Algebraic Geometry — Compactified Jacobians of Néron type, by Lucia Caporaso, communicated on 12 November 2010.

Abstract. — We characterize stable curves $X$ whose compactified degree-$d$ Jacobian is of Néron type. This means the following: for any one-parameter regular smoothing of $X$, the special fiber of the Néron model of the Jacobian is isomorphic to a dense open subset of the degree-$d$ compactified Jacobian of $X$. It is well known that compactified Jacobians of Néron type have the best modular properties, and that they are endowed with a mapping property useful for applications.

Key words: Stable curve, Picard scheme, Néron model, compactification, balanced line bundle.

Mathematic Subject Classification (2000):

1. Introduction and preliminaries

Let $X$ be a stable curve and $f: X \to B$ a one-parameter smoothing of $X$ with $X$ a nonsingular surface; $X$ is thus identified with the special fiber of $f$ and all other fibers are smooth curves. Let $N^d_f \to B$ be the Néron model of the degree-$d$ Jacobian of the generic fiber of $f$; its existence was proved by A. Néron in [N], and its connection with the Picard functor was established by M. Raynaud in [R]. So, $N^d_f \to B$ is a smooth and separated morphism, whose generic fiber is the degree-$d$ Jacobian of the generic fiber of $f$; the special fiber, denoted $N^d_X$, is isomorphic to a disjoint union of copies of the generalized Jacobian of $X$. $N^d_f \to B$ can be interpreted as the maximal separated quotient of the degree-$d$ Picard scheme $\text{Pic}^d_f \to B$. In particular, if $\text{Pic}^d_f \to B$ is separated, which happens if and only if $X$ is irreducible, then $N^d_f = \text{Pic}^d_f$ (we refer to [R], [BLR] or [Ar] for details).

The Néron model has a universal property, the Néron Mapping Property ([BLR, def. 1]), which determines it uniquely. Moreover, as $d$ varies in $\mathbb{Z}$, the special fibers, $N^d_X$, of $N^d_f \to B$ are all isomorphic.

By contrast, the compactified degree-$d$ Jacobian of a reducible curve $X$, denoted $P^d_X$, has a structure which varies with $d$. For example, the number of irreducible components, and the modular properties, depend on $d$; see Section 2 for details and references.

$P_X^d$ will be called of Néron type if its smooth locus is isomorphic to $N_X^d$. Compactified Jacobians of Néron type have the best modular properties. Moreover they inherit a mapping property from the universal property of the Néron model which provides a very useful tool; see for example [CE] for applications to Abel maps.
The purpose of this paper is to classify, for every \( d \), those stable curves \( X \in \mathcal{M}_g \) such that \( P_X^d \) is on Néron type. The question is interesting if \( g \geq 2 \), for otherwise \( P_X^d \) is always irreducible, and hence of Néron type.

Before stating our main result, we need a few words about compactified Jacobians. \( P_X^d \) parametrizes certain line bundles on quasistable curves having \( X \) as stabilization. These are the so-called “balanced” line bundles; among balanced line bundles there are some distinguished ones, called “strictly balanced”, which have better modular properties. In fact, to every balanced line bundle there corresponds a unique point in \( P_X^d \), but different balanced line bundles may determine the same point. On the other hand every point of \( P_X^d \) corresponds to a unique class of strictly balanced line bundles.

The curve \( X \) is called \( d \)-general if every balanced line bundle of degree \( d \) is strictly balanced. This is equivalent to the fact that \( P_X^d \) is a geometric GIT-quotient.

The property of being \( d \)-general depends only on the weighted dual graph of \( X \), and the locus of \( d \)-general curves in \( \mathcal{M}_g \) has been precisely described by M. Melo in [M].

Now, the degree-\( d \) compactified Jacobian of a \( d \)-general curve is of Néron type, by [C2, Thm. 6.1]. But, as we will prove, the converse does not hold.

More precisely, a stable curve \( X \) is called weakly \( d \)-general if a curve obtained by smoothing every separating node of \( X \), and maintaining all the non separating nodes, is \( d \)-general; see Definition 1.13.

Our main result, Theorem 2.9, states that \( P_X^d \) is of Néron type if and only if \( X \) is weakly \( d \)-general. The locus of weakly \( d \)-general curves in \( \mathcal{M}_g \) is precisely described in section 2.11; its complement turns out to have codimension at least 2.

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1.1. Notations and conventions

(1) We work over an algebraically closed field \( k \). The word “curve” means projective scheme of pure dimension one. The genus of a curve will be the arithmetic genus, unless otherwise specified.

(2) By \( X \) we will always denote a nodal curve of genus \( g \geq 2 \). For any subcurve \( Z \subset X \) we denote by \( g_Z \) its arithmetic genus, by \( Z^c := X \setminus Z \) and by \( \delta_Z := \#Z \cap Z^c \). We set \( w_Z := \deg_Z \omega_X = 2g_Z - 2 + \delta_Z \).

(3) A node \( n \) of a connected curve \( X \) is called separating if \( X \setminus \{n\} \) is not connected. The set of all separating nodes of \( X \) is denoted by \( X_{\text{sep}} \) and the set of all nodes of \( X \) by \( X_{\text{sing}} \).

(4) A nodal curve \( X \) of genus \( g \geq 2 \) is called stable if it is connected and if every component \( E \subset X \) such that \( E \cong \mathbb{P}^1 \) satisfies \( \delta_E \geq 3 \). \( X \) is called quasistable if it is connected, if every \( E \subset X \) such that \( E \cong \mathbb{P}^1 \) satisfies \( \delta_E \geq 2 \), and if two exceptional components never intersect, where an exceptional component is defined as an \( E \cong \mathbb{P}^1 \) such that \( \delta_E = 2 \). We denote by \( X_{\text{exc}} \) the union of the exceptional components of \( X \).
(5) Let $S \subset X_{\text{sing}}$ we denote by $v_S : X^*_S \to X$ the normalization of $X$ at $S$, and by $\hat{X}_S$ the quasistable curve obtained by “blowing-up” all the nodes in $S$, so that there is a natural surjective map

$$\hat{X}_S = \bigcup_{i=1}^S E_i \cup X^*_S \to X$$

restricting to $v_S$ on $X^*_S$ and contracting all the exceptional components $E_i$ of $X_S$. $\hat{X}_S$ is also called a quasistable curve of $X$.

(6) Let $C_1, \ldots, C_r$ be the irreducible components of $X$. Every line bundle $L \in \text{Pic } X$ has a multidegree $\deg L = (\deg C_1, \ldots, \deg C_r) \in \mathbb{Z}^r$. Let $d = (d_1, \ldots, d_r) \in \mathbb{Z}^r$, we set $|d| = \sum_i d_i$; for any subcurve $Z \subset X$ we abuse notation slightly and denote

$$d_Z := \sum_{C_i \subset Z} d_i.$$

1.2. Compactified Jacobians of Néron type. Let $X$ be any nodal connected curve and $f : \mathcal{X} \to B$ a one-parameter regular smoothing for $X$, i.e. $B$ is a smooth connected one-dimensional scheme with a marked point $b_0 \in B$, $\mathcal{X}$ is a regular surface, and $f$ is a projective morphism whose fiber over $b_0$ is $X$ and whose remaining fibers are smooth curves. We set $U := B \setminus \{b_0\}$ and let $U \to U$ be the family of smooth curves obtained by restricting $f$ to $U$. Consider the relative degree $d$ Picard scheme over $U$, denoted $\text{Pic}^d_U \to U$. Its Néron model over $B$ will be denoted by

$$N_f^d := N(\text{Pic}^d_U) \to B,$$

and its fiber over $b_0$ will be denoted by $N^d_X$; $N^d_X$ is isomorphic to a finite number of copies of the generalized Jacobian of $X$. The number of copies is independent of $d$; to compute it we introduce the so-called “degree class group”.

Let $g$ be the number of irreducible components of $X$. For every component $C_i$ of $X$ set $k_{i,j} := \#(C_i \cap C_j)$ if $j \neq i$, and $k_{i,i} = -\#(C_i \cap C_i \setminus C_j)$ so that the matrix $(k_{i,j})$ is symmetric matrix. Notice that for every regular smoothing $g : X \to B$ of $X$ as above, we have $\deg g_C(C_i) = k_{i,j}$. Hence this matrix is also related to $f$, although it does not depend on the choice of $f$ (as long as $X$ is regular).

We have $\sum_{j=1}^g k_{i,j} = 0$ for every $i$. Now, for every $i = 1, \ldots, g$ set $e_i := (k_{1,i}, \ldots, k_{r,i}) \in \mathbb{Z}^r$ and $Z := \{d \in \mathbb{Z}^r : |d| = 0\}$ so that $e_i \in Z$. We can now define the sublattice $\Lambda_X := \langle e_1, \ldots, e_g \rangle \subset \mathbb{Z}$.

The degree class group of $X$ is the group $\Delta_X := \mathbb{Z}/\Lambda_X$. It is not hard to prove that $\Delta_X$ is a finite group.

Let $d$ and $d'$ be in $\mathbb{Z}^r$; we say that they are equivalent if $d - d' \in \Lambda_X$. We denote by $\Delta^d_X$ the set of equivalence classes of multidegrees of total degree $d$; for a multidegree $d$ we write $[d]$ for its class. It is clear that $\Delta_X = \Delta_X^0$ and that

$$\# \Delta_X = \# \Delta^d_X.$$
Now back to $N^d_X$, the special fiber of (1.1); as we said it is a smooth, possibly non connected scheme of pure dimension $g$.

**Fact 1.3.** Under the above assumptions, the number of irreducible (i.e. connected) components of $N^d_X$ is equal to $\#\Delta_X$.

This is well known; see [R, 8.1.2] (where $\Delta_X$ is the same as $\ker B/\operatorname{Im} B$) or [BLR, thm. 9.6.1]. Using the standard notation of Néron models theory we have $\Delta_X = \Phi_{N^d_X}$, i.e. $\Delta_X$ is the “component group” of $N^d_X$.

For every stable curve $X$ and every $d$ we denote by $P^d_X$ the degree $d$ compactified Jacobian (or degree-$d$ compactified Picard scheme). $P^d_X$ has been constructed in [OS] for a fixed curve, and independently for families in [S] and in [C1] (the constructions of [OS] and [S] are here considered with respect to the canonical polarization); these three constructions give the same scheme by [Al], see also [LM]. We mention that another compactified Jacobian is constructed in [E], whose connection with the others is under investigation; see [MV]. An explicit description of $P^d_X$ will be recalled in 2.2. We here anticipate the fact that $P^d_X$ is a connected, projective scheme of pure dimension $g$. As we said in the introduction, several geometric and modular properties of $P^d_X$ depend on $d$.

**Definition 1.4.** Let $X$ be a stable curve and $P^d_X$ its degree-$d$ compactified Jacobian. We say that $P^d_X$ is of Néron type if the number of irreducible components of $P^d_X$ is equal to the number of irreducible components of $N^d_X$.

**Example 1.5.** A curve $X$ is called tree-like if every node of $X$ lying in two different irreducible components is a separating node.

The compactified Jacobian of a tree-like curve $X$ is easily seen to be always of Néron type. Indeed, $P^d_X$ is irreducible for every $d$; on the other hand $a D^d_X = 1$ so that $N^d_X$ is also irreducible.

Let now $\pi : \overline{P}^d_f \to B$ be the compactified degree-$d$ Picard scheme of a regular smoothing $f : \mathcal{X} \to B$ of a stable curve $X$, as defined in 1.2. So the fiber of $\pi$ over $b_0$ is $P^d_X$, and the restriction of $\pi$ over $U = B \setminus \{b_0\} \subset B$ is $\operatorname{Pic}^d_{\mathcal{X}}$. We denote $P^d_f \to B$ the smooth locus of $\pi$. By the Néron Mapping Property there exists a canonical $B$-morphism, $\chi_f$, from $P^d_f$ to the Néron model of $\operatorname{Pic}^d_{\mathcal{X}}$:

$$\chi_f : P^d_f \to N^d_f$$

extending the indentity map from the generic fiber of $\pi$ to the generic fiber of $N^d_f \to B$.

**Proposition 1.6.** With the above set up, $\overline{P}^d_f$ is of Néron type if and only if the map $\chi_f : P^d_f \to N^d_f$ is an isomorphism for every $f : \mathcal{X} \to B$ as above.

The proof, requiring a description of $\overline{P}^d_f$, will be given in subsection 2.6.
1.7. Smoothing separating nodes. A stable weighted graph of genus \( g \geq 2 \) is a pair \((\Gamma, w)\), where \( \Gamma \) is a graph and \( w : V(\Gamma) \to \mathbb{Z}_{\geq 0} \) a weight function. The genus of \((\Gamma, w)\) is the number \( g(\Gamma, w) \) defined as follows:

\[
g(\Gamma, w) = \sum_{v \in V(\Gamma)} w(v) + b_1(\Gamma).
\]

A weighted graph will be called \textit{stable} if every \( v \in V(\Gamma) \) such that \( w(v) = 0 \) has valency at least 3.

Let \( X \) be a nodal curve of genus \( g \), the weighted dual graph of \( X \) is the weighted graph \((\Gamma_X, w_X)\) such that \( \Gamma_X \) is the usual dual graph of \( X \) (the vertices of \( \Gamma_X \) are identified with the irreducible components of \( X \) and the edges are identified with the nodes of \( X \); an edge joins two, possibly equal, vertices if the corresponding node is in the intersection of the corresponding irreducible components), and \( w_X \) is the weight function on the set of irreducible components of \( X \), \( V(\Gamma_X) \), assigning to a vertex the geometric genus of the corresponding component. Hence

\[
g = \sum_{v \in V(\Gamma_X)} w_X(v) + b_1(\Gamma_X) = g(\Gamma_X, w_X).
\]

\( X \) is a stable curve if and only if \((\Gamma_X, w_X)\) is a stable weighted graph.

Now we ask: What happens to the weighted dual graph of \( X \) if we smooth all the separating nodes of \( X \)?

To answer this question, we introduce a new weighted graph, denoted by \((\Gamma^2, w^2)\), associated to a weighted graph \((\Gamma, w)\). \((\Gamma^2, w^2)\) is defined as follows. \( \Gamma^2 \) is the graph obtained by contracting every separating edge of \( \Gamma \) to a point. Therefore \( \Gamma^2 \) is 2-edge-connected, i.e. free from separating edges (this explains the notation). To define the weight function \( w^2 \), notice that there is a natural surjective map contracting the separating edges of \( \Gamma \)

\[
\sigma : \Gamma \to \Gamma^2,
\]

and an induced surjection on the set of vertices

\[
\phi : V(\Gamma) \to V(\Gamma^2); \quad v \mapsto \sigma(v).
\]

Now we define \( w^2 \) as follows. For every \( v^2 \in V(\Gamma^2) \)

\[
w^2(v^2) = \sum_{v \in \phi^{-1}(v^2)} w(v).
\]

As \( \sigma \) does not contract any cycle, \( b_1(\Gamma) = b_1(\Gamma^2) \) and \( g(\Gamma, w) = g(\Gamma^2, w^2) \).

Remark 1.8. If \((\Gamma, w)\) is the weighted dual graph of a curve \( X \), \((\Gamma^2, w^2)\) is the weighted dual graph of any curve obtained by smoothing every separating node
of $X$. We shall usually denote by $X^2$ such a curve. Of course $X$ and $X^2$ have the same genus.

1.9. $d$-general and weakly $d$-general curves. Let us recall the definitions of balanced and strictly balanced multidegrees.

**Definition 1.10.** Let $X$ be a quasistable curve of genus $g \geq 2$ and $L \in \text{Pic}^d X$. Let $d$ be the multidegree of $L$.

1. We say that $L$, or $d$, is balanced if for any subcurve (equivalently, for any connected subcurve) $Z \subseteq Y$ we have (notation in 1.1(2))

$$\deg_Z L \geq m_Z(d) := \frac{d_{\text{mv}} - \delta Z}{2y - 2},$$

and $\deg_Z L = 1$ if $Z$ is an exceptional component.

2. We say that $L$, or $d$, is strictly balanced if it is balanced and if strict inequality holds in (1.2) for every $Z \subseteq X$ such that $Z \cap Z' \neq X_{\text{exc}}$.

3. We denote

$$\overline{B}_d(X) = \{d : |d| = d \text{ balanced on } X\} \supset B_d(X) = \{d : \text{strictly balanced}\}.$$

The following trivial observations are useful.

**Remark 1.11.** (A) Let $Z = Z_1 \cup Z_2 \subseteq X$ be a disconnected subcurve. Then $m_Z(d) = m_{Z_1}(d) + m_{Z_2}(d)$.

(B) Suppose $X$ stable and $d \in \overline{B}_d(X)$. Then $d$ is not strictly balanced if and only if there exists a subcurve $Z \subseteq X$ such that $d_Z = m_Z(d)$, or equivalently, $d_Z. = m_Z(d) + \delta Z$.

**Remark 1.12.** Let $X$ be stable. By [C2, Prop. 4.12], every multidegree class in $\Delta^d_X$ has a balanced representative, which is unique if and only if it is strictly balanced. Therefore

$$\#B_d(X) \leq \# \Delta_X \leq \# \overline{B}_d(X).$$

The terminology “strictly balanced” is not to be confused with “stably balanced” (used elsewhere and unnecessary here). The two coincide for stable curves; in general, a stably balanced line bundle is strictly balanced, but the converse may fail. Let us explain the difference. The compactified Picard scheme of $X$, $\overline{P}^d_X$ is a GIT-quotient of a certain scheme by a certain reductive group $G$. Strictly balanced line bundles correspond to the GIT-semistable orbits that are closed in the GIT-semistable locus. Stably balanced line bundles correspond to GIT-stable points and balanced line bundles correspond to GIT-semistable points. As every point in $\overline{P}^d_X$ parametrizes a unique closed orbit, strictly balanced
line bundles of degree $d$ on quasistable curves of $X$ are bijectively parametrized by $P^d_X$. See Fact 2.2 below.

**Definition 1.13.** Let $X$ be a stable curve. We will say that $X$, or its weighted dual graph $(\Gamma_X, w_X)$, is $d$-general if $B_d(X) = B_d(\hat{X})$ (cf. [C2, 4.13]). (Equivalently, $X$ is $d$-general if the inequalities in Remark 1.12 are both equalities.)

We will say that $X$ is weakly $d$-general if $(\Gamma_X, w_X^2)$ is $d$-general.

**Remark 1.14.** The following facts are well known (see loc.cit.).

(1) The set of $d$-general stable curves is a nonempty open subset of $\mathcal{M}_g$.
(2) $(d - g + 1, 2g - 1) = 1$ if and only if every stable curve of genus $g$ is $d$-general.
(3) The property of being $d$-general depends only on the weighted dual graph (obvious).

**Example 1.15.** If $X_{sep} = \emptyset$, then $X$ is $d$-general if and only if it is weakly general.

If $X$ is tree-like, then $(\Gamma_X, w_X^2)$ has only one vertex, hence it is $d$-general for every $d$. Therefore tree-like curves are weakly $d$-general for every $d$.

## 2. Irreducible Components of Compactified Jacobians

### 2.1. Compactified degree-$d$ Jacobians

Let us describe the compactified Jacobian $P^d_X$ for any degree $d$. We use the set up of [C1] and [C2]; in these papers there is the assumption $g \geq 3$, but by [OS], [S] and [Al] we can extend our results to $g \geq 2$. A synthetic account of the modular properties of the compactified Jacobian for a curve or for a family can be found in [CE, 3.8 and 5.10].

**Fact 2.2.** Let $X$ be a stable curve of genus $g \geq 2$. Then $P^d_X$ is a connected, reduced, projective scheme of pure dimension $g$, admitting a canonical decomposition (notation in 1.1(5))

$$
P^d_X = \prod_{S \subseteq X_{\text{sing}}} P^d_S
$$

such that for every $S \subseteq X_{\text{sing}}$ and $d \in B_d(\hat{X}_S)$ there is a natural isomorphism

$$
P^d_S \cong \text{Pic}^{d^r}_S X^r_S
$$

where $d^r$ denotes the multidegree on $X^r_S \subseteq \hat{X}_S$ defined by restricting $d$.

Let $i(P^d_X)$ be the number of irreducible components of $P^d_X$; then

$$(2.1) \quad B_d(X) \leq i(P^d_X) \leq \# \Delta_X.$$
COROLLARY 2.3. Let $X$ be a stable curve.

(1) The decomposition of $\overline{P^d_X}$ in irreducible components is

$$\overline{P^d_X} = \bigcup_{(S, d) \in I^d_X} P^d_S,$$

where $I^d_X := \{(S, d) : S \subseteq X_{\text{sep}}, d \in B_d(X_S)\}$.

(2) Suppose that $X$ is $d$-general; then $P^d_X$ is of Néron type, and for every nonempty $S \subseteq X_{\text{sep}}$ we have $B_d(X_S) = \emptyset$.

PROOF. From Fact 2.2 we have that the irreducible components of $\overline{P^d_X}$ are the closures of subsets $P^d_S \cong \text{Pic}^d X^*_S$ where $S$ is such that $\dim \text{Pic}^d X^*_S = g$. Now, it is clear that

$$\dim \text{Pic}^d X^*_S = \dim (X^*_S)^g = g$$

if and only if $S \subseteq X_{\text{sep}}$.

Therefore the irreducible components of $\overline{P^d_X}$ correspond bijectively to pairs $(S, d)$ with $S \subseteq X_{\text{sep}}$ and $d \in B_d(X_S)$.

Now part (2). It is clear that the set $I^d_X$ contains a subset identifiable with $B_d(X)$, namely the subset $\{(0, d) : d \in B_d(X)\}$. If $X$ is $d$-general then $\# B_d(X) = \# \Delta_X$, hence by (2.1) we must have that $I^d_X$ contains no pairs other than those of type $(0, d)$. This concludes the proof.

□

LEMMA 2.4. Let $X$ be a stable curve and let $\mu \in \Delta^d_X$ be a multidegree class. Then there exists a unique $S(\mu) \subseteq X_{\text{sing}}$ and a unique $d(\mu) \in B_d(X_{S(\mu)})$ such that for every $d \in B_d(X)$ with $[d] = \mu$ the following properties hold.

(1) There is a canonical surjection

$$\text{Pic}^d X \to P^d_{S(\mu)} \cong \text{Pic}^{d(\mu)} X^*_S(\mu).$$

(2) We have

$$S(\mu) = \bigcup_{Z \subseteq X, d_Z = \mu} Z \cap Z^c.$$

PROOF. The proof is routine. Let us sketch it using the combinatorial results [C1, Lemma 5.1 and Lemma 6.1]. The terminology used in that paper differs from ours as follows: what we here call a “strictly balanced multidegree $d$ on a quasistable curve $X$” is there called an “extremal pair $(X, d)$”; cf. subsection 5.2 p. 631.

So, the pair $(\widehat{X_{S(d)}}, d(\mu))$ is the “extremal pair” associated to $\mu$. This means the following. For every balanced line bundle $L$ on $X$ such that $[\deg L] = \mu$ the point in $P^d_X$ associated to $L$ parametrizes a line bundle $\tilde{L} \in \text{Pic}^d X^*_S(\mu)$, and the restriction of $\tilde{L}$ to $X^*_S$ is uniquely determined by $L$. Conversely every line bundle in $\text{Pic}^{d(\mu)} X^*_S(\mu)$ is obtained in this way.
More precisely, as we said, $\overline{P}_X^d$ is a GIT quotient; let us denote it by $\overline{P}_X^d = V_X / G$, so that $V_X$ is made of GIT-semistable points. Let $O_G(L) \subset V_X$ be the orbit of $L$. Then the semistable closure of $O_G(L)$ contains a unique closed orbit $O_G(L)$ as above. Moreover for every $d' \in B_d(X)$ having class $\mu$ there exists $L' \in \text{Pic}^d X$ such that the above $O_G(L)$ lies in the closure of $O_G(L')$. Hence the maps $\text{Pic}^d X \to \overline{P}_X^d$ and $\text{Pic}^d X \to \overline{P}_X^d$ have the same image.

Using the notation of Fact 2.2, we have that for every balanced $d$ of class $\mu$ the canonical map $\text{Pic}^d X \to \overline{P}_X^d$ has image $P_{S(\mu)}^d$, so that the first part is proved.

Now (2). The previously mentioned Lemma 5.1 implies that for every $d \in B_d(X)$ and every $Z$ such that $d_Z = m_Z(d)$ we have $Z \cap Z^c = S(\mu)$. By the above Lemma 6.1 each $n \in S(\mu)$ is obtained in this way.

**Proposition 2.5.** Let $X$ be a stable curve. $\overline{P}_X^d$ is of Néron type if and only if for every $d \in B_d(X)$ and every connected $Z \subset X$ such that $d_Z = m_Z(d)$ we have

\[(2.2) \quad Z \cap Z^c = X_{\text{sep}}.\]

**Proof.** We begin by observing that, with the notation of Corollary 2.3 and Lemma 2.4, we have

\[I^d_X = \{(d(\mu), S(\mu)), \forall \mu \in \Delta^d_X \text{ such that } \dim P_{S(\mu)}^d = g\}.\]

Indeed, by Fact 2.2 the set on the right is clearly included in $I^d_X$. On the other hand let $(S, d) \in I^d_X$. To show that there exists $\mu \in \Delta^d_X$ such that $d = d(\mu)$ we can assume that $S \neq 0$ (otherwise it is obvious). So, $d$ is a strictly balanced multidegree of total degree $d$ on $X_S$. Let $n \in S$; by Corollary 2.3 the node $n$ is separating for $X$; let $X = Z \cup Z^c$ with $Z \cap Z^c = \{n\}$. Then $Z$ and $Z^c$ can be viewed as subcurves of $X_S$, where they do not intersect since the node $n$ is replaced by an exceptional component $E$. Now, $d_E = 1$, therefore $d_Z = m_Z(d)$ and $d_Z = m_{Z^c}(d)$. Let $C_Z \subset Z = X$ be the irreducible component intersecting $Z^c$ (so that $C_Z \cap X_S$ intersects $E$). Let $d^X_C$ be the multidegree on $X$ defined as follows: for every irreducible component $C \subset X$

\[d^X_C = \begin{cases} \frac{d_C + 1}{d_C} & \text{if } C = C_Z, \\ d_C & \text{otherwise}. \end{cases}\]

As $d$ is balanced on $X_S$ one easily checks that $d^X_C$ is balanced on $X$. Note that $d^X_C$ is not strictly balanced, since $d^X_{C^c} = m_{Z^c}(d)$ (see Remark 1.11). By iterating the above procedure for every node in $S$ we arrive at a balanced multidegree on $X$ whose class we denote by $\mu \in \Delta^d_X$. By Lemma 2.4 we have that $d = d(\mu)$.

Suppose that $P^d_X$ is of Néron type. By the previous discussion there is a natural bijection between $\Delta^d_X$ and $I^d_X$, mapping $\mu \in \Delta^d_X$ to $(S(\mu), d(\mu))$. By Corollary 2.3 we have $S(\mu) \subset X_{\text{sep}}$. Hence for every multidegree $d \in B_d(X)$ such that $[d] = \mu$ we have that condition (2) of that lemma holds. In particular every $Z$ as in our statement is such that $Z \cap Z^c = S(\mu) \subset X_{\text{sep}}$.\]
Conversely, if $P^d_X$ is not of Néron type there is a class $\mu \in \Delta^d_X$ such that
\[ g > \dim P^d_{S(\mu)} = \dim J(X^r_{S(\mu)}). \]

But then $S(\mu)$ contains some non separating node of $X$. Hence, by Lemma 2.4(2), there exists a connected subcurve $Z \subset X$ such that $d_Z = m_Z(d)$ and such that $Z \cap Z^c$ contains some non separating node.

2.6. Proof of Proposition 1.6. We generalize the proof of [C2, Thm. 6.1]. Let $f : \mathcal{X} \to B$ be a regular smoothing of $X$ as defined in subsection 1.2, and $\pi : \overline{P}^d_f \to B$ be the compactified degree-$d$ Picard scheme. Its smooth locus $P^d_f \to B$ is such that its fiber over $b_0$, denoted $P^d_X$, satisfies
\[ P^d_X = \coprod_{(S,d) \in T^d} P^d_S \]
\[ \text{(notation in 2.3)} \]
where each $P^d_S$ is irreducible of dimension $g$. If the morphism $\chi_f : P^d_f \to N^d_f$ is an isomorphism, then $P^d_X$ has as many irreducible components as $N^d_X$, hence the same holds for $\overline{P}^d_X$. So $\overline{P}^d_X$ is of Néron type.

Conversely, if $P^d_X$ is of Néron type, then $P^d_X$ has an irreducible component for every $\mu \in \Delta^d_X$ so that (2.3) takes the form
\[ P^d_X = \coprod_{\mu \in \Delta^d_X} P^d_{S(\mu)}. \]

Let us construct the inverse of $\chi_f$. We pick a balanced representative $g^\mu$ for every multidegree class $\mu \in \Delta^d_X$ (it exists by Remark 1.12). By [C2, Lemma 3.10] we have
\[ N^d_f \cong \coprod_{\mu \in \Delta^d_X} \text{Pic}^d_{g^\mu} \]
\[ \sim U \]
where $\sim_U$ denotes the gluing of the Picard schemes $\text{Pic}^d_{g^\mu}$ along their restrictions over $U$ (as $\text{Pic}^d_{g^\mu} = \text{Pic}^d_f$ for every $\mu$). Now, the Picard scheme $\text{Pic}^d_{g^\mu}$ is endowed with a Poincaré bundle, which is a relatively balanced line bundle on $\mathcal{X} \times_B \text{Pic}^d_f$. By the modular property of $P^d_f$ the Poincaré bundle induces a canonical $B$-morphism
\[ \psi^\mu_f : \text{Pic}^d_{g^\mu} \to P^d_{S(\mu)} \subset \overline{P}^d_f. \]

As $\mu$ varies, the restrictions of these morphisms over $U$ all coincide with the identity map $\text{Pic}^d_{g^\mu} \to \text{Pic}^d_f \subset \overline{P}^d_f$. Therefore the $\psi^\mu_f$ can be glued together to a morphism
\[ \psi_f : N^d_f \to P^d_f \subset \overline{P}^d_f. \]

It is clear that $\psi_f$ is the inverse of $\chi_f$. Proposition 1.6 is proved. \qed
2.7. The main result. From Proposition 2.5 we derive the following.

**Corollary 2.8.** Let $X$ be a stable curve free from separating nodes. Then $P_X^d$ is of Néron type if and only if $X$ is $d$-general.

**Proof.** By Corollary 2.3(2) there is only one implication to prove. Namely, suppose that $X$ is not $d$-general. Then there exists $d \in B_d(X) \setminus B_d(x)$, and hence a subcurve $Z \subseteq X$ such that $d_Z = m_Z(d)$ (see Remark 1.11). As $X_{\text{sep}} = \emptyset$, condition (2.2) of Proposition 2.5 cannot be satisfied. Therefore $P_X^d$ is not of Néron type.

We are ready to prove our main result.

**Theorem 2.9.** Let $X$ be a stable curve. Then $P_X^d$ is of Néron type if and only if $X$ is weakly $d$-general.

**Proof.** Observe that if $X$ is free from separating nodes we are done by Corollary 2.8. Let $(\Gamma, w)$ be the weighted graph of $X$ and consider the weighted graph $(\Gamma^2, w^2)$ defined in subsection 1.7. We denote by $X^2$ a stable curve whose weighted graph is $(\Gamma^2, w^2)$. By Remark 1.8 the curve $X^2$ can be viewed as a smoothing of $X$ at $X_{\text{sep}}$.

Recall that we denote by $\sigma : \Gamma \to \Gamma^2$ the contraction map and by

$$
\phi : V(\Gamma) \to V(\Gamma^2); \quad v \mapsto \sigma(v)
$$

the induced map on the vertices, i.e. on the irreducible components. The subcurves of $X$ naturally correspond to the so-called “induced” subgraphs of $\Gamma$, i.e. those subgraphs $\Gamma'$ such that if two vertices $v$, $w$ of $\Gamma$ are in $\Gamma'$, then every edge of $\Gamma$ joining $v$ with $w$ lies in $\Gamma'$. Similarly, the induced subgraphs of $\Gamma^2$ correspond to subcurves of $X^2$. If $Z^2$ is a subcurve of $X^2$, and $\Gamma_Z \subseteq \Gamma x^2$ its corresponding subgraph, we denote by $Z \subseteq X$ the subcurve associated to $\sigma^{-1}(\Gamma_Z)$ (it is obvious that the subgraph $\sigma^{-1}(\Gamma_Z)$ is induced if so is $\Gamma_Z$); we refer to $Z$ as the “preimage” of $Z^2$. Of course $\sigma(\Gamma_Z) = \Gamma_Z$.

For any $Z \subseteq X$ which is the preimage of a subcurve $Z^2 \subseteq X^2$ we have

$$(2.4) \quad Z \cap X_{\text{sep}} \subseteq Z_{\text{sep}}$$

or, equivalently, $Z \cap Z^e \cap X_{\text{sep}} = \emptyset$. Conversely, every $Z \subseteq X$ satisfying (2.4) is the preimage of some $Z^2 \subseteq X^2$.

Hence $Z^2$ can be viewed as a smoothing of $Z$ at its separating nodes that are also separating nodes of $X$, i.e. at $Z_{\text{sep}} \cap X_{\text{sep}}$. Thus, for every $Z^2$ with preimage $Z$ we have $g_Z = g_{Z^2}$ and $\delta_Z = \delta_{Z^2}$; hence for every $d \in \mathbb{Z}$

$$(2.5) \quad m_{Z^2}(d) = m_Z(d).$$
We shall now view multidegrees as an integer valued functions on the vertices. We claim that we have a surjection
\[ \alpha : B_d(X) \to B_d(X^2) \]
defined as follows: for every vertex \( v^2 \in V(\Gamma^2) \) we set
\[ \alpha(d)(v^2) := \sum_{v \in \varphi^{-1}(v^2)} d(v). \]

Let us first show that if \( d \) is balanced, so is \( \alpha(d) \). For every subcurve \( Z^2 \subset X^2 \) we have \( \alpha(d)_{Z^2} = d_Z \) where \( Z \subset X \) is the preimage of \( Z^2 \); by (2.5) the inequality (1.2) is satisfied on \( Z^2 \) if (and only if) it is satisfied on \( Z \).

Let us now show that \( \alpha \) is surjective. Let \( d^2 \) be a balanced multidegree on \( X^2 \); we know that \( X^2 \) can be chosen to be a smoothing of \( X \) at \( X_{\text{sep}} \). In other words, there exists a family of curves \( X_t \), all having \((\Gamma^2, w^2)\) as weighted graph, specializing to \( X \). But then there also exists a family of balanced line bundles \( L_t \) on \( X_t \) specializing to a line bundle of degree \( d \) on \( X \) (this follows from the construction of the universal compactified Picard scheme \( P_{d, g} \to \mathcal{M}_g \), see [C2, subsection 5.2]). By the definition of \( \alpha \), it is clear that the multidegree \( \deg L_t \) is such that \( \alpha(\deg L_t) = d^2 \).

We are ready to prove the Theorem. Assume that \( P_{d, g} \) is of Néron type. Our goal is to prove that \( X^2 \) is \( d \)-general. By contradiction, let \( Z^2 \subset X^2 \) be a connected subcurve such that for some \( d^2 \in B_d(X^2) \) we have \( d^2_Z = m_Z(d) \). Let \( Z \) be the preimage of \( Z^2 \), and let \( d \in B_d(X) \) be such that \( \alpha(d) = d^2 \). Then
\[ d_Z = d^2_Z = m_Z(d) = m_Z(d). \]

By Proposition 2.5 we obtain that \( Z \cap Z^c \subset X_{\text{sep}} \). This is in contradiction with (2.4); so we are done.

Conversely, let \( X \) be weakly \( d \)-general; i.e. \( B_d(X^2) = B_d(X^2) \). To show that \( P_{d, g} \) is of Néron type we use again Proposition 2.5, according to which it suffices to show that for every \( d \in B_d(X) \) and for every \( Z \subseteq X \) such that \( Z \cap Z^c \neq X_{\text{sep}} \) we have \( d_Z > m_Z(d) \).

By contradiction. Let \( Z \) be a connected subcurve such that \( Z \cap Z^c \neq X_{\text{sep}} \), and \( d_Z = m_Z(d) \) for some balanced multidegree \( d \) on \( X \). We choose \( Z \) maximal with respect to this properties. This choice yields
\[ Z \cap Z^c \cap X_{\text{sep}} = \emptyset. \]

Indeed, if \( Z \cap Z^c \) contains some \( n \in X_{\text{sep}} \), there exists a connected component \( Z' \) of \( Z^c \) such that \( Z \cap Z' = \{n\} \). Let \( W := Z \cup Z' \); then \( W \) is a connected curve containing \( Z \). Now, \( W \cap W^c = Z \cap Z^c - \{n\} \), hence \( W \cap W^c \neq X_{\text{sep}} \); moreover, using Remark 1.11 one easily checks that \( d_W = m_W(d) \). This contradicts the maximality of \( Z \).
By (2.6) we have that \( Z \cap X_{\sep} \) is all contained in \( Z_{\sep} \) therefore, as observed immediately after (2.4), the curve \( Z \) is the preimage of a subcurve \( Z^2 \subset X^2 \). Now let \( d^2 = \pi(d) \); so \( d^2 \in B_2(X^2) = B_4(X^2) \) by hypothesis. We have
\[
d_{Z^2} = d_Z = m_Z(d) = m_Z^2(d).
\]
This contradicts the fact that \( d^2 \) is strictly balanced.

**Corollary 2.10.** Let \( X \) be a stable curve of genus \( g \), and let \( d = g - 1 \). Then \( P^d_X \) is of Néron type if and only if \( X \) is a tree-like curve.

**Proof.** As \( d = g - 1 \), by [M, Remark 2.3] \( X \) is \( d \)-general if and only if \( X \) is irreducible. Hence \( X \) is weakly \( d \)-general if and only if \( X \) is tree-like.

2.11. The locus of weakly \( d \)-general curves in \( \overline{M}_g \). The locus of \( d \)-general curves in \( \overline{M}_g \) has been studied in details in [M] (also in [CE] if \( d = 1 \) for applications to Abel maps). A stable curve \( X \) which is not \( d \)-general is called \( d \)-special. The locus of \( d \)-special curves is a closed subscheme denoted \( \Sigma^d_g \subset \overline{M}_g \). By [M, Lemma 2.10], \( \Sigma^d_g \) is the closure of the locus of \( d \)-special curves made of two smooth components. Curves made of two smooth components are called vine curves.

We are going to exhibit a precise description of \( D^d_g \), the complement in \( \overline{M}_g \) of the locus of weakly \( d \)-general curves:

\[
D^d_g := \{ X \in \overline{M}_g : P^d_X \text{ not of Néron type} \}.
\]

In the following statement by \( \text{codim} \ D^d_g \) we mean the codimension of an irreducible component of maximal dimension.

**Proposition 2.12.** \( D^d_g \) is the closure of the locus of \( d \)-special vine curves with at least 2 nodes. Moreover

\[
\text{codim} \ D^d_g = \begin{cases} 
+\infty \ (\text{i.e.} \ D^d_g = \emptyset) & \text{if} \ (d - g + 1, 2g - 2) = 1, \\
3 & \text{if} \ (d - g + 1, 2g - 2) = 2 \text{ and } g \text{ is even}, \\
2 & \text{otherwise}.
\end{cases}
\]

**Proof.** By Theorem 2.9, we have that \( X \in D^d_g \) if and only if \( X \) is not weakly \( d \)-general, if and only if \( X^2 \) is not \( d \)-general (where \( X^2 \) is as in 1.8). This is equivalent to the fact that there exists \( d \in B_4(X^2) \) and a subcurve \( Z \subset X^2 \) such that \( d_Z = m_Z(d) \); as \( X^2 \) has no separating nodes, for every subcurve \( Z \subset X^2 \) we have \( d_Z \geq 2 \). This observation added to the proof of [M, Lemma 2.10] gives that \( X^2 \) (and every curve with the same weighted graph) lies in the closure of the locus of \( d \)-special vine curves with at least two nodes. Therefore the same holds for \( X \), since \( X \) is a specialization of curve with the same weighted graph as \( X^2 \).

Conversely, let \( X \) be in the closure of the locus of \( d \)-special vine curves with at least two nodes. Then \( X^2 \) is also in this closure, as such vine curves are obviously free from separating nodes. By [M, Lemma 2.10] the curve \( X^2 \) is \( d \)-special, hence \( X \) is not weakly \( d \)-general.
Let us turn to the codimension of $D^d_d$. The fact that if $(d - g + 1, 2g - 2) = 1$ then $D^d_d$ is empty is well known ([C2]). Conversely, assume $D^d_d = \emptyset$. By the previous part, the locus of $d$-special vine curves with at least two nodes is also empty. Now the proof of the numerical Lemma 6.3 in [C1] shows that this implies that $(d - g + 1, 2g - 2) = 1$. In fact, the proof of that Lemma shows that if there are no $d$-special vine curves with two or three nodes then $(d - g + 1, 2g - 2) = 1$.

Next, recall that the locus, $V_\delta$, of vine curves with $\delta$ nodes has pure codimension $\delta$, and notice that the sublocus of $d$-special curves is a union of irreducible components of $V_\delta$.

Now, again by the proof of the above Lemma 6.3, if $(d - g + 1, 2g - 2) \neq 1$ and if there are no $d$-special vine curves with two nodes, then $(d - g + 1, 2g - 2) = 2$, $g$ is even and every vine curve with three nodes, having one component of genus $g/2 - 1$, is $d$-special. This completes the proof of the Proposition. □

A precise description of the locus of $d$-special vine curves is given in [M, Prop. 2.13]. Her result combined with the previous proposition yields a more precise description of the locus of stable curves whose compactified degree-$d$ Jacobian is of Néron type, for every fixed $d$.

References

COMPACTIFIED JACOBIANS OF NÉRON TYPE


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