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Algebraic Geometry — *Compactified Jacobians of Néron type*, by Lucia Caporaso, communicated on 12 November 2010.

Abstract. — We characterize stable curves X whose compactified degree-d Jacobian is of Néron type. This means the following: for any one-parameter regular smoothing of X, the special fiber of the Néron model of the Jacobian is isomorphic to a dense open subset of the degree-d compactified Jacobian of X. It is well known that compactified Jacobians of Néron type have the best modular properties, and that they are endowed with a mapping property useful for applications.

Key words: Stable curve, Picard scheme, Néron model, compactification, balanced line bundle.

Mathematic Subject Classification (2000): [redacted].

1. Introduction and preliminaries

Let X be a stable curve and $f : \mathscr{X} \to B$ a one-parameter smoothing of X with $\mathscr{X}$ a nonsingular surface; X is thus identified with the special fiber of f and all other fibers are smooth curves. Let $N_f^d \to B$ be the Néron model of the degree-d Jacobian of the generic fiber of f; its existence was proved by A. Néron in [N], and its connection with the Picard functor was established by M. Raynaud in [R]. So, $N_f^d \to B$ is a smooth and separated morphism, whose generic fiber is the degree-d Jacobian of the generic fiber of f; the special fiber, denoted N_X^d, is isomorphic to a disjoint union of copies of the generalized Jacobian of X. $N_f^d \to B$ can be interpreted as the maximal separated quotient of the degree-d Picard scheme $\mathrm{Pic}_f^d \to B$. In particular, if $\mathrm{Pic}_f^d \to B$ is separated, which happens if and only if X is irreducible, then $N_f^d = \mathrm{Pic}_f^d$ (we refer to [R], [BLR] or [Ar] for details).

The Néron model has a universal property, the Néron Mapping Property ([BLR, def. 1]), which determines it uniquely. Moreover, as d varies in $\mathbb{Z}$, the special fibers, N_X^d, of $N_f^d \to B$ are all isomorphic.

By contrast, the compactified degree-d Jacobian of a reducible curve X, denoted $\overline{P_X^d}$, has a structure which varies with d. For example, the number of irreducible components, and the modular properties, depend on d; see Section 2 for details and references.

$\overline{P_X^d}$ will be called of *Néron type* if its smooth locus is isomorphic to N_X^d. Compactified Jacobians of Néron type have the best modular properties. Moreover they inherit a mapping property from the universal property of the Néron model which provides a very useful tool; see for example [CE] for applications to Abel maps.

The purpose of this paper is to classify, for every d, those stable curves $X \in \overline{M_g}$ such that $\overline{P_X^d}$ is on Néron type. The question is interesting if $g \geq 2$, for otherwise $\overline{P_X^d}$ is always irreducible, and hence of Néron type.

Before stating our main result, we need a few words about compactified Jacobians. $\overline{P_X^d}$ parametrizes certain line bundles on quasistable curves having X as stabilization. These are the so-called "balanced" line bundles; among balanced line bundles there are some distinguished ones, called "strictly balanced", which have better modular properties. In fact, to every balanced line bundle there corresponds a unique point in $\overline{P_X^d}$, but different balanced line bundles may determine the same point. On the other hand every point of $\overline{P_X^d}$ corresponds to a unique class of strictly balanced line bundles.

The curve X is called *d-general* if every balanced line bundle of degree d is strictly balanced. This is equivalent to the fact that $\overline{P_X^d}$ is a geometric GIT-quotient.

The property of being d-general depends only on the weighted dual graph of X, and the locus of d-general curves in $\overline{M_g}$ has been precisely described by M. Melo in [M].

Now, the degree-d compactified Jacobian of a d-general curve is of Néron type, by [C2, Thm. 6.1]. But, as we will prove, the converse does not hold.

More precisely, a stable curve X is called *weakly d-general* if a curve obtained by smoothing every separating node of X, and maintaining all the non separating nodes, is d-general; see Definition 1.13.

Our main result, Theorem 2.9, states that $\overline{P_X^d}$ is of Néron type if and only if X is weakly d-general. The locus of weakly d-general curves in $\overline{M_g}$ is precisely described in section 2.11; its complement turns out to have codimension at least 2.

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1.1. *Notations and conventions*

(1) We work over an algebraically closed field k. The word "curve" means projective scheme of pure dimension one. The genus of a curve will be the arithmetic genus, unless otherwise specified.

(2) By X we will always denote a nodal curve of genus $g \geq 2$. For any subcurve $Z \subset X$ we denote by g_Z its arithmetic genus, by $Z^c := \overline{X \setminus Z}$ and by $\delta_Z := \# Z \cap Z^c$. We set $w_Z := \deg_Z \omega_X = 2g_Z - 2 + \delta_Z$.

(3) A node n of a connected curve X is called *separating* if $X \setminus \{n\}$ is not connected. The set of all separating nodes of X is denoted by X_{sep} and the set of all nodes of X by X_{sing}.

(4) A nodal curve X of genus $g \geq 2$ is called *stable* if it is connected and if every component $E \subset X$ such that $E \cong \mathbb{P}^1$ satisfies $\delta_E \geq 3$. X is called *quasistable* if it is connected, if every $E \subset X$ such that $E \cong \mathbb{P}^1$ satisfies $\delta_E \geq 2$, and if two exceptional components never intersect, where an exceptional component is defined as an $E \cong \mathbb{P}^1$ such that $\delta_E = 2$. We denote by X_{exc} the union of the exceptional components of X.

(5) Let $S \subset X_{\text{sing}}$ we denote by $\nu_S : X_S^\nu \to X$ the normalization of X at S, and by $\hat{X}_S$ the quasistable curve obtained by "blowing-up" all the nodes in S, so that there is a natural surjective map

$$\hat{X}_S = \bigcup_{i=1}^{\#S} E_i \cup X_S^\nu \to X$$

restricting to ν_S on X_S^ν and contracting all the exceptional components E_i of $\hat{X}_S$. $\hat{X}_S$ is also called a *quasistable curve of* X.

(6) Let $C_1, \dots, C_\gamma$ be the irreducible components of X. Every line bundle $L \in \operatorname{Pic} X$ has a multidegree $\underline{\deg}\, L = (\deg_{C_1} L, \dots, \deg_{C_\gamma} L) \in \mathbb{Z}^\gamma$. Let $\underline{d} = (d_1, \dots, d_\gamma) \in \mathbb{Z}^\gamma$, we set $|\underline{d}| = \sum_1^\gamma d_i$; for any subcurve $Z \subset X$ we abuse notation slightly and denote

$$\underline{d}_Z := \sum_{C_i \subset Z} d_i.$$

1.2. *Compactified Jacobians of Néron type*. Let X be any nodal connected curve and $f : \mathscr{X} \to B$ a one-parameter *regular smoothing* for X, i.e. B is a smooth connected one-dimensional scheme with a marked point $b_0 \in B$, $\mathscr{X}$ is a regular surface, and f is a projective morphism whose fiber over b_0 is X and whose remaining fibers are smooth curves. We set $U := B \setminus \{b_0\}$ and let $f_U : \mathscr{X}_U \to U$ be the family of smooth curves obtained by restricting f to U. Consider the relative degree d Picard scheme over U, denoted $\operatorname{Pic}^d_{f_U} \to U$. Its Néron model over B will be denoted by

$$N_f^d := N(\operatorname{Pic}^d_{f_U}) \to B, \tag{1.1}$$

and its fiber over b_0 will be denoted by N_X^d; N_X^d is isomorphic to a finite number of copies of the generalized Jacobian of X. The number of copies is independent of d; to compute it we introduce the so-called "degree class group".

Let γ be the number of irreducible components of X. For every component C_i of X set $k_{i,j} := \#(C_i \cap C_j)$ if $j \neq i$, and $k_{i,i} = -\#(C_i \cap \overline{C \setminus C_i})$ so that the matrix $(k_{i,j})$ is symmetric matrix. Notice that for every regular smoothing $f : \mathscr{X} \to B$ of X as above, we have $\deg_{C_j} \mathscr{O}_{\mathscr{X}}(C_i) = k_{i,j}$. Hence this matrix is also related to f, although it does not depend on the choice of f (as long as $\mathscr{X}$ is regular).

We have $\sum_{j=1}^\gamma k_{i,j} = 0$ for every i. Now, for every $i = 1, \dots, \gamma$ set $\underline{c}_i := (k_{1,i}, \dots, k_{\gamma,i}) \in \mathbb{Z}^\gamma$ and $\mathbf{Z} := \{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = 0\}$ so that $\underline{c}_i \in \mathbf{Z}$. We can now define the sublattice $\Lambda_X := \langle \underline{c}_1, \dots, \underline{c}_\gamma \rangle \subset \mathbf{Z}$.

The *degree class group* of X is the group $\Delta_X := \mathbf{Z}/\Lambda_X$. It is not hard to prove that Δ_X is a finite group.

Let $\underline{d}$ and $\underline{d}'$ be in $\mathbb{Z}^\gamma$; we say that they are equivalent if $\underline{d} - \underline{d}' \in \Lambda_X$. We denote by Δ_X^d the set of equivalence classes of multidegrees of total degree d; for a multidegree $\underline{d}$ we write $[\underline{d}]$ for its class. It is clear that $\Delta_X = \Delta_X^0$ and that

$$\#\Delta_X = \#\Delta_X^d.$$

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Now back to N_X^d, the special fiber of (1.1); as we said it is a smooth, possibly non connected scheme of pure dimension g.

Fact 1.3. *Under the above assumptions, the number of irreducible* (*i.e. connected*) *components of* N_X^d *is equal to* $\#\Delta_X$.

This is well known; see [R, 8.1.2] (where Δ_X is the same as $\ker\beta/\operatorname{Im}\alpha$) or [BLR, thm. 9.6.1]. Using the standard notation of Néron models theory we have $\Delta_X = \Phi_{N_f^d}$, i.e. Δ_X is the "component group" of N_f^d.

For every stable curve X and every d we denote by $\overline{P_X^d}$ the degree d compactified Jacobian (or degree-d compactified Picard scheme). $\overline{P_X^d}$ has been constructed in [OS] for a fixed curve, and independently for families in [S] and in [C1] (the constructions of [OS] and [S] are here considered with respect to the canonical polarization); these three constructions give the same scheme by [Al], see also [LM]. We mention that another compactified Jacobian is constructed in [E], whose connection with the others is under investigation; see [MV]. An explicit description of $\overline{P_X^d}$ will be recalled in 2.2. We here anticipate the fact that $\overline{P_X^d}$ is a connected, projective scheme of pure dimension g. As we said in the introduction, several geometric and modular properties of $\overline{P_X^d}$ depend on d.

Definition 1.4. Let X be a stable curve and $\overline{P_X^d}$ its degree-d compactified Jacobian. We say that $\overline{P_X^d}$ is of *Néron type* if the number of irreducible components of $\overline{P_X^d}$ is equal to the number of irreducible components of N_X^d.

Example 1.5. A curve X is called *tree-like* if every node of X lying in two different irreducible components is a separating node.

The compactified Jacobian of a tree-like curve X is easily seen to be always of Néron type. Indeed, $\overline{P_X^d}$ is irreducible for every d; on the other hand $\#\Delta_X = 1$ so that N_X^d is also irreducible.

Let now $\pi : \overline{P_f^d} \to B$ be the compactified degree-d Picard scheme of a regular smoothing $f : \mathscr{X} \to B$ of a stable curve X, as defined in 1.2. So the fiber of π over b_0 is $\overline{P_X^d}$, and the restriction of π over $U = B \setminus \{b_0\} \subset B$ is $\operatorname{Pic}^d_{f_U}$. We denote $P_f^d \to B$ the smooth locus of π. By the Néron Mapping Property there exists a canonical B-morphism, χ_f, from P_f^d to the Néron model of $\operatorname{Pic}^d_{f_U}$:

$$\chi_f : P_f^d \to N_f^d$$

extending the indentity map from the generic fiber of π to the generic fiber of $N_f^d \to B$.

Proposition 1.6. *With the above set up,* $\overline{P_X^d}$ *is of Néron type if and only if the map* $\chi_f : P_f^d \to N_f^d$ *is an isomorphism for every* $f : \mathscr{X} \to B$ *as above.*

The proof, requiring a description of $\overline{P_f^d}$, will be given in subsection 2.6.

1.7. *Smoothing separating nodes.* A stable weighted graph of genus $g \geq 2$ is a pair (Γ, w), where Γ is a graph and $w : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ a *weight function*. The genus of (Γ, w) is the number $g_{(\Gamma, w)}$ defined as follows:

$$g_{(\Gamma, w)} = \sum_{v \in V(\Gamma)} w(v) + b_1(\Gamma).$$

A weigthed graph will be called *stable* if every $v \in V(\Gamma)$ such that $w(v) = 0$ has valency at least 3.

Let X be a nodal curve of genus g, the weighted dual graph of X is the weighted graph (Γ_X, w_X) such that Γ_X is the usual dual graph of X (the vertices of Γ_X are identified with the irreducible components of X and the edges are identified with the nodes of X; an edge joins two, possibly equal, vertices if the corresponding node is in the intersection of the corresponding irreducible components), and w_X is the *weight function* on the set of irreducible components of X, $V(\Gamma_X)$, assigning to a vertex the geometric genus of the corresponding component. Hence

$$g = \sum_{v \in V(\Gamma_X)} w_X(v) + b_1(\Gamma_X) = g_{(\Gamma_X, w_X)}.$$

X is a stable curve if and only if (Γ_X, w_X) is a stable weighted graph.

Now we ask: What happens to the weighted dual graph of X if we smooth all the separating nodes of X?

To answer this question, we introduce a new weighted graph, denoted by (Γ^2, w^2), associated to a weighted graph (Γ, w). (Γ^2, w^2) is defined as follows. Γ^2 is the graph obtained by contracting every separating edge of Γ to a point. Therefore Γ^2 is 2-edge-connected, i.e. free from separating edges (this explains the notation). To define the weight function w^2, notice that there is a natural surjective map contracting the separating edges of Γ

$$\sigma : \Gamma \to \Gamma^2,$$

and an induced surjection on the set of vertices

$$\phi : V(\Gamma) \to V(\Gamma^2); \quad v \mapsto \sigma(v).$$

Now we define w^2 as follows. For every $v^2 \in V(\Gamma^2)$

$$w^2(v^2) = \sum_{v \in \phi^{-1}(v^2)} w(v).$$

As σ does not contract any cycle, $b_1(\Gamma) = b_1(\Gamma^2)$ and $g_{(\Gamma, w)} = g_{(\Gamma^2, w^2)}$.

Remark 1.8. If (Γ, w) is the weighted dual graph of a curve X, (Γ^2, w^2) is the weighted dual graph of any curve obtained by smoothing every separating node

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of X. We shall usually denote by X^2 such a curve. Of course X and X^2 have the same genus.

1.9. *d-general and weakly d-general curves.* Let us recall the definitions of balanced and strictly balanced multidegrees.

Definition 1.10. Let X be a quasistable curve of genus $g \geq 2$ and $L \in \operatorname{Pic}^d X$. Let $\underline{d}$ be the multidegree of L.

(1) We say that L, or $\underline{d}$, is *balanced* if for any subcurve (equivalently, for any connected subcurve) $Z \subset Y$ we have (notation in 1.1(2))

$$\deg_Z L \geq m_Z(d) := \frac{dw_Z}{2g-2} - \frac{\delta_Z}{2}, \tag{1.2}$$

and $\deg_Z L = 1$ if Z is an exceptional component.
(2) We say that L, or $\underline{d}$, is *strictly balanced* if it is balanced and if strict inequality holds in (1.2) for every $Z \subsetneq X$ such that $Z \cap Z^c \not\subset X_{\text{exc}}$.
(3) We denote

$$\overline{B_d(X)} = \{\underline{d} : |\underline{d}| = d \text{ balanced on } X\} \supset B_d(X) = \{\underline{d} : \text{strictly balanced}\}.$$

The following trivial observations are useful.

Remark 1.11. (A) Let $Z = Z_1 \sqcup Z_2 \subset X$ be a disconnected subcurve. Then $m_Z(d) = m_{Z_1}(d) + m_{Z_2}(d)$.
(B) Suppose X stable and $\underline{d} \in \overline{B_d(X)}$. Then $\underline{d}$ is not strictly balanced if and only if there exists a subcurve $Z \subsetneq X$ such that $\underline{d}_Z = m_Z(d)$, or equivalently, $\underline{d}_{Z^c} = m_{Z^c}(d) + \delta_Z$.

Remark 1.12. Let X be stable. By [C2, Prop. 4.12], every multidegree class in Δ_X^d has a balanced representative, which is unique if and only if it is strictly balanced. Therefore

$$\# B_d(X) \leq \# \Delta_X \leq \# \overline{B_d(X)}.$$

The terminology "strictly balanced" is not to be confused with "stably balanced" (used elsewhere and unnecessary here). The two coincide for stable curves; in general, a stably balanced line bundle is strictly balanced, but the converse may fail. Let us explain the difference. The compactified Picard scheme of X, $\overline{P_X^d}$ is a GIT-quotient of a certain scheme by a certain reductive group G. Strictly balanced line bundles correspond to the GIT-semistable orbits that are closed in the GIT-semistable locus. Stably balanced line bundles correspond to GIT-stable points and balanced line bundles correspond to GIT-semistable points. As every point in $\overline{P_X^d}$ parametrizes a unique closed orbit, strictly balanced

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line bundles of degree d on quasistable curves of X are bijectively parametrized by $\overline{P_X^d}$. See Fact 2.2 below.

Definition 1.13. Let X be a stable curve. We will say that X, or its weighted dual graph (Γ_X, w_X), is *d-general* if $B_d(X) = \overline{B_d(X)}$ (cf. [C2, 4.13]). (Equivalently, X is d-general if the inequalities in Remark 1.12 are both equalities.)

We will say that X is *weakly d-general* if (Γ_X^2, w_X^2) is d-general.

Remark 1.14. The following facts are well known (see loc.cit.).

(1) The set of d-general stable curves is a nonempty open subset of $\overline{M_g}$.
(2) $(d-g+1, 2g-1) = 1$ if and only if every stable curve of genus g is d-general.
(3) The property of being d-general depends only on the weighted dual graph (obvious).

Example 1.15. If $X_{\mathrm{sep}} = \emptyset$, then X is d-general if and only if it is weakly general.

If X is tree-like, then (Γ_X^2, w_X^2) has only one vertex, hence it is d-general for every d. Therefore tree-like curves are weakly d-general for every d.

2. Irreducible components of compactified Jacobians

2.1. *Compactified degree-d Jacobians.* Let us describe the compactified Jacobian $\overline{P_X^d}$ for any degree d. We use the set up of [C1] and [C2]; in these papers there is the assumption $g \geq 3$, but by [OS], [S] and [Al] we can extend our results to $g \geq 2$. A synthetic account of the modular properties of the compactified Jacobian for a curve or for a family can be found in [CE, 3.8 and 5.10].

Fact 2.2. *Let X be a stable curve of genus $g \geq 2$. Then $\overline{P_X^d}$ is a connected, reduced, projective scheme of pure dimension g, admitting a canonical decomposition* (*notation in* 1.1(5))

$$\overline{P_X^d} = \coprod_{\substack{S \subset X_{\mathrm{sing}} \\ \underline{d} \in B_d(\hat{X}_S)}} P_S^{\underline{d}}$$

such that for every $S \subset X_{\mathrm{sing}}$ and $\underline{d} \in B_d(\hat{X}_S)$ there is a natural isomorphism

$$P_S^{\underline{d}} \cong \mathrm{Pic}^{\underline{d}^\nu} X_S^\nu$$

where $\underline{d}^\nu$ denotes the multidegree on $X_S^\nu \subset \hat{X}_S$ defined by restricting $\underline{d}$.

Let $\mathrm{i}(\overline{P_X^d})$ *be the number of irreducible components of* $\overline{P_X^d}$*; then*

$$B_d(X) \leq \mathrm{i}(\overline{P_X^d}) \leq \#\Delta_X. \tag{2.1}$$

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Corollary 2.3. *Let X be a stable curve.*

(1) *The decomposition of $\overline{P_X^d}$ in irreducible components is*

$$\overline{P_X^d} = \bigcup_{(S,\underline{d})\in I_X^d} \overline{P_S^{\underline{d}}}, \quad \textit{where } I_X^d := \{(S,\underline{d}) : S \subset X_{\rm sep}, \underline{d} \in B_d(\hat{X}_S)\}.$$

(2) *Suppose that X is d-general; then P_X^d is of Néron type, and for every nonempty $S \subset X_{\rm sep}$ we have $B_d(\hat{X}_S) = \emptyset$.*

Proof. From Fact 2.2 we have that the irreducible components of $\overline{P_X^d}$ are the closures of subsets $P_S^{\underline{d}} \cong \mathrm{Pic}^{\underline{d}^\nu} X_S^\nu$ where S is such that $\dim \mathrm{Pic}^{\underline{d}^\nu} X_S^\nu = g$. Now, it is clear that

$$\dim \mathrm{Pic}^{\underline{d}^\nu} X_S^\nu = \dim J(X_S^\nu) = g \quad \text{if and only if } S \subset X_{\rm sep}.$$

Therefore the irreducible components of $\overline{P_X^d}$ correspond bijectively to pairs $(S,\underline{d})$ with $S \subset X_{\rm sep}$ and $\underline{d} \in B_d(\hat{X}_S)$.

Now part (2). It is clear that the set I_X^d contains a subset identifiable with $B_d(X)$, namely the subset $\{(\emptyset,\underline{d}) : \underline{d} \in B_d(X)\}$. If X is d-general then $\#B_d(X) = \#\Delta_X$, hence by (2.1) we must have that I_X^d contains no pairs other than those of type $(\emptyset,\underline{d})$. This concludes the proof. □

Lemma 2.4. *Let X be a stable curve and let $\mu \in \Delta_X^d$ be a multidegree class. Then there exists a unique $S(\mu) \subset X_{\rm sing}$ and a unique $\underline{d}(\mu) \in B_d(\widehat{X_{S(\mu)}})$ such that for every $\underline{d} \in \overline{B_d(X)}$ with $[\underline{d}] = \mu$ the following properties hold.*

(1) *There is a canonical surjection*

$$\mathrm{Pic}^{\underline{d}} X \to P_{S(\mu)}^{\underline{d}(\mu)} \cong \mathrm{Pic}^{\underline{d}(\mu)^\nu} X_{S(\mu)}^\nu.$$

(2) *We have*

$$S(\mu) = \bigcup_{Z \subset X: \underline{d}_Z = m_Z(d)} Z \cap Z^c.$$

Proof. The proof is routine. Let us sketch it using the combinatorial results [C1, Lemma 5.1 and Lemma 6.1]. The terminology used in that paper differs from ours as follows: what we here call a "strictly balanced multidegree $\underline{d}$ on a quasistable curve X" is there called an "extremal pair $(X,\underline{d})$"; cf. subsection 5.2 p. 631.

So, the pair $(\widehat{X_{S(\mu)}}, \underline{d}(\mu))$ is the "extremal pair" associated to μ. This means the following. For every balanced line bundle L on X such that $[\underline{\deg} L] = \mu$ the point in $\overline{P_X^d}$ associated to L parametrizes a line bundle $\hat{L} \in \mathrm{Pic}^{\underline{d}} \widehat{X_{S(\mu)}}$, and the restriction of $\hat{L}$ to X_S^ν is uniquely determined by L. Conversely every line bundle in $\mathrm{Pic}^{\underline{d}(\mu)^\nu} X_{S(\mu)}^\nu$ is obtained in this way.

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More precisely, as we said, $\overline{P_X^d}$ is a GIT quotient; let us denote it by $\overline{P_X^d} = V_X/G$, so that V_X is made of GIT-semistable points. Let $O_G(L) \subset V_X$ be the orbit of L. Then the semistable closure of $O_G(L)$ contains a unique closed orbit $O_G(\hat{L})$ as above. Moreover for every $\underline{d}' \in B_d(X)$ having class μ there exists $L' \in \operatorname{Pic}^{\underline{d}'} X$ such that the above $O_G(\hat{L})$ lies in the closure of $O_G(L')$. Hence the maps $\operatorname{Pic}^{\underline{d}} X \to \overline{P_X^d}$ and $\operatorname{Pic}^{\underline{d}'} X \to \overline{P_X^d}$ have the same image.

Using the notation of Fact 2.2, we have that for every balanced $\underline{d}$ of class μ the canonical map $\operatorname{Pic}^{\underline{d}} X \to \overline{P_X^d}$ has image $P_{S(\mu)}^{\underline{d}(\mu)}$, so that the first part is proved.

Now (2). The previously mentioned Lemma 5.1 implies that for every $\underline{d} \in \overline{B_d(X)}$ and every Z such that $\underline{d}_Z = m_Z(d)$ we have $Z \cap Z^c \subset S(\mu)$. By the above Lemma 6.1 each $n \in S(\mu)$ is obtained in this way. □

Proposition 2.5. *Let X be a stable curve. $\overline{P_X^d}$ is of Néron type if and only if for every $\underline{d} \in \overline{B_d(X)}$ and every connected $Z \subsetneq X$ such that $\underline{d}_Z = m_Z(d)$ we have*

$$Z \cap Z^c \subset X_{\text{sep}}. \tag{2.2}$$

Proof. We begin by observing that, with the notation of Corollary 2.3 and Lemma 2.4, we have

$$I_X^d = \{(\underline{d}(\mu), S(\mu)), \forall \mu \in \Delta_X^d \text{ such that } \dim P_{S(\mu)}^{\underline{d}(\mu)} = g\}.$$

Indeed, by Fact 2.2 the set on the right is clearly included in I_X^d. On the other hand let $(S, \underline{d}) \in I_X^d$. To show that there exists $\mu \in \Delta_X^d$ such that $\underline{d} = \underline{d}(\mu)$ we can assume that $S \neq \emptyset$ (otherwise it is obvious). So, $\underline{d}$ is a strictly balanced multidegree of total degree d on $\hat{X}_S$. Let $n \in S$; by Corollary 2.3 the node n is separating for X; let $X = Z \cup Z^c$ with $Z \cap Z^c = \{n\}$. Then Z and Z^c can be viewed as subcurves of $\hat{X}_S$, where they do not intersect since the node n is replaced by an exceptional component E. Now, $\underline{d}_E = 1$, therefore $\underline{d}_Z = m_Z(d)$ and $\underline{d}_{Z^c} = m_{Z^c}(d)$. Let $C_Z \subset Z \subset X$ be the irreducible component intersecting Z^c (so that $C_Z \subset \hat{X}_S$ intersects E). Let $\underline{d}^X$ be the multidegree on X defined as follows: for every irreducible component $C \subset X$

$$\underline{d}_C^X = \begin{cases} \underline{d}_C + 1 & \text{if } C = C_Z, \\ \underline{d}_C & \text{otherwise.} \end{cases}$$

As $\underline{d}$ is balanced on $\hat{X}_S$ one easily cheks that $\underline{d}^X$ is balanced on X. Note that $\underline{d}^X$ is not strictly balanced, since $\underline{d}_{Z^c}^X = m_{Z^c}(d)$ (see Remark 1.11). By iterating the above procedure for every node in S we arrive at a balanced multidegree on X whose class we denote by $\mu \in \Delta_X^d$. By Lemma 2.4 we have that $\underline{d} = \underline{d}(\mu)$.

Suppose that P_X^d is of Néron type. By the previous discussion there is a natural bijection between Δ_X^d and I_X^d, mapping $\mu \in \Delta_X^d$ to $(S(\mu), \underline{d}(\mu))$. By Corollary 2.3 we have $S(\mu) \subset X_{\text{sep}}$. Hence for every multidegree $\underline{d} \in \overline{B_d(X)}$ such that $[\underline{d}] = \mu$ we have that condition (2) of that lemma holds. In particular every Z as in our statement is such that $Z \cap Z^c \subset S(\mu) \subset X_{\text{sep}}$.

Conversely, if P_X^d is not of Néron type there is a class $\mu \in \Delta_X^d$ such that

$$g > \dim P_{S(\mu)}^{d(\mu)} = \dim J(X_{S(\mu)}^{\nu}).$$

But then $S(\mu)$ contains some non separating node of X. Hence, by Lemma 2.4(2), there exists a connected subcurve $Z \subset X$ such that $\underline{d}_Z = m_Z(d)$ and such that $Z \cap Z^c$ contains some non separating node. □

2.6. *Proof of Proposition* 1.6. We generalize the proof of [C2, Thm. 6.1]. Let $f : \mathscr{X} \to B$ be a regular smoothing of X as defined in subsection 1.2, and $\pi : \overline{P_f^d} \to B$ be the compactified degree-d Picard scheme. Its smooth locus $P_f^d \to B$ is such that its fiber over b_0, denoted P_X^d, satisfies

$$P_X^d = \coprod_{(S,\underline{d}) \in I_X^d} P_S^{\underline{d}} \tag{2.3}$$

(notation in 2.3) where each $P_S^{\underline{d}}$ is irreducible of dimension g. If the morphism $\chi_f : P_f^d \to N_f^d$ is an isomorphism, then P_X^d has as many irreducible components as N_X^d, hence the same holds for $\overline{P_X^d}$. So $\overline{P_X^d}$ is of Néron type.

Conversely, if $\overline{P_X^d}$ is of Néron type, then P_X^d has an irreducible component for every $\mu \in \Delta_X^d$ so that (2.3) takes the form

$$P_X^d = \coprod_{\mu \in \Delta_X^d} P_{S(\mu)}^{d(\mu)}.$$

Let us construct the inverse of χ_f. We pick a balanced representative $\underline{d}^{\mu}$ for every multidegree class $\mu \in \Delta_X^d$ (it exists by Remark 1.12). By [C2, Lemma 3.10] we have

$$N_f^d \cong \frac{\coprod_{\mu \in \Delta_X^d} \mathrm{Pic}_f^{\underline{d}^{\mu}}}{\sim_U}$$

where $\sim_U$ denotes the gluing of the Picard schemes $\mathrm{Pic}_f^{\underline{d}^{\mu}}$ along their restrictions over U (as $\mathrm{Pic}_{f_U}^{\underline{d}^{\mu}} = \mathrm{Pic}_{f_U}^d$ for every μ). Now, the Picard scheme $\mathrm{Pic}_f^{\underline{d}^{\mu}}$ is endowed with a Poincaré bundle, which is a relatively balanced line bundle on $\mathscr{X} \times_B \mathrm{Pic}_f^{\underline{d}^{\mu}}$. By the modular property of $\overline{P_f^d}$ the Poincaré bundle induces a canonical B-morphism

$$\psi_f^{\mu} : \mathrm{Pic}_f^{\underline{d}^{\mu}} \to P_{S(\mu)}^{\underline{d}(\mu)} \subset \overline{P_f^d}.$$

As μ varies, the restrictions of these morphisms over U all coincide with the identity map $\mathrm{Pic}_{f_U}^d \to \mathrm{Pic}_{f_U}^d \subset \overline{P_f^d}$. Therefore the ψ_f^{μ} can be glued together to a morphism

$$\psi_f : N_f^d \to P_f^d \subset \overline{P_f^d}.$$

It is clear that ψ_f is the inverse of χ_f. Proposition 1.6 is proved. □

2.7. *The main result*. From Proposition 2.5 we derive the following.

COROLLARY 2.8. *Let X be a stable curve free from separating nodes. Then $\overline{P_X^d}$ is of Néron type if and only if X is d-general.*

PROOF. By Corollary 2.3(2) there is only one implication to prove. Namely, suppose that X is not d-general. Then there exists $\underline{d} \in \overline{B_d(X)} \setminus B_d(X)$, and hence a subcurve $Z \subset X$ such that $\underline{d}_Z = m_Z(d)$ (see Remark 1.11). As $X_{\mathrm{sep}} = \emptyset$, condition (2.2) of Proposition 2.5 cannot be satisfied. Therefore $\overline{P_X^d}$ is not of Néron type. □

We are ready to prove our main result.

THEOREM 2.9. *Let X be a stable curve. Then $\overline{P_X^d}$ is of Néron type if and only if X is weakly d-general.*

PROOF. Observe that if X is free from separating nodes we are done by Corollary 2.8. Let (Γ, w) be the weigthed graph of X and consider the weighted graph (Γ^2, w^2) defined in subsection 1.7. We denote by X^2 a stable curve whose weighted graph is (Γ^2, w^2). By Remark 1.8 the curve X^2 can be viewed as a smoothing of X at X_{sep}.

Recall that we denote by $\sigma : \Gamma \to \Gamma^2$ the contraction map and by

$$\phi : V(\Gamma) \to V(\Gamma^2); \quad v \mapsto \sigma(v)$$

the induced map on the vertices, i.e. on the irreducible components. The subcurves of X naturally correspond to the so-called "induced" subgraphs of Γ, i.e. those subgraphs Γ' such that if two vertices v, w of Γ are in Γ', then every edge of Γ joining v with w lies in Γ'. Similarly, the induced subgraphs of Γ^2 correspond to subcurves of X^2. If Z^2 is a subcurve of X^2, and $\Gamma_{Z^2} \subset \Gamma_{X^2}$ its corresponding subgraph, we denote by $Z \subset X$ the subcurve associated to $\sigma^{-1}(\Gamma_{Z^2})$ (it is obvious that the subgraph $\sigma^{-1}(\Gamma_{Z^2})$ is induced if so is Γ_{Z^2}); we refer to Z as the "preimage" of Z^2. Of course $\sigma(\Gamma_Z) = \Gamma_{Z^2}$.

For any $Z \subset X$ which is the preimage of a subcurve $Z^2 \subset X^2$ we have

$$Z \cap X_{\mathrm{sep}} \subset Z_{\mathrm{sep}} \tag{2.4}$$

or, equivalently, $Z \cap Z^c \cap X_{\mathrm{sep}} = \emptyset$. Conversely, every $Z \subset X$ satisfying (2.4) is the preimage of some $Z^2 \subset X^2$.

Hence Z^2 can be viewed as a smoothing of Z at its separating nodes that are also separating nodes of X, i.e. at $Z_{\mathrm{sep}} \cap X_{\mathrm{sep}}$. Thus, for every Z^2 with preimage Z we have $g_Z = g_{Z^2}$ and $\delta_Z = \delta_{Z^2}$; hence for every $d \in \mathbb{Z}$

$$m_{Z^2}(d) = m_Z(d). \tag{2.5}$$

We shall now view multidegrees as an integer valued functions on the vertices. We claim that we have a surjection

$$\alpha : \overline{B_d(X)} \to \overline{B_d(X^2)}$$

defined as follows: for every vertex $v^2 \in V(\Gamma^2)$ we set

$$\alpha(\underline{d})(v^2) := \sum_{v \in \phi^{-1}(v^2)} \underline{d}(v).$$

Let us first show that if $\underline{d}$ is balanced, so is $\alpha(\underline{d})$. For every subcurve $Z^2 \subset X^2$ we have $\alpha(\underline{d})_{Z^2} = \underline{d}_Z$ where $Z \subset X$ is the preimage of Z^2; by (2.5) the inequality (1.2) is satisfied on Z^2 if (and only if) it is satisfied on Z.

Let us now show that α is surjective. Let $\underline{d}^2$ be a balanced multidegree on X^2; we know that X^2 can be chosen to be a smoothing of X at X_{sep}. In other words there exists a family of curves X_t, all having (Γ^2, w^2) as weighted graph, specializing to X. But then there also exists a family of balanced line bundles L_t on X_t specializing to a line bundle of degree $\underline{d}$ on X (this follows from the construction of the universal compactified Picard scheme $\overline{P_{d,g}} \to \overline{M_g}$, see [C2, subsection 5.2]). By the definition of α, it is clear that the multidegree $\underline{\deg}\, L_t$ is such that $\alpha(\underline{\deg}\, L_t) = \underline{d}^2$.

We are ready to prove the Theorem. Assume that $\overline{P_X^d}$ is of Néron type. Our goal is to prove that X^2 is d-general. By contradiction, let $Z^2 \subset X^2$ be a connected subcurve such that for some $\underline{d}^2 \in \overline{B_d(X^2)}$ we have $\underline{d}^2_{Z^2} = m_{Z^2}(d)$. Let Z be the preimage of Z^2, and let $\underline{d} \in \overline{B_d(X)}$ be such that $\alpha(\underline{d}) = \underline{d}^2$. Then

$$\underline{d}_Z = \underline{d}^2_{Z^2} = m_{Z^2}(d) = m_Z(d).$$

By Proposition 2.5 we obtain that $Z \cap Z^c \subset X_{\text{sep}}$. This is in contradiction with (2.4); so we are done.

Conversely, let X be weakly d-general; i.e. $\overline{B_d(X^2)} = B_d(X^2)$. To show that $\overline{P_X^d}$ is of Néron type we use again Proposition 2.5, according to which it suffices to show that for every $\underline{d} \in \overline{B_d(X)}$ and for every $Z \subsetneq X$ such that $Z \cap Z^c \not\subset X_{\text{sep}}$ we have $\underline{d}_Z > m_Z(d)$.

By contradiction. Let Z be a connected subcurve such that $Z \cap Z^c \not\subset X_{\text{sep}}$, and $\underline{d}_Z = m_Z(d)$ for some balanced multidegree $\underline{d}$ on X. We choose Z maximal with respect to this properties. This choice yields

$$Z \cap Z^c \cap X_{\text{sep}} = \emptyset. \tag{2.6}$$

Indeed, if $Z \cap Z^c$ contains some $n \in X_{\text{sep}}$, there exists a connected component Z' of Z^c such that $Z \cap Z' = \{n\}$. Let $W := Z \cup Z'$; then W is a connected curve containing Z. Now, $W \cap W^c = Z \cap Z^c - \{n\}$, hence $W \cap W^c \not\subset X_{\text{sep}}$; moreover, using Remark 1.11 one easily checks that $\underline{d}_W = m_W(d)$. This contradicts the maximality of Z.

By (2.6) we have that $Z \cap X_{\mathrm{sep}}$ is all contained in Z_{sep} therefore, as observed immediately after (2.4), the curve Z is the preimage of a subcurve $Z^2 \subset X^2$. Now let $\underline{d}^2 = \alpha(\underline{d})$; so $\underline{d}^2 \in \overline{B_d(X^2)} = B_d(X^2)$ by hypothesis. We have

$$\underline{d}^2_{Z^2} = \underline{d}_Z = m_Z(d) = m_{Z^2}(d).$$

This contradicts the fact that $\underline{d}^2$ is strictly balanced. □

Corollary 2.10. *Let X be a stable curve of genus g, and let $d = g - 1$. Then $\overline{P_X^d}$ is of Néron type if and only if X is a tree-like curve.*

Proof. As $d = g - 1$, by [M, Remark 2.3] X is d-general if and only if X is irreducible. Hence X is weakly d-general if and only if X is tree-like. □

2.11. *The locus of weakly d-general curves in $\overline{M_g}$.* The locus of d-general curves in $\overline{M_g}$ has been studied in details in [M] (also in [CE] if $d = 1$ for applications to Abel maps). A stable curve X which is not d-general is called *d-special*. The locus of d-special curves is a closed subscheme denoted $\Sigma_g^d \subset \overline{M_g}$. By [M, Lemma 2.10], Σ_g^d is the closure of the locus of d-special curves made of two smooth components. Curves made of two smoth components are called *vine curves*.

We are going to exhibit a precise description of D_g^d, the complement in $\overline{M_g}$ of the locus of weakly d-general curves:

$$D_g^d := \{X \in \overline{M_g} : \overline{P_X^d} \text{ not of Néron type}\}.$$

In the following statement by $\operatorname{codim} D_g^d$ we mean the codimension of an irreducible component of maximal dimension.

Proposition 2.12. *D_g^d is the closure of the locus of d-special vine curves with at least 2 nodes. Moreover*

$$\operatorname{codim} D_g^d = \begin{cases} +\infty \ (i.e.\ D_g^d = \emptyset) & \text{if } (d - g + 1, 2g - 2) = 1, \\ 3 & \text{if } (d - g + 1, 2g - 2) = 2 \text{ and } g \text{ is even}, \\ 2 & \text{otherwise}. \end{cases}$$

Proof. By Theorem 2.9, we have that $X \in D_g^d$ if and only if X is not weakly d-general, if and only if X^2 is not d-general (where X^2 is as in 1.8). This is equivalent to the fact that there exists $\underline{d} \in B_d(X^2)$ and a subcurve $Z \subsetneq X^2$ such that $\underline{d}_Z = m_Z(d)$; as X^2 has no separating nodes, for every subcurve $Z \subsetneq X^2$ we have $\delta_Z \geq 2$. This observation added to the proof of [M, Lemma 2.10] gives that X^2 (and every curve with the same weighted graph) lies in the closure of the locus of d-special vine curves with at least two nodes. Therefore the same holds for X, since X is a specialization of curve with the same weighted graph as X^2.

Conversely, let X be in the closure of the locus of d-special vine curves with at least two nodes. Then X^2 is also in this closure, as such vine curves are obviously free from separating nodes. By [M, Lemma 2.10] the curve X^2 is d-special, hence X is not weakly d-general.

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Let us turn to the codimension of D_g^d. The fact that if $(d-g+1, 2g-2) = 1$ then D_g^d is empty is well known ([C2]). Conversely, assume $D_g^d = \emptyset$. By the previous part, the locus of d-special vine curves with at least two nodes is also empty. Now the proof of the numerical Lemma 6.3 in [C1] shows that this implies that $(d-g+1, 2g-2) = 1$. In fact, the proof of that Lemma shows that if there are no d-special vine curves with two or three nodes then $(d-g+1, 2g-2) = 1$.

Next, recall that the locus, V_δ, of vine curves with δ nodes has pure codimension δ, and notice that the sublocus of d-special curves is a union of irreducible components of V_δ.

Now, again by the proof of the above Lemma 6.3, if $(d-g+1, 2g-2) \neq 1$ and if there are no d-special vine curves with two nodes, then $(d-g+1, 2g-2) = 2$, g is even and every vine curve with three nodes, having one component of genus $g/2-1$, is d-special. This completes the proof of the Proposition. □

A precise description of the locus of d-special vine curves is given in [M, Prop. 2.13]. Her result combined with the previous proposition yields a more precise description of the locus of stable curves whose compactified degree-d Jacobian is of Néron type, for every fixed d.

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Dipartimento di Matematica
Università, Roma Tre
Largo San Leonardo Murialdo 1
00146 Roma, Italy
caporaso@mat.uniroma3.it