathcal{O}_{\mathcal{X}}(D)_{|X}$ only up to adding a multiple of the central fiber X of f; in fact $\mathcal{O}_{\mathcal{X}}(D)_{|X} \cong \mathcal{O}_{\mathcal{X}}(D + nX)_{|X}$ for any $n \in \mathbb{Z}$. Thus we have a surjective map $\operatorname{Tw} X \longrightarrow \overline{\mathcal{D}(X)}$ associating to any $T \in \operatorname{Tw} X$ the class of a $D \in \mathcal{D}(X)$ such that $T = \mathcal{O}_{\mathcal{X}}(D)_{|X}$. We shall denote such a class $Supp(T)$ and call it the *support of* T.

Let $D \in \mathcal{D}(X)$ and $T \in \operatorname{Tw} X$ such that $Supp(T) = \overline{D}$, then

$$\underline{\deg}\, T = ((D \cdot C_1), \ldots, (D \cdot C_\gamma)),$$

independently of the representative D for $\overline{D}$. Furthermore, this shows that $\underline{\deg}\, T$ does not depend on the regular smoothing defining T. In other words, the multidegree of a twister only depends on its support, so that we can unambiguously write

$$\underline{\deg}\, D := \underline{\deg}\, \mathcal{O}_{\mathcal{X}}(D)_{|X} = ((D \cdot C_1), \ldots, (D \cdot C_\gamma)); \tag{2}$$

for a subcurve $Z \subset X$ we set $\underline{\deg}_Z\, D := \underline{\deg}\, \mathcal{O}_{\mathcal{X}}(D)_{|Z}$. Summarizing, we have

$$\underline{\deg} : \operatorname{Tw} X \stackrel{Supp}{\longrightarrow} \overline{\mathcal{D}(X)} \stackrel{\underline{\deg}}{\longrightarrow} \mathbb{Z}^\gamma$$

where the second arrow is a group homomorphism. We denote by $\Lambda_X \subset \mathbb{Z}^\gamma$ the image of $\underline{\deg}$ above, i.e. the group of multidegrees of all twisters:

$$\Lambda_X := \{\underline{d} \in \mathbb{Z}^\gamma : \exists T \in \operatorname{Tw} X : \underline{\deg}\, T = \underline{d}\}. \tag{3}$$

More details about this set up will be in section 3.

Interpreting $\mathbb{Z}^\gamma$ as the set of all multidegrees on X, we get an equivalence relation on it, induced by X:
Let $\underline{d}, \underline{d}' \in \mathbb{Z}^\gamma$, define $\underline{d}$ equivalent to $\underline{d}'$ (in symbols $\underline{d} \equiv \underline{d}'$) if $\underline{d} - \underline{d}' \in \Lambda_X$
where Λ_X is defined in (3). Introduce the set of multidegree classes of total degree equal to a fixed d

$$\Delta_X^d := \frac{\{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = d\}}{\equiv}. \tag{4}$$

Δ_X^d is well known to be a finite set whose cardinality does not depend on d ([R70] 8.1.2, see also [C05] 3.7 for an overview). As we shall see, Δ_X^d is useful to control the non-separatedness of the Picard scheme.

For $\underline{d} \in \mathbb{Z}^\gamma$ we shall denote $[\underline{d}] \in \Delta_X^d$ its class.

1.0.3. Let $f : \mathcal{X} \to B$ be a regular smoothing of X. Then there exists a Picard scheme relative to f, denoted here Pic_f (an alternative notation is $\operatorname{Pic}_f = \operatorname{Pic}_{\mathcal{X}/B}$, which is not used in this paper). Pic_f is a scheme over B whose generic fiber is $\operatorname{Pic}_{\mathcal{X}_K}$, the Picard scheme of the curve $\mathcal{X}_K$. Pic_f has a basic moduli property which we need to recall. Let S be a B-scheme, then for any line bundle $\mathcal{L}$ on $S \times_B \mathcal{X}$ there exists a unique B-morphism

$$\mu_{\mathcal{L}} : S \longrightarrow \operatorname{Pic}_f \quad s \mapsto \mathcal{L}_{|k(s) \times_B \mathcal{X}} \tag{5}$$

which we refer to as the *moduli map of* $\mathcal{L}$. More details are in [GIT] 05 (d).

Pic_f decomposes into its connected components, $\operatorname{Pic}_f = \coprod_{d \in \mathbb{Z}} \operatorname{Pic}_f^d$, where $\operatorname{Pic}_f^d \to B$ parametrizes line bundles of relative degree d. The generic fiber of $\operatorname{Pic}_f^d \to B$ is denoted $\operatorname{Pic}_K^d := \operatorname{Pic}_{\mathcal{X}_K}^d$, an integral projective variety over K; the closed fiber is $\operatorname{Pic}^d X$, a reducible scheme if X is reducible (see 1.0.1).

Pic_f is smooth over B but not separated if X is reducible. The essential reason is the existence of twisters (see 1.0.2), which must all be identified in any separated completion of $\operatorname{Pic}_{\mathcal{X}_K}$ over B.

Using the moduli property of Pic_f one can construct Abel maps for smooth curves over any base scheme, see 2.0.4. For example let X be a smooth curve over k. As we already said, the d-th Abel map of X is the map $\alpha_X^d : X^d \to \operatorname{Pic}^d X$ such that $\alpha_X^d(p_1, \ldots, p_d) = \mathcal{O}_X(p_1 + \ldots + p_d)$.

We shall denote α_K^d the d-th Abel map of the generic fiber of f:

$$\alpha_K^d = \alpha_{\mathcal{X}_K}^d : \mathcal{X}_K^d \longrightarrow \operatorname{Pic}_K^d \tag{6}$$

Example 1.1. Let $X = C_1 \cup C_2$ be a curve having two smooth components meeting in only one node r. Let us examine the naive definition for the Abel map in degree 1, copying the smooth case. We get a map regular away from the node r:

$$\begin{array}{ccll} \dot{X} & \stackrel{\alpha}{\longrightarrow} & \operatorname{Pic}^1 X & = \coprod_{d_1+d_2=1} (\operatorname{Pic}^{d_1} C_1 \times \operatorname{Pic}^{d_2} C_2) \\ p & \mapsto & \mathcal{O}_X(p) & \end{array} \tag{7}$$

where $\dot{X} := X \smallsetminus \{r\}$. Let us illustrate some pathologies of definition (7). The two components of $\dot{X}$ are mapped to two different connected components of $\operatorname{Pic}^1 X$, namely $\alpha(C_1) \subset \operatorname{Pic}^1 C_1 \times \operatorname{Pic}^0 C_2$ while $\alpha(C_2) \subset \operatorname{Pic}^0 C_1 \times \operatorname{Pic}^1 C_2$. All the connected components $\operatorname{Pic}^{d_1} C_1 \times \operatorname{Pic}^{d_2} C_2$ of $\operatorname{Pic}^1 X$ are obviously projective, hence α cannot possibly be extended to a regular map from the whole of X to $\operatorname{Pic}^1 X$.

A second problem is the fact that $(1, 0) \equiv (0, 1)$; indeed $\#\Delta^1_X = 1$ (see (4)). Now if $\mathcal{X} \to B$ is a regular smoothing of X, any separated model of $\operatorname{Pic}^1_{\mathcal{X}_K}$ over B cannot contain components of $\operatorname{Pic} X$ corresponding to equivalent multidegrees (see below). So the target of the naive Abel map of the family fails to be separated.

A more satisfactory definition turns out to be 1.2 below; first some notation. Let $f : \mathcal{X} \to B$ be a regular smoothing of X, then $f_d : \mathcal{X}^d_B \to B$ denotes the d-th fibered power of $\mathcal{X}$ over B. The open subset of $\mathcal{X}^d_B$ where f_d is smooth is denoted by $\dot{\mathcal{X}}^d_B := \mathcal{X}^d_B \smallsetminus sing(f_d)$. Similarly, $\dot{X}^d := \{(p_1, \ldots, p_d) : p_i \in X \smallsetminus X_{\text{sing}}\}$ denotes the closed fiber of $\dot{\mathcal{X}}^d_B \to B$, where X_{sing} is the set of singular points of X.

Definition 1.2. A *d-th Abel map* for the curve X is a regular map $\beta : \dot{X}^d \longrightarrow \operatorname{Pic}^d X$ satisfying the following requirements.

(a) There exist a regular smoothing $f : \mathcal{X} \to B$ of X and a map $\beta_f : \dot{\mathcal{X}}^d_B \longrightarrow \operatorname{Pic}^d_f$ such that the restriction of β_f to the generic fiber is the d-th Abel map of $\mathcal{X}_K$ (i.e. $(\beta_f)_K = \alpha^d_K$ of (6)), and the restriction of β_f to the closed fiber is equal to β (i.e. $(\beta_f)_k = \beta$).

(b) If $\underline{d}, \underline{d}' \in \mathbb{Z}^\gamma$ are such that $\operatorname{Im} \beta \cap \operatorname{Pic}^{\underline{d}} X \neq \emptyset$ and $\operatorname{Im} \beta \cap \operatorname{Pic}^{\underline{d}'} X \neq \emptyset$, then $\underline{d} \not\equiv \underline{d}'$.

A d-th Abel map β is called *natural* if it is independent of the choice of f. More precisely, if for every regular smoothing f of X there exists a (necessarily unique) β_f as in (a), extending β.

Condition (b) will ensure that the image of an Abel map is contained in some separated model of Pic^d_K over B (cf. Example 1.1).

A concrete description of Abel maps will be given by Proposition 2.1. In Section 5 we shall relate our definition to others in the literature.

Although (see 2.2) every curve does have Abel maps, not all curves admit a natural one. Our main result, Theorem 1.5 characterizes in purely combinatorial terms those curves admitting a natural d-th Abel map. Before stating it we define the crucial combinatorial character. Denote $X^{\text{sep}}_{\text{sing}} \subset X_{\text{sing}}$ the set of *separating nodes* of X, i.e. $X^{\text{sep}}_{\text{sing}}$ is the set of nodes p of X such that $X \smallsetminus p$ is disconnected.

In the next definition, we use the notation of 1.0.1 and adopt the convention that if S is an empty set of integers, then $\inf\{n \in S\} = +\infty$.

Definition 1.3. The *essential connectivity* $\epsilon(X)$ of X is

$$\epsilon(X) := \inf\{k_Z,\ \forall Z :\ \emptyset \subsetneq Z \subsetneq X \text{ and } Z \cap Z' \cap X^{\text{sep}}_{\text{sing}} = \emptyset\}. \tag{8}$$

Equivalently

$$\epsilon(X) := \inf\{k_Z,\ \forall Z :\ \emptyset \subsetneq Z \subsetneq X \text{ and } Z \cap Z' \not\subset X^{\text{sep}}_{\text{sing}}\}. \tag{9}$$

Remark 1.4. So, if X is irreducible or of compact type then $\epsilon(X) = +\infty$.

An elementary arguments yields that to compute $\epsilon(X)$ it suffices to consider connected subcurves and that (8) and (9) are equivalent; we omit it since we shall only use version (9). An example: if $X = C_1 \cup C_2$ then either $\#(C_1 \cap C_2) = 1$ and $\epsilon(X) = +\infty$, or $\#(C_1 \cap C_2) \geq 2$ and $\epsilon(X) = \#(C_1 \cap C_2)$.

Theorem 1.5. *Let X be a nodal curve and d a positive integer. Then X admits a natural d-th Abel map if and only if $\epsilon(X) > d$.*

The proof of the theorem will be in Section 4.

2. Abel maps via Néron models

The goal of this section is to obtain a complete description of Abel maps, which will be done in Proposition 2.1. We shall use Néron models in the same spirit of [C05] (and of [E98] section 9). We refer to [BLR] chapter 9 or to [A86] section 1 for details.

Let $f : \mathcal{X} \to B$ be a regular smoothing of X; as we said in 1.0.3, if X is reducible Pic^d_f is not separated over B. To correct this we introduce the Néron model $\mathrm{N}(\operatorname{Pic}^d_K) \to B$ of its generic fiber Pic^d_K. This is a smooth, separated scheme over B uniquely determined by a universal property, the Néron mapping property ([A86] 1.1 or [BLR] 1.2.1). We shall denote

$$N^d_f := \mathrm{N}(\operatorname{Pic}^d_K).$$

Recall that there exists a unique surjective B-morphism

$$q_f : \operatorname{Pic}^d_f \longrightarrow N^d_f \tag{10}$$

which is the identity on the generic fiber (q_f is called "Ner" in [A86] diagram 1.21).

To describe the map (10), consider $\operatorname{Pic}^{\underline{d}}_f \subset \operatorname{Pic}^d_f$ the moduli scheme for degree-d line bundles having multidegree $\underline{d}$ on the closed fiber X of f. Then

$$\operatorname{Pic}^d_f = \frac{\coprod_{|\underline{d}|=d} \operatorname{Pic}^{\underline{d}}_f}{\sim_K}$$

where "$\sim_K$" denotes the natural gluing of the schemes $\operatorname{Pic}^{\underline{d}}_f$ along their generic fiber (which is the same for all $\underline{d}$: Pic^d_K). Similarly we have

$$\mathrm{N}^d_f \cong \frac{\coprod_{\delta \in \Delta^d_X} \operatorname{Pic}^{\delta}_f}{\sim_K} \tag{11}$$

where $\operatorname{Pic}^{\delta}_f := \operatorname{Pic}^{\underline{d}}_f$ for any representative $\underline{d}$ of δ. Such a definition does not depend on the choice of $\underline{d}$; in fact for every pair of equivalent multidegrees $\underline{d}, \underline{d}'$ there exists a unique isomorphism between $\operatorname{Pic}^{\underline{d}'}_f$ and $\operatorname{Pic}^{\underline{d}}_f$, determined by the unique $T \in \operatorname{Tw}_f X$ such that $\underline{\deg}\, T = \underline{d} - \underline{d}'$ (see 3.1 (ii)).

The map q_f restricted to $\operatorname{Pic}^{\underline{d}}_f \subset \operatorname{Pic}^d_f$ is the unique isomorphism $\operatorname{Pic}^{\underline{d}}_f \cong \operatorname{Pic}^{[\underline{d}]}_f$.

We denote N^d_X the closed fiber of $\mathrm{N}^d_f \to B$, which, if X is reducible, is the disjoint union of $\#\Delta^d_X$ copies of the generalized Jacobian of X.

Although N^d_f does not have good functorial properties, it has a geometric interpretation. Let $L, L' \in \operatorname{Pic} X$ and call them *f-twist equivalent* iff $L' \otimes L^{-1} \in \operatorname{Tw}_f X$. Then the fibers of $(q_f)_k : \operatorname{Pic}^d X \longrightarrow N^d_X$ are the classes of f-twist equivalent line bundles.

2.0.4. *Abel maps and Néron models.* To study Abel maps of singular curves, we use an approach analogous to [GIT] section 6, just like in [CE06]. With the notation introduced in 1.0.3, let $f:\mathcal{X}\to B$ be a regular smoothing of X. Consider the base change of f to $\dot{\mathcal{X}}^d_B\to B$, namely

$$\pi:\dot{\mathcal{X}}^d_B\times_B\mathcal{X}\longrightarrow\dot{\mathcal{X}}^d_B$$

so that π is the first projection. Define the "universal effective (Cartier) divisor" E_d on $\dot{\mathcal{X}}^d_B\times_B\mathcal{X}$ as the sum of the d natural sections $\sigma_1,\ldots,\sigma_d$ of π, that is, the sections $\sigma_i(p_1,\ldots,p_d)=((p_1,\ldots,p_d),p_i)$. Consider the moduli map of E_d, $\mu_{E_d}:\dot{\mathcal{X}}^d_B\to\operatorname{Pic}^d_f$ as defined in (5). By definition, its restriction to the generic fiber is its d-th Abel map, $\alpha^d_K:\mathcal{X}^d_K\to\operatorname{Pic}^d_K$, introduced in (6). Consider now the composition

$$\mathrm{N}(\alpha^d_K):\dot{\mathcal{X}}^d_B\stackrel{\mu_{E_d}}{\longrightarrow}\operatorname{Pic}^d_f\stackrel{q_f}{\longrightarrow}N^d_f \tag{12}$$

where the notation $\mathrm{N}(\alpha^d_K)$ is motivated as follows. To the map α^d_K one can apply the Néron mapping property: since $\dot{\mathcal{X}}^d_B$ is smooth over B and has $\mathcal{X}^d_K$ as generic fiber, there exists a unique extension of α^d_K to a morphism $\dot{\mathcal{X}}^d_B\to N^d_f$; such an extension, denoted $\mathrm{N}(\alpha^d_K)$, is unique and hence coincides with the map (12).

Finally, notice that if D is a Cartier divisor supported on the closed fiber of π (i.e. on $\dot{X}^d\times X$), the same construction, replacing E_d with E_d+D, gives again $\mathrm{N}(\alpha^d_K)$ (in other words: $\mathrm{N}(\alpha^d_K)=q_f\circ\mu_{E_d+D}=q_f\circ\mu_{E_d}$).

To use the morphism (12) we shall fix a choice of representatives for the multidegree classes, given by a map

$$\mathrm{r}:\Delta^d_X\longrightarrow\mathbb{Z}^\gamma\ \text{ such that }\ |\,\mathrm{r}(\delta)|=d\ \text{ and }\ [\mathrm{r}(\delta)]=\delta,\ \forall\delta\in\Delta^d_X.$$

Then there exists a unique isomorphism

$$\iota_{\mathrm{r}}:N^d_f\stackrel{\cong}{\longrightarrow}\frac{\coprod_{\delta\in\Delta^d_X}\operatorname{Pic}^{\mathrm{r}(\delta)}_f}{\sim_K}=:N^{d,\mathrm{r}}_f \tag{13}$$

So $N^{d,\mathrm{r}}_f$ defined above is a subset of Pic^d_f. Restricting $q_f:\operatorname{Pic}^d_f\to N^d_f$ to it we have $id_{N^{d,\mathrm{r}}_f}=\iota_{\mathrm{r}}\circ(q_f)_{|N^{d,\mathrm{r}}_f}$. The closed fiber of $N^{d,\mathrm{r}}_f$ is denoted

$$N^{d,\mathrm{r}}_X:=(N^{d,\mathrm{r}}_f)_k=\coprod_{\delta\in\Delta^d_X}\operatorname{Pic}^{\mathrm{r}(\delta)}X\subset\operatorname{Pic}^d X$$

and obviously does not depend on f. Now compose the map $\mathrm{N}(\alpha^d_K)$ of (12) with ι_{r} and call the composition $\alpha^{d,\mathrm{r}}_f$

$$\alpha^{d,\mathrm{r}}_f:\dot{\mathcal{X}}^d\stackrel{\mathrm{N}(\alpha^d_K)}{\longrightarrow}N^d_f\stackrel{\iota_{\mathrm{r}}}{\longrightarrow}N^{d,\mathrm{r}}_f. \tag{14}$$

Restricting to the closed fiber we get a d-th Abel map $\alpha^{d,\mathrm{r}}_{f,X}:\dot{X}^d\longrightarrow N^{d,\mathrm{r}}_X$.

To complete the picture, focus on the multidegrees that are attained by the divisor E_d restricted to the fibers of $\dot{\mathcal{X}}^d_B\times_B\mathcal{X}\to\dot{\mathcal{X}}^d_B$. We shall name them *partitional multidegrees* and denote their set $\operatorname{Part}(d,\gamma)$:

$$\operatorname{Part}(d,\gamma):=\{\underline{d}\in\mathbb{Z}^\gamma:|\underline{d}|=d\text{ and }\ d_i\geq 0\ \forall i=1,\ldots\gamma\}$$

Then the map $\alpha^{d,\mathrm{r}}_f$ factors

$$\alpha^{d,\mathrm{r}}_f:\dot{\mathcal{X}}^d\ \stackrel{\mu_{E_d}}{\longrightarrow}\ \frac{\coprod_{\underline{d}\in\operatorname{Part}(d,\gamma)}\operatorname{Pic}^{\underline{d}}_f}{\sim_K}\stackrel{q_f}{\longrightarrow}N^d_f\stackrel{\iota_{\mathrm{r}}}{\longrightarrow}N^{d,\mathrm{r}}_f \tag{15}$$

where, abusing notation, q_f is the restriction of $q_f:\operatorname{Pic}^d_f\longrightarrow N^d_f$. Therefore the image of $\alpha^{d,\mathrm{r}}_{f,X}$ is entirely contained in the union of the components of $N^{d,\mathrm{r}}_X$ that correspond to those multidegree classes containing some partitional representative.

The conclusion of the preceeding discussion is stated in part (i) of 2.1 below, where we also classify Abel maps as defined in 1.2.

Proposition 2.1. *(i) Let $f : \mathcal{X} \to B$ be a regular smoothing of X, d a positive integer and $\mathrm{r} : \Delta_X^d \longrightarrow \mathbb{Z}^\gamma$ a choice of representatives. Then there exists a unique morphism $\alpha_f^{d,\mathrm{r}} : \dot{\mathcal{X}}_B^d \longrightarrow N_f^{d,\mathrm{r}}$ whose restriction to the the generic fiber is the d-th Abel map of $\mathcal{X}_K$. The restriction of $\alpha_f^{d,\mathrm{r}}$ to the closed fiber is the d-th Abel map*

$$\alpha_{f,X}^{d,\mathrm{r}} : \dot{X}^d \longrightarrow \coprod_{\underline{d}\in \mathrm{Part}(d,\gamma)} \mathrm{Pic}^{\mathrm{r}([\underline{d}])} X \subset N_X^{d,\mathrm{r}}. \tag{16}$$

(ii) Conversely, every d-th Abel map of X equals $\alpha_{f,X}^{d,\mathrm{r}}$ for some f and r.

Proof. The first part has been proved before the statement. For the second, let $\beta : \dot{X}^d \to \mathrm{Pic}^d X$ be an Abel map as defined in 1.2. Then there exists a regular smoothing $f : \mathcal{X} \to B$ and a morphism $\beta_f : \dot{\mathcal{X}}_B^d \to \mathrm{Pic}_f^d$ extending β and restricting to the d-th Abel map α_K^d of $\mathcal{X}_K$ on the generic fiber.

Consider the composition $q_f \circ \beta_f : \dot{\mathcal{X}}_B^d \longrightarrow \mathrm{Pic}_f^d \longrightarrow N_f^d$. We claim that

$$q_f \circ \beta_f = \mathrm{N}(\alpha_K^d) \tag{17}$$

where $\mathrm{N}(\alpha_K^d)$ is defined in (12). Indeed the two maps $q_f \circ \beta_f$ and $\mathrm{N}(\alpha_K^d)$ coincide on the generic fiber $\mathcal{X}_K^d$ and hence, by the uniqueness part in the Néron mapping property, they are equal.

Now define $\mathrm{r} : \Delta_X^d \to \mathbb{Z}^\gamma$ as follows; pick $\delta \in \Delta_X^d$. If there exists a representative $\underline{d}^\delta$ for δ such that $\mathrm{Im}\,\beta \cap \mathrm{Pic}^{\underline{d}^\delta} X \neq \emptyset$, then such a $\underline{d}^\delta$ is unique by condition (b) of Definition 1.2; hence we can define $\mathrm{r}(\delta) = \underline{d}^\delta$. If instead there is no such representative for δ, we define $\mathrm{r}(\delta)$ however we like.

By construction

$$\mathrm{Im}\,\beta_f \subset \frac{\coprod_{\delta\in\Delta_X^d} \mathrm{Pic}_f^{\mathrm{r}(\delta)}}{\sim_K} = N_f^{d,\mathrm{r}}$$

Now consider the isomorphism ι_r (see (13)) $\iota_\mathrm{r} : N_f^d \stackrel{\cong}{\longrightarrow} N_f^{d,\mathrm{r}} \subset \mathrm{Pic}_f^d$.

Recall that $\iota_\mathrm{r} \circ (q_f)_{|N_f^{d,\mathrm{r}}} = id_{N_f^{d,\mathrm{r}}}$. Therefore

$$\beta_f = id_{N_f^{d,\mathrm{r}}} \circ \beta_f = \iota_\mathrm{r} \circ (q_f)_{|N_f^{d,\mathrm{r}}} \circ \beta_f = \iota_\mathrm{r} \circ \mathrm{N}(\alpha_K^d)$$

using (17) for the last equality. Now $\iota_\mathrm{r} \circ \mathrm{N}(\alpha_K^d) = \alpha_f^{d,\mathrm{r}}$ by definition (see (14)), therefore $\beta = (\beta_f)_k = \alpha_{f,X}^{d,\mathrm{r}}$ and we are done. ∎

Remark 2.2. We get that, for all d, every curve X has Abel maps, infinitely many of them if X is reducible (at least one for every r).

As a consequence of 2.1, to prove Theorem 1.5 it suffices to study the maps $\alpha_{f,X}^{d,\mathrm{r}}$. If r is fixed, the domain and the target $\coprod_{\underline{d}\in\mathrm{Part}(d,\gamma)} \mathrm{Pic}^{\mathrm{r}([\underline{d}])} X$ of $\alpha_{f,X}^{d,\mathrm{r}}$ do not depend on f, whereas the map itself does. In fact $\alpha_{f,X}^{d,\mathrm{r}}$ factors through certain isomorphisms $\mathrm{Pic}^{\underline{d}} X \to \mathrm{Pic}^{\mathrm{r}([\underline{d}])} X$; each of these isomorphisms is given by tensor product by the (unique) $T \in \mathrm{Tw}_f X$ of multidegree $\mathrm{r}([\underline{d}]) - \underline{d}$. The crux of the matter is that T may change as f varies (even though its multidegree and support do not change), in which case the Abel map $\alpha_{f,X}^{d,\mathrm{r}}$ will not be natural.

3. Spaces of twisters

The goal of this section is to characterize twisters that depend only on their support, hence only on their multidegree (see 3.1), and not on the regular smoothing defining them (see 3.10). Recall the set up of 1.0.2; we shall need the following well known facts, see for example [A86] p.220 diagram 1.21. The notation is in 1.0.2.

Lemma 3.1. *(i) The map* $\underline{\deg} : \overline{\mathcal{D}(X)} \longrightarrow \Lambda_X$ *is an isomorphism*

(ii) For any regular smoothing f *of* X *and any* $T, T' \in \operatorname{Tw}_f X$ *we have* $T = T' \Leftrightarrow \underline{\deg}\, T = \underline{\deg}\, T'$*. In particular* $\operatorname{Tw}_f X \cong \overline{\mathcal{D}(X)}$

Remark 3.2. For any $D \in \mathcal{D}(X)$ and $f : \mathcal{X} \longrightarrow B$ regular smoothing of X, there exists a unique $T \in \operatorname{Tw}_f X$ such that $Supp(T) = \overline{D}$. We will denote by D_f such a T, that is $D_f \cong \mathcal{O}_{\mathcal{X}}(D)_{|X}$.

Denote by $\underline{t} := \underline{\deg}\, D$; the lemma above implies that the set $\operatorname{Tw}_f X$ contains a unique element of multidegree $\underline{t}$. By contrast the set

$$\operatorname{Tw}^{\underline{t}} X := \{T \in \operatorname{Tw} X : \underline{\deg}\, T = \underline{t}\} = \{D_f,\ \forall f \text{ reg. sm. of } X\} \tag{18}$$

may be quite large. If $\# \operatorname{Tw}^{\underline{t}} X = 1$, i.e. if $D_f = D_{f'}$ for all regular smoothings f, f' of X, we will say that D_f *does not depend on the choice of* f.

3.0.5. *Level curves.* Let $D = \sum n_i C_i$, we can write

$$D = \sum_{m \in \mathbb{Z}} m D_m \tag{19}$$

with $D_m := \cup_{n_i = m} C_i$ (in particular $D_0 := \sum_{n_i=0} C_i$) so that the D_m are possibly empty subcurves of X having no components in common, and such that $\cup_{m\in\mathbb{Z}} D_m = X$. We call (19) the *level expression* of D and the non-empty curves D_m the *level curves* of D; of course they are uniquely determined.

Notice also that for any $n \in \mathbb{Z}$ the level curves of D and of $D + nX$ are the same, hence it makes sense to speak of level curves of a class $\overline{D} \in \overline{\mathcal{D}(X)}$. We can also define level curves of a twister $T \in \operatorname{Tw} X$ as the level curves of its support, $Supp(T) \in \overline{\mathcal{D}(X)}$. Similarly, for any $\underline{t} \in \Lambda_X$ the level curves of $\underline{t}$ can be defined via the isomorphism $\overline{\mathcal{D}(X)} \cong \Lambda_X$ of 3.1. We need the following

Lemma 3.3. *Let* $\underline{t} \in \Lambda_X$ *with* $\underline{t} \neq 0$*. There exists a unique* $D(\underline{t}) \in \mathcal{D}(X)$*, with* $\underline{\deg}\, D(\underline{t}) = \underline{t}$*, admitting an expression*

$$D(\underline{t}) = \sum_{1}^{\ell(\underline{t})} m_h Z_h(\underline{t}) \tag{20}$$

where $\ell(\underline{t})$*,* m_h *and* $Z_h(\underline{t})$ *are uniquely determined by the following properties.*

(a) $\ell(\underline{t}) \geq 1$ *and* $m_h \in \mathbb{Z}$ *with* $0 < m_1 < \ldots < m_{\ell(\underline{t})}$*;*

(b) the $Z_h(\underline{t})$ *are subcurves of* X *having no components in common;*

(c) the curve $Z_0(\underline{t}) := \overline{X \smallsetminus \cup_1^{\ell(\underline{t})} Z_h(\underline{t})}$ *is not empty.*

Moreover, on every subcurve $Y \subseteq Z_0(\underline{t})$ *we have*

$$|\underline{t}_Y| = \deg_Y D(\underline{t}) \geq -m_1 (Y \cdot Z_0(\underline{t})) \geq 0. \tag{21}$$

In particular, if $Y = Z_0(\underline{t})$ *we have* $|\underline{t}_Y| \geq m_1 k_{Z_0(\underline{t})} > 0$*.*

Proof. Pick any $D \in \mathcal{D}(X)$ with $\underline{\deg}\, D = \underline{t}$ and consider the level expression of D in (19). It has a finite number of nonzero summands, so let m_0 be the minimum integer for which D_{m_0} is not empty. Set $D(\underline{t}) := D - m_0 X$. Then the level expression of $D(\underline{t})$ can be written as in (20) and satisfies the conditions (a), (b),

(c) in the statement; note that $D_{m_0} = Z_0(\underline{t})$. We need to check condition (21). So let $Y \subseteq Z_0(\underline{t})$, then

$$|\underline{t}_Y| = (Y \cdot D(\underline{t})) = \sum_{1}^{\ell(\underline{t})} m_h (Y \cdot Z_h(\underline{t}));$$

now for every $h \geq 1$ we have $(Y \cdot Z_h(\underline{t})) \geq 0$ (because Y lies in the complementary curve of $Z_h(\underline{t})$). Therefore, since $m_1 \leq m_h$ if $h \geq 1$, we get

$$|\underline{t}_Y| \geq m_1 \sum_{1}^{\ell(\underline{t})} (Y \cdot Z_h(\underline{t})) = m_1 (Y \cdot \cup_1^{\ell(\underline{t})} Z_h(\underline{t})) = -m_1 (Y \cdot Z_0(\underline{t})) \geq 0$$

indeed the $Z_h(\underline{t})$ have no common components and $\cup_1^{\ell(\underline{t})} Z_h(\underline{t})$ is a reduced proper subcurve of X whose complement is, by definition, $Z_0(\underline{t})$. If $Y = Z_0(\underline{t})$ the last inequality is strict, so we are done. ■

Let $D \in \mathcal{D}(X)$; to the level expression $D = \sum_{m \in \mathbb{Z}} m D_m$ of (19) we can naturally associate a set of nodes $S(D)$ as follows.

(22) $$S(D) := \cup_{m \neq m'} (D_m \cap D_{m'}) \subset X_{\text{sing}}$$

Similarly, let $\underline{t} \in \Lambda_X$ and consider $D(\underline{t}) = \sum_1^{\ell(\underline{t})} m_h Z_h(\underline{t})$ of Lemma 3.3. Denote

(23) $$S(\underline{t}) := S(D(\underline{t})) = \bigcup_{0 \leq h < h' \leq \ell(\underline{t})} (Z_h(\underline{t}) \cap Z_{h'}(\underline{t})) \subset X_{sing}$$

Remark 3.4. For $D, D' \in \mathcal{D}(X)$ we have $S(D + D') \subset S(D) \cup S(D')$.

Remark 3.5. Fix $D \in \mathcal{D}(X)$ and let $S = S(D)$. Denote by $\nu_S : X_S^\nu \to X$ the normalization of X at the points in S. Then X_S^ν is the disjoint union of the level curves of D, i.e. $X_S^\nu = \coprod_{m \in \mathbb{Z}} D_m$.

For any regular smoothing $f : \mathcal{X} \to B$ of X consider the twister $D_f = \mathcal{O}_{\mathcal{X}}(D)_{|X}$ and its pull-back $\nu_S^*(D_f)$ to X_S^ν. Observe that $\nu_S^*(D_f)$ does not depend on the choice of f; indeed if f' is another regular smoothing of X, then D_f and $D_{f'}$ have the same restrictions to every curve D_m, hence their pull-backs to X_S^ν coincide. Therefore we can use the following non-ambiguous notation

(24) $$\mathcal{O}_{X_S^\nu}(D) := \nu_S^*(D_f) = \nu_S^* \mathcal{O}_{\mathcal{X}}(D)_{|X}$$

for any $f : \mathcal{X} \to B$ as above.

The following result due to Esteves and Medeiros, in [EM02], characterizes twisters among all line bundles on a nodal curve X.

Corollary 6.9 in [EM02] **p.297** *Fix X a nodal curve, $D \in \mathcal{D}(X)$ and $S = S(D)$; let $N \in \operatorname{Pic} X$. If $\nu_S^* N \cong \mathcal{O}_{X_S^\nu}(D)$ there exists a regular smoothing f of X such that $N \cong D_f$.*

As we mentioned, the converse also holds. The language used in [EM02], section 6, is different from ours; here is a small dictionary. Our D is $\tau_1 C_1 + \ldots + \tau_m C_m$ in 6.9 of [EM02]. Υ in [EM02] is the set of irreducible components of the curve. The partition $\mathcal{P}$ of Υ corresponds to our level expression of D so that a subset I of $\mathcal{P}$ corresponds to a non empty level curve D_m. Condition 6.9.1 is $\nu_S^* N \cong \mathcal{O}_{X_S^\nu}(D)$ above. Finally, [EM02] assumes characteristic 0, which is not needed for the proof given to this result.

3.0.6. *A useful graph.* Mantaining the hypothesis and notation above, we now introduce the graph $\Gamma(S)$, whose vertices are the connected components of X_S^ν and whose edges correspond to S. An edge e joins the two vertices corresponding to the two components passing through the node represented by e. So, $\Gamma(S)$ is the connected graph obtained from the standard dual graph of X by contracting to a point all the edges corresponding to nodes not in S. Let $b(S) = b_1(\Gamma(S), \mathbb{Z})$ be its first Betti number, so that

$$b(S) = \#S + 1 - \#(\text{connected components of } X_S^\nu).$$

Corollary 3.6. *Let $\underline{t} \in \Lambda_X$ and $S = S(\underline{t})$. Then there are bijections*

$$\operatorname{Tw}^{\underline{t}} X \leftrightarrow (\nu_S^*)^{-1}(\mathcal{O}_{X_S^\nu}) \leftrightarrow (k^*)^{b(S)}$$

where $\operatorname{Pic} X \xrightarrow{\nu_S^*} \operatorname{Pic} X_S^\nu$ *is the pull-back map.*

Proof. As observed in 3.5 (also in 6.9 of [EM02]) we have an injection $\operatorname{Tw}^{\underline{t}} X \hookrightarrow (\nu_S^*)^{-1}(\mathcal{O}_{X_S^\nu})$ for any fixed f, mapping $D_{f'} \in \operatorname{Tw}^{\underline{t}} X$ to $D_f \otimes D_{f'}^{-1}$. Surjectivity of such an injection follows from 6.9 of [EM02].

The second bijection (well known), follows from the exact sequence

$$1 \longrightarrow (k^*)^{b(S)} \longrightarrow \operatorname{Pic} X \xrightarrow{\nu_S^*} \operatorname{Pic} X_S^\nu \longrightarrow 1.$$

■

Definition 3.7. Let $Q \subset X$ be a (connected) complete subcurve. We say that Q is a *tail* of X if $Q \cap Q'$ is a separating node of X; we say that the node $Q \cap Q'$ *generates* the tails Q and Q'.

Let $D \in \mathcal{D}(X)$; we say that D is a *sum of tails* if there is an expression $D = \sum m_i Q_i + nX$ where the Q_i are tails of X.

Let $\overline{D} \in \overline{\mathcal{D}(X)}$, then $\overline{D}$ is a *sum of tails* if any of its representative in $\mathcal{D}(X)$ is. Let $T \in \operatorname{Tw} X$, we say that T is a *sum of tails* if $Supp(T)$ is.

Remark 3.8. The set $\overline{\mathcal{D}(X)}^0$ of elements in $\overline{\mathcal{D}(X)}$ that are sums of tails is clearly a subgroup. Let Λ_X^0 be its image via the multidegree map:

$$\Lambda_X^0 := \underline{\deg}(\overline{\mathcal{D}(X)}^0) \subset \Lambda_X. \tag{25}$$

Thus Λ_X^0 is the group of multidegrees of sums of tails.

Recall that $X_{\text{sing}}^{\text{sep}}$ is the set of separating nodes of X. If $r \in X_{\text{sing}}^{\text{sep}}$ and Q is one of the two tails generated by r, it is clear that $S(Q) = \{r\}$. More generally, we have

Lemma 3.9. *Let $D \in \mathcal{D}(X)$. D is a sum of tails if and only if $S(D) \subset X_{sing}^{sep}$. (Equivalently: let $\underline{t} \in \Lambda_X$. $\underline{t} \in \Lambda_X^0 \Leftrightarrow S(\underline{t}) \subset X_{sing}^{sep}$)*

Proof. We can work modulo adding to D a multiple of X. We can assume that $D \neq nX$, otherwise the statement is obvious since $S(D) = \emptyset$.

Let $D = \sum_1^l m_i Q_i$ be a sum of tails and let us use induction on l to prove that $S(D) \subset X_{\text{sing}}^{\text{sep}}$. If $l = 1$ then $D = mQ$ with Q a tail, hence $S(D) = Q \cap Q' = \{r\}$ where r is the separating node generating Q.

If $l > 1$ write $D = \sum_1^{l-1} m_i Q_i + m_l Q_l$; by 3.4 we have

$$S(D) \subset S(\sum_1^{l-1} m_i Q_i) \cup S(m_l Q_l);$$

by induction, each of the two sets on the right lies in $X_{\text{sing}}^{\text{sep}}$ so we are done.

Conversely, assume that $S(D) \subset X_{\text{sing}}^{\text{sep}}$. Consider the level expression $D = \sum_{m \in \mathbb{Z}} m D_m$ defined in (19); then every subcurve D_m meets its complement D'_m in separating nodes of X, by (22). Obviously it suffices to prove that every connected component of D_m is a sum of tails. To do that, it suffices to prove that if Z is a connected subcurve of X such that $Z \cap Z' \subset X^{\text{sep}}_{\text{sing}}$, then either Z or Z' is a union of tails.

Use induction on $\# Z \cap Z'$. If $\# Z \cap Z' = 1$ then Z and Z' are both tails. Suppose $\# Z \cap Z' > 1$ and let $r \in Z \cap Z'$. Since Z is connected Z is contained in one of the two tails Q_r and Q'_r generated by r, say $Z \subset Q_r$. Let $Y := \overline{Q_r \smallsetminus Z}$ so that $Z' = Y \cup Q'_r$; then $Z \cap Y$ is made of separating nodes for Q_r and of course $\# Z \cap Y = \# Z \cap Z' - 1$. Therefore we can use induction and conclude that either Z or Y is a union of tails of Q_r. If Y is a union of tails of Q_r then $Y \cup Q'_r = Z'$ is a union of tails of X and we are done. If Z is a union of tails of Q_r then Z is actually a tail in Q_r (Z is connected) hence Y is also a tail of Q_r and hence (arguing as before) Z' is a union of tails of X. ■

Corollary 3.10. *Fix the curve X and $\underline{t} \in \Lambda_X$. Then*

$$\# \operatorname{Tw}^{\underline{t}} X = 1 \Leftrightarrow S(\underline{t}) \subset X^{sep}_{sing} \Leftrightarrow \underline{t} \in \Lambda^0_X.$$

(Equivalently, with the terminology of 3.2: let $D \in \mathcal{D}(X)$; then D_f does not depend on f if and only if $S(D) \subset X^{sep}_{sing}$ if and only if D is a sum of tails.)

Proof. The second equivalence is 3.9, so we just need to prove the first. Fix $D \in \mathcal{D}(X)$ such that $\underline{\deg} D = \underline{t}$ and let $S := S(\underline{t}) = S(D)$.

By Corollary 3.6 we have $\# \operatorname{Tw}^{\underline{t}} X = 1$ if and only if $b(S) = 0$ where $b(S)$ is the first Betti number of the graph $\Gamma(S)$ defined in 3.0.6. Therefore $b(S) = 0$ if and only if $\Gamma(S)$ is a tree if and only if $S \subset X^{\text{sep}}_{\text{sing}}$. ■

4. Proof of the main Theorem

Fix a curve X, then by Proposition 2.1 every Abel map of X is of type $\alpha^{d,\mathrm{r}}_{f,X} : \dot{X}^d \to \operatorname{Pic}^d X$, for some r and f. To say that $\alpha^{d,\mathrm{r}}_{f,X}$ is natural (i.e. independent of the choice of f) is to say that for every regular smoothing f' of X we have $\alpha^{d,\mathrm{r}}_{f,X} = \alpha^{d,\mathrm{r}}_{f',X}$. We begin with a preliminary characterization.

Lemma 4.1. *The map $\alpha^{d,\mathrm{r}}_{f,X}$ is natural if and only if $\underline{d} - \mathrm{r}([\underline{d}]) \in \Lambda^0_X$ for every $\underline{d} \in \operatorname{Part}(d, \gamma)$.*

Proof. Let us revisit the factorization (15) of $\alpha^{d,\mathrm{r}}_f$ by writing it

$$\alpha^{d,\mathrm{r}}_f : \dot{\mathcal{X}}^d \stackrel{\mu}{\longrightarrow} \frac{\coprod_{\underline{d} \in \operatorname{Part}(d,\gamma)} \operatorname{Pic}^{\underline{d}}_f}{\sim_K} \stackrel{\cong}{\longrightarrow} \frac{\coprod_{\underline{d} \in \operatorname{Part}(d,\gamma)} \operatorname{Pic}^{\mathrm{r}([\underline{d}])}_f}{\sim_K}$$

where $\mu = \mu_{E_d}$ is the moduli map of the divisor $E_d \subset \dot{\mathcal{X}}^d \times_B \mathcal{X}$, and the isomorphism is the restriction of $\iota_{\mathrm{r}} \circ q_f$.

The restriction of μ to the closed fiber $\dot{X}^d$ is fixed, i.e. independent of f, indeed if $p_1, \ldots, p_d \in \dot{X}$ then $\mu(p_1, \ldots, p_d) = \mathcal{O}_X(\sum p_i)$. We obtain that $\alpha^{d,\mathrm{r}}_{f,X}$ is independent of f if and only if for every $\underline{d} \in \operatorname{Part}(d, \gamma)$ the isomorphism $\operatorname{Pic}^{\underline{d}} X \longrightarrow \operatorname{Pic}^{\mathrm{r}([\underline{d}])} X$ is independent of f. This map is given by tensor product by that twister $T \in \operatorname{Tw}_f X$ whose multidegree satisfies $\underline{\deg} T = \underline{d} - \mathrm{r}([\underline{d}])$; set $\underline{t} := \underline{d} - \mathrm{r}([\underline{d}])$. Recall that T is uniquely determined by f and $\underline{t}$ (see lemma 3.1 part (ii)).

We obtain that $\alpha^{d,\mathrm{r}}_{f,X}$ does not depend on the choice of f if and only if T is the same for every f, i.e. if and only if $\operatorname{Tw}^{\underline{t}} X = \{T\}$. We conclude by Corollary 3.10, which tells us that $\operatorname{Tw}^{\underline{t}} X = \{T\}$ if and only if $\underline{t} \in \Lambda^0_X$. ■

Remark 4.2. Thus, for any fixed d the set of natural d-th Abel maps is either empty or in bijective correspondence with $(\Lambda^0_X)^{\# \operatorname{Part}(d,\gamma)}$. So, if a natural d-th Abel map for X exists, it is unique if and only if X is free from separating nodes.

Proof of Theorem 1.5. We first prove that, if $\epsilon(X) > d$, then X admits a natural d-th Abel map, which is the harder part. Define a choice of representatives r as follows. Pick $\delta \in \Delta^d_X$; if δ does not contain any partitional representative, the way in which we define $\mathrm{r}(\delta)$ does not matter. If instead δ contains some partitional representative, we choose one of them, call it $\underline{d}^\delta$, and define $\mathrm{r}(\delta) = \underline{d}^\delta$. We claim that the Abel map $\alpha^{d,\mathrm{r}}_{f,X}$ is natural, i.e. it does not depend on f. To prove that it suffices to show that for every pair $\underline{d}, \underline{d}' \in \mathrm{Part}(d,\gamma)$ such that $\underline{d} \equiv \underline{d}'$ we have

$$\underline{d} - \underline{d}' \in \Lambda^0_X. \tag{26}$$

Indeed, for any $\underline{d} \in \mathrm{Part}(d,\gamma)$, denoting by $\delta \in \Delta^d_X$ its class, by (26) we get $\underline{d} - \mathrm{r}(\delta) = \underline{d} - \underline{d}^\delta \in \Lambda^0_X$. Hence, by 4.1, $\alpha^{d,\mathrm{r}}_{f,X}$ does not depend on f.

Let $\underline{t} := \underline{d} - \underline{d}'$ and consider $S(\underline{t})$ (see (23)). By 3.9, (26) is equivalent to

$$S(\underline{t}) \subset X^{\mathrm{sep}}_{\mathrm{sing}}. \tag{27}$$

In conclusion: we reduced ourselves to prove the following statement.
($\bullet$) *Let $\underline{t} \in \Lambda_X$ and assume that there exist $\underline{d}, \underline{d}' \in \mathrm{Part}(d,\gamma)$ such that $\underline{t} = \underline{d} - \underline{d}'$. Then $S(\underline{t}) \subset X^{sep}_{sing}$.*

We can of course assume that $\underline{t} \neq 0$. Let us introduce $D(\underline{t}) = \sum_1^{\ell(\underline{t})} m_h Z_h(\underline{t})$ described in lemma 3.3; then $S(\underline{t}) = S(D(\underline{t}))$. We shall prove ($\bullet$) using induction on $\ell(\underline{t})$. First, we simplify the notation by writing $Z_0 := Z_0(\underline{t})$; recall that Z_0, defined in 3.3 part (c), is not empty.

Now on with the induction: assume $\ell(\underline{t}) = 1$. Then $D(\underline{t}) = m_1 Z_1(\underline{t})$ with $m_1 > 0$ and $Z'_0 = Z_1(\underline{t})$.

By contradiction assume that $S(\underline{t}) \not\subset X^{\mathrm{sep}}_{\mathrm{sing}}$; then $Z_0 \cap Z'_0 \not\subset X^{\mathrm{sep}}_{\mathrm{sing}}$. Therefore, by the definition of essential connectivity (1.3 form (9)) we have

$$k_{Z_0} \geq \epsilon(X).$$

On the other hand by Lemma 3.3 (21) we have

$$|\underline{t}_{Z_0}| \geq k_{Z_0}.$$

Combining these two inequalities with the hypothesis $\epsilon(X) > d$ we get

$$|\underline{t}_{Z_0}| > d.$$

Restricting the equality $\underline{t} = \underline{d} - \underline{d}'$ to Z_0 and applying the above relation we obtain

$$|\underline{d}'_{Z_0}| = |\underline{d}_{Z_0}| - |\underline{t}_{Z_0}| < |\underline{d}_{Z_0}| - d.$$

Now $|\underline{d}_{Z_0}| \leq d$ because $\underline{d} \in \mathrm{Part}(d,\gamma)$; hence $|\underline{d}'_{Z_0}| < 0$, which is in contradiction with the fact that $\underline{d}' \in \mathrm{Part}(d,\gamma)$. This concludes the proof of the case $\ell(\underline{t}) = 1$.

Assume now that $\ell(\underline{t}) \geq 2$. Again by contradiction suppose that $S(\underline{t}) \not\subset X^{\mathrm{sep}}_{\mathrm{sing}}$. If, as in the preceeding case, $Z_0 \cap Z'_0 \not\subset X^{\mathrm{sep}}_{\mathrm{sing}}$, we can argue exactly as before to obtain a contradiction. So, suppose that $Z_0 \cap Z'_0$ is made of separating nodes; then $\underline{\deg}\, Z_0 \in \Lambda^0_X$ by 3.9. Set

$$\underline{u} := \underline{t} + \underline{\deg}\, m_1 Z_0;$$

then (Λ^0_X is a group) $\underline{u} \in \Lambda^0_X$ if and only if $\underline{t} \in \Lambda^0_X$. By 3.9 this is equivalent to saying that

$$S(\underline{u}) \subset X^{\mathrm{sep}}_{\mathrm{sing}} \Leftrightarrow S(\underline{t}) \subset X^{\mathrm{sep}}_{\mathrm{sing}}.$$

To reach a contradiction we shall prove that $S(\underline{u}) \subset X^{\mathrm{sep}}_{\mathrm{sing}}$. We have

$$\underline{u} = \underline{\deg}(m_1(Z_0 + Z_1(\underline{t})) + m_2 Z_2(\underline{t}) + \ldots m_{\ell(\underline{t})} Z_{\ell(\underline{t})}).$$

Therefore Lemma 3.3 applied to $\underline{u}$ gives that $Z_0(\underline{u}) = Z_0 + Z_1(\underline{t})$ and

$$D(\underline{u}) = (m_2 - m_1) Z_1(\underline{u}) + \ldots + (m_l - m_1) Z_{\ell(\underline{t})-1}(\underline{u}).$$

So $\ell(\underline{u}) = \ell(\underline{t}) - 1$. To be able to apply the induction hypothesis to $\underline{u}$ and conclude that $S(\underline{u}) \subset X_{\text{sing}}^{\text{sep}}$ we need to express $\underline{u}$ as the difference of two elements in $\text{Part}(d, \gamma)$. To do this let

$$\underline{d}'' := \underline{d} + \underline{\deg}\, m_1 Z_0 = \underline{d}' + \underline{t} + \underline{\deg}\, m_1 Z_0 = \underline{d}' + \underline{u}.$$

We claim that $\underline{d}'' \in \text{Part}(d, \gamma)$. In fact, let C be an irreducible component of X, then of course

$$\underline{d}''_C = \underline{d}_C + m_1(C \cdot Z_0)$$

and we need to show that $\underline{d}''_C \geq 0$. If $C \not\subset Z_0$ both summands on the right are nonnegative (recall that $\underline{d} \in \text{Part}(d, \gamma)$) so we are ok.

If $C \subset Z_0$, by (21) in lemma 3.3 we have that $\underline{t}_C \geq -m_1(C \cdot Z_0)$ hence

$$\underline{d}_C = \underline{d}'_C + \underline{t}_C \geq -m_1(C \cdot Z_0)$$

(as $\underline{d}' \in \text{Part}(d, \gamma)$). Hence

$$\underline{d}''_C = \underline{d}_C + m_1(C \cdot Z_0) \geq -m_1(C \cdot Z_0) + m_1(C \cdot Z_0) \geq 0.$$

This shows that $\underline{d}'' \in \text{Part}(d, \gamma)$. Since $\underline{u} = \underline{d}'' - \underline{d}'$ we can apply the induction hypothesis to $\underline{u}$ and obtain $S(\underline{u}) \subset X_{\text{sing}}^{\text{sep}}$. We are done with one half of the proof.

Now we prove the converse, so assume that X admits a natural d-th Abel map. By 2.1 this means that there exists r such that $\alpha_{f,X}^{d,\text{r}}$ is independent of f. By 4.1, $\forall \underline{d} \in \text{Part}(d, \gamma)$ we have that $\underline{d} - \text{r}([\underline{d}]) \in \Lambda_X^0$ and hence, for every $\underline{d}, \underline{d}' \in \text{Part}(d, \gamma)$ such that $\underline{d} \equiv \underline{d}'$

$$\underline{d} - \underline{d}' \in \Lambda_X^0 \tag{28}$$

(as $\underline{d} - \underline{d}' = \Big(\underline{d} - \text{r}([\underline{d}])\Big) + \Big(\text{r}([\underline{d}']) - \underline{d}'\Big)$ and both summands $\Big(\ldots\Big)$ lie in Λ_X^0).

To prove that $\epsilon(X) > d$ it suffices to show that if Z is a subcurve of X such that $k_Z \leq d$, then

$$Z \cap Z' \subset X_{\text{sing}}^{\text{sep}} \tag{29}$$

(using definition (9) of $\epsilon(X)$). So let Z be such a curve; up to renaming the components of X we have $Z = \cup_{h+1}^{\gamma} C_i$ where $1 \leq h < \gamma$. Thus $k_Z = \sum_1^h (C_i \cdot Z)$.

By the assumption that $k_Z \leq d$ there exist h integers $d_1, \ldots, d_h$ such that $d_i \geq (C_i \cdot Z)$ for all $i = 1 \ldots, h$, and such that $\sum_1^h d_i = d$. We have

$$(Z \cdot C_i) \begin{cases} \leq d_i & \text{if } i \leq h \ \ \text{(by definition)} \\ \leq 0 & \text{if } i > h \ \ \text{(because } C_i \subset Z) \end{cases}$$

Set $\underline{d} := (d_1, \ldots, d_h, 0 \ldots, 0)$ so that $\underline{d} \in \text{Part}(d, \gamma)$ (because $d_i \geq (C_i \cdot Z) \geq 0$ for $i \leq h$). Now define $\underline{d}' := \underline{d} - \underline{\deg}\, Z$ so that $\underline{d}' \equiv \underline{d}$. We have

$$\underline{d}' = \big(d_1 - (C_1 \cdot Z), \ldots, d_h - (C_h \cdot Z), -(C_{h+1} \cdot Z), \ldots, -(C_\gamma \cdot Z)\big)$$

therefore $\underline{d}' \in \text{Part}(d, \gamma)$. By (28) we obtain that $\underline{\deg}\, Z \in \Lambda_X^0$ which is to say that $S(Z) \subset X_{\text{sing}}^{\text{sep}}$ (by Lemma 3.9). But of course $S(Z) = Z \cap Z'$ and so we are done with the proof of theorem 1.5. ■

We highlight some remarkable special cases of the theorem.

Corollary 4.3. *(1) Let X be a curve of compact type. Then for all $d \geq 1$ every d-th Abel map of X is natural.*

(2) Let X be a nodal curve free from separating nodes and such that $\epsilon(X) > d$. Then X admits a unique natural d-th Abel map, described as follows:

$$\begin{array}{ccc} \dot{X}^d & \longrightarrow & \coprod_{\underline{d} \in \text{Part}(d,\gamma)} \text{Pic}^{\underline{d}} X \\ (p_1, \ldots, p_d) & \mapsto & \mathcal{O}_X(p_1 + \ldots + p_d) \end{array} \tag{30}$$

5. Abel maps and compactified Picard schemes

This final section is to establish some connection with the vast literature on compactified Jacobians. To keep it to a length comparable with the rest of the paper, various important, interesting facts have been left out; so, it may appear somewhat obscure to a reader who is not already acquainted with the theory of compactified Picard schemes. On the other hand it will hopefully be useful to somebody wishing to apply or generalize our results to study and compactify Abel maps within a particular compactified Picard scheme.

The generalized Jacobian of a nodal curve X fails to be projective, unless X is of compact type. The problem of constructing a compactification for it, with certain natural properties, has been studied for a long time by many authors and diverse solutions exist (see [C05] for a short overview and some guide to the rich bibliography). Such compactifications usually go under the generic name of "*compactified Jacobians*" or of "*compactified Picard schemes*".

Here we consider only compactifications compatible with the operation of smoothing the curve (over a local one-parameter base, see property P1 below). We shall list, rather loosely, some basic properties that are common to most such compactifications, in order to relate our notion of Abel maps to such constructions.

5.0.7. Given a nodal curve X a degree-d compactified Jacobian of X is a reduced complete scheme $\overline{J_X^d}$ satisfying the following properties (and others not needed here).

- **(P1)** For every $f : \mathcal{X} \to B$ regular smoothing of X, there exists a proper scheme over B, $\pi : \overline{J_f^d} \to B$ whose generic fiber is Pic^d_K and whose closed fiber is $\overline{J_X^d}$.
  Denote by $J_f^d \subset \overline{J_f^d}$ the smooth locus of π and let J_X^d be the closed fiber of $J_f^d \to B$; then J_X^d is an open dense subset of $\overline{J_X^d}$.
- **(P2)** There exists a canonical B-morphism $n_f : J_f^d \longrightarrow N_f^d$ which is the identity on the generic fiber. (This follows by the Néron mapping property)
- **(P3)** There exists a B-morphism $u_f : J_f^d \longrightarrow \text{Pic}_f^d$ such that $q_f \circ u_f = n_f$ and u_f induces an isomorphism of J_f^d with its image.
- **(P4)** The restriction $u_X : J_X^d \to \text{Pic}^d X$ of u_f induces an isomorphism of J_X^d with a finite number of copies of the generalized Jacobian of X.

The rest of the section applies to all compactified Jacobians $\overline{J_X^d}$ that satisfy all the properties in 5.0.7. For cases when the map n_f of (P2) is an isomorphism see [C05].

Our definition 1.2 of Abel maps does not involve compactified Jacobians, but only the Picard scheme. As we said in the introduction, one can define Abel maps having a specific compactified Jacobian as target (see [AK], [E01], [CE06], [Co06]). To compare such an approach to ours we now introduce an ad hoc terminology, slightly awkward but useful to make distinctions.

Definition 5.1. Let $\overline{J_X^d}$ be a compactified Jacobian for X as in 5.0.7.

- (i) A map $\zeta : \dot{X}^d \longrightarrow \overline{J_X^d}$ is a *pre-Abel map* of degree d if there exists a regular smoothing $f : \mathcal{X} \to B$ of X and a map $\zeta_f : \dot{\mathcal{X}}^d \longrightarrow \overline{J_f^d}$ such that $(\zeta_f)_K = \alpha_K^d$ and $(\zeta_f)_k = \zeta$.
- (ii) A pre-Abel map is called *nonsingular* if $\zeta(\dot{X}^d) \subset J_X^d$.

So, a nonsingular ζ maps the nonsingular locus of X^d to the nonsingular locus of $\overline{J_X^d}$. For example the pre-Abel maps studied in [AK] and [CE06] (called simply "Abel maps" in those papers) are nonsingular. This is almost immediate for the first paper, since the curves studied there are integral. The other paper explores the case of reducible stable curves, restricting to "d-general curves" (cf. [CE06] 3.6 and 3.10) when $d \geq 2$, and thus getting nonsingular Abel maps.

A nonsingular pre-Abel map determines an Abel map. More precisely:

Proposition 5.2. *Let $\zeta : \dot{X}^d \longrightarrow \overline{J_X^d}$ be a nonsingular pre-Abel map; the composition $u_X \circ \zeta : \dot{X}^d \longrightarrow \operatorname{Pic}^d X$ (notation of (P4)) is an Abel map.*

Proof. The nonsingularity assumption enables us to define $\beta := u_X \circ \zeta$ and

$$\beta_f := u_f \circ \zeta_f : \dot{\mathcal{X}}^d \longrightarrow \operatorname{Pic}^d_f$$

extending β (where ζ_f is given by definition 5.1). Thus to prove that β is an Abel map it remains to prove that condition (b) of 1.2 holds.

By P3 and P4 in 5.0.7 there exists a finite set S of multidegrees such that

$$u_f : J_f^d \xrightarrow{\cong} \frac{\coprod_{\underline{d} \in S} \operatorname{Pic}^{\underline{d}}_f}{\sim_K} \subset \operatorname{Pic}^d_f .$$

We shall from now on identify J_f^d with the image of u_f as indicated above. We must prove that for every pair of multidegrees $\underline{d}$ and $\underline{d}'$ both contained in S we have that $\underline{d} \not\equiv \underline{d}'$. By contradiction, suppose that there is a pair of distinct equivalent multidegrees $\underline{d}$ and $\underline{d}'$ in S; let $D \in \mathcal{D}(X)$ be such that $\underline{\deg} D = \underline{d}' - \underline{d}$. We shall arrive at a contradiction by showing that the map $J_f^d \to B$ is not separated.

Pick $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ having multidegree equal to $\underline{d}$ when restricted to X; in order for such an $\mathcal{L}$ to exist we may need to make a (étale) base change, but this will not affect the argument. The moduli map of $\mathcal{L}$, $\mu_{\mathcal{L}} : B \to \operatorname{Pic}^d_f$, has therefore image in $\operatorname{Pic}^{\underline{d}}_f$. Then we can view $\mu_{\mathcal{L}}$ as a map $\mu_{\mathcal{L}} : B \longrightarrow \operatorname{Pic}^{\underline{d}}_f \subset J^d_f$.

Set $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(D)$, so that $\mathcal{L}'$ restricted to X has multidegree $\underline{d}'$. Then, arguing as we did for $\mathcal{L}$, the moduli map of $\mathcal{L}'$ is $\mu_{\mathcal{L}'} : B \longrightarrow \operatorname{Pic}^{\underline{d}'}_f \subset J^d_f$. Now $\mu_{\mathcal{L}}$ and $\mu_{\mathcal{L}'}$ are different maps coinciding on the generic point of B (as $\mathcal{L}_{|\mathcal{X}_K} = \mathcal{L}'_{|\mathcal{X}_K}$). We have thus contradicted the fact that $J_f^d \to B$ is separated (by P1). ∎

Example 5.3. Consider again the Abel maps studied in [CE06]. If $d = 1$ our Theorem 1.5 implies that they are all natural, which was already known by the direct proof of loc.cit. 4.10 and 5.13. If $d \geq 2$ the question of which of them are natural is interesting (see loc.cit. 3.14) and open.

Example 5.4. Let $X = C_1 \cup C_2$ with $\#C_1 \cap C_2 = \delta \geq 2$. Let $\overline{J_X^d}$ be a compactified degree-d Jacobian having at most $\delta - 1$ irreducible components (this definitely occurs, see [C05] 6.5). Then for every $d \geq \delta$ a pre-Abel map $\dot{X}^d \to \overline{J_X^d}$ is singular.

To prove this, observe first that $\#\Delta_X^d = \delta$, so that δ is equal to the number of connected components of the Néron model (i.e. of N_X^d). The assumption on the number of irreducible components of $\overline{J_X^d}$, hence of J_X^d, implies that the image of $n_f : J_f^d \to N_f^d$ does not intersect some connected components of N_X^d.

Now, if $d \geq \delta$ every multidegree class in Δ_X^d contains some partitional representative, therefore the image of the map $\mathrm{N}(\alpha_K^d)$ (see (12)) does intersect every connected component of N_X^d. If there were a nonsingular pre-Abel map $\zeta : \dot{X}^d \longrightarrow J_X^d$, then we could factor $\mathrm{N}(\alpha_K^d) = n_f \circ \zeta_f$, which is impossible.

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